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We study and compute an infinite family of Hurwitz spaces parameterizing covers of $\mathbb{P}^1_{\mathbb{C}}$ branched at four points and deduce explicit regular $\mathcal{S}_n$ and $\mathcal{A}_n$-extensions over $\mathbb{Q}(T)$ with totally real fibers.

Introduction.

In this paper, we study a family of covers of the projective line suggested to us by Gunter Malle, namely those covers of even degree $n \geq 6$, ramified over four points, with monodromy $\mathcal{S}_n$ and having branch cycle description $\mathbf{C} = (C_1, C_2, C_3, C_4)$ of type:

$$\left((n-2), 3, 2^{\frac{n-2}{2}}, 2^{\frac{n}{2}}\right).$$

Malle suspected the Hurwitz curves have genus zero for every $n \geq 6$ and some covers in the family have totally real fibers. A similar family was suggested and extensively studied by Dèbes and Fried in [**DF94**]. Unfortunately, their Hurwitz spaces happen to have a quadratic genus in n and only provide the expected regular extensions for degrees 5 and 7 (see [**DF94**], Theorem 4.11). Their work uses braid action formulae (see [**FV91**]) and complex conjugation action formulae (see [**FD90**], Proposition 2.3).

In this paper, we first follow the lines of Dèbes and Fried method and show that Malle's expectations were right. Then the second half of the paper is devoted to the explicit calculation of the concerned universal family of covers. To this end, we use an explicit version of Harbater's deformation techniques ([**Har80**]) as proposed in [**CG94**, **Cou99**]. As far as we know, it is the first time such an advanced method is used for computing an infinite family of covers. We present numerical results, obtained with *Magma*, showing the efficiency of the proposed method compared to the classical ones involving Buchberger algorithm. Indeed our computation reduces to solving *linear systems*.

In the first section, we recall results from the theory of Hurwitz spaces and (non)-rigidity methods developed in [**FV91**, **Völ96**, **Ful69**, **MM99**]. The second section is devoted to the combinatoric study of our family and arithmetical consequences of it using the method of Dèbes and Fried. In the third section, we show the existence of totally real $\mathcal{S}_n$ and $\mathcal{A}_n$ residual $\mathbb{Q}$-extensions in that family. Related to this totally real specialization, we find a very special point of the boundary of our Hurwitz space that shows very useful when computing an explicit model. This is done in the last section, by a deformation method.

We notice that a fallout of our construction is the existence of totally real $\mathbb{Q}$-extensions with Galois group $\mathcal{S}_n$ and $\mathcal{A}_n$ with four rational branched points (with only three branched points such extentions does not exist, cf. [**Ser92**] §8.4.3.). However an astute and effective construction can be found in [**Mes90**] to realize $\mathcal{S}_n$, with an odd integer $n \geq 3$, as the regular Galois group of a degree n extension of $\mathbb{Q}(T)$ having totally real fibers; this construction leads to a number of branch points linear in n.

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1. General framework.

Let us observe that the covers of our family have monodromy group primitive with a three cycle so it is $\mathcal{A}_n$ or $\mathcal{S}_n$ and it must clearly be the later. Therefore this family is parameterized by a coarse moduli space called Hurwitz space, denoted $\mathcal{H}'_4(\mathcal{S}_n, \mathbf{C})$. It is a quasi-projective regular (not a priori connected) variety over $\overline{\mathbb{Q}}$ with the following properties (see [**FV91**, **Völ96**, **Ful69**]):

- Since any conjugacy class of $\mathcal{S}_n$ is rational, $\mathcal{H}'_4(\mathcal{S}_n, \mathbf{C})$ is defined over $\mathbb{Q}$.
- Let F_4 be the configuration space of 4 points, e.g., $(\mathbb{P}^1_{\mathbb{C}})^4 \setminus \mathcal{D}$ where $\mathcal{D}$ denotes the discriminant variety. The map:

$$\begin{aligned} \phi:\ \ \mathcal{H}'_4(\mathcal{S}_n, \mathbf{C}) &\longrightarrow F_4(\mathbb{P}^1_{\mathbb{C}}) \\ h\ \ &\longmapsto (z_1, z_2, z_3, z_4) \end{aligned}$$

  where z_1, z_2, z_3, z_4 are the branched points (in the given order) of the cover corresponding to the point h, is a finite étale morphism defined over $\mathbb{Q}$.
- Since $\mathcal{S}_n$ has no center, the covers in our family have no automorphism so the moduli space $\mathcal{H}'_4(\mathcal{S}_n, \mathbf{C})$ is a fine one and for any $h \in \mathcal{H}'_4(\mathcal{S}_n, \mathbf{C})$ the associated cover p_h can be defined over $\mathbb{Q}(h)$ the field of definition of the point h.

As in §4.2 of [**DF94**], rather than looking at the full moduli space, we concentrate on a curve in it. Let us fix three points $z_1, z_2, z_3 \in \mathbb{P}^1(\mathbb{Q})$ and consider the curve $\mathcal{H}'_{(z_1,z_2,z_3)}$ obtained by the pullback:

$$\begin{array}{ccc} \mathcal{H}'_{(z_1,z_2,z_3)} & \longrightarrow & \mathcal{H}'_4(\mathcal{S}_n,\mathbf{C}) \\ \varphi \big\downarrow & & \big\downarrow \phi \\ \mathbb{P}^1_{\mathbb{C}} \setminus \{z_1,z_2,z_3\} & \xrightarrow{\ i\ } & F_4(\mathbb{P}^1_{\mathbb{C}}) \end{array}$$

(the lower horizontal map i is $z \mapsto (z_1,z_2,z_3,z)$). If the three fixed points are rational, all the maps are defined over $\mathbb{Q}$ and the curve $\mathcal{H}'_{(z_1,z_2,z_3)}$ is also defined over $\mathbb{Q}$.

2. Combinatoric study of the Hurwitz curve.

In this section and the following, we will see that every element of our Nielsen class can be braid to another one $(\sigma_1,\sigma_2,\sigma_3,\sigma_4)$ such that the product $\sigma_1\sigma_2$ is an n-cycle. In this case, the Tchebycheff polynomial T_n appears by coalescing σ_1 and σ_2. This property make the computation relatively painless (see §4.2). Then we gather the more information we can about the cover $\varphi : \mathcal{H}'_{(z_1,z_2,z_3)} \longrightarrow \mathbb{P}^1_{\mathbb{C}} \smallsetminus \{z_1,z_2,z_3\}$ from its combinatoric description. This way we prove that the genus of $\mathcal{H}'_{(z_1,z_2,z_3)}$ is zero and have many rational points.

2.1. Nielsen classes description. Let us fix the branched points $\underline{z} = (z_1,z_2,z_3,z_4) \in F_4(\mathbb{P}^1_{\mathbb{C}})$ and an homotopic base of $\mathbb{P}^1_{\mathbb{C}} \setminus \{z_1,z_2,z_3,z_4\}$. Using the topological classification of covers, elements of the fiber $\phi^{-1}(\underline{z})$ are in bijection with $\mathrm{sni}^{\mathrm{ab}}(\mathcal{S}_n,\mathbf{C})$ the strict absolute Nielsen class of type $(\mathcal{S}_n,\mathbf{C})$, that is:

$$\mathrm{sni}^{\mathrm{ab}}(\mathcal{S}_n,\mathbf{C}) = \Big\{(\sigma_1,\sigma_2,\sigma_3,\sigma_4) \in (\mathcal{S}_n)^4,\ \sigma_1\sigma_2\sigma_3\sigma_4 = 1, \\ \sigma_i \in C_i\ \forall i,\ \langle \sigma_1,\sigma_2,\sigma_3,\sigma_4\rangle = \mathcal{S}_n\Big\}\Big/\sim$$

where $(\sigma_1,\sigma_2,\sigma_3,\sigma_4) \sim (\sigma'_1,\sigma'_2,\sigma'_3,\sigma'_4)$ means that there exists $\tau \in \mathcal{S}_n$ such that $\sigma'_i = \tau\sigma_i\tau^{-1}$ for all $1 \le i \le 4$.

We first enumerate the Nielsen class. To this end, we construct a representative system of this set of equivalence classes made of 4-uple $(\sigma_1,\sigma_2,\sigma_3,\sigma_4)$ such that the product $\sigma_1\sigma_2$ is a cycle.

Since any two $(n-2)$-cycles are $\mathcal{S}_n$-conjugate, every element of $\mathrm{sni}^{\mathrm{ab}}(\mathcal{S}_n,\mathbf{C})$ has a representative with $\sigma_1 = (1\dots n-2)$. Conjugating by a power of σ_1, we also assume that $\sigma_2 = (n-2\ \ k\ \ l)$. We now distinguish three cases.[1]

- *The case* $\{k,l\} = \{n-1,n\}$. Conjugating by a power of σ_1 and by $(n-1\ \ n)$, every element in that class has a unique representative with $\sigma_1 = (1\dots n-2)$, and $\sigma_2 = (n-2\ \ n-1\ \ n)$. So $\sigma_1\sigma_2 = (1\dots n)$

[1]Our convention for multiplying permutations is the following: $(12)(23) = (123)$.

and the enumeration reduces to finding all the permutations $(\sigma_4, \sigma_3) \in C_4 \times C_3$ such that $\sigma_4\sigma_3 = (1\ldots,n)$ and $\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4\rangle = \mathcal{S}_n$. We need a lemma used several times further. It deals with relations in the diedral group D_m.

Lemma 2.1. *Let $m \leq n$ be even and let c be an m-cycle of $\mathcal{S}_n$. There is a bijection between nontrivial cycles of $c^{\frac{m}{2}}$ (i.e., sets of the form $\{x, c^{\frac{m}{2}}(x)\}$ where x belongs to the support of c), and the decompositions $c = \sigma\tau$ with σ a product of $\frac{m}{2}$ transpositions and τ a product of $\frac{m}{2} - 1$ transpositions. Therefore there are exactly $\frac{m}{2}$ such decompositions of c.*

Proof. Let x be an element of the support of c then one can verify that:

$$c = \underbrace{\prod_{i=1}^{\frac{m}{2}} \left(c^{1-i}(x)\, c^{i}(x)\right)}_{\sigma_{c,x}} \underbrace{\prod_{j=1}^{\frac{m}{2}-1} \left(c^{j}(x)\, c^{-j}(x)\right)}_{\tau_{c,x}}. \tag{1}$$

This is a decomposition associated to the set $\{x, c^{\frac{m}{2}}(x)\}$. Two such decompositions associated to x and y are equal if and only if $\{x, c^{\frac{m}{2}}(x)\} = \{y, c^{\frac{m}{2}}(y)\}$. On the other hand, let $c = \sigma\tau$ be a decomposition as in the lemma then one can show that it can be written as in (1) by considering an element x of the support of σ not belonging to the support of τ. □

The group D_m is known to be the group of isometries of a regular polygon with m vertices. We can explain the previous relation geometrically: For each vertex x, the rotation (i.e., c) inducing an m-cycle on the vertices equals the composition of the unique reflection permuting the vertex x and its successor (i.e., $\sigma_{c,x}$) with the unique reflection fixing x (i.e., $\tau_{c,x}$).

Back to the enumeration of the Nielsen classes in the case where $\{k, l\} = \{n-1, n\}$, Lemma 2.1 shows that there are exactly $\frac{n}{2}$ such elements; this subset is denoted Class $\mathcal{A}$ in Table 1.

- *The case* $\#(\{k, l\} \cap \{n-1, n\}) = 1$. Conjugating by a power of σ_1 and by $(n-1\ \ n)$, every class has a unique representative with $k \in \{1, \ldots, n-3\}$, $\sigma_1 = (1\ldots n-2)$ and $\sigma_2 = (k\ \ \ n-2\ \ \ n-1)$. Inspect $(1\ldots k)(k+1\ldots n-1) = \sigma_1\sigma_2 = \sigma_4\sigma_3$. The right side fixes n, and so $\sigma_3(n) = \sigma_4(n)$, and $\sigma_4\sigma_3$ also fixes $\sigma_4(n)$. So, $k = 1$ and $(1\ \ n)$ appear in both σ_3 and σ_4; the rests of the decompositions are given by Lemma 2.1 with $c = (2\ldots n-1)$. These elements form the Class $\mathcal{B}$ in Table 1.
- *The case* $\{k, l\} \subset \{1, \ldots, n-3\}$. In that case $\sigma_1 = (1\ldots n-2)$ and $\sigma_2 = (k\ \ l\ \ n-2)$, then:

$$\begin{cases} \sigma_1\sigma_2 = (1\ldots k\ \ l+1\ldots n-2\ \ k+1\ldots l) & \text{if } k < l \\ \sigma_1\sigma_2 = (1\ldots l)(l+1\ldots k)(k+1\ldots n-2) & \text{if } k > l. \end{cases}$$

Since $\sigma_1\sigma_2 = \sigma_4\sigma_3$ fixes $n-1$ and n, transitivity on $\{1,\dots,n\}$ implies $(n-1\ \ n)$ is not a disjoint cycle in both σ_3 and σ_4. Therefore $\sigma_4\sigma_3$ has $n-1, \sigma_4(n-1), n$ and $\sigma_4(n)$ as fixed points. This implies $k > l$ and two of the disjoint cycle lengths l, $k-1$ and $n-2-k$ are 1. Each of the three possibilities for σ_2 $(n-3\ \ 1\ \ n-2)$ $(2\ \ 1\ \ n-2)$ and $(n-3\ \ n-4\ \ n-2)$, are conjugate under $\langle\sigma_1\rangle$. With no loss, the Nielsen class representative has $\sigma_2 = (n-3\ \ n-4\ \ n-2)$. Then $\sigma_1\sigma_2 = (1\dots n-4)$ and in the support of σ_3 and σ_4, we find $(n\ \ \ n-2)(n-1\ \ \ n-3)$ or $(n\ \ n-3)(n-1\ \ n-2)$ which are conjugate under $(n-1\ \ n)$. Again, Lemma 2.1 for $c = (1\dots n-4)$ concludes. These elements form the Class $\mathcal{C}$ in Table 1.

The whole enumeration can be found in Table 1. Note that the three pointed classes have the following cardinalities:

$$\#\mathcal{A} = \frac{n}{2}, \qquad \#\mathcal{B} = \frac{n}{2} - 1 \qquad \text{and} \qquad \#\mathcal{C} = \frac{n}{2} - 2$$

therefore:

$$\#\,\mathrm{sni}^{\mathrm{ab}}(\mathcal{S}_n, \mathbf{C}) = 3\left(\frac{n}{2} - 1\right).$$

Concerning the Hurwitz cover, we have shown that:

Fact 2.2. The degree of the Hurwitz cover ϕ (or φ) equals $3\left(\frac{n}{2}-1\right)$.

Class $\mathcal{A}$	$a_i = \left[(1\dots n-2), (n-2\ \ n-1\ \ n), \tau_{(1\dots n),i}, \sigma_{(1\dots n),i}\right]$ with $1 \le i \le \frac{n}{2}$
Class $\mathcal{B}$	$b_i = \left[(1\dots n-2), (1\ \ n-2\ \ n-1), \nu\tau_{(2\dots n-1),i}, \nu\sigma_{(2\dots n-1),i}\right]$ with $\nu = (1\ \ n)$ and $2 \le i \le \frac{n}{2}$
Class $\mathcal{C}$	$c_i = \left[(1\dots n-2), (n-2\ \ n-3\ \ n-4), \nu\tau_{(1\dots n-4),i}, \nu\sigma_{(1\dots n-4),i}\right]$ with $\nu = (n\ \ n-2)(n-1\ \ n-3)$ and $1 \le i \le \frac{n}{2} - 2$

Table 1. The Nielsen classes (same notations as in Lemma 2.1).

2.2. Braiding action. In this paragraph, we compute the action of braids on the Nielsen class given in Table 1.

2.2.1. Generator of the braid group and braiding action. Let us denote by $\mathcal{H}_4(\mathcal{S}_n, \mathbf{C})$ the Hurwitz space parameterizing the same set of isomorphism classes of covers as $\mathcal{H}'_4(\mathcal{S}_n, \mathbf{C})$ but without ordering the branch points. This space can be endowed with a topology which is constructed in the same way as the one of $\mathcal{H}'_4(\mathcal{S}_n, \mathbf{C})$ (see [**FV91**] or [**Ful69**]). On the other hand, it maps onto $C_4(\mathbb{P}^1_{\mathbb{C}})$, the quotient of $F_4(\mathbb{P}^1_{\mathbb{C}})$ by the action of $\mathcal{S}_4$ on the coordinates:

$$\begin{array}{ccc} \mathcal{H}'_4(\mathcal{S}_n,\mathbf{C}) & \to & \mathcal{H}_4(\mathcal{S}_n,\mathbf{C}) \\ \downarrow \phi & & \downarrow \phi' \\ F_4(\mathbb{P}^1_{\mathbb{C}}) & \longrightarrow & C_4(\mathbb{P}^1_{\mathbb{C}}). \end{array} \tag{2}$$

The fundamental group of $C_4(\mathbb{P}^1_{\mathbb{C}})$ is the Hurwitz braid group of index 4. It possesses a classical presentation (see [**Han89**] or [**Bir75**]):

$$\left\langle Q_1, Q_2, Q_3 \,\middle|\, \begin{array}{l} Q_1Q_3 = Q_3Q_1 \\ Q_1Q_2Q_1 = Q_2Q_1Q_2 \text{ and } Q_2Q_3Q_2 = Q_3Q_2Q_3 \\ Q_1Q_2Q_3^2Q_2Q_1 = 1 \text{ (sphere's relation)} \end{array} \right\rangle.$$

Denoting the fiber of ϕ' by $\mathrm{ni}^{\mathrm{ab}}(\mathcal{S}_n,\mathbf{C})$ the "unordered" Nielsen class associated to inertia's 4-uple $\mathbf{C}$, i.e., the quotient of $\mathrm{sni}^{\mathrm{ab}}(\mathcal{S}_n,\mathbf{C})$ by the action of $\mathcal{S}_4$ on the coordinates, we have:

Proposition 2.3 (Braiding action formula)**.** *For all $i = 1,2,3$, and for all $[\sigma_1,\sigma_2,\sigma_3,\sigma_4]$ in $\mathrm{ni}^{\mathrm{ab}}(\mathcal{S}_n,\mathbf{C})$, the monodromy (right) action of the braid Q_i for the previous generating system is given by the formula:*

$$[\sigma_1,\sigma_2,\sigma_3,\sigma_4].Q_i = [\ldots,\sigma_i\sigma_{i+1}\sigma_i^{-1},\sigma_i,\ldots]$$

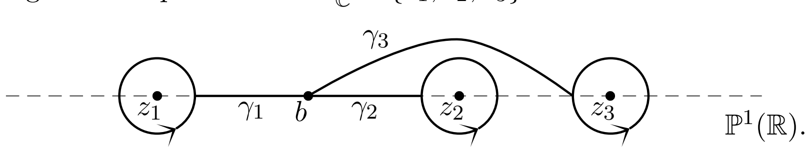
where $\sigma_i\sigma_{i+1}\sigma_i^{-1}$ is the i-th coordinate.

Generally speaking the diagram (2) permits us to express the monodromy of ϕ according to the one of ϕ'. This can be done explicitly because a presentation of the fundamental group of $F_4(\mathbb{P}^1_{\mathbb{C}})$ can be expressed in term of the one of $C_4(\mathbb{P}^1_{\mathbb{C}})$. We just give here the generators and refer to [**Han89**] for a complete system of relations:

$$t_{1,2} = {Q_1}^2, \qquad t_{2,3} = {Q_2}^2, \qquad t_{1,3} = Q_1{Q_2}^2{Q_1}^{-1}.$$

2.2.2. Monodromy action for the cover φ. Let us recall that we just defined φ in Section 1 to be the pullback of ϕ by the monomorphism denoted by $i : \mathbb{P}^1_{\mathbb{C}} \smallsetminus \{z_1,z_2,z_3\} \hookrightarrow \mathcal{U}^4$. We now want to study the monodromy of this cover.

Let us choose $z_1 < z_2 < z_3$ on the real line $\mathbb{P}^1(\mathbb{R})$ and let us define the following "homotopic base" of $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{z_1,z_2,z_3\}$:



Then, as explained in details in Theorem 4.5 of [**DF94**], for an adapted generating system of braids[2] Q_1, Q_2, Q_3 on $C_4(\mathbb{P}^1_{\mathbb{C}})$, we can compute the morphism i_* induced by i on the respective homotopic groups in term of those two sets of generators:

$$i_*(\gamma_1) = Q_1^2 = t_{1,2}, \quad i_*(\gamma_2) = Q_2^2 = t_{2,3}, \quad i_*(\gamma_3) = Q_2^{-1}Q_3^2Q_2 = (t_{1,2}t_{2,3})^{-1}.$$

With these formulas, the computation of the monodromy of φ is deduced from the one of ϕ. We summarize in the next:

Proposition 2.4. *Using the notations of the Table 1, the monodromy action for the cover φ is:*

- *For the path γ_1:*
$$\left(a_{\frac{n}{2}}\ a_{\frac{n}{2}-1}\ \dots\ a_1\right)\left(b_{\frac{n}{2}}\ b_{\frac{n}{2}-1}\ \dots\ b_2\right)\left(c_{\frac{n}{2}-2}\ c_{\frac{n}{2}-3}\ \dots\ c_1\right);$$
- *for the path γ_2:*
$$\left(a_{\frac{n}{2}-2}\ b_{\frac{n}{2}}\ b_{\frac{n}{2}-1}\ a_{\frac{n}{2}}\ c_{\frac{n}{2}-2}\right)\left(a_1\ c_{\frac{n}{2}-3}\right)\left(a_2\ c_{\frac{n}{2}-4}\right)\dots\left(a_{\frac{n}{2}-3}\ c_1\right)$$
$$\left(a_{\frac{n}{2}-1}\right)\begin{cases}\left(b_2\ b_{\frac{n}{2}-2}\right)\left(b_3\ b_{\frac{n}{2}-3}\right)\dots\left(b_{\frac{n}{4}-1}\ b_{\frac{n}{4}+1}\right)\left(b_{\frac{n}{4}}\right) & \text{if } 4 \mid n,\\ \left(b_2\ b_{\frac{n}{2}-2}\right)\left(b_3\ b_{\frac{n}{2}-3}\right)\dots\left(b_{\frac{n-2}{4}}\ b_{\frac{n+2}{4}}\right) & \text{if } 4 \nmid n;\end{cases}$$
- *for the path $\gamma_1.\gamma_2$:*
$$\left(a_{\frac{n}{2}-1}\ a_{\frac{n}{2}-2}\ b_{\frac{n}{2}-1}\right)\left(c_1\ a_{\frac{n}{2}-4}\right)\left(c_2\ a_{(\frac{n}{2}-5)}\right)\dots\left(c_{\frac{n}{2}-4}\ a_1\right)\left(c_{\frac{n}{2}-3}\ a_{\frac{n}{2}}\right)\left(c_{\frac{n}{2}-2}\ a_{\frac{n}{2}-3}\right)$$
$$\begin{cases}\left(b_{\frac{n}{2}}\ b_{\frac{n}{2}-2}\right)\left(b_{\frac{n}{2}-3}\ b_2\right)\left(b_{\frac{n}{2}-4}\ b_3\right)\dots\left(b_{\frac{n}{4}+1}\ b_{\frac{n}{4}-2}\right)\left(b_{\frac{n}{4}}\ b_{\frac{n}{4}-1}\right) & \text{if } 4 \mid n,\\ \left(b_{\frac{n}{2}}\ b_{\frac{n}{2}-2}\right)\left(b_{\frac{n}{2}-3}\ b_2\right)\dots\left(b_{\frac{n+2}{4}}\ b_{\frac{n-2}{4}-1}\right)\left(b_{\frac{n-2}{4}}\right) & \text{if } 4 \nmid n.\end{cases}$$

2.3. Ramification in the Hurwitz curve. In conclusion, the Hurwitz curve $\mathcal{H}'$ is a cover of $\mathbb{P}^1_{\mathbb{C}}$ of degree $3\left(\frac{n}{2}-1\right)$ ramified over three points, say $z_1, z_2, z_3 \in \mathbb{P}^1(\mathbb{Q})$ with ramification type described in Table 2.

Fact 2.5. The Hurwitz curve $\mathcal{H}'_{(z_1,z_2,z_3)}$ satisfies:

- It is irreducible;
- it is of genus zero and $\mathbb{Q}$-isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$.

So, our family contains infinitely many covers defined over $\mathbb{Q}$.

Proof. The irreducibility comes from the transitivity of the braiding action. The Riemann-Hurwitz formula shows that the genus of $\mathcal{H}'_{(z_1,z_2,z_3)}$ is zero. We can also note that, for example, the point of ramification index 5 must be a rational one (this ramification index is isolated). Being defined over $\mathbb{Q}$, of

[2]In our situation the 3 points z_1, z_2 and z_3 are on the real line, so the choice of Q_1, Q_2, Q_3 is the standard one (see [**Han89**] for example). In general the 3 generators Q_i must be precisely given. We point out that this choice just depends on a given path through z_1, z_2, z_3 in this order.

	z_1	z_2	z_3
if $4 \mid n$	$\frac{n}{2}$ $\frac{n}{2}-1$ $\frac{n}{2}-2$	5 1^2 $2^{\frac{3n}{4}-5}$	3 $2^{\frac{3n}{4}-3}$
if $4 \nmid n$	$\frac{n}{2}$ $\frac{n}{2}-1$ $\frac{n}{2}-2$	5 1 $2^{\frac{3(n-6)}{4}}$	3 1 $2^{\frac{3n-14}{4}}$

Table 2. Ramification types over z_1, z_2, z_3 in the Hurwitz curve (double line stands for repeated ramification points of same index).

genus zero and with a rational point, $\mathcal{H}'_{(z_1,z_2,z_3)}$ is necessarily $\mathbb{Q}$-isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$. In particular, there are covers in our family defined over $\mathbb{Q}$. □

3. Existence of totally real $\mathcal{S}_n$ and $\mathcal{A}_n$-extensions.

There is still a question left: Does our family contain elements defined over $\mathbb{Q}$ with totally real fibers?

In this section, we use complex conjugation action on fibers as describe by Dèbes and Fried (see Theorem 2.4 of [**FD90**] or Proposition 2.3 of [**DF94**]) and prove adapted formulae to our family and to our choice of homotopic basis.

3.1. Covers in the family with totally real fiber. We consider a finite algebraic cover $p : \mathcal{C} \to \mathbb{P}^1_{\mathbb{C}}$ ramified over four ordered real points $z_2 < z_3 < z_4 < z_1 \in \mathbb{P}^1(\mathbb{R})$, we fix $z_0 \in \mathbb{P}^1(\mathbb{R})$ a real base point between z_3 and z_4 and we denote by F the fiber $p^{-1}(z_0)$.

The complex conjugation will play a crucial role; let denote this conjugation by a bar; for example $\bar{p} : \bar{\mathcal{C}} \to \mathbb{P}^1_{\mathbb{C}}$ is the cover obtained from the first one by complex conjugation and $\overline{F}$ is its fiber above z_0. Let $c : F \to \overline{F}$ be the bijection induced by the complex conjugation.

- The complex conjugation also acts on the topological fundamental group by left composition (thank you complex conjugation for being continuous!). The fundamental group $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{z_1, z_2, z_3, z_4\}, z_0)$ is simply denoted by π_1. In the rest of this paper, when we refer to the standard homotopic basis of π_1, we always mean the one drawn in the Figure 1.

  So we have $\gamma_1\gamma_4\gamma_3\gamma_2 = 1$ and the complex conjugation acts as follows:

$$(3) \qquad \overline{\gamma_1} = \gamma_4^{-1}\gamma_1^{-1}\gamma_4 \qquad \overline{\gamma_4} = \gamma_4^{-1} \qquad \overline{\gamma_3} = \gamma_3^{-1} \qquad \overline{\gamma_2} = \gamma_3\gamma_2^{-1}\gamma_3^{-1}.$$

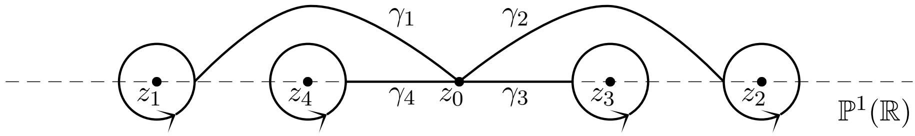
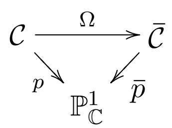


Figure 1. The standard homotopic basis.

- Since complex conjugation is a continuous morphism of $\mathbb{C}$, the monodromy of the cover $\bar{p}$ can be deduced from the monodromy of p. More precisely, if we denote by $\rho : \pi_1 \to \mathcal{S}_F$ and $\bar{\rho} : \pi_1 \to \mathcal{S}_{\overline{F}}$ the two monodromy (anti)morphisms, then we have:

$$\forall \gamma \in \pi_1, \qquad \bar{\rho}(\gamma) \circ c = c \circ \rho(\bar{\gamma}). \tag{4}$$

- From the Weil descent criterion (which boils down to the use of Artin theory because the extension $\mathbb{C}/\mathbb{R}$ is galois and finite) we know that the cover p can be defined over $\mathbb{R}$ if and only if there exists an isomorphism Ω such that:



and $\qquad \overline{\Omega} \circ \Omega = \mathrm{Id}$

(the last condition is a cocycle condition).

From these three points, we can deduce a completely combinatoric criterion for the descent to $\mathbb{R}$ and for the existence of totally real fibers:

Theorem 3.1. *Let $p : \mathcal{C} \to \mathbb{P}^1_{\mathbb{C}}$, z_0, and F be as above. Let $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \mathcal{S}_n^4$ be the branch cycle description of p — i.e., $\sigma_i = \rho(\gamma_i)$ where $\rho : \pi_1 \to \mathcal{S}_F$ is the monodromy morphism — and $G = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle \subset \mathcal{S}_F$ the monodromy group of p.*

1. *The cover p is defined over $\mathbb{R}$ if and only if there exists an involution $\tau \in \mathcal{S}_F$ such that:*

$$\sigma_4\sigma_1^{-1}\sigma_4^{-1} = {}^{\tau}\sigma_1, \qquad \sigma_4^{-1} = {}^{\tau}\sigma_4, \qquad \sigma_3^{-1} = {}^{\tau}\sigma_3, \qquad \sigma_3^{-1}\sigma_2^{-1}\sigma_3 = {}^{\tau}\sigma_2.$$

2. *If that is the case and if moreover the cover p has no automorphism — i.e., if the centralizator of G in $\mathcal{S}_F$ is trivial —, then the fiber F is totally real if and only if $\tau = \mathrm{Id}$.*

Proof. Corresponding to the isomorphism Ω of the Weil descent criterion there is a bijection $\omega : F \to \overline{F}$; in term of ω the conditions of the criterion are:

$$[\forall \gamma \in \pi_1, \quad \omega \circ \rho(\gamma) = \bar{\rho}(\gamma) \circ \omega] \quad \text{and} \quad \left[(c^{-1} \circ \omega)^2 = \mathrm{Id}\right]$$

(the last condition is just the cocycle one). Let $\tau = c^{-1} \circ \omega$. This is an involution of $\mathcal{S}_F$ which by (4) and (3) satisfies the expected relations on the σ_i (be careful: The monodromy is an anti-morphism) if and only if p is defined over $\mathbb{R}$.

Secondly, assuming that p is defined over $\mathbb{R}$, then the isomorphism Ω is an automorphism; so if moreover p has no automorphism, Ω, and therefore ω, must be identity. Furthermore, the conjugation c can now be viewed as a permutation of F. In conclusion, the bijection τ introduced in 1 satisfies $\tau = c^{-1}$ and the fiber F is totally real if and only if $\tau = \mathrm{Id}$. □

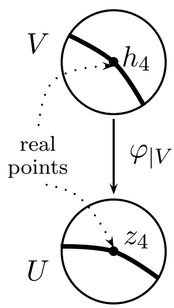
Having this result in mind, we can now go back to our family. Recall that the covers of our family have no automorphism (since the center of $\mathcal{S}_n$ is trivial). It is not difficult to verify that the Nielsen class:

$$a_{\frac{n}{2}-1} = \begin{cases} \sigma_1 = (12\ldots n-2) \\ \sigma_2 = (n-2 \;\; n-1 \;\; n) \\ \sigma_3 = \left[\prod_{i=1}^{\frac{n}{2}-2} (i \;\; n-i-2)\right] (n-2 \;\; n)(\frac{n}{2}-1)(n-1) \\ \sigma_4 = \left[\prod_{i=1}^{\frac{n}{2}-1} (i \;\; n-i-1)\right] (n-1 \;\; n) \end{cases}$$

satisfies the previous theorem and is the only one in that case. This calculation is very easy and is surely what motivated G. Malle to suggest to us this example.

Fact 3.2. Our family contains covers defined over $\mathbb{R}$ with an interval of non ramified points with totally real fibers.

3.2. Totally real $\mathcal{S}_n$-extensions. At this point we know that our family contains some covers defined over $\mathbb{Q}$ and some others with totally real fibers; we want to prove that there are covers satisfying both properties. We will have to move one of the ramification points.



Suppose that three of the ramification points $z_1, z_2, z_3 \in \mathbb{P}^1_{\mathbb{C}}(\mathbb{Q})$ are fixed and let the fourth one z_4 move on $\mathbb{P}^1(\mathbb{R})$ between z_1 and z_3. Thanks to the previous section, for all such z_4, there is a unique point $h_4 \in \varphi^{-1}(z_4)$ which represents a cover having totally real fibers above $]z_2, z_3[$ (namely the cover with branch cycle description $a_{\frac{n}{2}-1}$ with respect to a standard homotopic basis). We choose U and V two neighborhoods of z_4 and h_4 respectively such that $\varphi_{|V}$ becomes an homeomorphism from V to U. For every $z \in U \cap \mathbb{P}^1(\mathbb{R})$, the covering corresponding to $\varphi_{|V}^{-1}(z)$ satisfies the preceding descent criteria; so it is defined over $\mathbb{R}$ and has a real interval of specialization. Thus we have $\varphi_{|V}^{-1}(U \cap \mathbb{P}^1(\mathbb{R})) \subset V \cap \mathbb{P}^1(\mathbb{R})$. But rational points are dense in $\mathbb{P}^1(\mathbb{R})$ so we can find rational points in $V \cap \mathbb{P}^1(\mathbb{R})$; all the corresponding covers are defined over $\mathbb{Q}$ with a complete segment of totally real fibers:

Fact 3.3. Our family contains a rational nonempty interval of covers defined over $\mathbb{Q}$ each having an interval of totally real fibers.

Let $p : \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ be such a cover. Since an interval is not a thin set, by Hilbert irreducible theorem, one can find irreducible and totally real specializations.

Proposition 3.4. *There exists totally real $\mathcal{S}_n$-extensions of $\mathbb{Q}$ of degree n.*

3.3. Totally real $\mathcal{A}_n$-extensions. From the previous construction, we now want to deduce the same kind of result for the group $\mathcal{A}_n$. Let $\mathcal{C} \to \mathbb{P}^1_{\mathbb{Q}}$ be one of our covers over $\mathbb{Q}$ with totally real fibers and $\mathcal{C}^{\text{gal}} \to \mathbb{P}^1_{\mathbb{Q}}$ its Galois closure. The Galois group is still $\mathcal{S}_n$ because the arithmetic monodromy group contains the geometric one which is $\mathcal{S}_n$ too. We consider $\mathcal{D} = \mathcal{C}^{\text{gal}}/\mathcal{A}_n$ as in the following diagram:

$$\begin{array}{ccc} \mathcal{C}^{\text{gal}} & & \\ {\scriptstyle \mathcal{A}_n}\downarrow & \searrow & \\ \mathcal{D} & & \mathcal{C} \\ {\scriptstyle 2}\downarrow & \swarrow {\scriptstyle n} & \\ \mathbb{P}^1. & & \end{array}$$

Since two of the inertia permutations are even and the two others are odd, there are only two branched points in the cover $\mathcal{D} \to \mathbb{P}^1$, i.e., z_1 and z_3 or z_4 according to $n \equiv 0$ or $2 \pmod 4$. Therefore, by Riemann-Hurwitz formula, $\mathcal{D}$ is a genus zero curve; since it has at least one rational point (for example, one of the two ramified points), it is also $\mathbb{Q}$-isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$ (this kind of argument looks like the so-called "double group trick", see [**Ser92**]).

This gives $\mathcal{A}_n$ with totally real fibers. Moreover the conjugacy classes for $\mathcal{C}^{\text{gal}} \to \mathcal{D}$ are of the type:

$$\left(\left(\frac{n}{2}-1\right)^2, 3, 3, 2^{\frac{n-2}{2}}, 2^{\frac{n-2}{2}}\right) \qquad \text{or}$$
$$\left(\left(\frac{n}{2}-1\right)^2, 3, 3, 2^{\frac{n}{2}}, 2^{\frac{n}{2}}\right)$$

according to the parity of $\frac{n}{2}$. In conclusion:

Proposition 3.5. *There exist totally real $\mathcal{A}_n$-extensions of $\mathbb{Q}$ of degree n.*

At the end of this paper, we give an explicit version of both Propositions 3.4 and 3.5.

4. Explicit computation.

In this section, we compute the Hurwitz space and the universal curve for our family of covers. We fix once and for all three rational points $z_1 < z_3 < z_2$. To ease notations we denote by $\mathcal{H}$ the curve $\mathcal{H}'_{(z_1,z_2,z_3)}$.

4.1. The universal curve and a choice of coordinates. Because the covers in our family have no automorphism, by [**FV91**] or [**BF83**], there exists a universal curve $\mathcal{S}$ and a fibration:

$$\begin{array}{ccc} \mathcal{H} & \xleftarrow{\ \Pi\ } & \mathcal{S} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \Phi} \\ \mathbb{P}^1_{\mathbb{C}} \smallsetminus \{z_1, z_2, z_3\} & \xleftarrow{\ \pi\ } & F_2(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{z_1, z_2, z_3\}) \end{array}$$

where:

- $\mathcal{S}$ is a smooth quasi-projective surface over $\mathbb{Q}$,
- $F_2(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{z_1, z_2, z_3\})$ denotes the quasi-projective variety defined by the ordered pairs $(u, v) \in \mathbb{P}^1_{\mathbb{C}} \smallsetminus \{z_1, z_2, z_3\}$ with $u \neq v$,
- π is the morphism obtained by forgetting the second coordinate v,
- the vertical arrows are finite and étale morphisms of varieties, all defined over $\mathbb{Q}$.

The morphism π admits a well-known projective completion that is still denoted by π. This is the canonical morphism from the (projective) moduli space of curves of genus zero with 5 marked points $\mathcal{M}_{0,5}$ to the one with 4 marked points $\mathcal{M}_{0,4}$ (we refer to [**GHP88**] for a highly comprehensive study of the spaces $\mathcal{M}_{0,n}$ from an algebraic view point. A lot of ideas contained in this paper are used here). We define $\overline{\mathcal{S}}$ to be the normalization of $\mathcal{M}_{0,5}$ in $\mathbb{Q}(\mathcal{S})$. We obtain this way a commutative diagram between smooth projective varieties defined over $\mathbb{Q}$ which extends the previous one:

$$\begin{array}{ccc} \overline{\mathcal{H}} & \xleftarrow{\ \Pi\ } & \overline{\mathcal{S}} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \Phi} \\ \mathcal{M}_{0,4} & \xleftarrow{\ \pi\ } & \mathcal{M}_{0,5}. \end{array}$$

In order to choose a system of coordinates on the universal curve, we need to define another Hurwitz curve, a little bigger than $\mathcal{H}$, namely the moduli space of covers in our family with a marking of an unramified point above z_3. This space denoted by $\mathcal{H}^\bullet$ is a degree 2 cover of $\mathcal{H}$ by the forgetting map. We check that this cover is connected. There is a universal curve $\mathcal{S}^\bullet \to \mathcal{H}^\bullet$ obtained by base extension of $\mathcal{S} \to \mathcal{H}$. A normalization, as in the previous paragraph, provides the following commutative diagram of projective varieties over $\mathbb{Q}$:

$$\begin{array}{ccc} \overline{\mathcal{H}^\bullet} & \xleftarrow{\Pi^\bullet} & \overline{\mathcal{S}^\bullet} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \Phi} \\ \mathcal{M}_{0,4} & \xleftarrow{\pi} & \mathcal{M}_{0,5}. \end{array}$$

Now, we want to choose adapted coordinates on those spaces. To begin with, let's recall that there exist four sections s_1, s_2, s_3 and s_4 of π, corresponding to the four marked points (see [**GHP88**], Section 3). We define x and y to be the coordinates on $\mathcal{M}_{0,5}$ such that $x = y = \infty$ on s_1, $x = 1$ on s_2, $x = y = 0$ on s_4 and $y = 1$ on s_3. Then we set $\lambda := \frac{x}{y} \in \mathbb{C}(\mathcal{M}_{0,4})$.

We also need coordinates on the universal curve $\overline{\mathcal{S}^\bullet}$. The fibration $\overline{\mathcal{S}^\bullet} \to \overline{\mathcal{H}^\bullet}$ admits four sections corresponding to the four points A, B, C, D (see Table 3). This amounts to saying that these four points are defined over $\mathbb{C}(\overline{\mathcal{H}^\bullet})$. This is clear for A and B because they are isolated. This is also true for C and D by definition of $\overline{\mathcal{S}^\bullet}$. We define the function X to be the unique coordinate taking values $\infty, 1, 0$ at A, B, C respectively. We define Y to be the unique coordinate taking values $\infty, 0, 1$ at A, C, D respectively. We set $\mu := \frac{X}{Y} \in \mathbb{C}(\overline{\mathcal{H}^\bullet})$. This situation is summarized in Table 3.

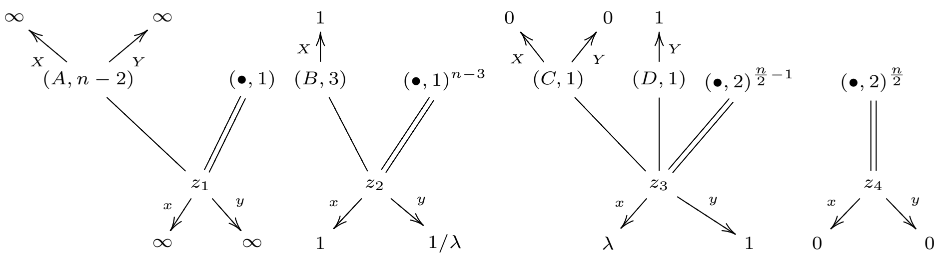


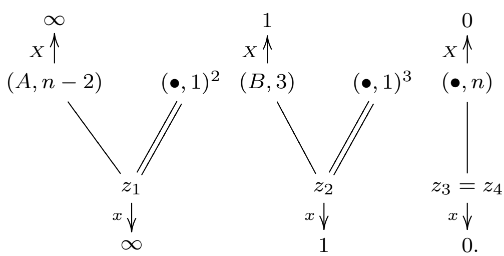
Table 3. Pointing the covers of $\mathcal{H}$ and choice of coordinates.

In order to compute an algebraic model for the cover Φ, we first exhibit an explicit model for a degenerate cover. In other words we compute the residual morphism induced by Φ on the fiber over a boundary point of $\mathcal{M}_{0,4}$ (that is corresponding to z_1, z_2 or z_3). Next, we can rebuilt the entire family by formal deformation of this residual morphism viewed as a morphism of curves on a formal power series ring.

4.2. Degenerate covers and their computation. Let b be a boundary point of $\mathcal{M}_{0,4}$. We fix a point, e.g., $b = z_3$. Since the key point of this section is the explicit algebraic structure of the compactification of moduli spaces of curves of genus zero, we will assume that the reader has some familiarities with these notions. We just recall (see [**GHP88**], Section 3) that the fiber $\mathcal{C}_b = \pi^{-1}(b)$ is a stable 4-pointed tree of projective lines made of two irreducible components. Let us denote by $\mathcal{C}_{b,1}$ the irreducible component of $\mathcal{C}_b$ containing the two closed points z_1 and z_2, and $\mathcal{C}_{b,2}$ the other one. For each point $h \in \overline{\mathcal{H}^\bullet}$ such that $\varphi(h) = b$, set $\mathcal{D}_h = (\Pi^\bullet)^{-1}(h)$. Then the restriction Φ_h of Φ on $\mathcal{D}_h$ is a cover of nodal curves (see for example Figure 2). We denote by $\Phi_{h,i}$ the restriction Φ_h to $\mathcal{D}_{h,i} = \Phi_h^{-1}(\mathcal{C}_i) \cap \mathcal{D}_h$.

For $i = 1, 2$ the morphism $\Phi_{h,i}$ is finite and the ramification locus is contained in the union of the two marked points and the singular point. Now the monodromy can classically be deduced (see [**Cou00**] for example) from the one of the nondegenerate covers in a small neighborhood of h. More precisely, let V be a small enough neighborhood of h (for the complex topology). If the branch cycle description of covers parameterized by $V \setminus \{h\}$ is given by a 4-uple $[\sigma_1, \sigma_2, \sigma_3, \sigma_4]$ then the branch cycle description of $\Phi_{h,1}$ (respectively $\Phi_{h,2}$) is $[\sigma_1\sigma_2, \sigma_3, \sigma_4]$ (respectively $[\sigma_1, \sigma_2, \sigma_3\sigma_4]$).

Let us now concentrate on the computation of an algebraic model for the covers $\Phi_{h,1}$ and $\Phi_{h,2}$. Recall, from the beginning of this section, that we have fixed three rational points $z_1 < z_3 < z_2$. For any $z_4 \in \,]z_1, z_3[$ let p_{z_4} be the cover of $\mathbb{P}^1 \setminus \{z_1, z_4, z_3, z_2\}$ with monodromy $a_{\frac{n}{2}-1}$ in the homotopic basis represented in Figure 1. Letting z_4 tends to z_3 we define a point h in the boundary of $\overline{\mathcal{H}^\bullet}$. For this h, the ramification data for $\Phi_{h,2}$ is:



This cover is a Padé approximant and for the couple of coordinates chosen, we have:

$$x = \Phi_{h,2}(X) = \frac{X^n}{\frac{n(n-1)}{2}\left(X^2 - 2\frac{n-2}{n-1}X + \frac{n-2}{n}\right)}. \tag{5}$$

Similarly for $\Phi_{h,1}$, we find:

$$y = \Phi_{h,1}(Y) = \frac{1}{2}T_n(2Y-1) + \frac{1}{2}, \tag{6}$$

where T_n denotes the Tchebycheff polynomial of order n.

Therefore a very simple model for this degenerate cover is known for every even n. We stress this decisive fact for the relevance of our approach.

4.3. Effective deformation and formal patching. In order to built the entire family, we now formally deform the previous degenerate cover. The deformation technique for covers appeared in [**Ful69**] and were then developed by D. Harbater for the study of the inverse Galois problem over complete local fields (see, e.g., [**Har80**, **Har87**]). In [**Wew99**], S. Wewers gives a presentation of the technique of deformation very well adapted to our purpose.

4.3.1. Computation of a formal model using effective deformation. Let us denote by R the complete local ring of $\overline{\mathcal{H}^\bullet}$ at the point h, namely $\mathbb{C}[[\mu]]$ because μ is a local parameter at h. By base extension $\mathcal{S}_R = \overline{\mathcal{S}^\bullet} \times_{\overline{\mathcal{H}^\bullet}} \mathrm{Spec}(R)$ and $\mathcal{C}_R = \mathcal{M}_{0,5} \times_{\mathcal{M}_{0,4}} \mathrm{Spec}(R)$ are *projective nodal curves* whose special fibers are nothing else than $\mathcal{D}_h$ and $\mathcal{C}_b$ respectively. The cover Φ induces a *tame admissible cover* from $\mathcal{S}_R$ to $\mathcal{C}_R$ which is a deformation of the previous degenerate cover Φ_h represented in Figure 2.

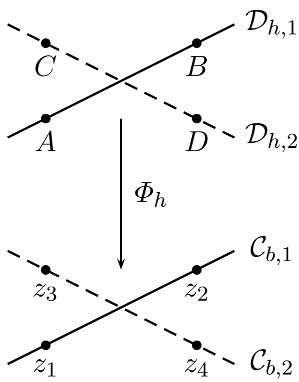


Figure 2. The degenerate cover Φ_h.

We describe the *deformation datum* associated to our deformation as it is explained in the paragraph of [**Wew99**] entitled *The main result* (pages 240-241). Using the preceding and putting $z = 1/y$, the curve $\mathcal{C}_R$ has affine equation $xz = \lambda$. So there is only one ordinary double point δ, i.e., $(0:0:1)$. Moreover, the complete local ring $\mathcal{O}_{\mathcal{C}_R,\delta} \simeq R[[x,z]]/\langle xz - \lambda\rangle$. We also have a *mark* on $\mathcal{C}_R$, namely the horizontal divisor defined by the four generic branched points

$$(0:1:0), \quad (1:\lambda:1), \quad (1:0:0), \quad (\lambda:1:1)$$

and this divisor is étale over $\mathrm{Spec}(R)$. The curve $\mathcal{S}_R$ has also a unique singular point Δ whose local ring is isomorphic to $R[[X,Z]]/\langle XZ - \mu\rangle$ where $Z = 1/Y$. Moreover, due to the ramification of the degenerate cover, we know that $X^n \sim x$, $Z^n \sim z$ and $\mu^n \sim \lambda$ (where $\sim$ means equal up to a unit factor). Therefore the *deformation datum* consists only in $\mu = XZ \in R$.

In concrete terms, effective patching permits us to compute a model of our family over a Puiseux series field $\mathbb{C}((\mu))$ where this field is nothing else than the completion of $\mathbb{C}(\overline{\mathcal{H}^\bullet})$ at the point h. We will call this model an formal one.

In view of our choice of coordinates, the model we are looking for has the following form:

$$S(X) = \frac{(X^{\frac{n}{2}} + \alpha_{\frac{n}{2}-1}X^{\frac{n}{2}-1} + \cdots + \alpha_0)^2}{\gamma(X^2+\beta_1 X+\beta_0)} = \frac{S_0(X)}{S_\infty(X)}$$
$$S(X) - 1 = \frac{(X-1)^3(X^{n-3} + \delta_{n-4}X^{n-4} + \cdots + \delta_0)}{\gamma(X^2+\beta_1 X+\beta_0)} = \frac{S_1(X)}{S_\infty(X)}$$
$$S(X) - \lambda = \frac{X(X-\varepsilon_0)(X^{\frac{n}{2}-1} + \eta_{\frac{n}{2}-2}X^{\frac{n}{2}-2} + \cdots + \eta_0)^2}{\gamma(X^2+\beta_1 X+\beta_0)} = \frac{S_\lambda(X)}{S_\infty(X)}$$

where the $2n$ coefficients $\alpha_i, \beta_j, \delta_k, \eta_l$, ε_0 and γ have to be found in $\mathbb{C}(\overline{\mathcal{H}^\bullet})$. From the three equalities above, one can deduce that $S_0(X) - S_1(X) - S_\infty(X) = 0$ and that $S_0(X) - S_\lambda(X) - \lambda S_\infty(X) = 0$. This gives us a system of $2n$ polynomials in $2n$ variables and with coefficients in $\mathbb{C}[\mu]$ (because λ is a polynomial in μ) which should be satisfied by our $2n$ coefficients.

The knowledge of the degenerate cover gives us first order μ-adic development for all the coefficients. Then computing higher orders is just a careful application of the Newton-Hensel lemma.

4.3.2. Computation of an algebraic model from the formal one. First of all, we change the coordinate X by an homography which fixes 1 and ∞ so as to cancel the coefficient $\alpha_{\frac{n}{2}-1}$ in the polynomial $S_0(X)$:

$$X \longleftarrow \left(1 + \frac{\alpha_{\frac{n}{2}-1}}{\frac{n}{2}}\right) X - \frac{\alpha_{\frac{n}{2}-1}}{\frac{n}{2}}.$$

This natural normalization turns all the coefficients in our model to be defined over $\mathbb{C}(\mathcal{H})$. Hopefully, we have noticed the all the new coefficients are now power series in μ^2; so we are well back on $\mathcal{H}$!

Then the last part of the computation consists in deriving an algebraic model over $\mathbb{C}(\mathcal{H})$ from the preceding formal one defined over $\mathbb{C}((\mu^2))$ (which, we recall; is the completion of $\mathbb{C}(\mathcal{H})$ with respect to a point of $\mathcal{H}$). Theoretically speaking, this steps is based on the Artin's algebraization theorem. From a computational point of view, using the known μ-adic approximations of the coefficients, we should:

- First deduce a generator of $\mathbb{C}(\mathcal{H})$;
- and then express all the coefficients as rational fractions in this generator.

Even if the first step can be done systematically (as is explained in [**Cou99**]), we just guess a generator T among the coefficients and compute all the other coefficients in function of T. If we know the μ-adic approximations of T and of every coefficients C with enough accuracy, finding such expression is just a matter of linear algebra; indeed, for increasing values of the degree d, we have to solve in $\alpha_i, \beta_j \in \mathbb{C}$ an equation like:

$$\alpha_d T^d + \cdots + \alpha_0 + C(\beta_d T^d + \cdots + \beta_0) = 0$$

which expanded in μ gives rise to a linear system in α_i, β_j. We stress that [**Cou99**] gives an upper bound for the degree d.

Remark. The computation of the last two steps could involve computations with complex numbers, and so numerical approximations, because nothing tells us that the model we have chosen is defined over $\mathbb{Q}$. But, luckily, it is!

5. Numerical results.

The two covers we are looking for are given by:

$$\begin{array}{rccc} \varphi: & \mathcal{H} & \longrightarrow & \mathcal{M}_{0,4} \\ & T & \longmapsto & H_n(T) \end{array} \qquad \begin{array}{rccc} \Phi: & \mathcal{S} & \longrightarrow & \mathcal{M}_{0,5} \\ & X & \longmapsto & S_n(T,X). \end{array}$$

For $n = 6$, we obtain:

$$S_6(T,X) = \frac{\left(X^3 + \frac{75T+120}{16}X + \frac{625T^3+3600T^2+6720T+4096}{96T+256}\right)^2}{\frac{3(25T+56)^3}{2^8(3T+8)}\left(X^2 + TX + \frac{25T^3+120T^2+192T+128}{36T+96}\right)}$$

$$S_6(T,X) - 1 = \frac{(X-1)^3\left(X^3 + 3X^2 + \frac{75T+168}{8}X + \frac{625T^3+4950T^2+12960T+11136}{48T+128}\right)}{\frac{3(25T+56)^3}{2^8(3T+8)}\left(X^2 + TX + \frac{25T^3+120T^2+192T+128}{36T+96}\right)}$$

$$\begin{aligned} &S_6(T,X) - H_6(T) \\ &\quad = \frac{\left(X^2 - \frac{5T+8}{2}X + \frac{125T^3+1050T^2+2720T+2176}{48T+128}\right)}{\frac{3(25T+56)^3}{2^8(3T+8)}\left(X^2 + TX + \frac{25T^3+120T^2+192T+128}{36T+96}\right)} \\ &\qquad \cdot \left(X^2 + \frac{5T+8}{4}X + \frac{25T^3+180T^2+424T+320}{24T+64}\right)^2 \end{aligned}$$

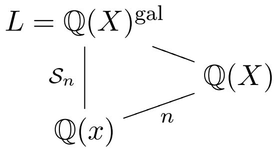
and:

$$H_6(T) = \frac{(T+8)(T+\frac{13}{5})^2(T+\frac{8}{5})^3}{-3\times 5(T+\frac{8}{3})(T+\frac{56}{25})^3}$$

$$H_6(T) - 1 = \frac{(T+2)(T+\frac{16}{5})^5}{-3\times 5(T+\frac{8}{3})(T+\frac{56}{25})^3}.$$

We manage similar computation in Magma for all n up to 20 in less than 20 minutes on an AMD 700Mhz.

In order to obtain totally real $\mathbb{Q}$-extension of degree n with Galois group $\mathcal{S}_n$ and $\mathcal{A}_n$, we have to specialize twice. The method points out a special value $t_h \in \mathbb{Q}$ of the parameter corresponding to the cover h we have deformed. Let us choose a close enough rational number $t_0 < t_h$ and set $t = t_0$. We get a regular $\mathcal{S}_n$-extension:



where: $x = S_n(t_0, X) \in \mathbb{Q}(X)$

with four ramified points $0, H_n(t_0), 1$ and ∞ in this order. Moreover, due to the criterion of Section 3.1, all the points in $]0, H_n(t_0)[$ have totally real fibers. By Hilbert irreducibility theorem, most rationals in this interval provide totally real $\mathcal{S}_n$-extensions of $\mathbb{Q}$. The case of $\mathcal{A}_n$ follows classically.

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