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We define A_k-moves for embeddings of a finite graph into the 3-sphere for each natural number k. Let A_k-equivalence denote an equivalence relation generated by A_k-moves and ambient isotopy. A_k-equivalence implies A_{k-1}-equivalence. Let $\mathcal{F}$ be an A_{k-1}-equivalence class of the embeddings of a finite graph into the 3-sphere. Let $\mathcal{G}$ be the quotient set of $\mathcal{F}$ under A_k-equivalence. We show that the set $\mathcal{G}$ forms an abelian group under a certain geometric operation. We define finite type invariants on $\mathcal{F}$ of order $(n; k)$. And we show that if any finite type invariant of order $(1; k)$ takes the same value on two elements of $\mathcal{F}$, then they are A_k-equivalent. A_k-move is a generalization of C_k-move defined by K. Habiro. Habiro showed that two oriented knots are the same up to C_k-move and ambient isotopy if and only if any Vassiliev invariant of order $\leq k-1$ takes the same value on them. The 'if' part does not hold for two-component links. Our result gives a sufficient condition for spatial graphs to be C_k-equivalent.

Introduction.

K. Habiro defined a local move, C_k*-move*, for each natural number k [**2**]. It is known that if two embeddings f and g of a graph into the three sphere are the same up to C_k-move and ambient isotopy, then g can be deformed into a band sum of f with certain $(k+1)$-component links and that changing position of a band and an arc, which is called *a band trivialization* of C_k-move, is realized by C_{k+1}-moves and ambient isotopy [**17**]. This is one of the most important properties of C_k-move. We consider local moves which have this property. We define A_1-move as the crossing change and A_{k+1}-move as a band trivialization of A_k-move; see Section 1 for the precise definition. So A_k-move is a generalization of C_k-move. In fact, the results for A_k-move in this paper hold for C_k-move.

Let A_k*-equivalence* denote an equivalence relation given by A_k-moves and ambient isotopy. Habiro showed that two oriented knots are C_k-equivalent if and only if they have the same Vassiliev invariants of order $\leq k-1$ [**3**], [**4**]. The 'only if' part of this result is true for A_k-move and for the embeddings of a graph, in particular for links (Theorem 5.1). However the 'if' part does not hold for two-component links. For example, the Whitehead link is not C_3-equivalent to a trivial link because they have different Arf invariants, see [**16**]. On the other hand, H. Murakami showed in [**7**] that the Vassiliev invariants of links of order ≤ 2 are determined by the linking numbers and the second coefficient of the Conway polynomial of each component. Hence, the values of any Vassiliev invariant of order ≤ 2 of these two links are the same. So we note that Vassiliev invariants of order $\leq k-1$ are not enough to characterize C_k-equivalent embeddings of a graph.

We will define in Section 1 a *finite type invariant of order* $(n;k)$ as a generalization of a Vassiliev invariant and see that if any finite type invariants of order $(1;k)$ takes the same value on two A_{k-1}-equivalent embeddings of a graph, then they are A_k-equivalent (Theorem 1.1). While a Vassiliev invariant is defined by the change in its value at every 'wall' corresponding to a crossing change, a finite type invariant of order $(n;k)$ is defined similarly by 'walls' corresponding to A_k-moves. A finite type invariant of order $(n;1)$ is a Vassiliev invariant of order $\leq n$.

It is shown that the set of C_k-equivalence classes of knots forms an abelian group under the connected sum [**3**], [**4**]. This is also true for A_k-equivalence classes. Since the connected sum is peculiar to knots, we cannot apply it to embeddings of a graph. In Section 2, we will define a certain geometric sum for the elements in an A_{k-1}-equivalence class of the embedding of a graph. Then we will see that the quotient set of the A_{k-1}-equivalence class under A_k-equivalence forms an abelian group (Theorem 2.4).

It is not essential that A_1-move is the crossing change. This is a big difference between A_k-move and C_k-move. We will study a generalization of A_k-move in Section 4. For example, if we put A_1-move to be the #-move defined by Murakami [**6**], then we get several results similar to that for original A_k-move.

1. A_k-moves and finite type invariants.

Let B^3 be the oriented unit 3-ball. A *tangle* is a disjoint union of properly embedded arcs in B^3. A tangle is *trivial* if it is contained in a properly embedded 2-disk in B^3. A *trivialization* of a tangle $T = t_1 \cup t_2 \cup \cdots \cup t_k$ is a choice of mutually disjoint disks $D_1, D_2, \ldots, D_k$ in B^3 such that $D_i = (D_i \cap \partial B^3) \cup t_i$ for $i = 1, 2, \ldots, k$. It can be shown that in general a trivialization is not unique up to ambient isotopy of B^3 fixed on the tangle.

Let T and S be tangles, and let $t_1, t_2, \ldots, t_k$ and $s_1, s_2, \ldots, s_k$ be the components of T and S respectively. Suppose that for each t_i there exists some s_j such that $\partial t_i = \partial s_j$. Then we call the ordered pair (T, S) a *local move*, which can be interpreted as substituting S for T. Two local moves (T, S) and (T', S') are *equivalent* if there exists an orientation preserving homeomorphism $h : B^3 \longrightarrow B^3$ such that $h(T) = T'$ and $h(S)$ is ambient isotopic to S' relative to ∂B^3. We consider local moves up to this equivalence.

Let (T, S) be a local move such that T and S are trivial tangles. First choose a trivialization $D_1, D_2, \ldots, D_k$ of T. Each D_i intersects ∂B^3 in an arc γ_i. Let E_i be a small regular neighbourhood of γ_i in ∂B^3. We devide the circle ∂E_i into two arcs α_i and β_i such that $\alpha_i \cap \beta_i = \partial\alpha_i = \partial\beta_i$. By slightly perturbing intα_i and intβ_i into the interior of B^3 on either side of D_i, we obtain properly embedded arcs $\widetilde{\alpha}_i$ and $\widetilde{\beta}_i$. We consider k local moves $(S \cup \widetilde{\alpha}_i, S \cup \widetilde{\beta}_i)$ $(i = 1, 2, \ldots, k)$ and call them the *band trivializations* of the local move (T, S) with respect to the trivialization $D_1, D_2, \ldots, D_k$. Note that both $S \cup \widetilde{\alpha}_i$ and $S \cup \widetilde{\beta}_i$ are trivial tangles.

We now inductively define a sequence of local moves on trivial tangles in B^3 which depend on the choice of trivialization. An A_1*-move* is the crossing change shown in Figure 1.1. Suppose that A_k-moves are defined and there are l A_k-moves $(T_1, S_1), (T_2, S_2), \ldots, (T_l, S_l)$ up to equivalence. For each A_k-move (T_i, S_i) $(i = 1, 2, \ldots, l)$, we choose a single trivialization $\tau_i = \{D_{i,1}, D_{i,2}, \ldots, D_{i,k+1}\}$ of T_i and fix it. (The choice of τ_i is independent of the trivialization that is chosen to define A_k-move (T_i, S_i).) Then the band trivializations of (T_i, S_i) with respect to the trivialization τ_i are called $A_{k+1}(\tau_i)$*-moves* and these $A_{k+1}(\tau_i)$-moves $(i = 1, 2, \ldots, l)$ are called $A_{k+1}(\tau_1, \tau_2, \ldots, \tau_l)$-moves. Note that the number of $A_{k+1}(\tau_1, \tau_2, \ldots, \tau_l)$-moves is at most $l(k+1)$ up to equivalence. Although the choice of trivializations is important for the definition of A_k-move, our proof is the same for every choice. Therefore the results of this paper hold for every choice of trivializations $\tau_1, \tau_2, \ldots, \tau_l$. So we denote $A_{k+1}(\tau_1, \tau_2, \ldots, \tau_l)$-move simply as A_{k+1}*-move*. It is known that C_k-move defined by Habiro is a special case of A_k-move for certain choices of trivializations; see [**2**], [**10**]. We will see that A_k-move, as well as C_k-move, has the property mentioned in Introduction (Proposition 2.1 and Lemma 2.2).

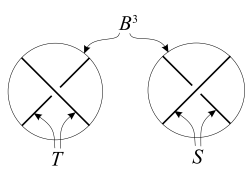
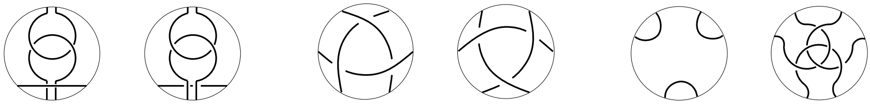


Figure 1.1.

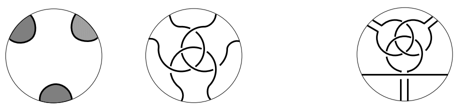
Examples. (1) The trivialization of a tangle in Figure 1.1 is unique up to ambient isotopy. Therefore we have any band trivialization of an A_1-move is equivalent to the local move in Figure 1.2-(i). Thus A_2-move is unique up to equivalence. It is not hard to see that an A_2-move is equivalent to the *delta move* in Figure 1.2-(ii) defined by H. Murakami and Y. Nakanishi [**8**], and then it is equivalent to the local move in Figure 1.2-(iii).

(2) If we choose a trivialization for the A_2-move as in Figure 1.3-(i), then, by the symmetry of the A_2-move, any A_3-move is equivalent to the local move in Figure 1.3-(ii).



(i) (ii) (iii)

Figure 1.2.



(i) (ii)

Figure 1.3.

A local move (S, T) is called the *inverse* of a local move (T, S). It is clear that the inverse of an A_1-move is again an A_1-move. By the definition of A_k-move, we see that the inverse of an A_k-move with $k \geq 2$ is equivalent to itself.

Let (T, S) be an A_k-move and $D_1, D_2, \dots, D_{k+1}$ the fixed trivialization of $T = t_1 \cup t_2 \cup \cdots \cup t_{k+1}$. We set $\alpha = \partial B^3 \cap (D_1 \cup D_2 \cup \cdots \cup D_{k+1})$ and $\beta = S$. A link L in S^3 is called *type* k if there is an orientation preserving embedding $\varphi : B^3 \longrightarrow S^3$ such that $L = \varphi(\alpha \cup \beta)$. Then the pair (α, β) is called a *link model of* L.

We now define an equivalence relation on spatial graphs by A_k-move. Let G be a finite graph. Let $V(G)$ denote the set of the vertices of G. Let $f, g : G \longrightarrow S^3$ be embeddings. We say that f and g are *related by an* A_k*-move* if there is an A_k-move (T, S) and an orientation preserving embedding $\varphi : B^3 \longrightarrow S^3$ such that:

(i) If $f(x) \neq g(x)$ then both $f(x)$ and $g(x)$ are contained in $\varphi(\mathrm{int} B^3)$,
(ii) $f(V(G)) = g(V(G))$ is disjoint from $\varphi(B^3)$, and
(iii) $f(G) \cap \varphi(B^3) = \varphi(T)$ and $g(G) \cap \varphi(B^3) = \varphi(S)$.

We also say that g is obtained from f by an *application* of (T, S). We define A_k*-equivalence* as an equivalence relation on the set of all embeddings of G into S^3 given by the relation above and ambient isotopy. For an embedding $f : G \longrightarrow S^3$, let $[f]_k$ denote the A_k-equivalence class of f. By the definition of A_k-move we see that an application of an A_{k+1}-move is realized by two applications of A_k-move and ambient isotopy. Thus A_{k+1}-equivalence implies A_k-equivalence. In other words we have $[f]_1 \supset [f]_2 \supset \cdots \supset [f]_k \supset [f]_{k+1} \supset \cdots$.

Let $f : G \longrightarrow S^3$ be an embedding, L_i links of type k and (α_i, β_i) their link models $(i = 1, 2, \ldots, n)$. Let $I = [0, 1]$ be the unit closed interval. An embedding $g : G \longrightarrow S^3$ is called a *band sum of* f *with* $L_1, L_2, \ldots, L_n$ if there are mutually disjoint embeddings $b_{ij} : I \times I \longrightarrow S^3$ $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, k+1)$ and mutually disjoint orientation preserving embeddings $\varphi_i : B^3 \longrightarrow S^3 - f(G)$ with $L_i = \varphi_i(\alpha_i \cup \beta_i)$ $(i = 1, 2, \ldots, n)$ such that the following (i) and (ii) hold:

(i) $b_{ij}(I \times I) \cap f(G) = b_{ij}(I \times I) \cap f(G - V(G)) = b_{ij}(I \times \{0\})$ and $b_{ij}(I \times I) \cap (\bigcup_l \varphi_l(B^3)) = b_{ij}(I \times \{1\})$ is a component of $\varphi_i(\alpha_i)$ for any i, j $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, k+1)$.

(ii) $f(x) = g(x)$ if $f(x)$ is not contained in $\bigcup_{i,j} b_{ij}(I \times \{0\})$ and

$$g(G) = \left(f(G) \cup \bigcup_i L_i - \bigcup_{i,j} b_{ij}(I \times \partial I) \right) \cup \bigcup_{i,j} b_{ij}(\partial I \times I).$$

Then we denote g by $F(f; \{L_1, L_2, \ldots, L_n\}, \{B_1, B_2, \ldots, B_n\})$, where $B_i = b_{i1}(I \times I) \cup b_{i2}(I \times I) \cup \cdots \cup b_{ik+1}(I \times I)$ $(i = 1, 2, \ldots, n)$. We call each $b_{ij}(I \times I)$ a *band*. We call each $\varphi_i(B^3)$ an *associated ball* of L_i. See Figure 1.4 for an example of a band sum of an embedding f with links L_1, L_2, L_3 of type 3.

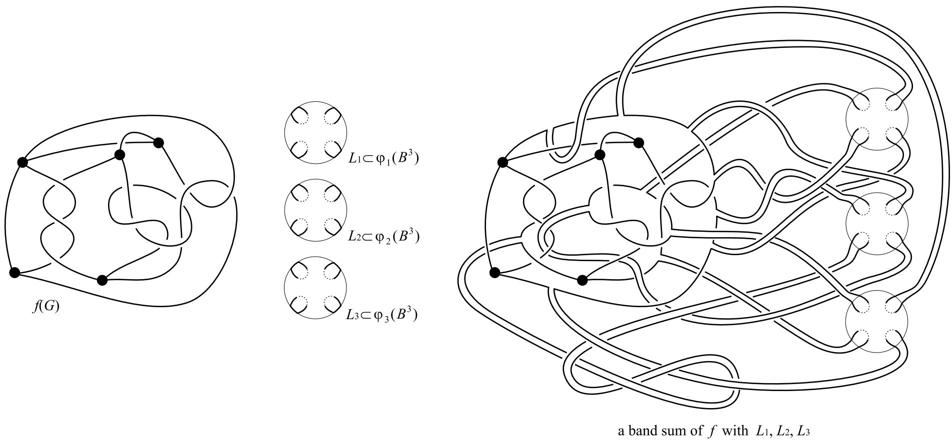


Figure 1.4.

Remark. It follows from the definition that if g is a band sum of f with some links of type k, then g is A_k-equivalent to f. The converse is also true and will be shown in Proposition 2.1. In Lemma 2.2, we show that the position of a band is changeable up to A_{k+1}-equivalence. The origin of the name 'band trivialization' comes from this fact.

Let $h : G \longrightarrow S^3$ be an embedding and H an abelian group. Let $\varphi : [h]_{k-1} \longrightarrow H$ be an invariant. We say that φ is a *finite type invariant of order* $(n; k)$ if for any embedding $f \in [h]_{k-1}$ and any band sum $F(f;\{L_1, L_2, \ldots, L_{n+1}\}, \{B_1, B_2, \ldots, B_{n+1}\})$ of f with links $L_1, L_2, \ldots, L_{n+1}$ of type $k-1$,

$$\sum_{X \subset \{1,2,\ldots,n+1\}} (-1)^{|X|} \varphi\left(F\left(f; \bigcup_{i\in X}\{L_i\}, \bigcup_{i\in X}\{B_i\}\right)\right) = 0 \in H,$$

where the sum is taken over all subsets, including the empty set, and $|X|$ is the number of the elements in X.

In the next section we show the following theorem:

Theorem 1.1. *Let $f, g : G \longrightarrow S^3$ be A_{k-1}-equivalent embeddings. Then they are A_k-equivalent if and only if $\varphi(f) = \varphi(g)$ for any finite type A_k-equivalence invariant φ of order $(1;k)$.*

Note that finite type invariants of order $(n;2)$ coincide with Vassiliev invariants of order n. It is shown in [**5**, Theorem 1.1, Theorem 1.3] that two embeddings of a finite graph G into S^3 are A_2-equivalent if and only if they have the same Wu invariant [**18**]. It follows from [**14**, Section 2] that Wu invariant is a finite type invariant of order $(1;2)$. Since two embeddings are always A_1-equivalent, we have the following corollary:

Corollary 1.2. *Let $f, g : G \longrightarrow S^3$ be embeddings. Then the following conditions are mutually equivalent:*

(i) *f and g are A_2-equivalent.*
(ii) *f and g have the same Wu invariant.*
(iii) *$\varphi(f) = \varphi(g)$ for any Vassiliev invariant φ of order 1.*

In Section 5 we show the following proposition:

Proposition 1.3. *Let φ be a Vassiliev invariant of order $(n+1)(k-1)-1$. Then φ is a finite type invariant of order $(n;k)$.*

2. A_k-equivalence group of spatial graphs.

The following proposition is a natural generalization of [**21**, Lemma] and stems from the fact that a knot with the unknotting number u can be unknotted by changing u crossings of a regular diagram of it [**12**] and [**19**].

Proposition 2.1. *Let $f, g : G \longrightarrow S^3$ be embeddings. If f and g are A_k-equivalent, then g is ambient isotopic to a band sum of f with some links of type k.*

Proof. We consider the embeddings up to ambient isotopy for simplicity. By the assumption there is a finite sequence of embeddings $f = f_0, f_1, \ldots, f_n = g$ and orientation preserving embeddings $\varphi_1, \varphi_2, \ldots, \varphi_n : B^3 \longrightarrow S^3$ such that $(\varphi_i^{-1}(f_{i-1}(G)), \varphi_i^{-1}(f_i(G)))$ is an A_k-move for each i. We shall prove this proposition by induction on n.

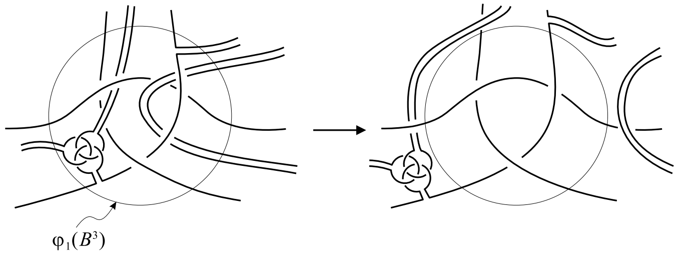


Figure 2.1.

First we consider the case $n = 1$. Let $D_1, D_2, \ldots, D_{k+1}$ be the fixed trivialization of the tangle $\varphi_1^{-1}(f_0(G))$ and $\gamma_j = D_j \cap \partial B^3$ ($j = 1, 2, \ldots, k+1$). Then $L = \bigcup_j \varphi_1(\gamma_j) \cup (\varphi_1(B^3) \cap f_1(G))$ is a link of type k. By taking a small one-sided collar for each $\varphi_1(\gamma_j)$ in $S^3 - \varphi_1(\mathrm{int}B^3)$, we have mutually disjoint embeddings $b_j : I \times I \longrightarrow S^3$ ($j = 1, 2, \ldots, k+1$) such that $b_j(I \times I) \cap \varphi_1(B^3) = b_j(I \times \{1\}) = \varphi_1(\gamma_j)$ and $b_j(I \times I) \cap f_0(G) = b_j(I \times I) \cap f_1(G) = b_j(\partial I \times I)$. Then we deform f_0 up to ambient isotopy along the disk $b_j(I \times I) \cup \varphi_1(D_j)$ such that $b_j(I \times I) \cap f_0(G) = b_j(I \times \{0\})$ for each j. Then we have a required band sum $g = F(f_0; \{L\}, \{B\})$, where $B = b_1(I \times I) \cup b_2(I \times I) \cup \cdots \cup b_{k+1}(I \times I)$.

Next suppose that $n > 1$. By the hypothesis of our induction, g is a band sum $F(f_1; \mathcal{L}, \mathcal{B})$, where $\mathcal{L} = \{L_1, L_2, \ldots, L_{n-1}\}$ is a set of links of type k, $\mathcal{B} = \{B_1, B_2, \ldots, B_{n-1}\}$ and each B_i is a union of bands attaching to L_i. Deform $F(f_1; \mathcal{L}, \mathcal{B})$ up to ambient isotopy keeping the image $f_1(G)$ so that neither the associated balls of $\mathcal{L}$ nor the bands in $\mathcal{B}$ intersect $\varphi_1(B^3)$. Note that this deformation is possible, since the tangle $\varphi_1^{-1}(f_1(G))$ is trivial. In fact, sweeping out the associated balls, band-slidings and sweeping out the bands are sufficient. See Figure 2.1. Then by the same arguments as that in the case $n = 1$, we find that f_1 is a band sum $F(f; \{L\}, \{B\})$. Then we have

$$F(F(f; \{L\}, \{B\}); \mathcal{L}, \mathcal{B}) = F(f; \{L\} \cup \mathcal{L}, \{B\} \cup \mathcal{B}).$$

This completes the proof. □

As we mentioned before, the origin of the name 'band trivialization' comes from the following lemma:

Lemma 2.2. *The moves in Figures* 2.2-(i), (ii), (iii) *and* (iv) *are realized by A_{k+1}-moves.*

Proof. The move in Figure 2.2-(i) is just a band trivialization of an A_k-move. Hence by the definition it is an A_{k+1}-move. It is easy to see that the moves in Figures 2.2-(ii) and (iii) are generated by the moves in Figure 2.2-(i). To see that the move in Figure 2.2-(iv) is realized by A_{k+1}-moves, we first

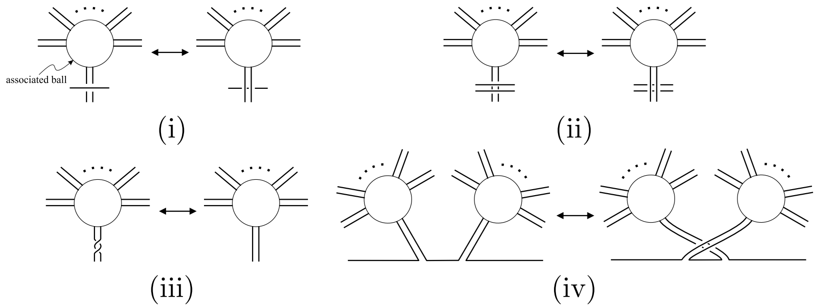


Figure 2.2.

slide the bands as illustrated in Figure 2.3, and then perform the moves in Figure 2.2-(i). □

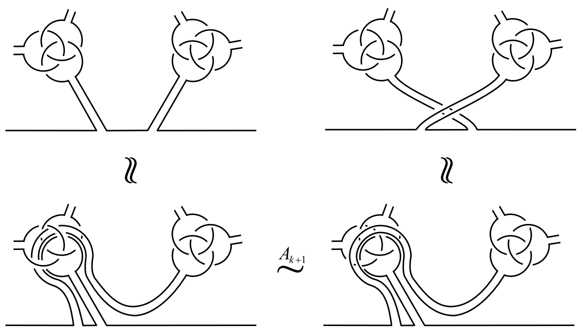


Figure 2.3.

Let $h : G \longrightarrow S^3$ be an embedding and let $[f_1]_k, [f_2]_k \in [h]_{k-1}/(A_k\text{-equivalence})$, where $[h]_{k-1}/(A_k\text{-equivalence})$ denotes the set of A_k-equivalence classes in $[h]_{k-1}$. Since both f_1 and f_2 are A_{k-1}-equivalent to h, by Proposition 2.1, there are band sums $F(h; \mathcal{L}_i, \mathcal{B}_i) \in [f_i]_k$ of h with links $\mathcal{L}_i$ of type $k-1 (i = 1, 2)$. Suppose that the bands in $\mathcal{B}_1$ and the associated balls of $\mathcal{L}_1$ are disjoint from the bands in $\mathcal{B}_2$ and the associated balls of $\mathcal{L}_2$. Note that up to slight ambient isotopy of $F(h; \mathcal{L}_2, \mathcal{B}_2)$ that preserves $h(G)$ we can always choose the bands and the associated balls so that they satisfy this condition. In the following we assume this condition without explicit mention. Then we have a new band sum $F(h; \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B}_1 \cup \mathcal{B}_2)$. We define

$$[f_1]_k +_h [f_2]_k = [F(h; \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B}_1 \cup \mathcal{B}_2)]_k.$$

Lemma 2.3. *The sum '$+_h$' above is well-defined.*

Proof. It is sufficient to show for two embeddings $F(h; \mathcal{L}_1, \mathcal{B}_1), F(h; \mathcal{L}'_1, \mathcal{B}'_1) \in [f_1]_k$ that $F(h; \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B}_1 \cup \mathcal{B}_2)$ and $F(h; \mathcal{L}'_1 \cup \mathcal{L}_2, \mathcal{B}'_1 \cup \mathcal{B}_2)$ are A_k-equivalent.

Consider a sequence of ambient isotopies and applications of A_k-moves that deforms $F(h;\mathcal{L}_1,\mathcal{B}_1)$ into $F(h;\mathcal{L}'_1,\mathcal{B}'_1)$. We consider this sequence of deformations together with the links in $\mathcal{L}_2$ and the bands in $\mathcal{B}_2$. Whenever we apply an A_k-move we deform the associated balls of $\mathcal{L}_2$ and the bands in $\mathcal{B}_2$ up to ambient isotopy so that they are away from the 3-ball within which the A_k-move is applied. Thus $F(h;\mathcal{L}_1\cup\mathcal{L}_2,\mathcal{B}_1\cup\mathcal{B}_2) = F(F(h;\mathcal{L}_1,\mathcal{B}_1);\mathcal{L}_2,\mathcal{B}_2)$ is A_k-equivalent to a band sum $F(F(h;\mathcal{L}'_1,\mathcal{B}'_1);\mathcal{L}'_2,\mathcal{B}'_2)$ for some $\mathcal{L}'_2$ and $\mathcal{B}'_2$. Compare the band sums $F(F(h;\mathcal{L}'_1,\mathcal{B}'_1);\mathcal{L}'_2,\mathcal{B}'_2)$ and $F(h;\mathcal{L}'_1\cup\mathcal{L}_2,\mathcal{B}'_1\cup\mathcal{B}_2) = F(F(h;\mathcal{L}'_1,\mathcal{B}'_1);\mathcal{L}_2,\mathcal{B}_2)$. We have that the links in $\mathcal{L}'_2$ are ambient isotopic to the links in $\mathcal{L}_2$. It follows from Lemma 2.2 that the bands in $\mathcal{B}'_2$ can be deformed into the position of the bands in $\mathcal{B}_2$ by band slidings and A_k-moves. Thus these two are A_k-equivalent. □

Theorem 2.4. *The set $[h]_{k-1}/(A_k$-equivalence$)$ forms an abelian group under '$+_h$' with the unit element $[h]_k$.*

We denote this group by $\mathcal{G}_k(h;G)$ and call it the *A_k-equivalence group* of the spatial embeddings of G with the unit element $[h]_k$.

Remark. Note that for any graph G and any embedding $h: G \longrightarrow S^3$, $[h]_1$ is equal to the set of all embeddings of G into S^3. In [**20**], the second author called $\mathcal{G}_2(h;G)$ a *graph homology group* and gave a practical method of calculating this group.

Proof. We consider embeddings up to ambient isotopy for simplicity. It is sufficient to show that for any $[f]_k \in [h]_{k-1}/(A_k$-equivalence), there is an inverse of $[f]_k$. Since f and h are A_{k-1}-equivalent, by Proposition 2.1, f and h are band sums $F(h;\mathcal{L},\mathcal{B})$ and $F(f;\mathcal{L}',\mathcal{B}')$ respectively, where $\mathcal{L}$ and $\mathcal{L}'$ are sets of links of type $k-1$. Thus we have $h = F(F(h;\mathcal{L},\mathcal{B});\mathcal{L}',\mathcal{B}')$. Then, by using Lemma 2.2, we deform the associated balls of $\mathcal{L}'$ and the bands in $\mathcal{B}'$ up to A_k-equivalence so that they are disjoint form the associated balls of $\mathcal{L}$ and the bands in $\mathcal{B}$. Thus we see that $h = F(F(h;\mathcal{L},\mathcal{B});\mathcal{L}',\mathcal{B}')$ is A_k-equivalent to a band sum $F(h;\mathcal{L}\cup\mathcal{L}'',\mathcal{B}\cup\mathcal{B}'')$ for some $\mathcal{L}''$ and $\mathcal{B}''$ (for example see Figure 2.4). Thus we have

$$\begin{aligned}[f]_k +_h [F(h;\mathcal{L}'',\mathcal{B}'')]_k &= [F(h;\mathcal{L},\mathcal{B})]_k +_h [F(h;\mathcal{L}'',\mathcal{B}'')]_k \\ &= [F(h;\mathcal{L}\cup\mathcal{L}'',\mathcal{B}\cup\mathcal{B}'')]_k \\ &= [h]_k.\end{aligned}$$

This implies that $[F(h;\mathcal{L}'',\mathcal{B}'')]_k$ is an inverse of $[f]_k$. □

Theorem 2.5. *Let $h_1, h_2 : G \longrightarrow S^3$ be A_{k-1}-equivalent embeddings. Then the groups $\mathcal{G}_k(h_1;G)$ and $\mathcal{G}_k(h_2;G)$ are isomorphic.*

Proof. We define a map $\phi : \mathcal{G}_k(h_1;G) \longrightarrow \mathcal{G}_k(h_2;G)$ by $\phi([f]_k) = [f]_k -_{h_2} [h_1]_k$, where $[x]_k -_{h_2} [y]_k$ denotes $[x]_k +_{h_2} (-[y]_k)$. Clearly this map is a bijection. We shall prove that ϕ is a homomorphism. Let $[f_i]_k \in \mathcal{G}_k(h_1;G)$ $(i =$

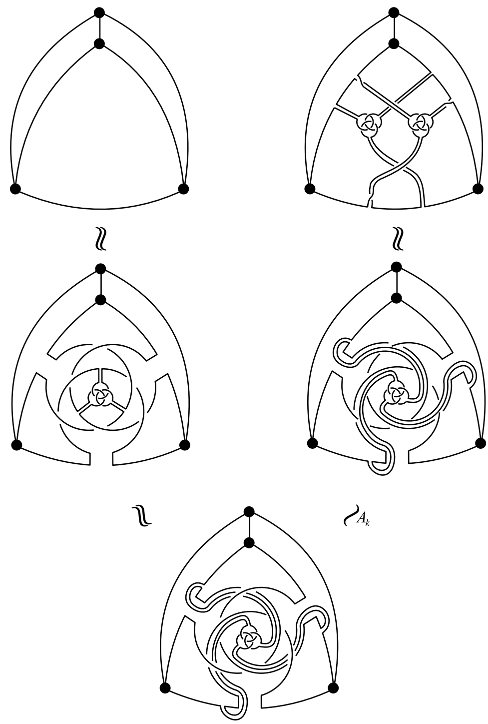


Figure 2.4.

1, 2). Then $f_i = F(h_1; \mathcal{L}_i, \mathcal{B}_i)$ where $\mathcal{L}_i$ is a set of links of type $k-1$ $(i = 1, 2)$. Since h_1 and h_2 are A_{k-1}-equivalent we see that $h_1 = F(h_2; \mathcal{L}, \mathcal{B})$ where $\mathcal{L}$ is a set of links of type $k-1$. Thus we have $f_i = F(F(h_2; \mathcal{L}, \mathcal{B}); \mathcal{L}_i, \mathcal{B}_i)$ $(i = 1, 2)$. By using Lemma 2.2, we deform f_i up to A_k-equivalence so that the associated balls of $\mathcal{L}_i$ and the bands in $\mathcal{B}_i$ are disjoint from the associated balls of $\mathcal{L}$ and the bands in $\mathcal{B}$ for $i = 1, 2$. We may further assume that the associated balls of $\mathcal{L}_1$ and the bands in $\mathcal{B}_1$ are disjoint from the associated balls of $\mathcal{L}_2$ and the bands in $\mathcal{B}_2$. Then we have

$$\begin{aligned}\phi([f_1]_k +_{h_1} [f_2]_k) &= \phi([F(F(h_2; \mathcal{L}, \mathcal{B}); \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B}_1 \cup \mathcal{B}_2)]_k) \\ &= [F(F(h_2; \mathcal{L}, \mathcal{B}); \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B}_1 \cup \mathcal{B}_2)]_k -_{h_2} [h_1]_k \\ &= [F(h_2; \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B} \cup \mathcal{B}_1 \cup \mathcal{B}_2)]_k -_{h_2} [F(h_2; \mathcal{L}, \mathcal{B})]_k \\ &= [F(h_2; \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B}_1 \cup \mathcal{B}_2)]_k,\end{aligned}$$

and for each i $(i = 1, 2)$,

$$\begin{aligned}\phi([f_i]_k) &= \phi([F(F(h_2; \mathcal{L}, \mathcal{B}); \mathcal{L}_i, \mathcal{B}_i)]_k) \\ &= [F(F(h_2; \mathcal{L}, \mathcal{B}); \mathcal{L}_i, \mathcal{B}_i)]_k -_{h_2} [h_1]_k \\ &= [F(h_2; \mathcal{L} \cup \mathcal{L}_i, \mathcal{B} \cup \mathcal{B}_i)]_k -_{h_2} [F(h_2; \mathcal{L}, \mathcal{B})]_k \\ &= [F(h_2; \mathcal{L}_i, \mathcal{B}_i)]_k.\end{aligned}$$

Thus we have $\phi([f_1]_k +_{h_1} [f_2]_k) = \phi([f_1]_k) +_{h_2} \phi([f_2]_k)$. □

Proposition 2.6. *The projection* $p : [h]_{k-1} \longrightarrow [h]_{k-1}/(A_k\text{-equivalence}) = \mathcal{G}_k(h;G)$ *is a finite type* A_k*-equivalence invariant of order* $(1;k)$*.*

Proof. It is clear that p is an A_k-equivalence invariant. We shall prove that p is finite type of order $(1;k)$. Let $f \in [h]_{k-1}$ be an embedding and $F(f;\{L_1,L_2\},\{B_1,B_2\})$ a band sum of f with links L_1, L_2 of type $k-1$. Then it is sufficient to show that

$$\sum_{X\subset\{1,2\}} (-1)^{|X|} p\left(F\left(f;\bigcup_{i\in X}\{L_i\},\bigcup_{i\in X}\{B_i\}\right)\right) = [h]_k.$$

Let $\phi : \mathcal{G}_k(f;G) \longrightarrow \mathcal{G}_k(h;G)$ be the isomorphism defined by $\phi([g]_k) = [g]_k -_h [f]_k$. Then we have

$$\begin{aligned}
&\phi\Big([F(f;\emptyset,\emptyset)]_k -_f [F(f;\{L_1\},\{B_1\})]_k \\
&\qquad -_f [F(f;\{L_2\},\{B_2\})]_k +_f [F(f;\{L_1,L_2\},\{B_1,B_2\})]_k\Big) \\
&= ([F(f;\emptyset,\emptyset)]_k -_h [f]_k) -_h ([F(f;\{L_1\},\{B_1\})]_k -_h [f]_k) \\
&\quad -_h ([F(f;\{L_2\},\{B_2\})]_k -_h [f]_k) +_h ([F(f;\{L_1,L_2\},\{B_1,B_2\})]_k -_h [f]_k) \\
&= [F(f;\emptyset,\emptyset)]_k -_h [F(f;\{L_1\},\{B_1\})]_k -_h [F(f;\{L_2\},\{B_2\})]_k \\
&\quad +_h [F(f;\{L_1,L_2\},\{B_1,B_2\})]_k \\
&= \sum_{X\subset\{1,2\}} (-1)^{|X|} p\left(F\left(f;\bigcup_{i\in X}\{L_i\},\bigcup_{i\in X}\{B_i\}\right)\right).
\end{aligned}$$

Since

$$[F(f;\{L_1,L_2\},\{B_1,B_2\})]_k = [F(f;\{L_1\},\{B_1\})]_k +_f [F(f;\{L_2\},\{B_2\})]_k,$$

we have

$$\begin{aligned}
&\phi\Big([F(f;\emptyset,\emptyset)]_k -_f [F(f;\{L_1\},\{B_1\})]_k -_f [F(f;\{L_2\},\{B_2\})]_k \\
&\qquad +_f [F(f;\{L_1,L_2\},\{B_1,B_2\})]_k\Big) \\
&= \phi([f]_k) \\
&= [h]_k.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.1*.* The 'only if' part is clear. We show the 'if' part. Let f and g be embeddings in $[h]_{k-1}$. Suppose that any finite type invariant of order $(1;k)$ takes the same value on f and g. Then by Proposition 2.6 we have $p(f) = p(g)$, where $p : [h]_{k-1} \longrightarrow [h]_{k-1}/(A_k\text{-equivalence}) = \mathcal{G}_k(h;G)$ is the projection. Hence we have $[f]_k = [g]_k$. This completes the proof. □

3. A_k-equivalence group of knots.

In this section we only consider the case that the graph G is homeomorphic to a disjoint union of circles. Let $G = S^1_1 \cup S^1_2 \cup \cdots \cup S^1_\mu$. Then there is a natural correspondence between the ambient isotopy classes of the embeddings of G into S^3 and the ambient isotopy classes of the ordered oriented μ-component links in S^3. Therefore instead of specifying an embedding $h : S^1_1 \cup S^1_2 \cup \cdots \cup S^1_\mu \longrightarrow S^3$, we denote by L the image $h(S^1_1 \cup S^1_2 \cup \cdots \cup S^1_\mu)$ and consider it together with the orientation of each component and the ordering of the components. Thus $\mathcal{G}_k(L)$ denotes the A_k-equivalence group $\mathcal{G}_k(h; S^1_1 \cup S^1_2 \cup \cdots \cup S^1_\mu)$ with the unit element $[h]_k$.

Theorem 3.1. *Let O be a trivial knot. Then for any oriented knot K, $\mathcal{G}_k(O)$ and $\mathcal{G}_k(K)$ are isomorphic.*

Remark. For a graph $G(\neq S^1)$ and embeddings $h, h' : G \longrightarrow S^3$, $\mathcal{G}_k(h; G)$ and $\mathcal{G}_k(h'; G)$ are not always isomorphic. In fact there are two-component links L_1 and L_2 such that $\mathcal{G}_3(L_1) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\mathcal{G}_3(L_2) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$ [**16**].

Proof. We define a map $\phi : \mathcal{G}_k(O) \longrightarrow \mathcal{G}_k(K)$ by

$$\phi([F(O; \mathcal{L}, \mathcal{B})]_k) = [K\#F(O; \mathcal{L}, \mathcal{B})]_k$$

for each $[F(O; \mathcal{L}, \mathcal{B})]_k \in \mathcal{G}_k(O)$, where $\mathcal{L}$ is a set of links of type $k-1$ and $\#$ means the connected sum of oriented knots. Clearly this is well-defined. By Lemma 2.2, any band sum $F(K; \mathcal{L}, \mathcal{B})$ of K with links $\mathcal{L}$ of type $k-1$ is A_k-equivalent to $K\#F(O; \mathcal{L}', \mathcal{B}')$ for some links $\mathcal{L}'$ of type $k-1$ and $\mathcal{B}'$. Hence ϕ is surjective. For $[F(O; \mathcal{L}_i, \mathcal{B}_i)]_k \in \mathcal{G}_k(O)$ $(i = 1, 2)$, we have

$$\begin{aligned}
&\phi([F(O; \mathcal{L}_1, \mathcal{B}_1)]_k +_O [F(O; \mathcal{L}_2, \mathcal{B}_2)]_k) \\
&= \phi([F(O; \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B}_1 \cup \mathcal{B}_2)]_k) \\
&= [K\#F(O; \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B}_1 \cup \mathcal{B}_2)]_k \\
&= [F(K; \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B}_1 \cup \mathcal{B}_2)]_k \\
&= [F(K; \mathcal{L}_1, \mathcal{B}_1)]_k +_K [F(K; \mathcal{L}_2, \mathcal{B}_2)]_k \\
&= [K\#F(O; \mathcal{L}_1, \mathcal{B}_1)]_k +_K [K\#F(O; \mathcal{L}_2, \mathcal{B}_2)]_k \\
&= \phi([F(O; \mathcal{L}_1, \mathcal{B}_1)]_k) +_K \phi([F(O; \mathcal{L}_2, \mathcal{B}_2)]_k).
\end{aligned}$$

This implies that ϕ is a homomorphism. In order to complete the proof, we show that ϕ is injective. Suppose that $[K\#F(O; \mathcal{L}, \mathcal{B})]_k = [K]_k$. By Lemma 3.2, there is a knot K' such that $[K'\#K]_k = [O]_k$. Then we have

$$\begin{aligned}
[F(O; \mathcal{L}, \mathcal{B})]_k &= [(K'\#K)\#F(O; \mathcal{L}, \mathcal{B})]_k \\
&= [K'\#(K\#F(O; \mathcal{L}, \mathcal{B}))]_k \\
&= [K'\#K]_k \\
&= [O]_k.
\end{aligned}$$

This implies that $\ker\phi = \{[O]_k\}$. □

Habiro originated 'clasper theory' and showed Lemma 3.2 for C_k-moves [**3**] and [**4**]. The following proof is a translation of his proof in terms of band sum description of knots:

Lemma 3.2. *For any knot K and any integer $k \geq 1$, there is a knot K' such that $K'\#K$ is A_k-equivalent to a trivial knot.*

Proof. We shall prove this by induction on k. The case $k = 1$ is clear. Suppose that there is a knot K' such that $K'\#K$ is A_{k-1}-equivalent to a trivial knot O ($k > 1$). By Proposition 2.1, we may assume that $O = F(K'\#K; \mathcal{L}, \mathcal{B})$, where $\mathcal{L}$ is a set of links of type $k-1$. Then, by Lemma 2.2, we see that $F(K'\#K; \mathcal{L}, \mathcal{B})$ is A_k-equivalent to some $K\#F(K'; \mathcal{L}, \mathcal{B}')$. This completes the proof. □

Let $\mathcal{K}_k$ be the set of A_k-equivalence classes of all oriented knots. For $[K]_k, [K']_k \in \mathcal{K}_k$, we define $[K]_k + [K']_k = [K\#K']_k$. Then the following, shown by Habiro [**3**],[**4**] in the case that A_k-moves coincide with C_k-moves, is an immediate consequence of Lemma 3.2.

Theorem 3.3. *The set $\mathcal{K}_k$ forms an abelian group under '$+$' with the unit element $[O]_k$, where O is a trivial knot.*

4. Generalized A_k-move.

In this section, we define a generalized A_k-move. For this move, several results similar to that in Sections 1, 2 and 3 hold.

Let T and S be trivial tangles such that (T, S) and (S, T) are equivalent. Let $t_1, t_2, \dots, t_n$ and $s_1, s_2, \dots, s_n$ be the components of T and S respectively. An $A_1(T, S)$*-move* is this local move (T, S). Suppose that $A_k(T, S)$-moves are defined. For each $A_k(T, S)$-move (T_k, S_k) we choose a trivialization of T_k and fix it. Then the band trivializations of (T_k, S_k) with respect to the trivialization are called $A_{k+1}(T, S)$*-moves*. Let (T_k, S_k) be an $A_k(T, S)$-move and $D_1, D_2, \dots, D_{n+k-1}$ the fixed trivialization of $T_k = t_1 \cup t_2 \cup \cdots \cup t_{n+k-1}$. We set $\alpha = \partial B^3 \cap (D_1 \cup D_2 \cup \cdots \cup D_{n+k-1})$ and $\beta = S_k$. A link L in S^3 is called *type* $(k; (T, S))$ if there is an orientation preserving embedding $\varphi : B^3 \longrightarrow S^3$ such that $L = \varphi(\alpha \cup \beta)$. Then the pair (α, β) is called a *link model of* L. As in Section 1, $A_k(T, S)$-move gives an equivalence relation, $A_k(T, S)$*-equivalence*, on the set of all embeddings of G into S^3. For an embedding $f : G \longrightarrow S^3$, let $[f]_k$ denote the $A_k(S, T)$-equivalence class of f. Let $h : G \longrightarrow S^3$ be an embedding and H an abelian group. Let $\varphi : [h]_{k-1} \longrightarrow H$ be an invariant. We can define that φ is a *finite type invariant of order* $(n; k; (T, S))$ as in Section 1.

By the arguments similar to that in Sections 1, 2 and 3, we have the following five theorems:

Theorem 4.1. *Let $f, g : G \longrightarrow S^3$ be $A_{k-1}(T,S)$-equivalent embeddings. Then they are $A_k(T,S)$-equivalent if and only if $\varphi(f) = \varphi(g)$ for any finite type $A_k(T,S)$-equivalence invariant φ of order $(1; k; (T,S))$.*

Let $h : G \longrightarrow S^3$ be an embedding. For $[f_1]_k, [f_2]_k \in [h]_{k-1}/(A_k(T,S)$-equivalence), we can define $[f_1]_k +_h [f_2]_k$ as in Section 2, and we have:

Theorem 4.2. *The set $[h]_{k-1}/(A_k(T,S)$-equivalence) forms an abelian group under '$+_h$' with the unit element $[h]_k$.*

We denote this group by $\mathcal{G}_{k(T,S)}(h; G)$ and call it the $A_k(T,S)$-*equivalence group* of the spatial embeddings of G with the unit element $[h]_k$.

Theorem 4.3. *Let $h_1, h_2 : G \longrightarrow S^3$ be $A_{k-1}(T,S)$-equivalent embeddings. Then the groups $\mathcal{G}_{k(T,S)}(h_1; G)$ and $\mathcal{G}_{k(T,S)}(h_2; G)$ are isomorphic.*

For an embedding $h : S^1 \longrightarrow S^3$, let $K = h(S^1)$ and let $\mathcal{G}_{k(T,S)}(K)$ denote the $A_k(T,S)$-equivalence group $\mathcal{G}_{k(T,S)}(h; S^1)$ with the unit element $[h]_k$.

Theorem 4.4. *Let O be a trivial knot. If any two knots are $A_1(T,S)$-equivalent, then for any oriented knot K, $\mathcal{G}_{k(T,S)}(O)$ and $\mathcal{G}_{k(T,S)}(K)$ are isomorphic.*

Let $\mathcal{K}_{k(T,S)}$ be the set of $A_k(T,S)$-equivalence classes of all oriented knots. For $[K]_k, [K']_k \in \mathcal{K}_{k(T,S)}$, we define $[K]_k + [K']_k = [K\#K']_k$.

Theorem 4.5. *If any two knots are $A_1(T,S)$-equivalent, then the set $\mathcal{K}_{k(T,S)}$ forms an abelian group under '$+$' with the unit element $[O]_k$, where O is a trivial knot.*

Remark. If (T,S) is the #-move defined by Murakami [6], then $\mathcal{K}_{k(T,S)}$ is an abelian group.

5. A_k-moves and Vassiliev invariants.

Let G be a finite graph. We give and fix orientations of the edges of G. Let $\mathcal{E}$ be the set of the ambient isotopy classes of the embeddings of G into S^3. Let $\mathbb{Z}\mathcal{E}$ be the free abelian group generated by the elements of $\mathcal{E}$. A *crossing vertex* is a double point of a map from G to S^3 as in Figure 5.1. An i-*singular embedding* is a map from G to S^3 whose multiple points are exactly i crossing vertices. By the formula in Figure 5.2 we identify an i-singular embedding with an element in $\mathbb{Z}\mathcal{E}$. Let $\mathcal{R}_i$ be the subgroup of $\mathbb{Z}\mathcal{E}$ generated by all i-singular embeddings. Note that $\mathcal{R}_i$ is independent of the choices of the edge orientations. Let H be an abelian group. Let $\varphi : \mathcal{E} \longrightarrow H$ be a map. Let $\widetilde{\varphi} : \mathbb{Z}\mathcal{E} \longrightarrow H$ be the natural extension of φ. We say that φ is a *Vassiliev invariant of order n* if $\widetilde{\varphi}(\mathcal{R}_{n+1}) = \{0\}$. Let $\iota : \mathcal{E} \longrightarrow \mathbb{Z}\mathcal{E}$ be the natural inclusion map and $\pi_i : \mathbb{Z}\mathcal{E} \longrightarrow \mathbb{Z}\mathcal{E}/\mathcal{R}_i$ the quotient homomorphism.

Let $u_{i-1} = \pi_i \circ \iota : \mathcal{E} \longrightarrow \mathbb{Z}\mathcal{E}/\mathcal{R}_i$ be the composition map. Then φ is a Vassiliev invariant of order n if and only if there is a homomorphism $\hat{\varphi} : \mathbb{Z}\mathcal{E}/\mathcal{R}_{n+1} \longrightarrow H$ such that $\varphi = \hat{\varphi} \circ u_n$. In the following we sometimes do not distinguish between an embedding and its ambient isotopy class so long as no confusion occurs.

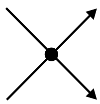


Figure 5.1.

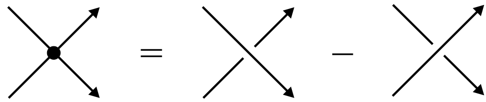


Figure 5.2.

Theorem 5.1. *Let $f, g : G \longrightarrow S^3$ be A_{k+1}-equivalent embeddings. Then $u_k(f) = u_k(g)$.*

By using induction on k, we see that an A_k-move (T, S) is a $(k+1)$-component Brunnian local move, i.e., $T - t$ and $S - s$ are ambient isotopic in B^3 relative ∂B^3 for any $t \in T$ and $s \in S$ with $\partial t = \partial s$ [**15**]. It is not hard to see that if two embeddings f and g are related by a $(k+1)$-component Brunnian local move, then f and g are *k-similar*, where k-similar is an equivalence relation defined by the first author [**13**]. Therefore, we note that Theorem 5.1 follows from [**1**] or [**9**]. However we give a self-contained proof here.

Let T be a tangle. Let $\mathcal{H}(T)$ be the set of all (possibly nontrivial) tangles that are homotopic to T relative to ∂B^3. Let $\mathcal{E}(T)$ be the quotient of $\mathcal{H}(T)$ by the ambient isotopy relative to ∂B^3. Then $\mathbb{Z}\mathcal{E}(T)$, i-singular tangles and $\mathcal{R}_i(T) \subset \mathbb{Z}\mathcal{E}(T)$ are defined as above.

Proof. It is sufficient to show for each A_k-move (T, S) that $T - S$ is an element in $\mathcal{R}_k(T)$. We show this by induction on k. The case $k = 1$ is clear. Recall that an A_k-move (T, S) is a band trivialization of an A_{k-1}-move, say (T', S'). Then we have that $T - S = X_1 - X_2$ where X_1 and X_2 are 1-singular tangles in Figure 5.3. Let Y_1 and Y_2 be 1-singular tangles in Figure 5.4. It is clear that $Y_1 - Y_2 = 0$. Thus we have $T - S = (X_1 - Y_1) - (X_2 - Y_2)$. By the induction hypothesis we see that $S' - T'$ is an element of $\mathcal{R}_{k-1}(S')$. Therefore both $X_1 - Y_1$ and $X_2 - Y_2$ are elements of $\mathcal{R}_k(T)$. □

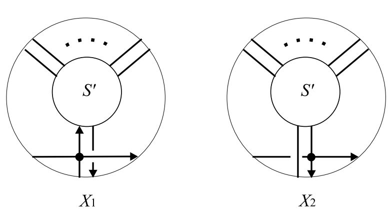


Figure 5.3.

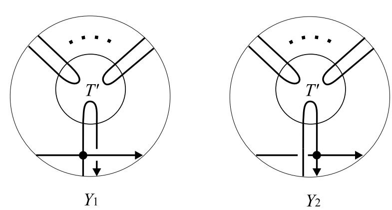


Figure 5.4.

Proof of Proposition 1.3. It is sufficient to show that

$$\sum_{X\subset\{1,2,\dots,n+1\}}(-1)^{|X|}F\left(f;\bigcup_{i\in X}\{L_i\},\bigcup_{i\in X}\{B_i\}\right)$$

is an element of $\mathcal{R}_{(n+1)(k-1)}$. We show this together with some additional claims by induction on n. First consider the case $n=0$. Then we have by Theorem 5.1 and its proof that $F(f;\emptyset,\emptyset)-F(f;\{L_1\},\{B_1\})$ is a sum of $(k-1)$-singular embeddings each of which has all crossing vertices in the associated ball. Note that these $(k-1)$-singular embeddings are natural extensions of the $(k-1)$-singular tangles that express the difference of the A_{k-1}-move, and these $(k-1)$-singular tangles depends only on the link L_1. Next we consider the general case. Note that

$$\begin{aligned}&\sum_{X\subset\{1,2,\dots,n+1\}}(-1)^{|X|}F\left(f;\bigcup_{i\in X}\{L_i\},\bigcup_{i\in X}\{B_i\}\right)\\&=\sum_{X\subset\{1,2,\dots,n\}}(-1)^{|X|}F\left(f;\bigcup_{i\in X}\{L_i\},\bigcup_{i\in X}\{B_i\}\right)\\&\quad-\sum_{X\subset\{1,2,\dots,n\}}(-1)^{|X|}F\left(f;\bigcup_{i\in X}\{L_i\}\cup\{L_{n+1}\},\bigcup_{i\in X}\{B_i\}\cup\{B_{n+1}\}\right).\end{aligned}$$

By the hypothesis we have both

$$\sum_{X\subset\{1,2,\dots,n\}}(-1)^{|X|}F\left(f;\bigcup_{i\in X}\{L_i\},\bigcup_{i\in X}\{B_i\}\right)$$

and

$$\sum_{X\subset\{1,2,\dots,n\}}(-1)^{|X|}F\left(f;\bigcup_{i\in X}\{L_i\}\cup\{L_{n+1}\},\bigcup_{i\in X}\{B_i\}\cup\{B_{n+1}\}\right)$$

are sums of $n(k-1)$-singular embeddings and they differ only by the band sum of L_{n+1}. Therefore we have

$$\sum_{X\subset\{1,2,\dots,n+1\}}(-1)^{|X|}F\left(f;\bigcup_{i\in X}\{L_i\},\bigcup_{i\in X}\{B_i\}\right)$$

is a sum of $(n(k-1)+(k-1))$-singular embeddings. □

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