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Consider a wildly ramified G-Galois cover of curves $\phi : Y \to X$ branched at only one point over an algebraically closed field k of characteristic p. In this paper, given G such that the Sylow p-subgroups of G have order p, I show it is possible to deform ϕ to increase the conductor at a wild ramification point. As a result, I prove that all sufficiently large conductors occur for covers $\phi : Y \to \mathbb{P}^1_k$ branched at only one point with inertia $\mathbb{Z}/p$. For the proof, I show there exists such a cover with small conductor under an additional hypothesis on G and then use deformation and formal patching to transform this cover.

1. Introduction.

1.1. Results. Let X be a smooth connected proper curve with marked points $\{x_i\}$ over an algebraically closed field k of characteristic p. Consider a Galois cover $\phi : Y \to X$ of smooth connected curves branched only at $\{x_i\}$. Abhyankar's Conjecture (proved by Raynaud [**10**] and Harbater [**4**]) determines exactly which groups G can be the Galois group of ϕ. An open problem is to determine which inertia groups and filtrations of higher ramification groups can be realized for such a cover ϕ. More simply, it is unknown which integers can be realized as the genus of Y.

The main results of this paper are for the case that $\phi : Y \to \mathbb{P}^1_k$ is branched at only one point. Such covers exist if and only if G is a quasi-p group, which means that G is generated by p-groups. Harbater [**3**] proved that the Sylow p-subgroups of G can be realized as the inertia groups of such a cover ϕ. Under the assumption that the Sylow p-subgroups of G have order p, the filtration of higher ramification groups is determined by one integer j for which $p \nmid j$, namely by the lower jump or conductor. In Theorem 3.2.4, I prove in this case that all sufficiently large conductors occur for such covers of the affine line. Theorem 3.2.4 involves the concept of the *p-weight* which is defined in Section 3.1.

Theorem 3.2.4. *Let G be a finite quasi-p group whose Sylow p-subgroups have order p. There exists an integer J depending explicitly on p, the p-weight of G, and the exponent of the normalizer of a Sylow p-subgroup of G with the following property: If $j \geq J$ and $p \nmid j$ then there exists a G-Galois* *cover $\phi : Y \to \mathbb{P}^1_k$ branched at only one point over which it has inertia group $\mathbb{Z}/p$ and conductor j.*

The first part of the proof is to show that all sufficiently large conductors will occur. To do this, I show the following more general result in Theorem 2.2.2: Suppose $\phi : Y \to X$ is a G-Galois cover with inertia group I of the form $\mathbb{Z}/p \rtimes \mu_m$ and conductor j at a ramification point; then it is possible to deform ϕ to increase this conductor. To do this, I construct a family of covers so that ϕ is isomorphic to the normalization of one fibre of the family. The techniques consist of local deformations and formal patching theorems of Harbater and Stevenson [**5**]. Since k is algebraically closed, it is then possible to use another fibre of this family to find another cover with the same group G and inertia group I but with a larger conductor. For certain applications, it is necessary to use ramification data of two covers and deform semi-stable curves in order to enlarge the Galois group and to change the inertia group, while simultaneously enlarging the conductor; see Theorem 2.3.7.

The second part of the proof is to find a relatively small integer J (depending only on the group theory of G) for which there exists a G-Galois cover $\phi : Y \to \mathbb{P}^1_k$ branched at only one point over which it has inertia group $\mathbb{Z}/p$ and conductor J. For this I use the following result which says roughly speaking that there exists such a cover of the affine line with very small conductor when G has p-weight one. (See 3.2.1 for the definition of $j_{\min}(I)$, which is a small set of integers depending only on I and not on G consisting of the minimal possible conductors for a cover of the affine line with inertia $I \simeq \mathbb{Z}/p \rtimes \mu_m$.)

Theorem 3.2.2. *Let G be a finite quasi-p group of p-weight one whose Sylow p-subgroups have order $p \neq 2$. For some $I \simeq \mathbb{Z}/p \rtimes \mu_m \subset G$ and some $j \in j_{\min}(I)$, there exists a G-Galois cover $\phi : Y \to \mathbb{P}^1_k$ of smooth connected curves branched at only one point over which it has inertia group I and conductor j. In particular,* genus $(Y) \leq 1 + |G|(p-1)/2p$.

This result was announced in [**7**]. The idea behind its proof is to reverse the process in Section 2 to decrease the conductor of a G-Galois cover $\phi : Y \to \mathbb{P}^1_k$ branched at only one point. This is done by analyzing the stable model of a family of covers with bad reduction over an equal characteristic discrete valuation ring. This is motivated by the work of Raynaud [**11**] in unequal characteristic.

1.2. Notation and background. Let k be an algebraically closed field of characteristic p. Let $R \simeq k[[t]]$ be an equal characteristic complete discrete valuation ring with residue field k and fraction field $K \simeq k((t))$. For each $m \in \mathbb{N}$ with $\gcd(m,p) = 1$, choose an mth root of unity $\zeta_m \in k$ such that

$\zeta_{m_2} = \zeta_{m_2 m_1}^{m_1}$. Let G be a finite group and let S be a chosen Sylow p-subgroup of G. The group G is *quasi-p* if it is generated by all its Sylow p-subgroups.

If X is a scheme over R, we assume that the morphism $f : X \to \mathrm{Spec}(R)$ is separated, flat and of finite type. If ξ is a point of a scheme X, the *germ* $\hat{X}_\xi$ of X at ξ is defined to be the spectrum of the complete local ring of functions of X at ξ. Suppose a scheme X is reduced and connected, but not necessarily irreducible. A morphism $\phi : Y \to X$ of schemes is a (possibly branched) *cover* if ϕ is finite and generically separable. A *G-Galois cover* is a cover $\phi : Y \to X$ along with a choice of homomorphism $G \to \mathrm{Aut}_X(Y)$ by which G acts simply transitively on each generic geometric fibre of ϕ (again allowing branching). If $\phi : Y \to X$ is a G-Galois cover and $G \subset G'$, define the *induced cover* $\mathrm{Ind}_G^{G'}(Y) \to X$ to be the disconnected G'-Galois cover consisting of $(G' : G)$ copies of Y indexed by the left cosets of G with the induced action of G'.

Consider a wildly ramified G-Galois cover of curves $\phi : Y \to X$ with branch locus B. See [**13**, Chapter IV] for information about the higher ramification groups of ϕ. In particular, suppose $\xi \in B$ is a closed point and $\eta \in \phi^{-1}(\xi)$. The inertia group I of ϕ at η is of the form $I = P \rtimes \mu_m$ where P is a p-group and $p \nmid m$. In the case that $I \simeq \mathbb{Z}/p \rtimes \mu_m$, the *conductor* of ϕ at η is the integer $j = \mathrm{val}(q(\pi_\eta) - \pi_\eta) - 1$ where π_η is a uniformizer of Y at η and q has order p in I. In this case, the conductor j is the unique *lower jump* in the filtration of higher ramification groups and the *upper jump* is $\sigma = j/m \in \mathbb{Q}$. Up to isomorphism, these objects do not depend on the choice of η above ξ. If ξ is not a closed point of X, the inertia group, filtration of higher ramification groups, and conductor for ϕ at η are the corresponding objects over the generic point of η.

This paper frequently uses the following technique of Harbater and Stevenson [**5**] (also see [**9**] for the case that $R \not\simeq k[[t]]$). Let X be a projective k-curve that is connected and reduced but not necessarily irreducible. Let $\mathbb{S}$ be a finite closed subset of X which contains the singular locus of X.

Definition 1.2.1. A *thickening problem of covers* for $(X, \mathbb{S})$ consists of the following data:

(a) A cover $f : Y \to X$ of geometrically connected reduced projective k-curves;
(b) for each $s \in \mathbb{S}$, a Noetherian normal complete local domain R_s containing R such that t is contained in the maximal ideal of R_s, along with a finite generically separable R_s-algebra A_s;
(c) for each $s \in \mathbb{S}$, a pair of k-algebra isomorphisms $F_s : R_s/(t) \to \hat{\mathcal{O}}_{X,s}$ and $E_s : A_s/(t) \to \hat{\mathcal{O}}_{Y,s}$ which are compatible with the inclusion morphisms.

Definition 1.2.2. A thickening problem is *G-Galois* if f and $R_s \subset A_s$ are G-Galois and the isomorphisms F_s are compatible with the G-Galois action (for all $s \in \mathbb{S}$). A *thickening* of X is a projective normal R-curve X^* such that $X^*_k \simeq X$. A thickening problem is *relative* if the data for the problem also includes a thickening X^* of X, so that X^* is a trivial deformation of X away from $\mathbb{S}$ and so that the pullback of X^* over the complete local ring at a point $s \in \mathbb{S}$ is isomorphic to R_s.

Definition 1.2.3. A *solution* to a thickening problem of covers is a cover $f^* : Y^* \to X^*$ of projective normal R-curves, where X^* is a thickening of X, whose closed fibre is isomorphic to f, whose pullback to the formal completion of X^* along $X' = X - \mathbb{S}$ is a trivial deformation of the restriction of f over X', and whose pullback over the complete local ring at a point $s \in \mathbb{S}$ is isomorphic to $R_s \subset A_s$ (and such that everything is compatible with the isomorphisms above). (Note that X^* is a thickening of X.)

Theorem 1.2.4 (Harbater, Stevenson)**.** *Every* (G*-Galois*) *thickening problem for covers has a* (G*-Galois*) *solution. The solution is unique if the thickening problem is relative.*

Proof. [5, Theorem 4]. □

2. Deformation of covers.

Consider a G-Galois cover $\phi : Y \to X$ of smooth connected proper k-curves. Let ξ be in the branch locus of ϕ and let $\eta \in \phi^{-1}(\xi)$. The goal in this section is to deform the cover ϕ with precise control over the ramification behavior near ξ. To do this it is first necessary to deform the I-Galois cover $\hat{\phi} : \hat{Y}_\eta \to \hat{X}_\xi$ of germs of curves near ξ with such control. We assume throughout that p strictly divides the order of the inertia group I.

2.1. Covers of complete local rings. Let I be a semi-direct product $\mathbb{Z}/p \rtimes \mu_m$ with $p \nmid m$. Let n' be the order of the prime-to-p part of the center of I. Let $U = \mathrm{Spec}(k[[u]])$ and let b be the closed point of U. The next results describe the structure of I-Galois covers $\phi : X \to U$ of germs of curves with lower jump j in the filtration of higher ramification groups above b.

Definition 2.1.1. Suppose $\phi : X \to U$ is an I-Galois cover of germs of curves such that X is connected but not necessarily normal. Let r_1 be the number of connected components of the normalization of X and assume that $p \nmid r_1$. Define the *inertia group* of ϕ to be the inertia group I_1 of a closed point in the normalization. The order of I is $pm = pm_1r_1$ where pm_1 is the order of I_1. The *conductor* (respectively *upper jump*) of ϕ is defined to be the conductor (respectively upper jump) of a ramification point in the normalization.

Lemma 2.1.2. *Let $I \simeq \mathbb{Z}/p \rtimes \mu_m$ and let $I_1 \subset I$ have index r_1 with $p \nmid r_1$. Write $m = r_1 m_1$. Suppose that $\phi : X \to U$ is an I-Galois cover of connected germs of curves with inertia I_1 and conductor j.*

i) *There exists an automorphism A of U such that the equations for $A^*\phi$ away from b are:*

$$u_1^m = u^{r_1},\ x^p - x = u_1^{-j}.$$

ii) *The Galois action on the generic fibre is given by the following equations for some γ with $\gcd(\gamma, m) = 1$; (after possibly changing the choice of q):*

$$c(u_1) = \zeta_m^{\gamma} u_1,\ c(x) = \zeta_m^{-\gamma j} x,\ q(u_1) = u_1,\ q(x) = x + 1.$$

iii) *The conductor j satisfies $p \nmid j$ and $\gcd(j, m) = n'$. The upper jump is $\sigma = j/m_1$.*

Proof. First consider the case that Y is normal (i.e., $r_1 = 1$). By [**8**, Lemma 1.4.2], there exists $f(u) \in k[[u]]^*$ with degree j so that the equations for ϕ away from $u = 0$ are $u_1^m = u$ and $x^p - x = u_1^{-j} f(u)$. The proof uses Kummer theory and Artin-Schreier Theory. To finish the proof of i) consider the automorphism A of $k[[u]]$ such that $A(u) = uf(u)^{m/j}$. This automorphism exists since $f(u)$ is a jth power in $k[[u]]^*$. Then $A(u_1) = u_1 f(u)^{1/j}$. After this automorphism of the base, the equations for $A^*\phi$ are $u_1^m = u$ and $x^p - x = u_1^{-j}$.

Now consider the case that $r_1 \neq 1$. The normalization of ϕ is a disconnected cover whose components are Galois with group I_1. Thus the normalization has equations:

$$v^{r_1} = 1,\ u_1^{m_1} = vu,\ x^p - x = u_1^{-j}.$$

The equations for $A^*\phi$ in i) are a blow-down of these by the relation $uv = u_1^{m_1}$. Properties ii)-iii) follow directly from [**13**, Chapter IV] and [**8**, Lemma 1.4.2]. □

The next lemma allows one to induce a given I_1-Galois cover up to a reducible connected I-Galois cover if the restriction from Lemma 2.1.2 (iii) is satisfied. This will be used in Proposition 2.3.4 to patch together covers with different inertia groups.

Lemma 2.1.3. *Suppose $I_1 \subset I \simeq \mathbb{Z}/p \rtimes \mu_m$ with index r_1 where $p \nmid r_1$. Let $m = m_1 r_1$. Suppose there exists an I_1-Galois cover ϕ of connected normal germs of curves with conductor j. Assume $n' = \gcd(m, j)$. Then there exists a connected reducible I-Galois cover ϕ^{ind} with conductor j which is isomorphic to $\mathrm{Ind}_{I_1}^{I}(\phi)$ away from the closed point.*

Proof. Let $c_1 = c^{r_1}$. By Lemma 2.1.2, there is an automorphism A of U so that the equations for $A^*\phi$ are $u_1^{m_1} = u$, $x^p - x = u_1^{-j}$; and its I_1-Galois action is given by $c_1(u_1) = \zeta_{m_1}^{\gamma} u_1$, $c_1(x) = \zeta_{m_1}^{-\gamma j} x$, $q(u_1) = u_1$ and $q(x) = x + 1$ for some γ with $\gcd(\gamma, m_1) = 1$. The equations for $\mathrm{Ind}_{I_1}^{I}(A^*\phi)$ are $v^{r_1} = 1$, $u_1^{m_1} = uv$ and $x^p - x = u_1^{-j}$.

Let ϕ_A^{ind} be the blow down of $\mathrm{Ind}_{I_1}^{I}(A^*\phi)$ which identifies the r_1 ramification points. This yields a connected reducible I-Galois cover ϕ_A^{ind} whose equations and I-Galois action are the same as in Lemma 2.1.2 (i) and (ii). This Galois action is well-defined by the condition on n'. Let $\phi^{\mathrm{ind}} = (A^{-1})^*\phi_A^{\mathrm{ind}}$ and note that ϕ^{ind} is isomorphic to $\mathrm{Ind}_{I_1}^{I}(\phi)$ away from the closed point by construction. □

2.2. Deformation of smooth curves. In this section, we show that it is possible to increase the conductor at a branch point while preserving the inertia and Galois group. This result was announced in [**7**]. Let $R = k[[t]]$ and $K = k((t))$. Let b be the closed point of $U = \mathrm{Spec}(k[[u]])$. Let $U_R = \mathrm{Spec}(R[[u]])$ and $U_K = U_R \times_R K = \mathrm{Spec}(k[[u,t]][t^{-1}])$.

Proposition 2.2.1. *Let $I \simeq \mathbb{Z}/p \rtimes \mu_m$. Suppose there exists an I-Galois cover $\phi : X \to U$ of normal connected germs of curves with conductor j. Then for $i \in \mathbb{N}$ with $p \nmid (j + im)$, there exists an I-Galois cover $\phi_R : X_R \to U_R$ of irreducible germs of R-curves, whose branch locus consists of only the R-point $b_R = b \times_k R$, such that:*

1. *The normalization of the special fibre of ϕ_R $(t = 0)$ is isomorphic to ϕ away from b.*
2. *The generic fibre $\phi_K : X_K \to U_K$ of ϕ_R is an I-Galois cover of normal connected curves whose branch locus consists of only the K-point $b_K = b_R \times_R K$ over which it has inertia I and conductor $j + im$.*

Proof. After an automorphism A of $k[[u]]$, the equations for $A^*\phi$ are given by: $u_1^m = u$, $x^p - x = u_1^{-j}$. Consider the normal cover $\phi_R' : X_R' \to U_R$ given generically by the equations:

$$u_1^m = u, x^p - x = u_1^{-(j+im)}(t + u^i).$$

The I-Galois action on the variables is given by the same expressions as on the closed fibre and the cover is irreducible. The curve X_R' is singular only above the point $(u,t) = (0,0)$. The normalization of the special fibre agrees with $A^*\phi$. The cover ϕ_R' is branched only at the R-point $u = 0$ since $u_1 = 0$ is the only pole of the function $u_1^{-(j+im)}(t + u^i)$. Taking the restriction of ϕ_K over $\mathrm{Spec}(K[[u]])$ where $t + u^i$ is a unit, we see that ϕ_K has inertia I and conductor $j + im$ over b_K. Pulling back the cover ϕ_R' by the automorphism $A^{-1} \times_k R$ of $R[[u]]$ changes none of these properties and thus yields the cover ϕ_R. □

Let X_k be a proper k-curve. The next theorem uses Propositon 2.2.1 and Theorem 1.2.4 to deform a given cover of X_k to a family of covers ϕ_R of X_k. This family can be defined over a variety Θ of finite type over k. We then specialize to a fibre of the family over another k-point of Θ to get a cover ϕ' with new ramification data.

Theorem 2.2.2. *Suppose there exists a G-Galois cover $\phi : Y \to X_k$ of smooth connected curves with branch locus B. Suppose ϕ has inertia group $I \simeq \mathbb{Z}/p \rtimes \mu_m$ and conductor j above $\xi_1 \in B$ with $p \nmid m$. Let $i \in \mathbb{N}^+$ be such that $p \nmid (j + im)$. Then there exist G-Galois covers $\phi_R : Y_R \to X_R$ and $\phi' : Y' \to X_k$ such that:*

1. *The curves Y_R and Y' are irreducible and Y_K and Y' are smooth and connected.*
2. *After normalization, the special fibre ϕ_k of ϕ_R is isomorphic to ϕ away from ξ_1.*
3. *The branch locus of the cover ϕ_R (respectively ϕ') consists exactly of the R-points $\xi_R = \xi \times_k R$ (respectively the k-points ξ) for $\xi \in B$.*
4. *For $\xi \in B$, $\xi \neq \xi_1$, the ramification behavior for ϕ_R (respectively ϕ') at ξ_R (respectively ξ) is identical to that of ϕ at ξ.*
5. *The cover ϕ_K (respectively ϕ') has inertia I and conductor $j + im$ at the K-point $\xi_{1,K}$ (respectively at ξ_1).*
6. *The genus of Y' and of the fibres of Y_R is $g'_Y = \mathrm{genus}(Y) + i|G|(p - 1)/2p$.*

Proof. In the notation of Theorem 1.2.4, let $X^* = X_R$ and let $\mathbb{S} = \{\xi_1\}$. Let $\eta \in \phi^{-1}(\xi_1)$. Consider the I-Galois cover $\hat{\phi} : \hat{Y}_\eta \to \hat{X}_{\xi_1}$. Applying Proposition 2.2.1 to $\hat{\phi}$, there exists a deformation $\hat{\phi}_R : \hat{Y}_R \to \hat{X}_R$ of $\hat{\phi}$ with the desired properties. In particular, $\hat{\phi}_K$ has inertia $I \simeq \mathbb{Z}/p \rtimes \mu_m$ and conductor $j + im$ over $\xi_{1,K}$. Consider the inclusion $R_s \to A_s$ of rings corresponding to the disconnected G-Galois cover $\mathrm{Ind}_I^G(\hat{\phi}_R)$.

The covers ϕ_k and $\mathrm{Ind}_I^G(\hat{\phi}_R)$ and the isomorphism given by Proposition 2.2.1 (1) constitute a relative G-Galois thickening problem as in Definition 1.2.1. The (unique) solution to this thickening problem (Theorem 1.2.4) yields the G-Galois cover $\phi_R : Y_R \to X_R$. Recall from Definition 1.2.3 that the cover ϕ_R is isomorphic to $\mathrm{Ind}_I^G(\hat{\phi}_R)$ over $\hat{X}_{R,\xi_1}$. Also, ϕ_R is isomorphic to the trivial deformation $\phi_{tr} : Y_{tr} \to X_{tr}$ of ϕ away from ξ_1. Thus Y_R is irreducible since Y is irreducible and Y_K is smooth since $Y_{tr,K}$ and $\hat{Y}_K$ are smooth.

The data for the cover ϕ_R is contained in a subring $\Theta \subset R$ of finite type over k, with $\Theta \neq k$ since the family is nonconstant. Since k is algebraically closed, there exist infinitely many k-points of $\mathrm{Spec}(\Theta)$. The closure L of the locus of k-points x of $\mathrm{Spec}(\Theta)$ over which the fibre ϕ_x is not a G-Galois cover of smooth connected curves is closed, [**2**, Proposition 9.29]. Furthermore,

$L \neq \mathrm{Spec}(\Theta)$ since Y_K is smooth and irreducible. Let $\phi' : Y' \to X_k$ be the fibre over a k-point not in L. Note that Y' is smooth and irreducible by definition. The other properties follow immediately from the compatibility of ϕ_R with $\mathrm{Ind}_I^G(\hat{\phi}_R)$ over $\hat{X}_{R,\xi_1}$ and with $\phi_{tr} : Y_{tr} \to X_{tr}$ away from ξ_1.

The genus of Y' and of the fibres of Y_R increases because of the extra contribution to the Riemann-Hurwitz formula. In particular, there are $|G|/mp$ ramification points above $\xi_{1,K}$, each of which has im extra nontrivial higher ramification groups. Thus the degree of ramification $\mathrm{Deg}(\xi_1)$ over $\xi_{1,K}$ increases by $|G|(im)(p-1)/mp$. □

Theorem 2.2.2 can be used to increase the conductor of a cover of proper curves while preserving the inertia and Galois group. For some applications, however, it is necessary to change the Galois group, the prime-to-p part of the inertia group, or the congruence value of the conductor. To do this, it is necessary to deform covers of semistable curves.

2.3. Deformation of semi-stable curves. In this section, we deform covers of semi-stable curves with control over the ramification information. The motivation for Theorem 2.3.7 is that it allows us to use two wildly ramified covers to produce another whose Galois group, inertia group and conductor at a point are determined from the given ones. In Section 3, we use this theorem to produce a cover with complicated Galois group and relatively small conductor. See [**1**] for an application in which the prime-to-p part of the inertia group, and the congruence value of the conductor are changed using this theorem. Since the notation involved in Theorem 2.3.7 is complicated, we will start with a corollary.

Corollary 2.3.1. *Let G be a group generated by subgroups G_1 and G_2. Let $\phi_1 : X \to \mathbb{P}^1_k$ and $\phi_2 : Y \to \mathbb{P}^1_k$ be Galois covers of smooth connected curves each branched at only one point. Suppose ϕ_i has group G_i, inertia I_i, and conductor j_i respectively. Then there exists a G-Galois cover $\phi' : Y' \to \mathbb{P}^1_k$ of smooth connected curves, branched at only one point with inertia I' and conductor j' as follows:*

1. **Purely wild:** *Suppose $I_1 = I_2 = \mathbb{Z}/p$. Then $I' = \mathbb{Z}/p$ and $j' = j_1 + j_2 + \epsilon$. Here $\epsilon = 0$ if $p \nmid (j_1 + j_2)$; if $p | (j_1 + j_2)$, then $\epsilon = 2$ if $p \neq 2$ and $\epsilon = 3$ if $p = 2$.*
2. **Admissible:** *Suppose $I_1 \simeq I_2$ and $j_1 = \gamma j_2$ for some $\gamma \equiv -1 \bmod m$. Write $\gamma = \nu m - 1$. Assume that $p \nmid (j_1 + j_2)$. Then $I' = \mathbb{Z}/p$ and $j' = (j_1 + j_2)/m = \nu j_2$.*
3. **Different inertia:** *More generally, suppose $I_1 \subset I_2 = \mathbb{Z}/p \rtimes \mu_m$ with index r for some r with $p \nmid r$. Let $e = j_1 r + j_2$ and let $g = \gcd(m, e/\gcd(j_1, j_2))$. Assume that $j_1 = \gamma j_2$ for some γ such that $\gcd(\gamma, m) = 1$. Assume that $p \nmid e$. Then I' is the unique subgroup of I_2 with order pm/g and $j' = e/g$.*

The second case is called admissible since the prime-to-p ramification disappears.

Proof. The proof is immediate from Theorem 2.3.7 and Proposition 2.3.5. $\square$

Notation 2.3.2. Let $U = \mathrm{Spec}(k[[u]])$ and $V = \mathrm{Spec}(k[[v]])$. For $e \in \mathbb{N}^+$, we define $\Omega^e_{uv} = k[[u,v,t]]/(uv - t^e)$ and let $S^e_{uv} = \mathrm{Spec}(\Omega^e_{uv})$. Let $\iota_u : U \to S^e_{uv}$ and let $\iota_v : V \to S^e_{uv}$ be the natural inclusions. Let $b \in S^e_{uv}$ be the k-point $(u,v,t) = (0,0,0)$. Suppose $I \simeq \mathbb{Z}/p \rtimes \mu_m = \mathbb{Z}/p \rtimes \langle c \rangle$. Let n' be the order of the prime-to-p part of the center of I. For $i = 1, 2$, suppose $I_i = \mathbb{Z}/p \rtimes \mu_{m_i} \subset I$ with index r_i where $p \nmid r_i$. Note that $m = m_1 r_1 = m_2 r_2$.

Let $\phi_1 : X \to U$, $\phi_2 : Y \to V$ be Galois covers of normal connected germs of curves. Suppose the cover ϕ_i has inertia (and Galois) group $I_i \simeq \mathbb{Z}/p \rtimes \mu_{m_i}$ and conductor j_i for $i = 1, 2$. Let $g' = \gcd(j_1, j_2)$. Let $e = j_1 r_1 + j_2 r_2$, and let $g = \gcd(m, e/g')$. Let $e' = e/g'$, $j'_1 = j_1/g'$ and $j'_2 = j_2/g'$.

Numerical hypotheses: Suppose that $n' = \gcd(m, j_1) = \gcd(m, j_2)$. Suppose $p \nmid e$. Suppose that $j'_1 \equiv \gamma j'_2 \bmod m$ for some γ such that $\gcd(\gamma, m) = 1$ and $1 \leq \gamma < m$. Suppose that $\gcd(j'_2, m) = 1$.

The first condition is necessary to dominate each cover by an I-Galois cover. The other three conditions imply that $p \nmid (e/g)$; $\gcd(j'_1, m) = 1$; and $g = \gcd(m, \gamma r_1 + r_2)$. Also, when $j_1 = \gamma j_2$ and $p \nmid (\gamma + 1)$ for some γ with $\gcd(\gamma, m) = 1$ then the three last conditions are satisfied.

Proposition 2.3.4 constructs an I-Galois cover $\phi_R : W_R \to S^{e'}_{uv}$ with specified ramification from the covers ϕ_1 and ϕ_2. Although W_R will be flat over R and normal, its special fibre W_k will be singular at the point $w = \phi_R^{-1}(b)$.

The following lemma will be used in the proof of Proposition 2.3.4:

Lemma 2.3.3. *Suppose $\ell = \ell_1 + \ell_2$ with $\ell_i \in \mathbb{N}^+$ and $a \in k$. Then $d_0 \in \Omega^\ell_{uv}$ where*

$$d_0 = \frac{(u + at^{\ell_2})^\ell - u^\ell - (at^{\ell_2})^\ell}{u^{\ell_2} t}.$$

Proof. It is sufficient to show that the binomial coefficient $c_i = u^i t^{\ell_2(\ell - i)}/u^{\ell_2} t \in \Omega^\ell_{uv}$ for $1 \leq i \leq \ell - 1$. Since $t^\ell = uv$,

$$c_i = u^{i-\ell_2}(uv)^{\ell_2} t^{-\ell_2 i - 1} = u^i v^{\ell_2} t^{-\ell_2 i - 1}.$$

Since Ω^ℓ_{uv} is normal, it is sufficient to check that $c_i^\ell \in \Omega^\ell_{uv}$. Here

$$c_i^\ell = u^{\ell i} v^{\ell \ell_2} (uv)^{-\ell_2 i - 1} = u^{i(\ell - \ell_2) - 1} v^{\ell_2(\ell - i) - 1}.$$

Thus $c_i^\ell \in \Omega^\ell_{uv}$ since $\ell - \ell_2 \geq 1$ and $\ell - i \geq 1$. $\square$

Proposition 2.3.4. *Consider the pair (ϕ_1, ϕ_2) from in Notation 2.3.2 satisfying the numerical hypotheses. There exists an I-Galois cover $\phi_R : W_R \to S^{e'}$ of (possibly disconnected) germs of R-curves and an isomorphism $i : S^{e'}_{uv} \to S^{e'}$ such that:*

1. *The branch locus of ϕ_R consists of one R-point, denoted b_R, which specializes to b.*
2. *After normalization, the pullbacks of the special fibre of ϕ_R to U and V, namely $\iota_u^* i^* \phi_k$ and $\iota_v^* i^* \phi_k$, are isomorphic to a disjoint union of copies of respectively ϕ_1 and ϕ_2 away from the branch point b.*
3. *The generic fibre $\phi_K : X_K \to S^{e'}_K = S^{e'} \times_R K$ of ϕ_R is an I-Galois cover of (possibly disconnected) germs of curves branched at exactly the K-point $b_K = b_R \times_R K$.*
4. *The cover ϕ_K has g ramification points above b_K, each with inertia group $I_K \simeq \mathbb{Z}/p \rtimes \langle c^g \rangle = \mathbb{Z}/p \rtimes \mu_{m/g}$ and conductor $e/g = (j_1 r_1 + j_2 r_2)/g$.*
5. *The curve W_R is irreducible if and only if $\gcd(r_1, r_2) = 1$. The curve W_K is irreducible if and only if $g = 1$.*

Proof. The proof is to construct ϕ_R and then verify its properties.

The equations for ϕ_1^{ind} and ϕ_2^{ind}: Applying Lemma 2.1.2, there exist automorphisms A_u of $k[[u]]$ and A_v of $k[[v]]$ which fix the closed points of U and V and such that the pullbacks $A_u^* \phi_1$ and $A_v^* \phi_2$ are given by the equations in Lemma 2.1.2 (i).

Since $n' = \gcd(m, j_1) = \gcd(m, j_2)$, Lemma 2.1.3 implies that there exist connected reducible I-Galois covers ϕ_1^{ind} and ϕ_2^{ind} which are isomorphic respectively to $A_u^* \phi_1$ and $A_v^* \phi_2$ away from the branch point. The equations for these covers are:

$$\phi_1^{\mathrm{ind}} : u_1^m = u^{r_1},\ x^p - x = u_1^{-j_1};$$
$$\phi_2^{\mathrm{ind}} : v_1^m = v^{r_2},\ y^p - y = v_1^{-j_2}.$$

After possibly changing c and q once and for all, the I-Galois action of ϕ_1^{ind} and ϕ_2^{ind} is given (for some γ such that $\gcd(\gamma, m) = 1$ and some $a \in \mathbb{F}_p$) by:

$$c(u_1) = \zeta_m u_1, \quad c(x) = \zeta_m^{-j_1} x, \quad q(x) = x + 1,$$
$$c(v_1) = \zeta_m^{\gamma} v_1, \quad c(y) = \zeta_m^{-\gamma j_2} y, \quad q(y) = y + a.$$

Note that the normalization of ϕ_1^{ind} (resp. ϕ_2^{ind}) is isomorphic to a disjoint union of copies of $A_u^* \phi_1$ (resp. $A_v^* \phi_2$) away from the closed point.

The equations for ϕ_R: Let $g' = \gcd(j_1, j_2)$ and $e' = j_1' r_1 + j_2' r_2$ be as in Notation 2.3.2. There exists $a_1 \in k$ such that $a_1^{g'} = a^m$ since k is algebraically closed. Consider the cover $\phi'_R : W_R \to S^{e'}_{uv}$ given by:

$$w_1^m = u^{j'_1 r_1} + a_1 v^{j'_2 r_2} + d_0 t,\ z^p - z = (1 + d_1 t) w_1^{-g'}.$$

For any choice of the variables $d_0, d_1 \in \Omega^{e'}_{uv}$, the cover ϕ'_R reduces to ϕ_1^{ind} and ϕ_2^{ind} on the components of the special fibre. To see this, note that $\mathrm{mod}(v, t)$ the equations for $\iota_u^* i^* \phi'_R$ are $w_1^m = u^{j'_1 r_1}$ and $z^p - z = w_1^{-g'}$. After making the identification $w_1 \mapsto u_1^{j'_1}$ and $z \mapsto x$, a normalization of these equations is isomorphic to ϕ_1^{ind}. (Specifically, we take a normalization of $u_1^{j'_1 m} = u^{j'_1 r_1}$ and $x^p - x = u_1^{-g' j'_1} = u_1^{-j_1}$.) Likewise, $\mathrm{mod}(u, t)$ the equations for $\iota_v^* i^* \phi'_R$ are $w_1^m = a_1 v^{j'_2 r_2}$ and $z^p - z = w_1^{-g'}$. After making the identification $w_1 \mapsto a_1^{1/m} v_1^{j'_2}$ and $z \mapsto y/a$, this simplifies to $v_1^{j'_2 m} = v^{j'_2 r_2}$ and $y^p - y = a(a_1^{1/m} v_1^{j'_2})^{-g'} = v_1^{-j_2}$. A normalization of these equations is isomorphic to ϕ_2^{ind}. By the numerical hypotheses that $j'_1 \equiv \gamma j'_2 \bmod m$ and $\gcd(j'_1, m) = 1$, there is a well-defined I-Galois action on ϕ'_R which reduces correctly:

$$c(w_1) = \zeta_m^{j'_1} w_1,\ c(z) = \zeta_m^{-j_1} z,\ q(z) = z + 1.$$

In conclusion, after normalization, the pullbacks of the special fibre of ϕ'_R to U and V, namely $\iota_u^* i^* \phi_k$ and $\iota_v^* i^* \phi_k$, are isomorphic to a disjoint union of copies of respectively $A_u^* \phi_1$ and $A_v^* \phi_2$ away from the branch point b.

The cover ϕ_R will be the composition of ϕ'_R with a change of base. Namely, suppose $A_u^{-1}(u) = u d_u$ and $A_v^{-1}(v) = v d_v$ for $d_u \in k[[u]]^*$ and $d_v \in k[[v]]^*$. Recall that $\Omega^{e'}_{uv} = k[[u, v, t]]/(uv - t^{e'})$. Let $\Omega^{e'} = k[[u, v, t]]/(uv d_u d_v - t^{e'})$ and let $S^{e'} = \mathrm{Spec}(\Omega^{e'})$. There exists an isomorphism $A : \Omega^{e'} \to \Omega^{e'}_{uv}$ which reduces to A_u on U and A_v on V.

Consider the pullback of the cover ϕ'_R by A. In other words, consider the cover $\phi_R : W_R \to S^{e'}$ corresponding to the composition $\Omega^{e'} \xrightarrow{A} \Omega^{e'}_{uv} \to \mathcal{O}_{W'}$. Most properties for ϕ_R in Proposition 2.3.4 are automatic by the construction of ϕ'_R. Since A is an isomorphism, to finish the proof it is sufficient to verify Properties 3)-5) for ϕ'_R.

The branch locus: Note that $e' = j'_1 r_1 + j'_2 r_2$ is the sum of two positive integers. Let

$$d_0 = \frac{(u + a_2 t^{j'_2 r_2})^{e'} - u^{e'} - a_1 t^{j'_2 r_2 e'}}{u^{j'_2 r_2} t}.$$

By Lemma 2.3.3, $d_0 \in \Omega^{e'}_{uv}$. Rewrite the first equation for ϕ'_R as:

$$w_1^m = u^{-j'_2 r_2}(u^{e'} + a_1 (uv)^{j'_2 r_2} + d_0 t u^{j'_2 r_2}).$$

Since $uv = t^{e'}$, this simplifies to: $w_1^m = u^{-j'_2 r_2}(u^{e'} + a_1 (t^{j'_2 r_2})^{e'} + d_0 t u^{j'_2 r_2})$.

For this choice of d_0 and for $a_2 = a_1^{1/e}$, the equations on the generic fibre are:

$$w_1^m = u^{-j_2' r_2}(u + a_2 t^{j_2' r_2})^{e'},\ z^p - z = (1 + d_1 t)w_1^{-g'}.$$

Note that in $\Omega_{uv,K}^{e'}$ the function u has no zero or pole and $1 + d_1 t$ has no poles. Thus ϕ_K' has a unique branch point given by the coordinates $u = -a_2 t^{j_2' r_2}$ and $v = -(t^{e' - j_2' r_2})/a_2 = -t^{j_1' r_1}/a_2$. This K-point specializes to the branch point $(u, v) = (0, 0)$ of ϕ_k. Thus ϕ_R is branched at exactly one R-point for this choice of d_0.

Irreducibility: Consider the equations for the cover ϕ_R':

$$w_1^m = u^{-j_2' r_2}(u + a_2 t^{j_2' r_2})^{e'},\ z^p - z = (1 + d_1 t)w_1^{-g'}.$$

Note that if m, e' and $j_2' r_2$ share no common factors then the first equation is irreducible; the second is irreducible since the right-hand side is not of the form $\alpha^p - \alpha$. Recall that j_1' and j_2' are relatively prime to m and thus to r_2. Since $\gcd(m, j_2' r_2) = r_2$, the curve W_R is irreducible if and only if $1 = \gcd(e', r_2) = \gcd(r_1, r_2)$. Let $g = \gcd(m, j_1' r_1 + j_2' r_2) = \gcd(m, e')$. We see that W_K is irreducible if and only if $g = 1$ from the following equations for the normalization of ϕ_K':

$$w_2^g = 1,\ w_1^{m/g} = u^{-j_2' r_2/g}(u + a_2 t^{j_2' r_2})^{e'/g},\ z^p - z = (1 + d_1 t)w_1^{-g'}.$$

Ramification information: The first equation indicates that normalization of the generic fibre has g components and thus g points above the branch point each with inertia group $I_K = \mathbb{Z}/p \rtimes \langle c^g \rangle$. The second equation indicates that w_1 is an (e'/g)th power of a uniformizer. Thus the third equation indicates that the lower conductor on the generic fibre is $e'g'/g = e/g$. Thus the cover of the generic fibre has inertia group $I_K \simeq \mathbb{Z}/p \rtimes \langle c^g \rangle = \mathbb{Z}/p \rtimes \mu_{m/g}$ and has conductor $e/g = (j_1 r_1 + j_2 r_2)/g$. □

It is more difficult to deform the covers ϕ_1 and ϕ_2 together if p divides $(j_1 r_1 + j_2 r_2)/g$. This is done in the following proposition in the case that $m = 1$ and $p | (j_1 + j_2)$. The conductor on the generic fibre will be slightly bigger than $(j_1 r_1 + j_2 r_2)/g$.

Proposition 2.3.5. *Let $\phi_1 : X \to U$, $\phi_2 : Y \to V$ be $\mathbb{Z}/p$-Galois covers of normal connected germs of curves with conductors j_1 and j_2 respectively. Suppose that $p | (j_1 + j_2)$. Let $e = j_1 + j_2 + \epsilon$ where $\epsilon = 2$ if $p \neq 2$ and $\epsilon = 3$ if $p = 2$. Then there exists a $\mathbb{Z}/p$-Galois cover $\phi_R : W_R \to S^e \simeq S_{uv}^e$ of irreducible germs of R-curves such that:*

1. *The branch locus of the cover ϕ_R consists of exactly one R-point, denoted b_R.*

2. *After normalization, the pullbacks of the special fibre of ϕ_R to U and V, namely $\iota_u^* i^* \phi_k$ and $\iota_v^* i^* \phi_k$, are isomorphic to ϕ_1 and ϕ_2 away from the branch point.*
3. *The generic fibre $\phi_K : X_K \to S_K^e = S^e \times_R K$ of ϕ_R is an I-Galois cover of smooth irreducible germs of curves whose branch locus consists of the point $b_K = b_R \times_R K$.*
4. *The cover ϕ_K has inertia $\mathbb{Z}/p$ and conductor e over the unique branch point b_K.*

Proof. The proof is essentially the same as for Proposition 2.3.4. For $p \neq 2$ one can deform the equations $x^p - x = u^{-j_1}$ and $y^p - y = v^{-j_2}$ using

$$w = u^{j_1+1} + a_1 v^{j_2+1} + d_0 t,\ \ z^p - z = (u + v + d_1 t)/w.$$

(Here $a_1 = a^m$.) These equations and the Galois action reduce correctly on the components of the special fibre. In particular,

$$\mathrm{mod}(v,t) : w_1 \mapsto u_1^{j_1+1},\ z \mapsto x; \mathrm{mod}(u,t) : w_1 \mapsto a v_1^{j_2+1},\ z \mapsto y/a.$$

Since $uv = t^e = t^{j_1+j_2+2}$, for the same choice of d_0 as in the proof of Proposition 2.3.4, the equations can be rewritten as:

$$w = u^{-(j_2+1)}(u + t^{j_2+1})^{j_1+j_2+2},\ z^p - z = (u + v + d_1 t)/w.$$

Since $p \nmid e$, the conductor is equal to e. $\square$

Notation 2.3.6. Let G be a quasi-p group with Sylow p-subgroup S. Assume $S \simeq \mathbb{Z}/p$. Let $I \simeq \mathbb{Z}/p \rtimes \mu_m \subset G$ and let $r = |G|/mp$. Let $\phi_1 : X \to \mathbb{P}_k^1$ and $\phi_2 : Y \to \mathbb{P}_k^1$ be two (possibly disconnected) G-Galois covers each branched at only one point. Let u (respectively v) be a local parameter at the branch point of ϕ_1 (respectively ϕ_2). Suppose the cover ϕ_i has inertia $I_i \simeq \mathbb{Z}/p \rtimes \mu_{m_i} \subset I$ and conductor j_i for $i = 1, 2$ above $u = 0$ and $v = 0$ respectively. Let the genus of X and Y be g_1 and g_2 respectively.

Let $P_R^{e'}$ be an R-curve whose generic fibre is isomorphic to $\mathbb{P}_K^1$, whose special fibre consists of two projective lines P_u and P_v meeting transversally at a point b where $(u, v) = (0, 0)$, and which satisfies $\hat{P}_{R,b} \simeq S_{uv}^{e'}$.

The next theorem uses Propositon 2.2.1 and Theorem 1.2.4 to deform the covers ϕ_1 and ϕ_2 to a family of covers ϕ_R of $P_R^{e'}$ branched at only one R-point. This family can be defined over a variety Θ of finite type over k. We then specialize to a fibre of the family over another k-point of Θ to get a cover ϕ' with new ramification data.

Theorem 2.3.7. *Consider the pair (ϕ_1, ϕ_2) as in Notation 2.3.2 and 2.3.6. Suppose that ϕ_1 and ϕ_2 satisfy the numerical hypotheses. Then there exist G-Galois covers of curves $\phi_R : Y_R \to P_R^{e'}$ and $\phi' : Y' \to \mathbb{P}_k^1$ such that:*

1. *After normalization, the pullbacks of the special fibre of ϕ_R to P_u and P_v are isomorphic respectively to ϕ_1 and ϕ_2 away from b.*

2. *The branch locus of the cover ϕ_R (respectively ϕ') consists of exactly one R-point denoted b_R which specializes to b (respectively of exactly one point b').*
3. *There are g ramification points of ϕ_K (respectively ϕ') above the branch point b_K (respectively above b'). These have inertia group $I_K \simeq \mathbb{Z}/p \rtimes \langle c^g \rangle = \mathbb{Z}/p \rtimes \mu_{m/g}$ and conductor $e/g = (j_1 r_1 + j_2 r_2)/g$.*
4. *The curves Y_K and Y' are smooth of genus $g_1 + g_2 - 1 + |G| - r(pm + g - r_1 - r_2)/2$.*
5. *Suppose G_1 and G_2 are the stabilizers of a connected component of X and Y respectively. Then Y_R and Y' are connected if G_1 and G_2 generate G.*

Proof. After doing the appropriate set-up, the proof is identical to that of Theorem 2.2.2. Let $X^* = P_R^{e'}$ and let $\mathbb{S} = \{b\}$. Let $S \simeq \mathbb{Z}/p$ be a chosen Sylow p-subgroup of G. Then there exist points $\eta_1 \in \phi_1^{-1}(u)$ and $\eta_2 \in \phi_2^{-1}(v)$ with inertia groups $I_i \simeq S \rtimes \mu_{m_i}$. Consider the I_i-Galois covers of germs of curves $\hat{\phi}_1 : \hat{X}_{\eta_1} \to U$ and $\hat{\phi}_2 : \hat{Y}_{\eta_2} \to V$.

Applying Proposition 2.3.4 to the pair $(\hat{\phi}_1, \hat{\phi}_2)$ we see there exists an I-Galois cover $\hat{\phi}_R : W_R \to S^{e'}$ with all the desired properties at the unique branch point b_K. Namely, there are g ramification points above the branch point over K. The inertia group at one of these points is of the form $I_K \simeq \mathbb{Z}/p \rtimes \langle c^g \rangle$ and has conductor e/g. Consider the inclusion $R_s \to A_s$ of rings corresponding to the disconnected G-Galois cover $\mathrm{Ind}_I^G(\hat{\phi}_R)$.

Consider the cover ϕ_k of the special fibre of $P_R^{e'}$ which restricts to ϕ_1 over P_u and to ϕ_2 over P_v. The covers ϕ_k and $\mathrm{Ind}_I^G(\hat{\phi}_R)$ and the isomorphisms given by Proposition 2.3.4 (2) constitute a relative G-Galois thickening problem as in Definition 1.2.1. The (unique) solution to this thickening problem (Theorem 1.2.4) yields the G-Galois cover ϕ_R. Recall from Definition 1.2.3 that the cover ϕ_R is isomorphic to $\mathrm{Ind}_I^G(\hat{\phi}_R)$ over $\hat{P}_{R,b}$. Thus the deformation ϕ_R has the desired properties near the branch point b_R. Also, the cover ϕ_R is isomorphic to the trivial deformation ϕ_{tr} of ϕ away from b, which completes the proof of Properties 1)-3) for ϕ_R. The fact that Y_K is smooth follows because both W_K and ϕ_{tr} are smooth.

If g' is the genus of the fibres of Y_K, the Riemann-Hurwitz formula implies:

$$2g_1 - 2 = -2|G| + rr_1[(pm_1 - 1) + j_1(p-1)],$$
$$2g_2 - 2 = -2|G| + rr_2[(pm_2 - 1) + j_2(p-1)],$$
$$2g' - 2 = -2|G| + rg[(pm/g - 1) + e(p-1)/g].$$

Using the fact that $e = j_1 r_1 + j_2 r_2$ it follows that:

$$g' = g_1 + g_2 - 1 + |G| - r(pm + g - r_1 - r_2)/2.$$

Note that $pm + g - r_1 - r_2$ is always even. Finally, if G_1 and G_2 generate G then the special fibre of Y_R is connected. Thus Y_K is connected.

As in the proof of Theorem 2.2.2, the cover ϕ_R can be defined over Spec(Θ) for some $k \neq \Theta \subset R$ of finite type over k. Also one can choose a fibre $\phi' : Y' \to \mathbb{P}^1_k$ over a k-point of Θ so that ϕ' is a G-Galois cover and Y' is smooth with the same number of connected components as Y_K, [**2**, Proposition 9.29]. Properties (2)-(5) for ϕ' follow immediately from the corresponding properties for ϕ_K. □

Remark 2.3.8. These patching results do not need to be restricted to the case $X = \mathbb{P}^1_k$ or the case of only one branch point. In general, one can consider G-Galois covers $\phi_1 : Y_1 \to X_1$ and $\phi_2 : Y_2 \to X_2$. Suppose ϕ_1 and ϕ_2 each have a branch point with inertia group contained in $I \simeq \mathbb{Z}/p \rtimes \mu_m$ whose ramification data satisfies the numerical hypotheses. Using Proposition 2.3.4, one can construct a cover $\phi : Z \to W$ with specified inertia behavior above one point. The genus of W will be the sum of genus(X_1) and genus(X_2) and ϕ will be branched at $|B_1| + |B_2| - 1$ points.

Remark 2.3.9. One would like to know whether the above constructions are optimal in the following sense: Given the covers ϕ_1 and ϕ_2, with upper jumps $\sigma_1 = j_1/m_1$ and $\sigma_2 = j_2/m_2$, in Theorem 2.3.7 we construct a deformation so that the generic fibre is a cover ϕ_K branched at exactly one point with upper jump $\sigma_\eta = \sigma_1 + \sigma_2$; would it have been possible to get any smaller upper jump on the generic fibre? The key formula [**7**, Theorem 3.11] indicates that the result in Theorem 2.3.7 is almost optimal since the upper jump on the generic fibre must satisfy $\sigma_\eta \geq \sigma_1 + \sigma_2 - 1$. In other words, the singularity for ϕ_k is not too severe. The following lemma gives another way to measure this singularity.

Consider the cover $\phi_R : Y_R \to \mathbb{P}^1_R$ constructed in Theorem 2.2.2 (respectively $\phi_R : Y_R \to P^{e'}_R$ constructed in Theorem 2.3.7 or Corollary 2.3.1). Choose $y \in \phi_R^{-1}(\infty_k)$ (respectively $\phi_R^{-1}(0,0)$). Let $\pi_y : \widetilde{Y}_{y,k} \to \hat{Y}_{y,k}$ be the normalization of $\hat{Y}_{y,k}$ and let $\hat{\mathcal{O}}_y \to \widetilde{\mathcal{O}}_y$ be the corresponding extension of rings. Let $\delta_y = \dim_k(\widetilde{\mathcal{O}}_y/\mathcal{O}_y)$ and let $m_y = \#\pi_Y^{-1}(y)$. Let $\mu_y = 2\delta_y - m_y + 1$.

Lemma 2.3.10.

i) *In Theorem* 2.2.2, $\mu_y = (j_\eta - j_b)(p-1) = im(p-1)$.
ii) *In Theorem* 2.3.7 *and Corollary* 2.3.1 (2-3), $\mu_y = 1 + mp - g$.
iii) *In Corollary* 2.3.1 (1), $\mu_y = p + \epsilon(p-1)$.

Proof. The proof uses a formula of Kato [**6**] which compares the local ramification and the singularities for the cover $\hat{\phi}_R$ of germs of curves. The details are omitted. □

3. Applications to ramification questions.

Let G be a finite quasi-p group. Let S be a chosen Sylow p-subgroup of G and suppose S has order p. Let k be an algebraically closed field of characteristic

p. All covers in this section are smooth, connected and proper. We now show that for all sufficiently large $j \in \mathbb{N}$ with $p \nmid j$, there exists a G-Galois cover $\phi : Y \to \mathbb{P}^1_k$ branched at only one point with inertia $\mathbb{Z}/p$ and conductor j. The method is to first show the existence of such a cover with small conductor under an additional hypothesis on G. We then use group theory to determine which conductors are sufficiently large enough to realize with formal patching.

3.1. *P*-Weight. In this section, we measure the complexity of the group G.

Definition 3.1.1. Let $G(S) \subset G$ be the subgroup generated by all proper quasi-p subgroups G' such that $G' \cap S$ is a Sylow p-subgroup of G'. The group G is *p-pure* if $G(S) \neq G$.

Note that this definition is independent of the choice of S. This condition was introduced in [**10**]. Note that when $|S| = p$, then G is p-pure if and only if G is not generated by all proper quasi-p subgroups $G' \subset G$ such that $S \subset G'$. Some examples of p-pure quasi-p groups with $|S| = p$ are $\mathrm{PSL}_2(\mathbb{F}_p)$, and the semi-direct product $(\mathbb{Z}/r\mathbb{Z})^l \rtimes \mathbb{Z}/p$ where the action is irreducible. When $p = 11$, M_{11} and M_{22} are quasi-11 and 11-pure. Every finite quasi-p group can be generated from p-pure ones.

Definition 3.1.2. Consider all subgroups $G' \subset G$ such that G' is quasi-p and p-pure and such that $G' \cap S$ is a Sylow p-subgroup of G'. The *p-weight* $\omega(G)$ of G is the minimal number of such subgroups G' of G which are needed to generate G.

Lemma 3.1.3. *The p-weight $\omega(G)$ of G is a finite number independent of the choice of S.*

Proof. The proof uses induction on $|G|$ to show that G can be generated by p-pure quasi-p subgroups G'' with $G'' \cap S = \mathrm{Syl}_p(G'')$. This statement is true if $G \simeq \mathbb{Z}/p$. For then G contains no proper quasi-p subgroups and so $\{1\} = G(S) \neq G$. Thus $G \simeq \mathbb{Z}/p$ is p-pure and $\omega(G) = 1$.

Now given G, suppose that the hypothesis is true for any quasi-p group G' such that $p \leq |G'| < |G|$. If $G(S) \neq G$ then G is p-pure and so $\omega(G) = 1$.

If $G(S) = G$ then by definition G is generated by its proper quasi-p subgroups $G' \subset G$ with $G' \cap S = \mathrm{Syl}_p(G')$. Since $|G'| < |G|$ the induction hypothesis states that each G' is generated by p-pure quasi-p subgroups G'' with $G'' \cap \mathrm{Syl}_p(G') = \mathrm{Syl}_p(G'')$. Note that $\mathrm{Syl}_p(G'') = G'' \cap (G' \cap S) = G'' \cap S$. Thus each G'' satisfies the necessary conditions and the collection of G'' generate G. Thus the p-weight is the minimum size among all sets of p-pure quasi-p subgroups G'' with $S \cap G'' = \mathrm{Syl}_p(G'')$ which generate G.

To show that $\omega(G)$ is independent of S, consider another Sylow p-subgroup S_0 of G. Let $\omega_0(G)$ be the p-weight with respect to S_0. Since the Sylow

p-subgroups are all conjugate, there exists some $g \in G$ with $S_0 = gSg^{-1}$. Suppose G is generated by a set $\{G''\}$ of p-pure quasi-p subgroups with $S \cap G'' = \mathrm{Syl}_p(G'')$. Note that $S_0 \cap (gG''g^{-1}) = \mathrm{Syl}_p(gG''g^{-1})$. Each subgroup $gG''g^{-1}$ is still quasi-p and p-pure (with respect to its Sylow). Also G is generated by the set of $gG''g^{-1}$. Thus $\omega_0(G) \leq \omega(G)$. Reversing the roles of S and S_0, $\omega(G) \leq \omega_0(G)$. □

3.2. Conductors. Let $\phi : Y \to \mathbb{P}^1_k$ be a G-Galois cover which is branched at only one point. Such a cover exists if and only if G is a quasi-p group which means that G is generated by p-groups, [**10**]. Suppose ϕ has inertia group $I \simeq \mathbb{Z}/p \rtimes \mu_m$ and conductor j. When $G \neq \mathbb{Z}/p$, there is a small set of values $j_{\min}(I)$, depending only on I, consisting of the minimal possible conductors for ϕ. Let n' be the order of the prime-to-p part of the center of I. Let n be such that $m = nn'$.

Definition 3.2.1. Define $j_{\min}(I) = \{j_{\min}(I, a) |\ 1 \leq a \leq n,\ \gcd(a, n) = 1\}$ where $j_{\min}(I, a) = 2m + n'$ if $a = 1$ and $n = p - 1$ and $j_{\min}(I, a) = m + an'$ otherwise.

A geometric interpretation for the set $j_{\min}(I)$ is that ϕ has a non-isotrivial deformation in equal characteristic p if and only if $j \notin j_{\min}(I)$, [**8**, Theorem 3.1.11]. Suppose $1 \leq a \leq n$ and $j \equiv an' \bmod m$. If $G \neq \mathbb{Z}/p$ then $j \geq j_{\min}(I, a)$, by [**8**, Lemma 1.4.3]. Note that if $j \in j_{\min}(I)$ then $p \nmid j$ and $j \leq m(2 + 1/(p - 1))$.

Theorem 3.2.2. *Let G be a finite p-pure quasi-p group whose Sylow p-subgroups have order $p \neq 2$. For some $I \simeq \mathbb{Z}/p \rtimes \mu_m \subset G$ and some $j \in j_{\min}(I)$, there exists a G-Galois cover $\phi : Y \to \mathbb{P}^1_k$ of smooth connected curves branched at only one point over which it has inertia group I and conductor j. In particular,* $\mathrm{genus}(Y) \leq 1 + |G|(p - 1)/2p$.

Proof. For the convenience of the reader we briefly recall the outline of the proof from [**7**, Theorem 4.5]. By Abhyankar's Conjecture [**10**, 6.5.3], for some I_0 of the form $\mathbb{Z}/p \rtimes \mu_{m_0}$ and some j_0, there exists a G-Galois cover $\phi_0 : Y_0 \to \mathbb{P}^1_k$ with group G which is branched at only one point with inertia group I_0 and conductor j_0. If $j_0 \notin j_{\min}(I)$, there exists a non-isotrivial deformation of ϕ_0 in equal characteristic p by [**8**, Theorem 3.1.11]. This deformation yields a cover ϕ_K with bad reduction by [**8**, Theorem 3.3.7].

Let $\phi : Y \to X$ be the stable model of ϕ_K. See [**7**, Section 3] for information on the structure of ϕ, which is very similar to that in the unequal characteristic case in [**10**, Section 6], [**11**, Sections 2-3], and [**12**]. In particular, the special fibre X_k is a tree of projective lines and the restriction of ϕ over any terminal component of X_k is separable. Since G is p-pure and has no (nontrivial) normal p-subgroups, for some terminal component P_b of X_k, the curve $Y_b = \phi^{-1}(P_b)$ is connected. For this component, the restriction $\phi_b : Y_b \to P_b \simeq \mathbb{P}^1_k$ is a G-Galois cover branched at only one point. If ϕ_b has inertia group $I_b \simeq \mathbb{Z}/p \rtimes \mu_{m_b} \subset N_G(S)$ and conductor j_b then $j_b/m_b < j_0/m_0$ by [7, Theorem 3.11]. We reiterate this process until finding such a cover with inertia group $I_b = \mathbb{Z}/p \rtimes \mu_{m_b}$ and conductor j_b satisfying $j_b/m_b \leq 2 + 1/(p-1)$ which implies $j_b \in j_{\min}(I)$. The condition on genus(Y) follows directly from Definition 3.2.1 and the Riemann-Hurwitz formula. $\square$

Lemma 3.2.3. *Suppose there exists a G-Galois cover $\phi : Y \to \mathbb{P}^1_k$ branched at only one point with inertia group $I \simeq \mathbb{Z}/p \rtimes \mathbb{Z}/m$ and conductor j. Then for any $I' \subset I$ of the form $I' = \mathbb{Z}/p \rtimes \mu_{m'}$ and for any $j' \equiv j \bmod m'$ with $j' \geq j$ and $p \nmid j'$, there exists a G-Galois cover $\phi' : Y \to \mathbb{P}^1_k$ of smooth connected curves branched at only one point with inertia I' and conductor j'.*

Proof. Let r be the index of I' in I. Consider the cover $f : X \to \mathbb{P}^1_k$ which is cyclic of order r and branched at 0 and ∞. By Abhyankar's Lemma, the cover $f^*\phi$ is a G-Galois cover $\phi' : Y' \to \mathbb{P}^1_k$ which is branched at only one point with inertia group $\mathbb{Z}/p \rtimes \mu_{m'}$. The cover $f^*\phi$ is connected since f and ϕ are disjoint. The conductor of $f^*\phi$ still equals j. Thus the statement is immediate from Theorem 2.2.2. $\square$

Let G be a quasi-p group with $|S| = p$ and with p-weight ω. By Lemma 3.1.3, G can be generated by a collection of ω proper p-pure quasi-p subgroups G' such that $G' \cap S = \mathrm{Syl}_p(G')$. We give sufficient conditions for the conductor of a G-Galois cover $\phi : Y \to \mathbb{P}^1_k$ branched at only one point with inertia $\mathbb{Z}/p$.

Theorem 3.2.4. *Let G be a finite quasi-p group whose Sylow p-subgroups have order $p \neq 2$. Let ω be the p-weight of G. Let m_e be the exponent of the normalizer $N_G(S)$ of S in G divided by p. Let $j \in \mathbb{N}^+$ satisfy $\gcd(j,p) = 1$. Suppose $j \geq m_e(2 + 1/(p-1))\omega$ if $p \nmid \omega$ and $j \geq m_e(2 + 1/(p-1))\omega + 2$ if $p|\omega$. Then there exists a G-Galois cover $\phi : Y \to \mathbb{P}^1_k$ of smooth connected curves which is branched at only one point over which it has inertia $\mathbb{Z}/p$ and conductor j.*

Proof. Note that $m_e(2 + 1/(p-1))$ is not necessarily an integer. In this proof the phrase "cover of this type" indicates that the cover in question is a smooth connected cover of the projective line branched at only one point with inertia $I = \mathbb{Z}/p$.

By Theorem 2.2.2, given G as above it is sufficient to prove the following: For some $J \in \mathbb{Z}$ such that $p \nmid J$ and $J \leq m_e(2 + 1/(p-1))$ there exists a G-Galois cover ϕ of this type with conductor $j = J\omega$ if $p \nmid \omega$ and conductor $j = J\omega + 2$ if $p|\omega$. The proof will proceed by induction on ω.

If $\omega = 1$ then G is quasi-p and p-pure. By Theorem 3.2.2, for some $I \simeq \mathbb{Z}/p \rtimes \mu_m \subset G$ there exists a G-Galois cover $\phi : Y \to \mathbb{P}^1_k$ branched at only one point with inertia group I and conductor $j \in j_{\min}(I)$. Recall that if $j \in j_{\min}(I)$ then $j = m(2 + 1/(p-1))$ or $j \leq 2m$. Also $m \leq m_e$. Let $I' = \mathbb{Z}/p$. By Lemma 3.2.3, there exists a G-Galois cover $\phi' : Y \to \mathbb{P}^1_k$ branched at only one point with inertia $\mathbb{Z}/p$ and conductor j. Choose $J = j$ and note that $p \nmid J$ and $J \leq m_e(2 + 1/(p-1))$.

Now suppose that $\omega > 1$. By the inductive hypothesis, for all quasi-p groups G' having p-weight ω' where $\omega' < \omega$ and $p \nmid \omega'$, there exists J' such that $\gcd(J', p) = 1$ and such that $J' \leq m_{e'}(2 + 1/(p-1))$ (where $m_{e'}$ is the exponent of $N_{G'}(S)$ divided by p) and there exists a G'-Galois cover ϕ' of this type with conductor $j' = J'\omega'$.

Choose $w_1 \geq 1$ and $w_2 \geq 1$ such that $p \nmid w_1 w_2$ and $w_1 + w_2 = \omega$. (If $\omega \not\equiv 1 \bmod p$, then choose $w_1 = 1$ and $w_2 = \omega - 1$. If $\omega \equiv 1 \bmod p$, then choose $w_1 = 2$ and $w_2 = \omega - 2$.)

Since G has p-weight $\omega > 1$, G can be generated by ω proper p-pure quasi-p groups $G'_1, \ldots, G'_\omega$ with $S \subset G'_i$ for all $1 \leq i \leq \omega$. Let $G_1 \subset G$ be the subgroup generated by G'_i for $1 \leq i \leq w_1$. Let $G_2 \subset G$ be the subgroup generated by G'_i for $w_1 + 1 \leq i \leq w_2$. Then G_1 and G_2 are quasi-p groups since they are generated by quasi-p groups. For $i = 1, 2$, the order of a Sylow p-subgroup of G_i is p since $S \subset G_i \subset G$. By construction, $\omega(G_1) \leq w_1$ and $\omega(G_2) \leq w_2$. But in fact, $\omega(G_1) = w_1$ and $\omega(G_2) = w_2$ since the p-weight of G is only ω. For $i = 1, 2$, let m_{e_i} be the exponent of $N_{G_i}(S)$ divided by p.

By the inductive hypothesis, for $i = 1, 2$, there exists J_i such that $p \nmid J_i$ and $J_i \leq m_{e_i}(2 + 1/(p-1))$ and there exists a G_i-Galois cover ϕ_i of this type with conductor $J_i w_i$. Let m_e be the exponent of the normalizer of $N_G(S)$ divided by p and note that $m_e \geq m_{e_i}$. Let $J = \max\{J_1, J_2\}$ and note that $p \nmid J$ and $J \leq m_e(2 + 1/(p-1))$. Using Theorem 2.2.2 to increase the conductor of ϕ_i for $i = 1, 2$ we find a G_i-Galois cover ϕ'_i of this type with conductor Jw_i.

By Corollary 2.3.1 (1), in the case that $p \nmid \omega$, there exists a G-Galois cover ϕ of this type with conductor $j = Jw_1 + Jw_2 = J\omega$; in the case that $p|\omega$, there exists a G-Galois cover ϕ of this type with conductor $j = Jw_1 + Jw_2 + 2 = J\omega + 2$. (In particular, the covers are connected since G_1 and G_2 generate G, Theorem 2.3.7 (5).) This completes the proof by induction. $\square$

Example 3.2.5. Let $G = \mathrm{PSL}_2(\mathbb{F}_p)$. Then G is quasi-p and p-pure and its Sylow p-subgroups have order p. The normalizer of a Sylow is of the form $I^* \simeq \mathbb{Z}/p \rtimes \mu_{m^*}$ where $m^* = (p-1)/2$ and μ_{m^*} acts faithfully on $\mathbb{Z}/p$. By Corollary 3.2.2, for some $m|m^*$, there exists a G-Galois cover $\phi : Y \to \mathbb{P}^1_k$ of smooth connected curves branched only at ∞ with inertia $I \simeq \mathbb{Z}/p \rtimes \mu_m$ and conductor $j \in j_{\min}(I)$. In this case $j \leq 2m \leq 2m^* < p$.

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