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THE FUNDAMENTAL GROUP OF THE COMPLEMENT OF A RESULTANT HYPERSURFACE

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Dedicated to Professor Tatsuo Suwa on his sixtieth birthday

We prove that the complement of a generalized resultant hypersurface has an abelian fundamental group.

1. Introduction

Let X be a non-singular irreducible complex projective variety of dimension $n \geq 1$, and let $L_0, \dots, L_n$ be very ample line bundles on X. We denote by V_ν the vector space $H^0(X, L_\nu)$, and set

$$V := V_0 \times \cdots \times V_n.$$

For $f_\nu \in V_\nu$, we put

$$(f_\nu) := \{x \in X \mid f_\nu(x) = 0\}.$$

The resultant variety R of V is defined to be

$$\{f = (f_0, \dots, f_n) \in V \mid (f_0) \cap \cdots \cap (f_n) \neq \emptyset\}.$$

It is known that R is an irreducible hypersurface of V ([**GKZ**, Chapter 3, Proposition 3.1]). Therefore we will call R *the resultant hypersurface.*

When X is the n-dimensional projective space $\mathbb{P}^n$, the resultant hypersurface R is the classical resultant of $(n+1)$ forms in $(n+1)$-variables. See [**GKZ**] or [**CLO**] for other properties of the resultant hypersurfaces.

In this paper, we prove the following:

Theorem 1. *The fundamental group of $V \setminus R$ is an infinite cyclic group.*

In the case where $X = \mathbb{P}^1$, Theorem 1 follows from the result of [**C**], in which Choudary showed that the classical resultant hypersurface $R_{p,q}$ of polynomials of degree p and q has only normal crossings as its singularities in codimension 1, and proved the commutativity of $\pi_1(\mathbb{C}^{p+q} \setminus R_{p,q})$ by Zariski hyperplane section theorem [**Z**] and Fulton-Deligne's Theorem ([**D**], [**F**], [**FL**]) on Zariski conjecture.

The generalized resultant hypersurface R can have singularities in codimension 1 worse than normal crossings. For example, let $X \subset \mathbb{P}^2$ be a non-singular projective plane curve of degree $d \geq 3$, and let L_0 and L_1 be the line bundles corresponding to a hyperplane section of X in $\mathbb{P}^2$. Then a general fiber of the projection $R \to V_0$ consists of d hyperplanes in V_1 passing through a fixed linear subspace of codimension 2.

In fact, as the proof in the next section shows, the case where we cannot apply Fulton-Deligne's Theorem in a straightforward way (combined with Nori's lemma [**N**, Lemma 1.5 (C)] and Zariski hyperplane section theorem) is always reduced to this example.

The fundamental group of the complement to the *discriminant* hypersurface of a linear system $|L|$ on a non-singular complex projective variety X was studied by Dolgachev and Libgober in [**DL**]. We will explain the relation between the resultant hypersurface and the discriminant hypersurface in the case where $X = \mathbb{P}^n$ and $L = \mathcal{O}_X(d)$, where $n \geq 2$ and $d \geq 2$. We put $L_0 := L$ and $L_i := \mathcal{O}_X(d-1)$ $(i = 1, \dots, n)$. The discriminant hypersurface $D \subset |L_0|$ is the projectivization of the hypersurface

$$\widetilde{D} := \{f_0 \in V_0 \mid f_0 = 0 \text{ or } (f_0 \neq 0 \text{ and the divisor } (f_0) \text{ is singular})\}$$

in the vector space V_0 of homogeneous polynomials of degree d in $(n+1)$-variables. Let $(x_0 : x_1 : \cdots : x_n)$ be a homogeneous coordinate system of $X = \mathbb{P}^n$. We define a linear map φ from V_0 to V by

$$\varphi(f_0) := \left(f_0, \frac{\partial f_0}{\partial x_1}, \dots, \frac{\partial f_0}{\partial x_n}\right).$$

Then we have

$$\widetilde{D} = \varphi^{-1}(\varphi(V_0) \cap R);$$

that is, the discriminant hypersurface $\widetilde{D}$ is a linear section of the resultant hypersurface R. Note that, since the image $\varphi(V_0)$ of φ is *not* a general linear subspace of V, the non-commutativity of $\pi_1(|L_0| \setminus D)$ for many n and d (for example, see [**DL**, Section 4]) does not contradict to our theorem.

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2. Proof of Theorem 1.

First note that it is enough to prove that $\pi_1(V \setminus R)$ is abelian, because R is irreducible.

For ν with $0 \leq \nu \leq n$, we put

$$V'_\nu := V_0 \times \cdots \times V_\nu, \qquad V''_\nu := V_{\nu+1} \times \cdots \times V_n,$$

and denote by

$$\bar{p}_\nu : V \to V''_\nu$$

the natural projection. For a point g of V''_ν, we denote by $R_\nu(g)$ the intersection of R with the fiber $\bar{p}_\nu^{-1}(g)$, and consider $R_\nu(g)$ as a Zariski closed subset of V'_ν. When $\nu = n$, V''_n is the zero-dimensional vector space $\{0\}$, and we have $R_n(0) = R$. Let

$$p_\nu : V \setminus R \to V''_\nu$$

be the restriction of $\bar{p}_\nu$ to $V \setminus R$. Then we have

$$p_\nu^{-1}(g) = V'_\nu \setminus R_\nu(g).$$

Claim 2. If $g \in V''_\nu$ is general, the inclusion of $p_\nu^{-1}(g)$ into $V \setminus R$ induces a surjective homomorphism from $\pi_1(V'_\nu \setminus R_\nu(g))$ to $\pi_1(V \setminus R)$.

Proof of Claim 2. For $g = (g_{\nu+1}, \dots, g_n) \in V''_\nu$, let $W_\nu(g)$ denote the subscheme of X defined by

$$g_{\nu+1} = \cdots = g_n = 0,$$

which is of dimension ν if g is general in V''_ν. We consider the universal family

$$\begin{array}{ccc} \mathcal{W}_\nu & \xrightarrow{\psi_\nu} & X \\ {\scriptstyle \phi_\nu}\downarrow & & \\ V''_\nu & & \end{array}$$

of the subschemes $W_\nu(g)$, where

$$\mathcal{W}_\nu := \{(g, x) \in V''_\nu \times X \mid g_{\nu+1}(x) = \cdots = g_n(x) = 0\}.$$

The projection $\psi_\nu : \mathcal{W}_\nu \to X$ is smooth, and every fiber of ψ_ν is a linear subspace of V''_ν with codimension $n - \nu$. Hence $\mathcal{W}_\nu$ is non-singular, irreducible and of dimension equal to $\dim V''_\nu + \nu$. On the other hand, the projection $\phi_\nu : \mathcal{W}_\nu \to V''_\nu$ is surjective. Therefore there exists a Zariski closed subset Ξ of V''_ν with codimension ≥ 2 such that

$$\dim W_\nu(g) = \nu \quad \text{for all} \quad g \in V''_\nu \setminus \Xi.$$

If $g \in V''_\nu \setminus \Xi$, then $R_\nu(g)$ is a proper Zariski closed subset of V'_ν.

A general fiber of $p_\nu : V \setminus R \to V''_\nu$ is irreducible. If $g \in V''_\nu \setminus \Xi$, then $p_\nu^{-1}(g)$ has at least one point at which p_ν is smooth. Therefore Claim 2 follows from Nori's lemma [**N**, Lemma 1.5 (C)]. □

We choose and fix a general point

$$g = (g_1, \dots, g_n)$$

of V''_0. We put

$$\begin{aligned} d &:= c_1(L_1)c_1(L_2)\dots c_1(L_n), \\ d' &:= c_1(L_0)c_1(L_2)\dots c_1(L_n), \end{aligned}$$

where c_1 denote the first Chern class. Both of d and d' are positive integers. Then $W_0(g)$ consists of d distinct points $a_1, \dots, a_d$ of X, and $R_0(g)$ consists of d distinct hyperplanes $H_1, \dots, H_d$ of $V'_0 = V_0$, where

$$H_i := \{f_0 \in V_0 \mid f_0(a_i) = 0\}.$$

If $d \le 2$, then $\pi_1(V_0 \setminus R_0(g))$ is obviously abelian. Hence $\pi_1(V \setminus R)$ is abelian by Claim 2. Suppose that $\dim V_\nu = 2$ for some ν . Then we have $n = 1$, $X = \mathbb{P}^1$ and $\deg(L_\nu) = 1$. Interchanging L_ν and L_1, we will have $d = 1$, and can show the commutativity of $\pi_1(V \setminus R)$ by the above argument. From now on, we will assume

$$\dim V_\nu \ge 3 \quad \text{for} \quad \nu = 0, \dots, n.$$

Moreover, by interchanging L_0 and L_1 if necessary, we can assume

$$d' \le d.$$

By the above argument, we can also assume

$$3 \le d.$$

Suppose that $R_0(g)$ satisfies the following:

$$a_i \ne a_j \ne a_k \ne a_i \quad \Longrightarrow \quad \dim(H_i \cap H_j \cap H_k) = \dim V_0 - 3.$$

Let $A \subset V_0$ be a general affine plane. Then $A \cap R_0(g)$ is a nodal affine plane curve consisting of d lines, no pairs of which are parallel. Hence $\pi_1(A \setminus (A \cap R_0(g)))$ is abelian by Fulton-Deligne's Theorem ([**D**], [**F**], [**FL**]) on Zariski conjecture. By Zariski hyperplane section theorem [**Z**], the inclusion

$$A \setminus (A \cap R_0(g)) \hookrightarrow V_0 \setminus R_0(g)$$

induces an isomorphism on the fundamental groups. Hence $\pi_1(V_0 \setminus R_0(g))$ is also abelian, and thus the commutativity of $\pi_1(V \setminus R)$ follows from Claim 2.

Suppose, conversely, that there exist three distinct points a_i, a_j and a_k of $W_0(g)$ such that

$$\dim(H_i \cap H_j \cap H_k) = \dim V_0 - 2. \tag{2.1}$$

Let U be a Zariski open dense subset of V_0'' containing the point g such that the projection $\phi_0 : \mathcal{W}_0 \to V_0''$ is étale over U. We have the monodromy action

$$\mu : \pi_1(U, g) \to \mathfrak{S}(W_0(g))$$

of $\pi_1(U, g)$ on the finite set $W_0(g)$, where $\mathfrak{S}(W_0(g))$ is the full symmetric group of $W_0(g)$. Since the action μ is doubly transitive, and the image of μ contains a transposition, we see that μ is surjective ([**H**, Uniform Position Lemma]). Since g is general in V_0'', we can conclude that (2.1) holds for any choice of distinct three points a_i, a_j, a_k of $W_0(g)$. This means that, if a divisor $D \in |L_0|$ of X contains distinct two points of $W_0(g)$, then D contains every point of $W_0(g)$.

When $n = 1$, we put $h := 0 \in V_1'' = \{0\}$ and $C := X$. In this case, we have $p_1^{-1}(h) = V_1' \setminus R_1(h) = V \setminus R$. When $n > 1$, we put

$$h := (g_2, \dots, g_n),$$

which is a general point of V_1'', and put

$$C := W_1(h).$$

We show that

$$\pi_1(p_1^{-1}(h)) = \pi_1(V_1' \setminus R_1(h))$$

is abelian. The proof of Theorem 1 will then be completed by Claim 2.

First we will show that C is a projective plane curve. The curve C is non-singular and irreducible. The line bundles $L_0|_C$ and $L_1|_C$ on C are very ample of degree d' and d, respectively. Since the restriction $g_1|_C$ of g_1 to C is a general element of $H^0(C, L_1|_C)$, and $d' \le d$ has been assumed, we see from the above consideration that the following holds:

> Let D_1 be a general divisor in the complete linear system $|L_1|_C|$ on C. If a divisor D_0 in the complete linear system $|L_0|_C|$ has at least two common points with D_1, then $D_0 = D_1$ holds.

In particular, we have $d = d'$ and $|L_1|_C| = |L_0|_C|$. We will denote by P the dual projective space of the complete linear system $|L_1|_C| = |L_0|_C|$, and let

$$\Psi \,:\, C \,\to\, P$$

be the embedding of C by $|L_1|_C| = |L_0|_C|$. Let H be a general hyperplane of P. If b_1 and b_2 are points of $\Psi^{-1}(H)$, then H is the only hyperplane containing $\Psi(b_1)$ and $\Psi(b_2)$. Therefore we have

$$\dim P = 2,$$

and C can be regarded as a non-singular projective plane curve on P via Ψ. The complete linear system $|L_1|_C| = |L_0|_C|$ is the linear system of intersections with lines in P.

We put

$$V_C := H^0(P, \mathcal{O}_P(1)).$$

For $\lambda \in V_C$, let (λ) denote the linear subspace of P defined by $\lambda = 0$. We denote by S the hypersurface

$$\{(\lambda_0, \lambda_1) \in V_C \times V_C \;\mid\; (\lambda_0) \cap (\lambda_1) \cap C \neq \emptyset\}$$

of $V_C \times V_C$, and put

$$(V_C \times V_C)^\circ := (V_C \times V_C) \setminus S.$$

The restriction map

$$(f_0, f_1) \;\mapsto\; (f_0|_C, f_1|_C)$$

gives a morphism

$$p_1^{-1}(h) = V_1' \setminus R_1(h) \;\to\; (V_C \times V_C)^\circ,$$

which is locally trivial with fibers isomorphic to a vector space. Hence $\pi_1(p_1^{-1}(h))$ is isomorphic to $\pi_1((V_C \times V_C)^\circ)$. Therefore it is enough to show the following:

Claim 3. The fundamental group of $(V_C \times V_C)^\circ$ is abelian.

Proof of Claim 3. We denote by

$$\rho : (V_C \times V_C)^\circ \to P \setminus C$$

the morphism given by

$$\rho(\lambda_0, \lambda_1) := \text{the intersection point of the lines } (\lambda_0) \text{ and } (\lambda_1).$$

Then ρ is locally trivial, and its fiber is isomorphic to $\mathrm{GL}(2, \mathbb{C})$. We choose a general line $L_\infty \subset P$, and fix affine coordinates (x, y) on $P \setminus L_\infty$. Then ρ has a section

$$\sigma : P \setminus (C \cup L_\infty) \to (V_C \times V_C)^\circ \setminus \rho^{-1}(L_\infty)$$

over the affine part $P \setminus (C \cup L_\infty)$ of $P \setminus C$ defined by

$$\sigma(a, b) := (x - a, y - b),$$

where $x - a$ and $y - b$ are considered as linear forms on P. In particular, the fundamental group of $(V_C \times V_C)^\circ \setminus \rho^{-1}(L_\infty)$ is the semi-direct product

$$\pi_1(\mathrm{GL}(2, \mathbb{C})) \rtimes \pi_1(P \setminus (C \cup L_\infty))$$

constructed from the monodromy action of $\pi_1(P \setminus (C \cup L_\infty))$ on $\pi_1(\mathrm{GL}(2, \mathbb{C}))$ associated with the section σ. Since $\pi_1(\mathrm{GL}(2, \mathbb{C})) \cong \mathbb{Z}$ has a canonical positive generator, this monodromy action is trivial. Hence we have

$$\pi_1((V_C \times V_C)^\circ \setminus \rho^{-1}(L_\infty)) \cong \pi_1(\mathrm{GL}(2, \mathbb{C})) \times \pi_1(P \setminus (C \cup L_\infty)).$$

Since $C \cup L_\infty$ is a nodal curve, $\pi_1(P \setminus (C \cup L_\infty))$ is abelian. Therefore

$$\pi_1((V_C \times V_C)^\circ \setminus \rho^{-1}(L_\infty))$$

is also abelian. Since the inclusion of $(V_C \times V_C)^\circ \setminus \rho^{-1}(L_\infty)$ into $(V_C \times V_C)^\circ$ induces a surjective homomorphism on the fundamental groups, we get the commutativity of $\pi_1((V_C \times V_C)^\circ)$. □

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