Pacific Journal of Mathematics

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Volume 211 No. 1 September 2003

PACIFIC JOURNAL OF MATHEMATICS
Vol. 211, No. 1, 2003

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We consider counterexamples in the field of random iteration to two well-known theorems of classical complex dynamics - Sullivan's non-wandering theorem and the classification of periodic Fatou components. Random iteration which was first introduced by Fornaess and Sibony (1991) is a generalization of standard complex dynamics where instead of considering iterates of a fixed rational function, one allows the mappings to vary at each stage of the iterative process. In this setting one can produce oscillatory behaviour of a type forbidden in classical rational iteration. The technique of the proof requires us to extend the classical notion of conjugacy between dynamical systems to random iteration and we prove some basic results concerning conjugacy in this setting.

1. Introduction.

We consider a sequence of rational functions $\{R_n\}_{n=1}^{\infty} = \{R_1, R_2, R_3, \dots\}$ of some fixed degree $d \geq 2$. Let $Q_n(z)$ be the composition of the first n of these functions in the natural order, i.e.,

$$Q_n = R_n \circ R_{n-1} \circ \cdots\cdots \circ R_2 \circ R_1.$$

We will also be interested in the compositions

$$Q_{m,n} = R_n \circ R_{n-1} \circ \cdots\cdots \circ R_{m+2} \circ R_{m+1}.$$

We now define the *Fatou set* $\mathcal{F}$ for such a sequence of rational functions as

$$\mathcal{F} = \{z \in \overline{\mathbb{C}} : \{Q_n\}_{n=1}^{\infty} \text{ is a normal family on some neighbourhood of } z\}$$

and the *Julia set* $\mathcal{J}$ is then simply the complement of the Fatou set in $\overline{\mathbb{C}}$. These definitions were first introduced by Fornaess and Sibony in [**9**]. Note that, if $\{R_n\}_{n=1}^{\infty}$ is a constant sequence, then these definitions coincide with the standard ones. One important consequence of this definition of Julia and Fatou sets is that we can formulate an analogue of the principle of complete invariance in standard rational iteration. In order to do this, we need to introduce the following terminology:

Let $\{R_n\}_{n=1}^{\infty}$ be as above and for any fixed $n \geq 0$, let us define the *n-th iterated Julia set* $\mathcal{J}_n$ and *n-th iterated Fatou set* $\mathcal{F}_n$ to be the Julia and

Fatou sets for the sequence $\{R_{n+1}, R_{n+2}, R_{n+3}, \dots\}$ which we obtain from our original sequence simply by deleting the first n members. Note that with these definitions, $\mathcal{J}_0 = \mathcal{J}$ and $\mathcal{F}_0 = \mathcal{F}$. We then have the following:

Theorem 1.1. *For any $m < n \in \mathbb{Z}_+$, $Q_{m,n}(\mathcal{J}_m) = \mathcal{J}_n$ and $Q_{m,n}(\mathcal{F}_m) = \mathcal{F}_n$, with Fatou components of $\mathcal{F}_m$ being mapped surjectively onto those of $\mathcal{F}_n$.*

The proof is a straightforward adaptation of the standard classical proof.

The notation introduced above can also be extended to sets and points. For a set U which we introduce at stage m and for $n > m$, we set $U_n = Q_{m,n}(U)$ and for a point x which is introduced at stage m, we set $x_n = Q_{m,n}(x)$.

It turns out that the above situation using rational functions is somewhat too general for proving significant results. In fact, even if one restricts oneself to sequences of polynomials, one obtains pathologies which show that this situation is still too far from traditional complex dynamics to develop a meaningful theory [4, 6]. The most natural restriction one can probably make was introduced by Fornaess and Sibony [9] who considered sequences of monic polynomials with uniformly bounded coefficients, i.e., sequences of the form

$$R_n(z) = P_n(z) = z^d + a_{d-1,n}z^{d-1} + \cdots\cdots + a_{1,n}z + a_{0,n}$$

and where we can find some $M \geq 0$ such that $|a_{i,n}| \leq M$ for $0 \leq i \leq d-1$ and all $n \geq 1$. From now on, we shall call such sequences *bounded monic polynomial sequences*.

We will also be interested in the more general setting where we consider sequences of polynomials which need no longer be monic but where we still retain some degree of control over the leading coefficients. More specifically we will consider sequences of the form

$$P_n(z) = a_{d,n}z^d + a_{d-1,n}z^{d-1} + \cdots\cdots + a_{1,n}z + a_{0,n}$$

where as before we can find some $M \geq 0$ such that $|a_{i,n}| \leq M$ for $0 \leq i \leq d-1$ and all $n \geq 1$ and we can also find $K \geq 1$ such that $1/K \leq |a_{d,n}| \leq K$ for all $n \geq 1$. From now on, we shall call such sequences *bounded polynomial sequences*. This definition clearly contains the previous one as a special case and one of the advantages of both definitions is that we can find some radius R depending only on the coefficient bounds M, K above so that for any sequence $\{P_n\}_{n=1}^{\infty}$ as above, it is easy to see that

$$|Q_n(z)| \to \infty \qquad \text{as} \quad n \to \infty, \quad |z| > R$$

which shows in particular that as for classical polynomial Julia sets, there will be a basin at infinity $\mathcal{A}_\infty$ on which all points escape to infinity under iteration. Such a radius will be called an *escape radius* for the coefficient bound M and one can employ the maximum principle to show that $\mathcal{A}_\infty$ is completely invariant just as in the classical case. The complement of the basin at infinity $\mathcal{A}_\infty$ is called the *filled Julia set* $\mathcal{K}$. As in the classical case, it is simply the set of points whose orbits remain bounded under iteration and also as in the classical case, it follows by Montel's theorem that $\partial\mathcal{K} = \mathcal{J}$. Clearly $\mathcal{K} \subset D(0, R)$ where R is an escape radius for $\{P_n\}_{n=1}^\infty$ and $D(0, R)$ denotes the closed disk of radius R about 0.

Using monic sequences, one can construct the Green's function with pole at infinity for the outer domain $\mathcal{A}_\infty$ by analogy with the classical result as was done by Fornaess and Sibony [**9**]. The Julia set is perfect, regular, and has logarithmic capacity one. The proofs still work for general bounded sequences, only one no longer has such a nice formula for the capacity. However, bounded sequences allow one greater freedom in choosing polynomials for special counterexamples as we shall see later. We now turn to stating the main result of this paper. Classically, we have the following two well-known theorems from complex dynamics:

Theorem 1.2 (Classification of Periodic Fatou Components)**.** *Let R be a rational function and let U be a* (*classical*) *Fatou component for R which is periodic, i.e., $R^{\circ n}(U) = U$ for some $n \geq 1$. Then we have one of the following four possibilities for U:*

1. *U contains a point of an attracting or superattracting cycle.*
2. *U is a basin of a parabolic periodic point lying on ∂U.*
3. *U is a Siegel disk.*
4. *U is a Herman ring.*

Theorem 1.3 (Sullivan)**.** *Let R be a rational function and let U be a* (*classical*) *Fatou component for R. Then U is eventually periodic.*

The first of these results immediately implies that for a periodic Fatou component, all normal limit functions on that component are either constant or univalent. If we combine this with Sullivan's non-wandering theorem, we see that for any given Fatou component the normal limit functions will be either all constant or all nonconstant. A second simple consequence of the non-wandering theorem is that for a given Fatou component, the diameters of the iterates of that component must eventually stabilize in view of the eventual periodicity. For random iteration, however, neither of these results are true as the following theorem shows:

Theorem 1.4. *There exists a bounded monic sequence of polynomials for which there is a Fatou component V with the following properties:*

1. *There are both constant and nonconstant normal limit functions on V.*
2. $\limsup_{n\to\infty} \operatorname{diam}(V_n) > 0$ *but* $\liminf_{n\to\infty} \operatorname{diam}(V_n) = 0$.

This result therefore both gives a direct counterexample to an immediate consequence of Sullivan's theorem and to a simple consequence of Sullivan's theorem and the classification of periodic Fatou components. In addition, the same argument yields a sequence with a Fatou component which satisfies both 1. and 2. For a further counterexample to a simple classical consequence of these results together with the Fatou-Shishikura-Epstein bound on the number of non-repelling cycles, see [**7**]. Finally, we remark that for the situation of iteration with an entire function, examples of wandering domains exist and even examples of wandering domains on which all the iterates are univalent [**8**].

In order to be able to say that there is a bounded monic polynomial sequence with the properties we need, we will need to develop and make precise the notion of conjugacy between random sequences of polynomials and the next section of this paper is devoted to this aim. We will first prove the result for a bounded non-monic sequence of polynomials and then argue by conjugacy that there is a monic sequence with the desired properties.

2. Analytic conjugacy.

We start by recalling the classical definition of analytic conjugacy between polynomials. Two polynomials P^1 and P^2 are conjugate on $\overline{\mathbb{C}}$ if there exists an affine linear mapping $\varphi(z) = \alpha z + \beta$ such that $\varphi \circ P^1 \circ \varphi^{-1} = P^2$.

Our version of this for random iteration is as follows: We start by considering two sequences $\{P_n\}_{n=1}^{\infty}$, $\{\widetilde{P}_n\}_{n=1}^{\infty}$ of polynomials of degree ≥ 2 acting on $\overline{\mathbb{C}}$. We say that two such sequences are conjugate if there exists a sequence of affine linear mappings $\varphi_n(z) = \alpha_n z + \beta_n$ such that $\varphi_n \circ P_n \circ \varphi_{n-1}^{-1} = \widetilde{P}_n$ for every $n \geq 1$. In order to be able to make meaningful comparisons between different sequences, we need the mappings and their inverses to form an equicontinuous family, i.e., there must exist $K \geq 1$, $M > 0$ such that $1/K \leq |\alpha_n| \leq K$, $|\beta_n| \leq M$ for all $n \geq 0$. We note that this definition has appeared earlier in the literature, for example in the paper of Kolyada and Snoha [**10**] where they make use of it in dealing with topological entropy for sequences of mappings of a compact topological space. The following result is immediate from the definitions:

Proposition 2.1. *Let $\{P_n\}_{n=1}^{\infty}$ and $\{\widetilde{P}_n\}_{n=1}^{\infty}$ be two bounded polynomial sequences which are analytically conjugate in the random sense above. Then for any $n \geq 0$, x is in the nth iterated Fatou set $\mathcal{F}_n$ for $\{P_n\}_{n=1}^{\infty}$ if and only if $\varphi_n(x)$ is in the nth iterated Fatou set $\widetilde{\mathcal{F}}_n$ for $\{\widetilde{P}_n\}_{n=1}^{\infty}$. Also U is a Fatou component of $\mathcal{F}_n$ if and only if $\varphi_n(U)$ is a Fatou component of $\widetilde{\mathcal{F}}_n$.*

What is the relationship between classical and random conjugacy in this situation? Clearly any classical conjugacy also gives a conjugacy in the random sense but is the converse true? The answer to this question is not in general. However, in the generic case, we will see that not only are two polynomials which are conjugate in the random sense classically conjugate but also that every random conjugacy between them must in fact also be a classical conjugacy.

Let P^1 and P^2 be two polynomials of the same degree $d \geq 2$ which are conjugate in the random sense via a sequence of affine linear mappings $\varphi_n = \alpha_n z + \beta_n$, $n \geq 0$ and where the constants α_n and β_n are bounded as in the definition. By considering the fact that these mappings must map the barycentre of the set of critical points for P^1 to that of the critical points of P^2, and by classically conjugating with suitable translations, one finds that we may take $\beta_n = 0$ for every $n \geq 0$.

So let us assume from now on that $\varphi_n = \alpha_n z$ for every $n \geq 0$. Let us now turn our attention to the zeros of P^1 and P^2. Each mapping $\alpha_n z$ must map the zeros of P^1 onto those of P^2 and its inverse must map those of P^2 onto those of P^1. Clearly, we may assume that after conjugating P^2 with a rotation and a dilation if necessary, that the zeros of P^1 and those of P^2 do in fact coincide. Let us for now assume that there are zeros of P^1 and P^2 which do not lie at 0. It then follows that we can find $1 \leq q \leq d$ and $r \geq 0$, $s \geq 1$ with $qs + r = d$ together with nonzero constants $a_1, a_2, \ldots, a_s$ and c_1, c_2 so that

$$P^i(z) = c_i z^r \prod_{i=1}^{s} (z^q - a_i), \qquad i = 1, 2.$$

The number q is the degree of rotational symmetry of this set of zeros about 0 and if there is no nontrivial symmetry of this kind, this is equivalent to having $q = 1$.

It follows easily from this that we must have $\alpha_n = e^{2\pi i p_n/q}$ where p_n is a sequence of integers all of which can be taken from among the finite set $\{0, 1, 2, \ldots, q-1\}$. From this we see that $c_2/c_1 = e^{2\pi i p/q}$ for some fixed $0 \leq p < q$. The equation relating p_n and p_{n+1} is therefore

$$p_{n+1} = p + r p_n \qquad \text{in} \quad \mathbb{Z}_q$$

and each initial choice of p_0 gives a potentially different random conjugacy between P^1 and P^2. One observes at this point that the above equation shows that the mappings in any conjugacy are in fact determined by a discrete classical dynamical system on $\mathbb{Z}_q$ given by the mapping $f(i) = p + ri$, $i \in \mathbb{Z}_q$. Hence, finding out what types of conjugacies are permitted between P^1 and P^2 is the same as knowing all possible types of orbit for every element of $\mathbb{Z}_q$ under this dynamical system.

Conjugacies which correspond to fixed points of f will be referred to as *fixed conjugacies* and clearly coincide with classical conjugacies. Periodic orbits of points in $\mathbb{Z}_q$ give rise to conjugacies which we shall refer to as *periodic conjugacies*. On the other hand, since $\mathbb{Z}_q$ is a finite set, every orbit is preperiodic and so we can talk of *preperiodic* and *strictly preperiodic conjugacies*. With this terminology in hand one can deduce the following:

1. P^1 and P^2 are classically conjugate if and only if $r-1$ divides p in $\mathbb{Z}_q$ and all conjugacies between P^1 and P^2 are classical if and only if $p = 0$ and $r = 1$, both equations being over $\mathbb{Z}_q$.
2. There are periodic conjugacies of period $k \geq 1$ if and only if there exist solutions of the equation

$$(r^k - 1)a + (r^{k-1} + r^{k-2} + \cdots + r + 1)p = 0$$

which are not solutions of the same equation for some smaller value of k (including $k = 1$).
3. There are strictly preperiodic conjugacies if and only if $q > 1$ and r divides 0 in $\mathbb{Z}_q$.

This finishes the case when there are nonzero roots of P^1 and P^2. We now consider the case when all the zeros of P^1 and P^2 are at 0. In this case, by classically conjugating both polynomials if needed, we may assume that $P^1(z) = P^2(z) = z^d$. Here P^1 and P^2 are obviously classically conjugate but there are other types of conjugacy as well. For example, the sequence of mappings $\{\omega z, \overline{\omega} z, \omega z, \overline{\omega} z, \ldots\}$ where $\omega = e^{2\pi i/(d+1)}$ provides a conjugacy of period 2 while the sequence $\{\omega^{1/d} z, \omega z, \overline{\omega} z, \omega z, \overline{\omega} z, \ldots\}$ provides a strictly preperiodic conjugacy provided we choose a branch of $z^{1/d}$ other than that which gives us $\overline{\omega}$ as a dth root of ω. Finally, if a is any irrational number and we let $\alpha = e^{2\pi i a}$, then the sequence $\{\alpha z, \alpha^d z, \alpha^{d^2} z, \alpha^{d^3} z, \ldots\}$ gives a conjugacy where the sequence of mappings clearly never repeats. Following our scheme above, we shall call this kind of conjugacy an *aperiodic conjugacy*.

In view of what we have proved above, we see that the notion of conjugacy for random iteration will be strong enough for us to make many meaningful statements concerning similarities in the behaviour of different random dynamical systems. Our main result concerning conjugacy which we will make use of in constructing the counterexample we will outline in the next section is stated below.

Theorem 2.1. *Let $\{P_m\}_{m=1}^{\infty}$ be a bounded sequence of polynomials defined on all of $\overline{\mathbb{C}}$. Then $\{P_m\}_{m=1}^{\infty}$ is analytically conjugate to a bounded monic sequence of polynomials $\{\widetilde{P}_m\}_{m=1}^{\infty}$. Moreover, we may require that each $\widetilde{P}_m$ have a critical point at* 0 *for each m.*

Proof. Let K and M be the bounds respectively for the leading and the other coefficients of our sequence $\{P_m\}_{m=1}^{\infty}$ as given in the definition. The argument will proceed by constructing a 'partial conjugacy' $\varphi_0^n, \varphi_1^n, \ldots, \varphi_n^n$ which satisfies $\varphi_m^n \circ P_m \circ (\varphi_{m-1}^n)^{-1} = \widetilde{P}_m^n$ for each $1 \leq m \leq n$ where $\widetilde{P}_m^n$ is monic and has a critical point at 0. This step can be thought of as a version of the 'pullback argument' which is a standard approach to constructing conjugacies in complex dynamics. We will then let $n \to \infty$ and apply a limiting argument to obtain the full result.

We begin by fixing $n \geq 0$, and setting $\varphi_n^n(z) = z$. Our induction assumption concerning the constants α_m^n is that $K^{-1/(d-1)} \leq |\alpha_m^n| \leq K^{1/(d-1)}$ for each $1 \leq m \leq n$ (here d is simply the degree of each of the polynomials in our sequence $\{P_m\}_{m=1}^{\infty}$). If $n = 0$, this condition is trivially satisfied and we have now of course finished constructing the partial conjugacy, otherwise we can assume from now on that $n \geq 1$. One can easily compute that in order for the leading coefficient of each of the polynomials $\widetilde{P}_m = \varphi_m^n \circ P_m \circ (\varphi_{m-1}^n)^{-1}$ to be 1, we must have that

$$|\alpha_{n-i}^n| = |a_{d,n}|^{1/d^i} |a_{d,n-1}|^{1/d^{i-1}} \cdots\cdots |a_{d,n-i+1}|^{1/d}, \qquad 0 \leq i < n$$

from which it follows immediately that $K^{\frac{-1}{d-1}} \leq |\alpha_{n-i}^n| \leq K^{\frac{1}{d-1}}$.

Turning now to the constant coefficients, it is again fairly easy to calculate that the requirement that the linear term of $\widetilde{P}_m$ be zero (which guarantees that there will be a critical point at 0) is equivalent to the condition that β_{n-i}^n, $1 \leq i \leq n$ satisfy the polynomial equation

$$\frac{da_{d,n-i+1}(-\beta_{n-i}^n)^{n-1}}{\alpha_{n-i}^n} + \frac{(d-1)a_{d-1,n-i+1}(-\beta_{n-i}^n)^{n-2}}{\alpha_{n-i}^{n-1}} + \cdots\cdots + \frac{a_{1,n-i+1}}{\alpha_{n-i}} = 0.$$

If follows from the bounds on our sequence $\{P_n\}_{n=1}^{\infty}$ together with the bounds we have just established for the linear terms α_m^n, that if we use Rouché's theorem to compare the above polynomial with $\frac{(-1)^{n-1}da_{d,n-i+1}}{\alpha_{n-i}^n} z^{n-1}$, then we can deduce that the constants β_m^n can be bounded uniformly in terms of the bounds K and M for $\{P_n\}_{n=1}^{\infty}$.

We therefore obtain a sequence of sequences of affine linear mappings $\{\varphi_m^n = \alpha_m^n z + \beta_m^n\}_{m=0}^{\infty}$ where the constants α_m^n and β_m^n are uniformly bounded. If we fix $m \geq 0$, we can find a subsequence n_k so that $\alpha_m^{n_k}$ and $\beta_m^{n_k}$ will converge. It then follows from the standard Cantor diagonalization procedure that we can find a subsequence n_k so that for every fixed m, $\alpha_m^{n_k}$ and $\beta_m^{n_k}$ will converge to some limits α_m and β_m respectively. If we now let $\varphi_m = \alpha_m z + \beta_m$, it follows that we can find a monic sequence of polynomials $\{\widetilde{P}_m\}_{m=1}^{\infty}$ each member of which has a critical point at 0 such that the sequence of functions $\{\varphi_m\}_{m=0}^{\infty}$ will provide the desired conjugacy between $\{P_m\}_{m=1}^{\infty}$ and $\{\widetilde{P}_m\}_{m=1}^{\infty}$ with the necessary properties.

From Theorem 2.1 we obtain the following corollary:

Corollary 2.1. *Let $\{P_n\}_{n=1}^{\infty}$ be a bounded sequence of quadratic polynomials. Then $\{P_n\}_{n=1}^{\infty}$ is conjugate to a sequence $\{\widetilde{P}_n\}_{n=1}^{\infty}$ of monic quadratic polynomials of the form $\widetilde{P}_n(z) = z^2 + c_n$ where $\{c_n\}_{n=1}^{\infty}$ is a sequence in l_∞. Also, for any conjugacy between two sequences of this type, the conjugating maps $\varphi_n = \alpha_n z + \beta_n$ must satisfy $|\alpha_n| = 1$, $\beta_n = 0$ for every n.*

Proof. The existence of the desired conjugacy follows immediately from Theorem 2.1.

For the uniqueness part, note that the linear coefficients α_n satisfy

$$\alpha_{n-1}^2 = \alpha_n, \qquad n \in \mathbb{N}$$

from which the result on the linear terms follows. For the constant terms, the fact that $\beta_n = 0$ for every n follows from the fact that each quadratic polynomial for the two sequences has a unique critical point at 0 together with the fact that the conjugacy will map critical points for the polynomials of one sequence to those of the other sequence.

This of course is the random analogue of the classical fact that any quadratic polynomial is conjugate to a polynomial of the form z^2+c and that no two distinct polynomials of this form can be conjugate to each other. There is therefore little loss of generality when working with quadratic polynomials, in restricting oneself to sequences of the form $P_n(z) = z^2 + c_n$ with $\{c_n\}_{n=1}^{\infty}$ a bounded sequence. For more on sequences of this type see [**1**, **2**, **3**] and [**6**].

Finally, we remark that using our earlier arguments, one may show that any two quadratic polynomials are conjugate in the random sense if and only if they are classically conjugate and if P^1 and P^2 are not classically conjugate to z^2, the mappings of any random conjugacy will all be a classical conjugacy except possibly at stage 0.

3. Proof of the main result.

We start by noting that it will suffice to construct a bounded polynomial sequence which has a Fatou component with the desired properties. It will then follow immediately from Theorem 2.1 and Proposition 2.1 that there is a monic polynomial sequence with a suitable Fatou component. The construction of the sequence which has the properties we require for Theorem 1.4 hinges mainly on the dynamics of the polynomial $P(z) = \lambda z(1-z)$ where $\lambda = e^{2\pi i\varphi}$ and $\varphi = (\sqrt{5}-1)/2$, the golden ratio. As is well-known, this polynomial gives rise to the famous golden mean Siegel disk. The origin is a neutral fixed point for P with multiplier λ and it lies in an invariant Fatou component U on which the dynamics are (classically) conjugate to a rotation by $2\pi\varphi$ which is of course an irrational multiple of 2π. It follows immediately from this we can find high iterates of P which come arbitrarily close to the identity on U (in the sense of uniform convergence on compact subsets of U). Our construction will rely heavily on this fact which we will use to control the distortion of a disk in U under the iterates of our sequence. The idea is to put the disk through ever longer cycles of shrinking and expanding where we first shrink it down to some arbitrarily small size and then expand it again to be approximately the same size and shape as it was originally. Our construction is by induction and we proceed to outline it below.

Induction Step: Stage 1. Let $D = D(0, r)$ be a closed disk of radius r and centre 0 for which $D(0, 2r)$ lies in U and let $\mu = r/2$. Note that this forces r to be less than 1/4 as it is well-known that the critical point for P at 1/2 lies on ∂U. We will construct our sequence of polynomials so that the interior of this disk will lie in a Fatou component for our sequence with the desired properties. Note that if we dilate D by a factor of $1/\mu = 2/r$, we obtain the disk $D(0, 2)$ whose boundary $C(0, 2)$, the circle centered at 0 of radius 2 lies entirely in the basin at infinity for P. On the other hand, if we dilate the disk $D(0, r^2/2)$ by the same factor, we simply obtain D which lies within U. The three polynomials we will use to construct our sequence are $P(z)$, $P(\mu z) = \lambda\mu z(1 - \mu z)$ and $P(z/\mu) = (\lambda/\mu)z(1 - z/\mu)$, the last two being P precomposed with dilations of μ and $1/\mu$ respectively. All sequences formed from these three polynomials are clearly bounded polynomial sequences in the sense of the definition given in Section 1, and we may therefore find an escape radius R so that if $|z| > R$, $Q_n(z) \to \infty$ as $n \to \infty$ where Q_n is any arbitrary composition of polynomials chosen from among these three. We will make use of these observations later when it comes to proving the second part of Theorem 1.4.

Our first induction step consists of first shrinking D by a factor of μ under the dilation μz, applying P $m_{1,1}$ times where $m_{1,1}$ is a natural number to be determined, then applying a dilation by a factor of $1/\mu$ and finally again applying P $m_{1,2}$ times where $m_{1,2}$ is also to be determined. We now define the first $m_{1,1} + m_{1,2}$ members of our sequence of polynomials by setting $P_1(z) = P(\mu z)$, $P_n(z) = P(z)$ for $1 < n \leq m_{1,1}$, $P_{m_{1,1}+1}(z) = P(z/\mu)$ and finally $P_n(z) = P(z)$ for $m_{1,1} + 1 < n \leq m_{1,1} + m_{1,2}$. Clearly it follows that $Q_{m_{1,1}}(z) = P^{\circ m_{1,2}} \circ z/\mu \circ P^{\circ m_{1,1}}(z) \circ \mu z$ and so this polynomial has the same effect on D as the process we outlined above.

The dilations clearly do not introduce any distortion on the image of D while by choosing $m_{1,1}$ and $m_{1,2}$ large enough, we may assume that $P^{\circ m_{1,1}}$ and $P^{\circ m_{1,2}}$ are as close to the identity on $D(0, 2r)$ as we wish. It follows from these two facts that we may make the image of D under $Q_{m_{1,1}}$ (i.e., $D_{m_{1,1}}$) as close to $D(0, r^2/2)$ and the image under $Q_{m_{1,1}+m_{1,2}}$ as close to D as we wish. To be more precise, we specify that if $\partial D = C(0, r)$, then we require that the Hausdorff distance between $C(0, r^2/2)$ and $Q_{m_{1,1}}(C(0, r))$ is $\leq r^2/8$ and that between $C(0, r)$ and $Q_{m_{1,1}+m_{1,2}}(C(0, r))$ is $\leq r(1/4 + 1/8) = 3r/8$. In addition, if we consider the circle $C(0, r)$ at time $m_{1,1}$ which certainly contains $Q_{m_{1,1}}(C(0, r))$, the image of this circle under the dilation z/μ is $C(0, 2)$ and so by making $m_{1,2}$ as large as we wish, we may ensure that the iterate of the circle under $Q_{m_{1,1}, m_{1,1}+m_{1,2}}$ lies entirely outside $D(0, R)$ and so is guaranteed to escape to infinity regardless of how we choose from among our three polynomials for the construction of the rest of the sequence. Finally, we let the number of polynomials in our sequence specified so far be N_1 so that in this case we simply have that $N_1 = m_{1,1} + m_{1,2}$.

Induction Hypothesis: Stage n. We suppose now that we have constructed the first N_n members of our sequence of polynomials. Each of these has again been chosen from among the three polynomials $P(z), P(\mu z)$ and $P(z/\mu)$. The origin has therefore remained fixed under our sequence so far and the effect of the compositions of our polynomials has been that of iterates of P interspersed with dilations by either μ or $1/\mu$. Our first assumption concerning these is that for each $1 \le i \le n$, Q_{N_{i-1},N_i} consists of a composition of $N_i - N_{i-1}$ polynomials (where in the case $i = 1$ we set $N_0 = 1$) which can be expressed as i dilations by μ each of which is followed by some iterate of P followed by i dilations by $1/\mu$ each of which is again followed by an iterate of P. Let us denote the number of these iterates of P by $m_{i,1}, m_{i,2}, \ldots, m_{i,i}, \ldots, m_{i,2i}$ and for $1 \le j \le 2i$, let us denote the sum of the first j of these numbers by $M_{i,j}$, i.e., we set $M_{i,j} = \sum_{k=1}^{j} m_{i,k}$. Our second assumption concerning the polynomials chosen so far is that all iterates of P chosen so far are close enough to the identity so that the Hausdorff distance between $Q_{N_{i-1}+M_{i,i}}(C(0,r))$ and $C(0,\mu^i r)$ is $\le \frac{\mu^i r}{4}\sum_{j=0}^{2i-1} 2^{-j} = \mu^{i+1}\sum_{j=1}^{2i} 2^{-i}$ and that between $Q_{N_i}(C(0,r))$ and $C(0,r)$ is $\le \frac{r}{4}\sum_{j=0}^{2i} 2^{-j}$. Finally, we make the assumption that at stage $N_{i-1} + M_{i,i}$, all points on the circle $C(0,\mu^{i-1}r)$ (which encloses a disk which contains $D_{N_{i-1}+M_{i,i}}$ in view of our assumptions above) are guaranteed to escape to infinity by ensuring that the image of this circle at time N_n lies entirely outside $D(0,R)$ where R is the escape radius from above.

Induction Step: Stage n+1. The $(n+1)$st step of the induction consists of defining the next members of our sequence to be $n+1$ steps each of which is a dilation by μ followed by a suitable iterate of P followed by a further $n+1$ steps each of which is a dilation by $1/\mu$ each of which is again followed by a suitable iterate of P. To be more precise, let $m_{n+1,1}, m_{n+1,2}, \ldots, m_{n+1,2n+2}$ be natural numbers to be determined and let $M_{n+1,i} = \sum_{j=1}^{i} m_{n+1,j}$ for each $1 \le i \le 2n+2$. Now let $N_{n+1} = N_n + M_{n+1,2n+2}$ and for $N_n + 1 \le n \le N_{n+1}$, define P_n by

$$P_n(z) = \begin{cases} P(\mu z) & n = N_n + 1 \\ P(\mu z) & n = N_n + M_{n+1,i} + 1, \quad 1 \le i \le n \\ P(z/\mu) & n = N_n + M_{n+1,i}, \quad n+1 \le i \le 2n+1 \\ P(z) & \text{otherwise.} \end{cases}$$

We can clearly choose the integers $m_{n+1,i}$ so that the corresponding iterates $P^{\circ m_{n+1,i}}$ are as close to the identity on $D(0,2r)$ as we wish. It then follows easily from the induction hypothesis and our initial assumption that $D(0,2r) \subset U$ that we can ensure that the Hausdorff distance between $Q_{N_n+M_{n+1,n+1}}(C(0,r))$ and $C(0,\mu^{n+1}r)$ is $\leq \mu^{n+2}\sum_{i=1}^{2n+2} 2^{-i}$ while that between $Q_{N_{n+1}}(C(0,r))$ and $C(0,r)$ is $\leq \frac{r}{4}\sum_{i=0}^{2n+2} 2^{-i}$. Finally, for the same reasons, we may assume that if we consider the circle $C(0,\mu^n r)$ at time $N_n + M_{n+1,n+1}$ which certainly encloses a disk which contains

$$Q_{N_n+M_{n+1,n+1}}(C(0,r))$$

from above, if we now iterate with $Q_{N_n+M_{n+1,n+1},N_n+M_{n+1,2n+1}}$ we will obtain a curve which we can make as close to $C(0,r)$ in the Hausdorff topology as we desire. If we now dilate by a factor of $1/\mu$ and iterate with P the remaining $m_{n+1,2n+2}$ times and $m_{n+1,2n+2}$ is large enough, we can then ensure that the image of this circle at time $N_{n+1} = N_n + M_{2n+2}$ lies entirely outside $D(0,R)$ and so is guaranteed to escape to infinity regardless of how we construct our sequence in the future.

This completes the induction and the construction of our sequence. Our assumption concerning the distortion of D under iteration shows us that at the times N_n, the diameter of D_{N_n} is at most $r(1 + 1/4\sum_{i=0}^{2n} 2^{-i}) < 3r/2$ which implies that the orbit of D must remain bounded under iteration and hence its interior must be contained in a Fatou component V for our sequence. On the other hand, one easily checks that $|Q'_{N_n}(0)| = 1$ from which it follows immediately that there is a nonconstant normal limit function on V. However, at the times $N_n + M_{n+1}$, $|Q'_{N_n+M_{n+1}}(0)| = \mu^{n+1}$ and so there must also be a constant normal limit function on V. Hence $\{P_n\}_{n=1}^{\infty}$ is a bounded polynomial sequence which has a Fatou component with the required properties for the first part of Theorem 1.4. For the second part of the result, we see from above that at the times N_n, one checks similarly to above that $\operatorname{diam} V_{N_n} \geq r$ for all $n \geq 0$ while the condition about the circles $C(0,\mu^n r)$ at times $N_n + M_{n+1,n+1}$ escaping to infinity ensures that $\operatorname{diam} V_{N_n+M_{n+1,n+1}} \leq \mu^n r$. This shows that V has the required properties for the second part of the theorem. Finally, in view of our earlier remarks at the beginning of this section, it follows that there is also a monic sequence of polynomials which has a Fatou component with the desired properties for both parts of the theorem and with this the proof is complete.

Before finishing, we make a few remarks. The technique for constructing the desired monic sequence of polynomials is indirect, relying as it does on the notion of conjugacy. It appears that it is in fact rather difficult to proceed directly to construct such a sequence without using non-monic polynomials and appealing to conjugacy to obtain a monic sequence. One could for example try using two monic polynomials, one for the ‘contraction’ and the other for the ‘expansion’ parts of the cycles above which correspond to the steps in the induction. However, computer experiments show that attempts to use the dynamics of either hyperbolic or parabolic examples to do this do not seem to work. Roughly what happens is that although one can control the distortion of a Fatou component over one such cycle, the distortions one picks up from many of these cycles are not summable.

References

[1] R. Brück, *Geometric properties of Julia sets of the composition of polynomials of the form $z^2 + c_n$*, Pacific J. Math., **198** (2001), 347-371, MR 2002d:37078.

[2] ———, *Connectedness and stability of Julia sets of the composition of polynomials of the form $z^2 + c_n$*, J. London Math. Soc. (2), **61**(**2**) (2000), 462-470, MR 2001e:30040.

[3] R. Brück, M. Büger and S. Reitz, *Random iterations of polynomials of the form $z^2 + c_n$: Connectedness of Julia sets*, Ergodic Theory Dynam. Systems, **19** (1999), 1221-1231, MR 2000h:37064, Zbl 0942.37041.

[4] M. Büger, *Self-similarity of Julia sets of the composition of polynomials*, Ergodic Theory Dynam. Systems, **17** (1997), 1289-1297, MR 98k:5819, Zbl 0894.58029.

[5] L. Carleson and T.W. Gamelin, *Complex Dynamics*, Springer Verlag, Universitext: Tracts in Mathematics, 1993, MR 94h:30033, Zbl 0782.30022.

[6] M. Comerford, *Properties of Julia Sets Arising from Arbitrary Compositions of Monic Polynomials*, Ph.D. Thesis, Yale University, May 2001.

[7] ———, *Infinitely many grand orbits*, Michigan Math. J., **51** (2003), 47-57, CMP 1 960 920.

[8] A.E. Eremenko and M.Ju. Lyubich, *Examples of entire functions with pathological dynamics*, J. London Math. Soc., **36** (1987), 458-468, MR 89e:30047, Zbl 0601.30033.

[9] J.E. Fornaess and N. Sibony, *Random iterations of rational functions*, Ergodic Theory Dynam. Systems, **11** (1991), 687-708, MR 93c:58173, Zbl 0753.30019.

[10] S. Kolyada and L. Snoha, *Topological entropy of nonautonomous dynamical systems*, Random Comput. Dynam., **4** (1996), 205-233, MR 98f:58126, Zbl 0909.54012.

[11] J. Milnor, *Dynamics in One Complex Variable. Introductory Lectures*, Friedr. Vieweg & Sohn, Braunschweig, 1999. MR 2002i:37057, Zbl 0946.30013.

[12] N. Steinmetz, *Rational Iteration*, Walter de Gruyter, 1993, MR 94h:30035, Zbl 0773.58010.

[13] D. Sullivan, *Quasiconformal homeomorphisms and dynamics* I. *Solution of the Fatou-Julia problem on wandering domains*, Ann. of Math., **122** (1985), 401-418, MR 87i:58103, Zbl 0589.30022.

Received July 25, 2002 and revised October 30, 2002.

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