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A PHRAGMÈN–LINDELÖF THEOREM AND THE BEHAVIOR AT INFINITY OF SOLUTIONS OF NON-HYPERBOLIC EQUATIONS

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We prove a Phragmèn–Lindelöf theorem which yields the behavior at infinity of bounded solutions of Dirichlet problems for non-hyperbolic (e.g., elliptic, parabolic) quasilinear second-order partial differential equations in terms of particular solutions of appropriate ordinary differential equations.

1. Introduction.

Many types of Phragmèn-Lindelöf Theorem have appeared in the literature since Edvard Phragmèn and Ernst Lindelöf's famous 1908 article ([**20**]; also see [**3**], Ch. 3). When Ω is an unbounded domain and $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of a Dirichlet problem on Ω for a second-order elliptic or non-hyperbolic equation, a fundamental question is that of the behavior of $f(X)$ as $|X|$ goes to infinity. A Phragmèn-Lindelöf theorem "at infinity" establishes the existence of asymptotic limits of f at infinity and offers insight into the nature of these limits when f lies in an appropriate class of solutions. The goal of this note is to obtain a comprehensive Phragmèn-Lindelöf theory at infinity for bounded solutions of Dirichlet problems in certain types of domains using the "local barrier functions" constructed in [**14**] and solutions of boundary value problems for ordinary differential equations.

Let Ω be an open set in $\mathbf{R}^n$. Suppose $(a_{ij}(X, z, P))$ is any $n \times n$ (symmetric) matrix with trace one which is positive semidefinite for $X \in \Omega$, $z \in \mathbf{R}$, and $P \in \mathbf{R}^n$ and whose entries satisfy $a_{ij} \in C^0(\Omega \times \mathbf{R} \times \mathbf{R}^n)$. Assume further that $a_{nn}(X, z, P) \geq \sigma_1(|P|)$ for some positive continuous function σ_1 defined on $[1, \infty)$. Let b be a function in $C^0(\Omega \times \mathbf{R} \times \mathbf{R}^n)$. Let Q be the non-hyperbolic operator defined by

$$(1) \quad Qu(X) = \sum_{i,j=1}^{n} a_{ij}(X, u(X), Du(X)) D_{ij} u(X) + b(X, u(X), Du(X)).$$

For convenience, let us write elements $X = (x_1, \dots, x_n)$ of $\mathbf{R}^n$ as $(\mathbf{x}, y)$, where

$$\mathbf{x} = (x_1, \dots, x_{n-1}) \quad \text{and} \quad y = x_n$$

and, for each $M > 0$, let S_M denote the set

$$\{X = (x_1, \dots, x_n) \in \mathbf{R}^n \mid |x_n| < M\}.$$

Let us assume here that $\Omega \subset S_M$ for some $M > 0$. If b/a_{nn} have appropriate limits at infinity (i.e., (7)), $\phi \in C^0(\mathbf{R}^n)$, $\omega \in S^{n-2}$ is a direction of Ω at infinity (i.e., (6)) and Assumptions 1 and 2 in §2 are satisfied, we will prove that every bounded solution $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of the Dirichlet problem

$$Qf = 0 \quad \text{in } \Omega \tag{2}$$

and

$$f = \phi \quad \text{on } \partial\Omega \tag{3}$$

satisfies

$$f(\mathbf{x}, y) \to k_\omega(y) \qquad \text{for} \quad X = (\mathbf{x}, y) \in \overline{\Omega} \tag{4}$$

as $|X| \to \infty$ with $\frac{\mathbf{x}}{|\mathbf{x}|} \to \omega$, where k_ω is a solution of a related boundary value problem (e.g., (13)).

We will use local barrier functions and solutions of ordinary differential equations to obtain (4). Rescaling (and truncating) the graph of a barrier function w while leaving unaltered a solution k of an appropriate ordinary differential equation and comparing a bounded solution f of the Dirichlet problem with $w + k$ is a principal technique we will use. As a consequence of the facts that our fundamental comparisons are made in barrier domains of the form $U = \{X \in \mathbf{R}^n \mid C_1 < X \cdot \nu < C_2, |X - (X \cdot \nu)\nu| < h(X \cdot \nu)\}$ for $\nu \in S^{n-1}$ and $C_1 < C_2$ and the ability to rescale w (and so h) improves our estimates, domains in slabs are of particular interest. Our results are significant for:

(i) The generality of allowable domains Ω,
(ii) the generality of allowable operators Q, and
(iii) the simplification achieved by approximating $f(\mathbf{x}, y)$ by $k(y)$ for $|\mathbf{x}|$ large when f is an "unknown" solution of (2) & (3) and k is a "known" solution of a boundary value problem for an ordinary differential equation (e.g., (13)).

Theorem 2.2 complements other Phragmèn-Lindelöf principles at infinity in which the domain has different geometric constraints, for example being required to lie in a cone (e.g., [**1**], [**17**]). The results here hold for a large class of operators, including uniformly elliptic operators, degenerate elliptic operators and parabolic operators. These results can be used to investigate other questions, such as the effect on the behavior at infinity of a solution f when the coefficients of Q are perturbed. Finally, the approximation (near infinity) of the solution of a partial differential equation by the solution of an ordinary differential equation (i.e., (iii)) is a very useful technique which is often used, sometimes without justification, in continuum mechanics (e.g., [9]).

Previous results on Phragmèn-Lindelöf theorems at infinity generally concern limited classes of operators and/or limited types of domains. The cases in which Ω is (or is contained in) a strip in $\mathbf{R}^2$ or a cylinder in $\mathbf{R}^3$ have generated particular interest, in part because of applications of Phragmèn-Lindelöf principles and their companion "spatial decay estimates" to problems in continuum mechanics (e.g., [7]; also see references in [14], [15]). Classes of operators in previous articles include (linear and nonlinear) uniformly elliptic operators or divergence structure operators (e.g., [1], [2], [11], [12], [21]). Theorems containing decay estimates usually concern limited classes of operators in special geometries, including strips (e.g., [10], [11]) and cylinders (e.g., [2], [8]).

In [14], Dirichlet problems in domains $\Omega \subset S_M$ for quasilinear elliptic second-order partial differential equations which do not have lower-order terms are studied. It is shown there that $f(\mathbf{x}, y) \to \Phi(\omega)$ as $|\mathbf{x}| \to \infty$ with $\frac{\mathbf{x}}{|\mathbf{x}|} \to \omega$ when ω is a "direction of Ω at infinity," f is a solution of the Dirichlet problem with Dirichlet data ϕ, $\phi(\mathbf{x}, y) \to \Phi(\omega)$ as $|\mathbf{x}| \to \infty$ with $\frac{\mathbf{x}}{|\mathbf{x}|} \to \omega$, and $b \equiv 0$. In addition, if $\Omega = S_M$, $\phi(\mathbf{x}, M) \to \Phi_1(\omega)$ and $\phi(\mathbf{x}, -M) \to \Phi_2(\omega)$ as $|\mathbf{x}| \to \infty$ with $\frac{\mathbf{x}}{|\mathbf{x}|} \to \omega$, and the other conditions remain unchanged, it is proven that

$$f(\mathbf{x}, y) \to \frac{1}{2M}\big(\Phi_1(\omega) - \Phi_2(\omega)\big)(y + M) + \Phi_2(\omega)$$

as $|\mathbf{x}| \to \infty$ with $\frac{\mathbf{x}}{|\mathbf{x}|} \to \omega$. The results in [14] are significant for the generality of operators Q and domains Ω allowed and especially for the construction of new barrier functions. The inclusion of lower-order terms here complicates the arguments used in [14] in a subtle but significant way; we compensate for this in the Proof of Theorem 2.2 by assuming our solutions are bounded. All arguments occurring here are "local" with respect to the direction ω.

2. Main result.

We will assume from now on that the coefficients of Q have been normalized so that

$$\sum_{i=1}^{n} a_{ii}(X, z, P) = 1 \quad \text{for} \quad (X, z, P) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \tag{5}$$

and satisfy the conditions mentioned previously (i.e., before (1)). We will set $I_M = (-M, M)$ and

$$\pi(\Omega) = \{(x_1, \ldots, x_{n-1}) : \exists_{y \in [-M,M]}\ (x_1, \ldots, x_{n-1}, y) \in \Omega\}.$$

Let $T(\Omega)$ represent the set of directions $\omega \in S^{n-2}$ at infinity of Ω (actually $\pi(\Omega)$); that is

$$T(\Omega) = \cap_{N=1}^{\infty} \overline{\cup_{r\ge N}\{\omega \in S^{n-2} : r\omega \in \pi(\Omega)\}}. \tag{6}$$

Notice that $\omega \in T(\Omega)$ if and only if there exists a sequence $\{(\mathbf{x_j}, y_j)\}$ in Ω with $|\mathbf{x_j}| \to \infty$ and $\frac{\mathbf{x_j}}{|\mathbf{x_j}|} \to \omega$ as $j \to \infty$.

For $\omega \in T(\Omega)$, consider the following assumptions:

Assumption 1. For some open subset O of S^{n-2} with $\omega \in O$, there exists a function $E \in C^0(O \times I_M \times \mathbf{R}^2)$ such that $E\left(\frac{\mathbf{x}}{|\mathbf{x}|}, y, z, q\right)$ is nonincreasing in z and

$$\frac{b(\mathbf{x}, y, z, \mathbf{p}, q)}{a_{nn}(\mathbf{x}, y, z, \mathbf{p}, q)} \to E(\sigma, y, z, q) \tag{7}$$

as $|\mathbf{x}| \to \infty$ with $\frac{\mathbf{x}}{|\mathbf{x}|} \to \sigma$ and $|\mathbf{p}| \to 0$ uniformly for $|y| < M$, $\sigma \in O$, and $z, q \in \mathbf{R}$.

Assumption 2. There exists a function k mapping $\overline{I_M} \times T$ into $\mathbf{R}$ such that

$$\phi(\mathbf{x}, y) \to k(y, \omega) \tag{8}$$

uniformly as $|\mathbf{x}| \to \infty$ and $\frac{\mathbf{x}}{|\mathbf{x}|} \to \omega$ for $(\mathbf{x}, y) \in \partial\Omega$ and, for each $\alpha > 0$, there exist $\delta = \delta_{\alpha,\omega} > 0$ and functions k_1 and k_2 in $C^1(\overline{I_M}) \cap C^2(I_M)$ such that for each $y \in I_M$

$$|k_1(y) - k(y, \omega)| \le \alpha, \tag{9}$$

$$|k_2(y) - k(y, \omega)| \le \alpha, \tag{10}$$

$$k_1''(y) + E(\omega, y, k_1(y), k_1'(y)) \ge \delta, \tag{11}$$

and

$$k_2''(y) + E(\omega, y, k_2(y), k_2'(y)) \le -\delta. \tag{12}$$

Remark 2.1. In what might be the most common situation in which Assumptions 1 and 2 are satisfied for all $\omega \in T(\Omega)$, we would have $\Omega = U \times I_M$ for some open subset U of $\mathbf{R}^{n-1}$, $E \in C^0(S^{n-2} \times I_M \times \mathbf{R}^2)$, $k \in C^0(\overline{I_M} \times T(\Omega))$, $k_\omega \in C^2(I_M)$,

$$k_\omega''(y) + E(\omega, y, k_\omega(y), k_\omega'(y)) = 0 \quad \text{for} \quad |y| < M, \omega \in T(\Omega), \tag{13}$$

where k_ω is defined by $k_\omega(y) = k(y, \omega)$ for $\omega \in T(\Omega)$, and, for each $\omega \in T(\Omega)$, functions k_1 and k_2 respectively satisfying (9)-(12).

Theorem 2.2. *Let $M > 0$, $\Omega \subset S_M$, and $\omega \in T(\Omega)$. Suppose:*

1) *$f \in C^2(\Omega) \cap C^0(\overline{\Omega}) \cap L^\infty(\Omega)$ satisfies (2) & (3);*
2) *Assumptions 1 and 2 are satisfied for ω;*

3) *there exist* $L \geq 0$ *and a positive continuous function* σ_1 *on* $[1,\infty)$ *such that*

$$a_{nn}(\mathbf{x}, y, z, \mathbf{p}, q) \geq \sigma_1(|\mathbf{p}|^2 + q^2) \tag{14}$$

whenever $\mathbf{x}, \mathbf{p} \in \mathbf{R}^{n-1}$, $y, z, q \in \mathbf{R}$ *with* $|\mathbf{x}| \geq L$ *and* $|y| < M$;

4) Q *satisfies* (5).

Then

$$\lim_{j\to\infty} |f(\mathbf{x}_j, y_j) - k(y_j, \omega)| = 0 \tag{15}$$

uniformly for sequences $\{(\mathbf{x}_j, y_j)\}$ *in* $\overline{\Omega}$ *with* $|\mathbf{x}_j| \to \infty$ *and* $\frac{\mathbf{x}_j}{|\mathbf{x}_j|} \to \omega$ *as* $j \to \infty$.

When Q is of a particular type (e.g., uniformly elliptic), arguments exist which show that a solution f of (2) & (3) is bounded whenever it satisfies an appropriate (for Q) growth condition. For such operators, we may assume that the hypothesis $f \in L^\infty(\Omega)$ in Theorem 2.2 is replaced by this growth condition without changing the conclusion of the theorem. From the Proof of Theorem 2.2, it follows that $f \in L^\infty(\Omega)$ can be replaced by f is "bounded in the direction ω" in the sense that there exist $\delta > 0$, $R > 0$, and $J \geq 0$ such that $|f(\mathbf{x}, y)| \leq J$ if $(\mathbf{x}, y) \in \overline{\Omega}$, $|\mathbf{x}| \geq R$, and $\left|\frac{\mathbf{x}}{|\mathbf{x}|} - \omega\right| < \delta$. Finally, the necessity of the nondegerancy condition on a_{nn} (i.e., (14)) is illustrated by Example 4 of [**15**].

We shall also prove the following consequence of Theorem 2.2:

Theorem 2.3. *Let* $\omega \in T$. *Suppose:*

1) $f \in C^2(\Omega) \cap C^0(\overline{\Omega}) \cap L^\infty(\Omega)$ *satisfies* (2) & (3);
2) *Assumption* 1 *is satisfied for* ω;
3) $E = E(y, z, q)$ *is a nonincreasing function of* z *for each* $(y, q) \in \overline{I} \times \mathbf{R}$;
4) $E, \frac{\partial E}{\partial z}, \frac{\partial E}{\partial q} \in C^0(\overline{I} \times \mathbf{R}^2)$;
5) *there exists* $k \in C^2(\overline{I_M})$ *such that*

$$k''(y) + E(\omega, y, k(y), k'(y)) = 0 \qquad \text{for} \quad |y| < M$$

and $\phi(\mathbf{x}, y) \to k(y)$ *uniformly as* $|\mathbf{x}| \to \infty$ *and* $\frac{\mathbf{x}}{|\mathbf{x}|} \to \omega$ *for* $(\mathbf{x}, y) \in \partial\Omega$;

6) *there exist* $L \geq 0$ *and a positive continuous function* σ_1 *on* $[1,\infty)$ *such that*

$$a_{nn}(\mathbf{x}, t, z, \mathbf{p}, q) \geq \sigma_1(|\mathbf{p}|^2 + q^2)$$

whenever $\mathbf{x}, \mathbf{p} \in \mathbf{R}^{n-1}$, $z, t, q \in \mathbf{R}$ *with* $|\mathbf{x}| \geq L$ *and* $|t| \leq M$;

7) Q *satisfies* (5).

Then

$$\lim_{j\to\infty} |f(\mathbf{x}_j, y_j) - k(y_j, \omega)| = 0 \tag{16}$$

uniformly for sequences $\{(\mathbf{x}_j, y_j)\}$ *in* $\overline{\Omega}$ *with* $|\mathbf{x}_j| \to \infty$ *and* $\frac{\mathbf{x}_j}{|\mathbf{x}_j|} \to \omega$ *as* $j \to \infty$.

3. Barrier functions.

Let us review the construction of barrier functions in [14]. This idea originated from the following fact: If $w = w(\mathbf{x}, y)$ in $C^2(\mathbf{R}^n)$ satisfies $Qw = 0$, Q is elliptic (non-hyperbolic), and $g = g(\mathbf{x}, z)$ is a function in $C^2(\mathbf{R}^n)$ for which $g_z \neq 0$ and

$$g(\mathbf{x}, w(\mathbf{x}, y)) = y,$$

then g will satisfy an equation of the form $Q^{\#}g = 0$ for an elliptic (respectively non-hyperbolic) operator $Q^{\#}$, where Q and $Q^{\#}$ are related by the equation

$$Qw(\mathbf{x}, y) = \frac{-1}{g_z^3(\mathbf{x}, w(\mathbf{x}, y))} Q^{\#}g(\mathbf{x}, w(\mathbf{x}, y)). \tag{17}$$

In particular, if $g_z > 0$ and $Q^{\#}g > 0$, then $Qw < 0$. A computation shows that $Q^{\#}$ is defined by

$$Q^{\#}v(\mathbf{x}, z) = \sum_{i,j=1}^{n} A_{ij}(\mathbf{x}, z, v, Dv) D_{ij}v + B(\mathbf{x}, z, v, Dv) \tag{18}$$

for $v = v(\mathbf{x}, z)$ in $C^2(R^n)$ with $\frac{\partial v}{\partial z} \neq 0$, where

$$A_{ij}(\mathbf{x}, z, t, \mathbf{p}, q) = q^2 a_{ij}, \qquad 1 \leq i, j \leq n-1, \tag{19}$$

$$A_{in}(\mathbf{x}, z, t, \mathbf{p}, q) = q a_{in} - \sum_{j=1}^{n-1} p_j q a_{ij}, \qquad 1 \leq i \leq n-1, \tag{20}$$

$$A_{nn}(\mathbf{x}, z, t, \mathbf{p}, q) = a_{nn} - 2\sum_{j=1}^{n-1} p_j a_{jn} + \sum_{i,j=1}^{n-1} p_i p_j a_{ij}, \tag{21}$$

and

$$B(\mathbf{x}, z, t, \mathbf{p}, q) = -q^3 b\left(\mathbf{x}, t, z, -\frac{\mathbf{p}}{q}, \frac{1}{q}\right). \tag{22}$$

Here a_{ij} means $a_{ij}(\mathbf{x}, t, z, -\frac{\mathbf{p}}{q}, \frac{1}{q})$ for $1 \leq i, j \leq n$, $\mathbf{p} = (p_1, \ldots, p_{n-1}) \in \mathbf{R}^{n-1}$, $t \in \mathbf{R}$, $q \neq 0$, $D_i = \frac{\partial}{\partial x_i}$ for $1 \leq i \leq n-1$, $D_n = \frac{\partial}{\partial z}$, $Dv = (D_1 v, \ldots, D_n v)$, and $D_{ij} = D_i D_j$ for $1 \leq i, j \leq n$. The construction of barriers for Q is somewhat similar to the constructions of barriers for the operator $Q^{\#}$ given in [13] and [22].

In this note, we will be unable to use barriers specifically tailored to our operator as was done in [**14**]. Instead, in the construction in §7 of [**14**], we will set $\sigma \equiv 1$ and obtain the functions

$$\chi(\alpha) = \begin{cases} \frac{1}{2} - \ln(\alpha) & \text{if } 0 < \alpha < 1 \\ \frac{1}{2\alpha^2} & \text{if } 1 \le \alpha < \infty \end{cases}$$

and

$$\eta(\beta) = \begin{cases} \frac{1}{\sqrt{2\beta}} & \text{if } 0 < \beta < \frac{1}{2} \\ e^{\frac{1}{2}-\beta} & \text{if } \frac{1}{2} \le \beta < \infty. \end{cases}$$

Also let us define for $H \ge 1$ the number

$$A(H) = 2M \left(\int_1^{e^{\chi(H)}} \eta(\ln(t))\, dt \right)^{-1}.$$

Then for $a > 0$, $H \ge 1$, $\mathbf{x}_0 \in \mathbf{R}^{n-1}$, and $\Gamma \in \mathbf{R}$, the construction in §7 of [**14**] yields the functions $h_a = h_{a,H}$, $g_a = g_{a,\mathbf{x}_0,\Gamma,M,H}$, and $w_a = w_{a,\mathbf{x}_0,\Gamma,H}$ defined by

$$h_a(r) = \begin{cases} a\sqrt{e}\left(\frac{1}{2H^2} - \frac{1}{2} \right) + \frac{a}{\sqrt{2}}\left(\lambda(\sqrt{e}) - \lambda\left(\frac{r}{a}\right)\right) & \text{if } a < r < a\sqrt{e} \\ a\sqrt{e}\left(\frac{1}{2H^2} - \ln\left(\frac{r}{a}\right) \right) & \text{if } a\sqrt{e} \le r < ae^{\chi(H)}, \end{cases}$$

$$g_a(\mathbf{x}, z) = h_a\left(\sqrt{|\mathbf{x} - \mathbf{x}_0|^2 + (z-\Gamma)^2}\right) - M,$$

and

$$w_a(\mathbf{x}, y) = \Gamma - \sqrt{(h_a^{-1}(y+M))^2 - |\mathbf{x} - \mathbf{x}_0|^2},$$

where λ satisfies $\lambda'(t) = \frac{1}{\sqrt{\ln(t)}}$.

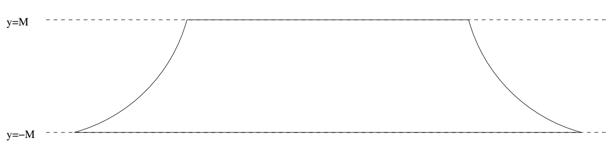

Figure 1. $\Omega_{a,\mathbf{x}_0,H}$.

The domain of $w_{a,\mathbf{x}_0,\Gamma,H}$ is a set $\Omega_{a,\mathbf{x}_0,H} \subset S_M$, illustrated in Figure 1 when $n = 2$, which is relatively compact in $\mathbf{R}^n$ and whose central axis (of symmetry) is $\{(\mathbf{x}_0, y) : \ |y| < M\}$. As the parameter a becomes larger, the domain $\Omega_{a,\mathbf{x}_0,H}$ becomes larger but the variation of w_a along the axis of symmetry decreases and goes to zero as a goes to infinity. This "rescaling" of the barrier w_a by increasing a allows increasingly better estimates of a solution along the central axis; this fact plays a key role in the use of these barriers. As a goes to infinity, w_a also goes to infinity on $\partial\Omega_{a,\mathbf{x}_0,H} \cap S_M$. We assume a solution f is bounded in order to use this fact to help show that $f \le w_a + k_2$ in the Proof of Theorem 2.2; a careful examination of the growth rate of w_a on $\partial\Omega_{a,\mathbf{x}_0,H} \cap S_M$ might allow the growth hypothesis on f (i.e., f is bounded) to be relaxed (e.g., Theorem 2.5 of [**14**]).

4. Proofs of Theorems 2.2 & 2.3.

Proof of Theorem 2.2. We may assume that the set O mentioned in Assumption 1 is all of S^{n-2}. As described in the previous section, in the construction in §7, [**14**] we let $\sigma \equiv 1$ (we ignore (7.1), [**14**]), so that $\Psi \equiv 1$. Let $\omega \in T$, $\epsilon > 0$, and $\alpha = \epsilon$. Let $\delta = \delta_{\alpha,\omega}$, k_1 and k_2 be as given in Assumption 2. From Assumption 2 and the continuity of $k(y,\omega)$, we see that there exist $\delta_1 > 0$ and R_1 such that if $(\mathbf{x}, y) \in \partial\Omega$, $|\mathbf{x}| \ge R_1$, $|y| \le M$, and $\left|\frac{\mathbf{x}}{|\mathbf{x}|} - \omega\right| < \delta_1$, we have

$$|\phi(\mathbf{x}, y) - k(y,\omega)| < \epsilon. \tag{23}$$

Assumption 1 implies there exist $\delta_2 > 0$ and R_2 such that

$$\left|\frac{b(\mathbf{x}, y, z, \mathbf{p}, q)}{a_{nn}(\mathbf{x}, y, z, \mathbf{p}, q)} - E(\omega, y, z, q)\right| \le \frac{\delta}{4} \tag{24}$$

if $|\mathbf{x}| \ge R_2$, $|\mathbf{p}| \le \delta_2$, and $\left|\frac{\mathbf{x}}{|\mathbf{x}|} - \omega\right| \le 2\delta_2$. Consider the compact set

$$K = \{(\mathbf{p}, q) \in \mathbf{R}^n : |\mathbf{p}|^2 + q^2 \le 2(1 + \|k_2'\|_\infty^2)\}.$$

From (14), we see that there exists $\mu(K) > 0$ such that

$$a_{nn}(\mathbf{x}, y, z, \mathbf{p}, q) \ge \mu(K) \qquad \text{if} \quad (\mathbf{p}, q) \in K, \mathbf{x} \in \mathbf{R}^{n-1} \quad \text{and} \quad y, z \in \mathbf{R}.$$

Set $N = \|f - k_2\|_\infty$. Since for fixed ω, $E(\omega, y, z, q)$ is uniformly continuous for (y, z, q) in a fixed compact set, there exists $\delta_3 > 0$ such that

$$|E(\omega, y, z, t + q) - E(\omega, y, z, q)| \le \frac{\delta}{4} \tag{25}$$

for $|t| \le \delta_3$, $|y| < M$, $|z| < N$, and $|q|^2 \le 2(1 + \|k_2'\|_\infty^2)$. Let us set $\delta_0 = \min\{1, \delta_1, \delta_2, \delta_3\}$ and choose $H \ge 2$ such that $\frac{2M}{H} < \epsilon$, $\chi(H) \le \ln(2)$,

$$A(H) \ge 16N, \qquad A(H) \ge \frac{5}{\mu(K)\delta}, \tag{26}$$

and

$$\frac{2\sqrt{2NA(H)e^{\chi(H)}}}{A(H)} + \frac{2}{H} < \delta_0, \tag{27}$$

where $A(H)$ is given in (7.8), [**14**]. There exists $R_3 > 0$ such that if $|\mathbf{x}_0| \ge R_3$, $|\mathbf{x} - \mathbf{x}_0| \le A(H)e^{\chi(H)}$, and $\left|\frac{\mathbf{x}_0}{|\mathbf{x}_0|} - \omega\right| < \delta_0$, then $\left|\frac{\mathbf{x}}{|\mathbf{x}|} - \omega\right| < 2\delta_0$. Set $R_0 = \max\{R_1, R_2, R_3\} + A(H)e^{\chi(H)}$.

Now define

$$W = \left\{ \mathbf{x} \mid |\mathbf{x}| > R_0, \quad \left| \frac{\mathbf{x}}{|\mathbf{x}|} - \omega \right| < \delta_0 \right\}.$$

We claim that if $(\mathbf{x}_0, y) \in \overline{\Omega}$ and $\mathbf{x}_0 \in W$, then

$$f(\mathbf{x}_0, y) < k(y, \omega) + 2\epsilon. \tag{28}$$

Throughout the remainder of this proof, let $\mathbf{x}_0$ represent a point in W such that $(\mathbf{x}_0, y) \in \overline{\Omega}$ for some $y \in I_M$.

Let $w(\mathbf{x}, y) = w_{a,\mathbf{x}_0,\gamma,H}(\mathbf{x}, y)$ be the upper barrier given by (7.14), [**14**] with $\gamma = 2\epsilon$ and $a = A(H)$; a formula for w_a is given in the previous section. Notice then that $w \geq \gamma = 2\epsilon$ on $\Omega_{a,\mathbf{x}_0,H}$. Now set

$$\Omega_1 = \left\{ (\mathbf{x}, y) \in \Omega_{a,\mathbf{x}_0,H} \cap \Omega \ : \ |\mathbf{x} - \mathbf{x}_0| < \sqrt{2NA(H)e^{\chi(H)} - N^2} \right\}, \tag{29}$$

which is illustrated by the shaded region in Figure 2 when $n = 2$,

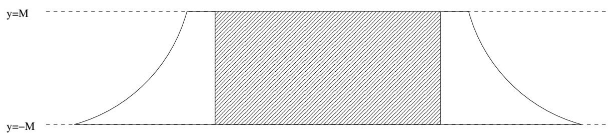

Figure 2. Ω_1.

and define $u_2 \in C^1(\overline{\Omega_1}) \cap C^2(\Omega_1)$ by

$$u_2(\mathbf{x}, y) = w(\mathbf{x}, y) + k_2(y).$$

Notice that if $(\mathbf{x}, y) \in \Omega_1$, then $|\mathbf{x}| \geq \max\{R_1, R_2, R_3\}$, $h_a'(h_a^{-1}(y + M)) \geq H \geq 2$, $A(H) < h_a^{-1}(y + M) < A(H)e^{\chi(H)}$, and $\left| \frac{\mathbf{x}}{|\mathbf{x}|} - \omega \right| < 2\delta_0$.

Let $\zeta \geq 0$. We claim that

$$Q(u_2 + \zeta) < 0 \ \text{ in } \ \Omega_1. \tag{30}$$

From §7, [**14**], we find that

$$\frac{\partial^2 w}{\partial x_i \partial x_j}(\mathbf{x}, y) = \frac{\delta_{ij} S^2 + (x_i - x_i^{(0)})(x_j - x_j^{(0)})}{S^3} \quad \text{for} \quad 1 \leq i, j \leq n - 1,$$

$$\frac{\partial^2 w}{\partial x_i \partial y}(\mathbf{x}, y) = \frac{-(x_i - x_i^{(0)}) Z}{S^3 h_a'(Z)} \quad \text{for} \quad 1 \leq i \leq n - 1,$$

$$\frac{\partial^2 w}{\partial y^2}(\mathbf{x}, y) = \frac{S^2 \big(Z\ h_a''(Z) - h_a'(Z) \big) + Z^2 h_a'(Z)}{S^3 (h_a'(Z))^3}$$

where $\mathbf{x}_0 = \left(x_1^{(0)}, \ldots, x_{n-1}^{(0)}\right)$,

$$Z = h_a^{-1}(y+M), \qquad \text{and}$$
$$S = \sqrt{(h_a^{-1}(y+M))^2 - |\mathbf{x}-\mathbf{x}_0|^2} = \sqrt{Z^2 - |\mathbf{x}-\mathbf{x}_0|^2}.$$

Since $A(H) < Z < 2A(H)$, $|\mathbf{x}-\mathbf{x}_0|^2 < 2NA(H)e^{\chi(H)} - N^2 \leq 4NA(H) - N^2$, and $A(H) \geq 16N$, it is easy to see that $2S^2 \geq (A(H))^2$. Notice then that

$$|Dw(\mathbf{x},y)| \leq \frac{|\mathbf{x}-\mathbf{x}_0|}{S} + \frac{Z}{S|h_a'(Z)|} \leq \frac{2\sqrt{2NA(H)e^{\chi(H)} - N^2}}{A(H)} + \frac{2}{H}$$

and so (27) implies $|Dw(\mathbf{x},y)| < \delta_0$.

If we set $\xi_i = \frac{x_i - x_i^{(0)}}{S}$ for $1 \leq i \leq n-1$, $\xi_n = \frac{-Z}{Sh_a'(Z)}$, and $\xi = (\xi_1, \ldots, \xi_n)$, then $|\xi| \leq 1$ and

$$\frac{1}{S}\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},y,u_2+\zeta,Du_2)\xi_i\xi_j \leq \frac{1}{S}.$$

Since

$$\sum_{i,j=1}^{n-1} a_{ij}(\mathbf{x},y,u_2+\zeta,Du_2)\frac{\delta_{ij}}{S} = \frac{1}{S}(1 - a_{nn}(\mathbf{x},y,u_2+\zeta,Du_2))$$

and

$$\frac{Zh_a''(Z)}{S(h_a'(Z))^3} = -\frac{1}{S},$$

we have

$$\begin{aligned}&\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},y,u_2+\zeta,Du_2)D_{ij}w(\mathbf{x},y)\\ &\leq \frac{2}{S} - \frac{1}{S}\left(2 + \frac{1}{(h_a'(Z))^2}\right)a_{nn}(\mathbf{x},y,u_2+\zeta,Du_2) < \frac{2}{S}.\end{aligned}$$

Since $|Dw(\mathbf{x},y)| < \delta_0$ when $(\mathbf{x},y) \in \Omega_1$, we have $Du_2(\mathbf{x},y) \in K$ and so $a_{nn}(\mathbf{x},y,u_2+\zeta,Du_2) \geq \mu(K)$ if $(\mathbf{x},y) \in \Omega_1$. Set $\mu = \mu(K)$. From (26), we obtain $\frac{2}{S} \leq \frac{\mu\delta}{2}$. Notice that

$$E(\omega,y,u_2+\zeta,q) \leq E(\omega,y,u_2,q) \leq E(\omega,y,k_2,q) \tag{31}$$

for all $y \in I_M$ and $q \in \mathbf{R}$ since $\zeta \geq 0$ and $u_2 = w + k_2 \geq 2\epsilon + k_2 > k_2$. Using (24), (25) and (31), we have

$$\begin{aligned}&Q(w+k_2+\zeta)(\mathbf{x},y)\\ &= \sum_{i,j=1}^{n-1} a_{ij}D_{ij}w + 2\sum_{i=1}^{n-1} a_{in}D_{in}w + (D_{nn}w + k_2''(y))a_{nn} + b\end{aligned}$$

$$\begin{aligned}
&< \frac{\mu\delta}{2} + \left(k_2''(y) + \frac{b(u_2+\zeta, Du_2)}{a_{nn}(u_2+\zeta, Du_2)} \right) a_{nn}(u_2+\zeta, Du_2) \\
&= \frac{\mu\delta}{2} + \Bigg[\frac{b(u_2+\zeta, Du_2)}{a_{nn}(u_2+\zeta, Du_2)} - E(u_2+\zeta, D_n u_2) \\
&\quad + E(u_2+\zeta, D_n u_2) - E(u_2, D_n u_2) + E(u_2, D_n u_2) - E(u_2, k_2') \\
&\quad + E(u_2, k_2') - E(k_2, k_2') + E(k_2, k_2') + k_2''(y) \Bigg] a_{nn} \\
&\le \frac{\mu\delta}{2} + \left[\frac{\delta}{4} + \frac{\delta}{4} - \delta \right] a_{nn}(Du_2) \\
&\le \frac{\mu\delta}{2} - \frac{\mu\delta}{2} = 0,
\end{aligned}$$

where we abbreviate $a_{nn} = a_{nn}(u_2+\zeta, Du_2) = a_{nn}(\mathbf{x}, y, u_2+\zeta, Du_2)$, $b = b(u_2+\zeta, Du_2) = b(\mathbf{x}, y, u_2+\zeta, Du_2)$, $E(u_2, D_n u_2) = E(\omega, y, u_2, D_n u_2) = E(\omega, y, u_2, D_n w + k_2'(y))$, $E(u_2, k_2'(y)) = E(\omega, y, u_2, k_2'(y))$ and $E(k_2(y), k_2'(y)) = E(\omega, y, k_2(y), k_2'(y))$.

If $(\mathbf{x}, y) \in \partial\Omega \cap \partial\Omega_1$, from (7.3), [**14**] and (23) we have

$$f(\mathbf{x}, y) = \phi(\mathbf{x}, y) < k_\omega(y) + \epsilon \le k_2(y) + 2\epsilon = k_2(y) + \gamma \le w(\mathbf{x}, y) + k_2(y).$$

Thus

$$f(\mathbf{x}, y) - u_2(\mathbf{x}, y) < 0 \text{ on } \partial\Omega \cap \partial\Omega_1.$$

If $(\mathbf{x}, y) \in \Omega \cap \partial\Omega_1$, then $|\mathbf{x} - \mathbf{x}_0| = \sqrt{2NA(H)e^{\chi(H)} - N^2}$ and so

$$\begin{aligned}
w(\mathbf{x}, y) &= 2\epsilon + A(H)e^{\chi(H)} - \sqrt{\left(h_a^{-1}(y+M)\right)^2 - |\mathbf{x} - \mathbf{x}_0|^2} \\
&\ge 2\epsilon + A(H)e^{\chi(H)} \\
&\quad - \sqrt{\left(A(H)e^{\chi(H)}\right)^2 - 2NA(H)e^{\chi(H)} + N^2} \\
&= 2\epsilon + N.
\end{aligned}$$

Hence

$$f(\mathbf{x}, y) - k_2(y) \le \|f - k_2\|_\infty = N \le w(\mathbf{x}, y) - 2\epsilon < w(\mathbf{x}, y)$$

and so $f(\mathbf{x}, y) < u_2(\mathbf{x}, y)$ for $(\mathbf{x}, y) \in \Omega \cap \partial\Omega_1$.

Let $U_0 = \{(\mathbf{x}, y) \in \Omega_1 \ : \ f(\mathbf{x}, y) > u_2(\mathbf{x}, y)\}$. Since $f < u_2$ on $\partial\Omega_1$, U_0 is a relatively compact subset of Ω_1 and $f = u_2$ on $\Omega_1 \cap \partial U_0$. Now define

$$Ru(\mathbf{x}, y) = \sum_{ij=1}^{n} \bar{a}_{ij}(\mathbf{x}, y, Du) D_{ij} u(\mathbf{x}, y) + \bar{b}(\mathbf{x}, y, Du)$$

by setting $\bar{a}_{ij}(\mathbf{x}, y, q) = a_{ij}(\mathbf{x}, y, f(\mathbf{x}, y), q)$ and $\bar{b}(\mathbf{x}, y, q) = b(\mathbf{x}, y, f(\mathbf{x}, y), q)$. Let $(\mathbf{x}_1, y_1)$ be an arbitrary point in U_0 and set $\zeta = f(\mathbf{x}_1, y_1) - u_2(\mathbf{x}_1, y_1) > 0$.

Since $Q(u_2+\zeta)<0$ on Ω_1, we have

$$Ru_2(\mathbf{x}_1,y_1)=Q(u_2+\zeta)(\mathbf{x}_1,y_1)<0.$$

Since $(\mathbf{x}_1,y_1)$ is an arbitrary point in U_0, we have $Ru_2<0$ in U_0. Recalling that the ellipticity of R is not needed in Theorem 10.1 of [4] (as noted in the proof of Theorem 3.1 of [4]), we see that $f\le u_2$ on U_0. Hence $U_0=\emptyset$ and so

$$f(\mathbf{x},y)\le u_2(\mathbf{x},y)\ \text{ on }\ \Omega_1.$$

Therefore,

$$f(\mathbf{x}_0,y)\le w(\mathbf{x}_0,y)+k_2(y)\le \frac{2M}{H}+k_2(y)<2\epsilon+k_\omega(y)$$

or $f(\mathbf{x}_0,y)-k(y,\omega)<2\epsilon$.

Together with a similar argument using lower barriers and $k_1(y)$ (i.e., $u_1(\mathbf{x},y)=l_a(\mathbf{x},y)+k_1(y)$ with $\Psi(\rho)=1$), we then find that

$$|f(\mathbf{x}_0,y)-k(y,\omega)|<2\epsilon.$$

Since $\mathbf{x}_0\in W$ is arbitrary, we finally have

$$|f(\mathbf{x},y)-k(y,\omega)|\le 2\epsilon\ \text{ for }\ (\mathbf{x},y)\in\Omega\ \text{ with }\ \mathbf{x}\in W. \tag{32}$$

Now if $\frac{\mathbf{x}_j}{|\mathbf{x}_j|}\to\omega$ as $j\to\infty$, there exists $N>0$ such that $\mathbf{x}_j\in W$. Then from (32), for $(\mathbf{x}_j,y_j)\in\Omega$, we have

$$|f(\mathbf{x}_j,y_j)-k(y_j,\omega)|\le 2\epsilon\ \text{ if }\ j\ge N.$$

Since $\epsilon>0$ is arbitrary, the conclusion of Theorem 2.2 follows.

Proof of Theorem 2.3. Consider first the following:

Lemma 4.1. *Suppose $M>0$, $I=(a,b)\subset I_M$, $E=E(y,z,q)$ is a nonincreasing function of z for each $(y,q)\in\overline{I}\times\mathbf{R}$, and E, $\frac{\partial E}{\partial z}$, $\frac{\partial E}{\partial q}\in C^0(\overline{I}\times\mathbf{R}^2)$. Suppose also that there exists $k\in C^2(\overline{I})$ which satisfies*

$$k''(y)+E(y,k(y),k'(y))=0\qquad\text{for }\ y\in I.$$

Then for each $\delta_1>0$, there is a number $\beta>0$ such that if $c\in\mathbf{R}$ with $|c|<\beta$, then there exists $k_{(c)}\in C^2(\overline{I})$ satisfying

$$k''_{(c)}(y)+E(y,k_{(c)}(y),k'_{(c)}(y))=c,\quad k_{(c)}(a)=k(a),\quad k_{(c)}(b)=k(b),$$

and

$$|k(y)-k_{(c)}(y)|\le\delta_1\quad\text{for}\quad y\in I.$$

Using Lemma 4.1, whose proof is given in the appendix, we see that the hypotheses of Theorem 2.2 are satisfied and then Theorem 2.3 is proven.

5. Examples.

There are many common examples of operators of the form (1), normalized to satisfy (5), which satisfy (14). Some of these are the (normalized a la (5)) Laplace, Poisson, minimal surface, prescribed mean curvature, p-Laplace (for C^2 solutions), and heat (e.g., with $t = x_1$) operators. A C^2 solution of a fully nonlinear equation may also be considered here when the appropiate (normalized) quasilinear operator (i.e., [**4**] (17.10)) satisfies the hypotheses of our Theorems (i.e., p. 444, [**4**]).

Example 5.1. Suppose $n = 2$, $\Omega = \{(x, y) : x > 0, -M < y < M\}$ for some $M > 0$, $h \in C^0(\overline{\Omega} \times \mathbf{R}^3)$, $h(x, y, z, p, q) = m(y) + o(1)$ as $x \to \infty$ uniformly for $|y| \le M$ and $z, p, q \in \mathbf{R}$, $\max_{y \in \overline{I_M}} |\int_0^y m(s)ds| = \alpha_0 < 1$, $\phi(x, \pm M) \to 0$ as $x \to \infty$, and Q is a mean curvature operator with $Qu(x, y)$ equal to

$$\frac{(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy}}{2 + u_x^2 + u_y^2} - \frac{h(x, y, u, u_x, u_y)(1 + u_x^2 + u_y^2)^{\frac{3}{2}}}{2 + u_x^2 + u_y^2}.$$

When f is a solution of (2) & (3), ϕ is bounded, and

$$2M|h(x, y, f(x, y), f_x(x, y), f_y(x, y))| \le \beta_0 < 1$$

for all $(x, y) \in \Omega$, the use of comparison arguments with Delaunay surfaces shows that f is bounded. Notice that $a_{22}(x, y, z, p, q) = \frac{1+p^2}{2+p^2+q^2}$ and $E(\omega, y, q) = -m(y)\big(1 + q^2\big)^{\frac{3}{2}}$. The Dirichlet problem for (13) is

(33)
$$k''(y) = m(y)\big(1 + (k'(y))^2\big)^{\frac{3}{2}} \text{ for } |y| \le M \text{ with } k(-M) = k(M) = 0.$$

Suppose $m(y) > 0$ for $y \in \mathbf{R}$ and $\int_{-M}^{M} m(y)\, dy < 1$. Theorem 2.4 of [**18**] implies (33) has a unique classical solution $k(y)$ and Theorem 2.3 implies

$$\lim_{x \to \infty} f(x, y) = k(y) \tag{34}$$

when $|y| \le M$ for any bounded solution f of (2) & (3). On the other hand, if we had $m(y) > 0$ for $y \in \mathbf{R}$ and $\int_{-M}^{M} m(y)\, dy \ge 1$, Theorem 2.3 (ii) of [**18**] would imply (33) has no solution in $C^1(\overline{I_M}) \cap C^2(I_M)$.

Example 5.2. Suppose $n = 3$, $M = 1$, $\Omega = S_M$, Q is defined by

$$Qu(x_1, x_2, y) = \frac{1}{3}\big(u_{x_1x_1} + u_{x_2x_2} + u_{yy}\big) - \frac{1}{3}\left(u_y^2 + \frac{x_1^2}{1 + |\mathbf{x}|^2}\right),$$

and $\phi(\mathbf{x}, \pm 1) = 0$ for $|\mathbf{x}| \ge 1$. Notice that $E(\omega, y, z, q) = -(q^2 + \omega_1^2)$ for $\omega = (\omega_1, \omega_2) \in S^1$. The Dirichlet problem (13) here is $k''_\omega(y) = (k'_\omega(y))^2 + \omega_1^2$ with $k_\omega(-1) = k_\omega(1) = 0$ and its solution is

$$k(y, \omega) = \ln\big(\sec(\omega_1 y)\big) - \ln\big(\sec(\omega_1)\big).$$

If $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a bounded solution of (2) & (3), Theorem 2.3 implies, for example, that

$$\lim_{r\to\infty} f(r\omega, y) = k(y,\omega) = \ln\big(\sec(\omega_1 y)\big) - \ln\big(\sec(\omega_1)\big)$$

for $|y| \le 1$.

In the next example, the domain has the form $\Omega = U \times I_M$ with U a subset of $\mathbf{R}^2$ which contains the first quadrant of the plane and whose boundary oscillates sinusoidally in the second and fourth quadrants. The conclusion of Theorem 2.2 is satisfied for directions into the (open) first quadrant but not for directions into the (closed) second or fourth quadrant. The behavior at infinity of a bounded solution in an "oscillatory direction" ω for such a domain is an open question.

Example 5.3. Let $n = 3$, $M = 1$, and set $h(r) = \frac{\pi}{4}(1 + \sin(r))$. Let

$$\Omega = \left\{ (r\cos(\theta), r\sin(\theta), y) \in \mathbf{R}^3 : r > 0, -h(r) < \theta < \frac{\pi}{2} + h(r), |y| < 1 \right\}.$$

Notice that the set of directions at infinity for Ω is $T = \{(\cos(\theta), \sin(\theta)) : \theta \in [-\frac{\pi}{2}, \pi]\}$. Set $T_0 = \{(\cos(\theta), \sin(\theta)) : \theta \in (0, \frac{\pi}{2})\}$ and $T_1 = T \setminus T_0$. Define Q by $Qu = \frac{1}{3}(\triangle u - u)$ and $\phi \equiv \cosh(1)$; notice that $E(\omega, y, z, q) = -z$. Let $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ denote a bounded solution of (2) & (3).

Suppose first that $\omega \in T_0$. Then (8) and (13) yield

$$k''_\omega(y) - k_\omega(y) = 0 \quad \text{for} \quad -1 < y < 1, \qquad k_\omega(\pm 1) = \cosh(1),$$

and so $k_\omega(y) = \cosh(y)$. Setting $k_1(y) = \cosh((1+\epsilon)y)$ and $k_2(y) = \cosh((1-\epsilon)y)$ for $\epsilon > 0$ sufficiently small shows that Assumption 2 is satisfied. Theorem 2.2 then implies

$$f(\mathbf{x}, y) \to \cosh(y)$$

as $|\mathbf{x}| \to \infty$ with $\frac{\mathbf{x}}{|\mathbf{x}|} \to \omega$ uniformly for $|y| \le 1$.

Suppose second that $\omega \in T_1$. Notice that (8) requires $k_\omega(y) = \cosh(1)$ for all $y \in [-1, 1]$. However this constant function is not a solution of (13). In fact, it is impossible to obtain a function $k_1(y)$ which satisfies (9) and (11) when α is sufficiently small. (Notice that (9) implies $k_1(\pm 1) \le \cosh(1) + \alpha$ and (11) then implies $k_1(y) \le \left(1 + \frac{\alpha}{\cosh(1)}\right)\cosh(y)$. On the other hand, (9) implies $k_1(y) \ge \cosh(1) - \alpha$ and so $\cosh(1) - \alpha \le k_1(0) \le 1 + \frac{\alpha}{\cosh(1)}$; this is impossible for $\alpha > 0$ sufficiently small.) This means that the hypotheses of Theorem 2.2 are not satisfied when $\omega \in T_1$. (If the Dirichlet data had satisfied $\phi(\mathbf{x}, y) \to \cosh(y)$ as $\mathbf{x} \to \infty$, then Theorem 2.2 would have been applicable for all directions $\omega \in T$ and our conclusion would be that $f(\mathbf{x}, y) \to \cosh(y)$ as $|\mathbf{x}| \to \infty$ uniformly for $|y| \le 1$.) If we set $k_1(y) = \cosh(y)$ and $k_2(y) = \cosh(1)$, the comparison argument in the Proof of Theorem 2.2 shows that for $|y| \le 1$ and $\omega \in T_1$,

$$\cosh(y) \le \liminf_{r\to\infty} f(r\omega, y) \le \limsup_{r\to\infty} f(r\omega, y) \le \cosh(1).$$

A general characterization of the behavior of a bounded solution of (2) & (3) when the boundary oscillates in a manner similar to that considered here would be very interesting.

Using our techniques, some structural conditions on Q which imply that all solutions $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of (2) & (3) are bounded can be obtained. Although we lmit our discussion here primarily to domains in slabs S_M, the geometric condition that $\Omega \subset S_M$ can be weakened substantially (e.g., §5, [**14**]) without changing the conclusion that $f(\mathbf{x}, y) \to k_\omega(y)$ as $|\mathbf{x}| \to \infty$. Theorem 2.2 can also be applied to determine the asymptotic behavior of solutions of Dirichlet problems in exterior domains for certain types of operators.

Appendix.

Proof of Lemma 4.1. We may assume $I = I_M$. Suppose first that a function $k_c(y)$ satisfying the conclusion of Lemma 4.1 did exist. If we were to set $s(y) = k_c(y) - k(y)$, then $s(y)$ would satisfy $|s(y)| \le \delta_1$ for $|y| \le M$ and

$$s''(y) + k''(y) + E(y, k(y) + s(y), k'(y) + s'(y)) = c \qquad \text{with } \; s(-M) = 0, \quad s(M) = 0.$$

If we define $G(y, z, q) = E(y, k(y)+z, k'(y)+q) - E(y, k(y), k'(y))$ and recall that $k''(y) + E(y, k(y), k'(y)) = 0$, we see that $s(y)$ would satisfy

$$s(-M) = 0, \qquad s(M) = 0, \tag{35}$$

$$s''(y) + G(y, s(y), s'(y)) = c, \tag{36}$$

and

$$|s(y)| \le \delta_1 \quad \text{for} \quad |y| \le M. \tag{37}$$

Conversely, if we find a function $s \in C^2(\overline{I})$ which satisfies (36), (37), $s(-M) = 0$, and $s(M) = 0$, then the function $k_c(y) = k(y) + s(y)$ satisfies the conclusion of Lemma 4.1.

Let us rewrite (36) as

$$\begin{aligned} s''(y) &+ \frac{\partial G}{\partial q}(y,0,0)s'(y) + \frac{\partial G}{\partial z}(y,0,0)s(y) \\ &= c + \frac{\partial G}{\partial q}(y,0,0)s'(y) + \frac{\partial G}{\partial z}(y,0,0)s(y) - G(y, s(y), s'(y)). \end{aligned}$$

We define a sequence $\{s_n\}$ by

$$s_1(y) = 0 \quad on \quad |y| \le M$$

and, for $n \geq 1$,

$$(38)\qquad s''_{n+1}(y) + \frac{\partial G}{\partial q}(y,0,0)s'_{n+1}(y) + \frac{\partial G}{\partial z}(y,0,0)s_{n+1}(y) = c + \frac{\partial G}{\partial q}(y,0,0)s'_n(y) + \frac{\partial G}{\partial z}(y,0,0)s_n(y) - G(y, s_n(y), s'_n(y))$$

with

$$(39)\qquad s_{n+1}(-M) = 0, \quad s_{n+1}(M) = 0.$$

We claim that when $|c|$ is small enough, the sequence $s_n(y)$ converges uniformly on $[-M, M]$ to a function $s \in C^2(\bar{I})$ which satisfies (36), (37), and (35).

We require several estimates. Consider the boundary value problem

$$w''(y) + \frac{\partial G}{\partial q}(y,0,0)w'(y) + \frac{\partial G}{\partial z}(y,0,0)w(y) = h(y), \qquad w(-M) = 0, \quad w(M) = 0.$$

Since $E(y,z,q)$ is non-increasing on z, $\frac{\partial G}{\partial z}(y,0,0) = \frac{\partial E}{\partial z}(y,k(y),k'(y)) \leq 0$ for all $|y| \leq M$. Now we apply Theorem 3.7 (or the proof of Theorem 3.7) in [4] to conclude that there is a constant c_1 depending on E and $k(y)$ such that

$$(40)\qquad |w(y)| \leq c_1 |h|_{C^0([-M,M])} \quad on \quad |y| \leq M;$$

for notational simplicity, we will write $\|u\|$ or $\|u(y)\|$ for the value of the supremum norm $|u|_{C^0([-M,M])}$ of a function u. Using the equation

$$(41)\qquad w''(y) + \frac{\partial G}{\partial q}(y,0,0)w'(y) = h(y) - \frac{\partial G}{\partial z}(y,0,0)w(y),$$

we see that

$$w'(y) = \int_{-M}^{y} \exp\left(\int_y^t \frac{\partial G}{\partial q}(\alpha,0,0)d\alpha\right)\left(h(t) - \frac{\partial G}{\partial z}(t,0,0)w(t)\right)dt + B\exp\left(-\int_{-M}^{y} \frac{\partial G}{\partial q}(\alpha,0,0)d\alpha\right)$$

where

$$B = -\frac{\int_{-M}^{M}\int_{-M}^{y} \exp\left(\int_y^t \frac{\partial G}{\partial q}(\alpha,0,0)d\alpha\right)\left(h(t) - \frac{\partial G}{\partial z}(t,0,0)w(t)\right)dt\,dy}{\int_{-M}^{M} \exp\left(-\int_{-M}^{y} \frac{\partial G}{\partial q}(\alpha,0,0)d\alpha\right)dy}.$$

Using (40), we see that for some constant c_2,

$$(42)\qquad |w'(y)| \leq c_2\|h\| \quad \text{for} \quad |y| \leq M.$$

Using (40), (42), and

$$w''(y) = h(y) - \frac{\partial G}{\partial q}(y,0,0)w'(y) - \frac{\partial G}{\partial z}(y,0,0)w(y),$$

we conclude that there is a constant c_3 depending only on M, G (hence E, k) such that

$$|w(y)| \le c_3\|h\| \quad \text{for} \quad |y| \le M; \tag{43}$$

$$|w'(y)| \le c_3\|h\| \quad \text{for} \quad |y| \le M; \tag{44}$$

$$|w''(y)| \le c_3\|h\| \quad \text{for} \quad |y| \le M. \tag{45}$$

Setting $w(y) = s_{n+1}(y)$ in (43)-(45), for $|y| \le M$ we obtain

$$\begin{aligned}
|s_{n+1}(y)| &\le c_3\biggl(|c| + \biggl\|\frac{\partial G}{\partial q}(y,0,0)s_n'(y) + \frac{\partial G}{\partial z}(y,0,0)s_n(y) \\
&\qquad - G(y, s_n(y), s_n'(y))\biggr\|\biggr) \\
|s_{n+1}'(y)| &\le c_3\biggl(|c| + \biggl\|\frac{\partial G}{\partial q}(y,0,0)s_n'(y) + \frac{\partial G}{\partial z}(y,0,0)s_n(y) \\
&\qquad - G(y, s_n(y), s_n'(y))\biggr\|\biggr) \\
|s_{n+1}''(y)| &\le c_3\biggl(|c| + \biggl\|\frac{\partial G}{\partial q}(y,0,0)s_n'(y) + \frac{\partial G}{\partial z}(y,0,0)s_n(y) \\
&\qquad - G(y, s_n(y), s_n'(y))\biggr\|\biggr).
\end{aligned}$$

Now since $\frac{\partial E}{\partial q}(y,z,q)$ and $\frac{\partial E}{\partial z}(y,z,q)$ are continuous on $[-M,M]\times R^2$, $\frac{\partial E}{\partial q}(y,z,q)$ and $\frac{\partial E}{\partial z}(y,z,q)$ are uniformly continuous on $[-M,M]\times D$ for any compact set $D \subset R^2$. Since $k(y) \in C^2([-M,M])$, it follows that $\frac{\partial G}{\partial q}(y,z,q)$ and $\frac{\partial G}{\partial z}(y,z,q)$ are uniformly continuous on $[-M,M]\times[-1,1]\times[-1,1]$. Then there is a constant $\gamma > 0$ such that

$$\left|\frac{\partial G}{\partial q}(y,0,0) - \frac{\partial G}{\partial q}(y,\alpha_1,\alpha_2)\right| \le \frac{1}{8c_3} \tag{46}$$

$$\left|\frac{\partial G}{\partial z}(y,0,0) - \frac{\partial G}{\partial z}(y,\alpha_1,\alpha_2)\right| \le \frac{1}{8c_3} \tag{47}$$

for all $|\alpha_1| \le \gamma$, $|\alpha_2| \le \gamma$, $|y| \le M$. Now set

$$\delta = \min\{\delta_1, \gamma\}, \quad \delta_2 = \frac{3}{4c_3}\delta.$$

We claim that for $|c| \le \delta_2$,

$$|s_n(y)| \le \delta, \quad |s_n'(y)| \le \delta, \quad |s_n''(y)| \le \delta \tag{48}$$

for all n, $|y| \le M$. Indeed, if $n = 1$, (48) is obvious since $s_1(y) = 0$. Now we assume (48) holds for all integers up to n. In order to prove (48) for $n+1$, we need only show that

$$c_3 \left(|c| + \left\| \frac{\partial G}{\partial q}(y,0,0)s_n'(y) + \frac{\partial G}{\partial z}(y,0,0)s_n(y) - G(y, s_n(y), s_n'(y)) \right\| \right) \le \delta.$$

Since $G(y, 0, 0) = 0$, we have from the mean value theorem

$$\begin{aligned} &G(y, s_n(y), s_n'(y)) \\ &= G(y, s_n(y), s_n'(y)) - G(y,0,0) \\ &= \frac{\partial G}{\partial q}(y, \theta(y)s_n(y), \theta(y)s_n'(y))s_n'(y) + \frac{\partial G}{\partial z}(y, \theta(y)s_n(y), \theta(y)s_n'(y))s_n(y) \end{aligned}$$

for some function $\theta(y)$ on $[-M, M]$ with $|\theta(y)| \le 1$. Then from (46), (47) and (48), we have

$$\begin{aligned} &\left\| \frac{\partial G}{\partial q}(y,0,0)s_n'(y) + \frac{\partial G}{\partial z}(y,0,0)s_n(y) - G(y, s_n(y), s_n'(y)) \right\| \\ &\le \left\| \frac{\partial G}{\partial q}(y,0,0) - \frac{\partial G}{\partial q}(y, \theta(y)s_n(y), \theta(y)s_n'(y)) \right\| \cdot \|s_n'\| \\ &\quad + \left\| \frac{\partial G}{\partial z}(y,0,0) - \frac{\partial G}{\partial z}(y, \theta(y)s_n(y), \theta(y)s_n'(y)) \right\| \cdot \|s_n\| \le \frac{1}{4c_3}\delta. \end{aligned}$$

Hence

$$\begin{aligned} &c_3 \left(|c| + \left\| \frac{\partial G}{\partial q}(y,0,0)s_n'(y) + \frac{\partial G}{\partial z}(y,0,0)s_n'(y) - G(y, s_n(y), s_n'(y)) \right\| \right) \\ &\le c_3 \left(|c| + \frac{1}{4c_3}\delta \right) \le c_3 \left(\delta_2 + \frac{1}{4c_3}\delta \right) = \delta. \end{aligned}$$

This implies (48) holds for all n.

We claim that the sequence $s_n(y)$ converges uniformly on $[-M, M]$. Let us set $w(y) = s_{n+1}(y) - s_n(y)$. From (38), we have

$$\begin{aligned} &w''(y) + \frac{\partial G}{\partial q}(y,0,0)w'(y) + \frac{\partial G}{\partial z}(y,0,0)w(y) = h(y), \\ &\qquad\qquad\qquad\qquad w(-M) = 0, \quad w(M) = 0, \end{aligned}$$

where the function h is defined by

$$\begin{aligned} h(y) = &\frac{\partial G}{\partial q}(y,0,0)(s_n(y) - s_{n-1}(y))' + \frac{\partial G}{\partial z}(y,0,0)(s_n(y) - s_{n-1}(y)) \\ &- (G(y, s_n(y), s_n'(y)) - G(y, s_{n-1}(y), s_{n-1}'(y))). \end{aligned}$$

Thus from (43)-(45), we have the inequalities

$$|s_{n+1}(y) - s_n(y)| \quad \text{and} \quad |s'_{n+1}(y) - s'_n(y)| \quad \text{and} \quad |s''_{n+1}(y) - s''_n(y)|$$
$$\le c_3 \bigg| \frac{\partial G}{\partial q}(y,0,0)(s_n(y) - s_{n-1}(y))' + \frac{\partial G}{\partial z}(y,0,0)(s_n(y) - s_{n-1}(y))$$
$$-(G(y, s_n(y), s'_n(y)) - G(y, s_{n-1}(y), s'_{n-1}(y)))\bigg|_{C^0([-M,M])}.$$

Setting $r(y) = s_n(y) - s_{n-1}(y)$ for a moment and using the mean value theorem, we see that the sum above is bounded by

$$c_3 \bigg[\bigg\| \frac{\partial G}{\partial q}(y,0,0) - \frac{\partial G}{\partial q}(y, s_n(y) + \eta(y)r(y), s'_n(y) + \eta(y)r'(y)) \bigg\| \; \|r'(y)\|$$
$$+ \bigg\| \frac{\partial G}{\partial z}(y,0,0) - \frac{\partial G}{\partial z}(y, s_n(y) + \eta(y)r(y), s'_n(y) + \eta(y)r'(y)) \bigg\| \; \|r(y)\| \bigg],$$

for some functions $\eta(y)$ on $[-M, M]$ with $|\eta(y)| \le 1$. From (46) and (47), we see that this sum is bounded by

$$\frac{1}{8}\|s'_n(y) - s'_{n-1}(y)\| + \frac{1}{8}\|s_n(y) - s_{n-1}(y)\|;$$

hence for $|y| \le M$,

$$|s_{n+1}(y) - s_n(y)| \le \frac{1}{8}\|s'_n - s'_{n-1}\| + \frac{1}{8}\|s_n - s_{n-1}\|,$$
$$|s'_{n+1}(y) - s'_n(y)| \le \frac{1}{8}\|s'_n - s'_{n-1}\| + \frac{1}{8}\|s_n - s_{n-1}\|, \qquad \text{and}$$
$$|s''_{n+1}(y) - s''_n(y)| \le \frac{1}{8}\|s'_n - s'_{n-1}\| + \frac{1}{8}\|s_n - s_{n-1}\|.$$

Setting

$$C = \|s_2(y) - s_1(y)\| + \|s'_2(y) - s'_1(y)\| + \|s''_2(y) - s''_1(y)\|$$

and using the inequalities above, we observe that an induction argument yields

$$|s_{n+1} - s_n|_{C^0([-M,M])} \le C \left(\frac{1}{4}\right)^{n-1},$$
$$|s'_{n+1} - s'_n|_{C^0([-M,M])} \le C \left(\frac{1}{4}\right)^{n-1},$$
$$|s''_{n+1} - s''_n|_{C^0([-M,M])} \le C \left(\frac{1}{4}\right)^{n-1},$$

for all $n \ge 2$.

Therefore $s_n(y)$, $s'_n(y)$ and $s''_n(y)$ converge uniformly on $[-M, M]$. Let

$$s(y) = \lim_{n \longrightarrow \infty} s_n(y).$$

Then it is easy to see $s(y)$ is in $C^2([-M, M])$ and satisfies (36)-(37). This completes the proof.

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