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QUANTUM LENS SPACES AND GRAPH ALGEBRAS

Jeong Hee Hong and Wojciech Szymański

We construct the C^*-algebra $C(L_q(p;m_1,\dots,m_n))$ of continuous functions on the quantum lens space as the fixed point algebra for a suitable action of $\mathbb{Z}_p$ on the algebra $C(S_q^{2n-1})$, corresponding to the quantum odd dimensional sphere. We show that $C(L_q(p;m_1,\dots,m_n))$ is isomorphic to the graph algebra $C^*\left(L_{2n-1}^{(p;m_1,\dots,m_n)}\right)$. This allows us to determine the ideal structure and, at least in principle, calculate the K-groups of $C(L_q(p;m_1,\dots,m_n))$. Passing to the limit with natural imbeddings of the quantum lens spaces we construct the quantum infinite lens space, or the quantum Eilenberg-MacLane space of type $(\mathbb{Z}_p,1)$.

0. Introduction.

Classical lens spaces $L(p;m_1,\dots,m_n)$ are defined as the orbit spaces of suitable free actions of finite cyclic groups on odd dimensional spheres (e.g., see [**13**]). In the present article, we define and investigate their quantum analogues. The C^*-algebras of continuous functions on the quantum lens spaces were introduced earlier by Matsumoto and Tomiyama in [**18**], but our construction leads to different (in general) algebras. (The very special case of the quantum 3-dimensional real projective space was investigated by Podleś [**20**] and Lance [**17**], in the context of the quantum $SO(3)$ group.) The starting point for us is the C^*-algebra $C(S_q^{2n-1})$, $q \in (0,1)$, of continuous functions on the quantum odd dimensional sphere. If $n = 2$ then $C(S_q^3)$ is nothing but $C(SU_q(2))$ of Woronowicz [**27**]. The construction in higher dimensions is due to Vaksman and Soibelman [**26**], and from a somewhat different perspective to Nagy [**19**]. (See also the closely related construction of representations of the twisted canonical commutation relations due to Pusz and Woronowicz [**21**].) We define the C^*-algebra $C(L_q(p;m_1,\dots,m_n))$ of continuous functions on the quantum lens space as the fixed point algebra for a suitable action of the finite cyclic group $\mathbb{Z}_p$ on $C(S_q^{2n-1})$. This definition depends on the deformation parameter $q \in (0,1)$, as well as on positive integers $p \geq 2$ and $m_1,\dots,m_n$. We normally assume that each of $m_1,\dots,m_n$ is relatively prime to p. On the classical level, this guarantees freeness of the action. In the special case $p = 2$, $m_1 = \cdots = m_n = 1$ we recover quantum odd dimensional real projective spaces, defined and investigated in our earlier article [**11**].

The key technical result on which this article depends is Theorem 4.4 of [**11**], which gives an explicit isomorphism between $C(S_q^{2n-1})$ and the C^*-algebra $C^*(L_{2n-1})$ corresponding to the directed graph L_{2n-1}. Thus, $C(L_q(p; m_1, \ldots, m_n))$ is isomorphic to the fixed point algebra $C^*(L_{2n-1})^\Lambda$, corresponding to a suitable action $\Lambda : \mathbb{Z}_p \to \mathrm{Aut}(C^*(L_{2n-1}))$. This allows us to employ in our investigations of the quantum lens spaces the huge and comprehensive machinery developed for dealing with Cuntz-Krieger algebras of directed graphs (cf. [**6**, **5**, **16**, **12**, **15**, **14**, **2**, **9**, **24**, **10**, **22**, **25**, **1**] and references there).

In order to understand the fixed point algebra $C^*(L_{2n-1})^\Lambda$ we first look at the crossed product $C^*(L_{2n-1}) \rtimes_\Lambda \mathbb{Z}_p$. By virtue of the results of [**14**] this crossed product itself is isomorphic to the C^*-algebra of the skew product graph $L_{2n-1} \times_c \mathbb{Z}_p$, corresponding to a suitable labelling c of the edges of L_{2n-1} by elements of $\mathbb{Z}_p$. The action Λ is saturated and, hence, $C^*(L_{2n-1})^\Lambda$ is isomorphic to a full corner of $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$. This allows us, at least in principle, to calculate the K-groups of $C(L_q(p; m_1, \ldots, m_n))$.

Our main result, Theorem 2.5, shows that $C(L_q(p; m_1, \ldots, m_n))$ itself is isomorphic to the graph algebra $C^*\left(L_{2n-1}^{(p;m_1,\ldots,m_n)}\right)$, corresponding to a finite graph $L_{2n-1}^{(p;m_1,\ldots,m_n)}$. As a corollary, we easily deduce the ideal structure of $C(L_q(p; m_1, \ldots, m_n))$. We believe that on the basis of Theorem 2.5 one should be able to determine isomorphisms between the C^*-algebras of continuous functions on the quantum lens spaces, but this work is not carried in the present article.

1. Preliminaries.

1.1. Definition. We recall the definition of the C^*-algebra corresponding to a directed graph [**9**]. Let $E = (E^0, E^1, r, s)$ be a directed graph with (at most) countably many vertices E^0 and edges E^1, and range and source functions $r, s : E^1 \to E^0$, respectively. $C^*(E)$ is, by definition, the universal C^*-algebra generated by families of projections $\{P_v \mid v \in E^0\}$ and partial isometries $\{S_e \mid e \in E^1\}$, subject to the following relations:

(G1) $P_v P_w = 0$ for $v, w \in E^0$, $v \neq w$.
(G2) $S_e^* S_f = 0$ for $e, f \in E^1$, $e \neq f$.
(G3) $S_e^* S_e = P_{r(e)}$ for $e \in E^1$.
(G4) $S_e S_e^* \leq P_{s(e)}$ for $e \in E^1$.
(G5) $P_v = \sum_{e \in E^1 : \, s(e) = v} S_e S_e^*$ for $v \in E^0$, provided $\{e \in E^1 \mid s(e) = v\}$ is finite and nonempty.

Universality in this definition means that if $\{Q_v \mid v \in E^0\}$ and $\{T_e \mid e \in E^1\}$ are families of projections and partial isometries, respectively, satisfying Conditions (G1)–(G5), then there exists a C^*-algebra homomorphism from $C^*(E)$ to the C^*-algebra generated by $\{Q_v \mid v \in E^0\}$ and $\{T_e \mid e \in E^1\}$ such that $P_v \mapsto Q_v$ and $S_e \mapsto T_e$ for $v \in E^0$, $e \in E^1$.

It follows from the universal property that there exists the gauge action $\gamma : \mathbb{T} \to \mathrm{Aut}(C^*(E))$ such that $\gamma_t(P_v) = P_v$ and $\gamma_t(S_e) = tS_e$, for all $v \in E^0$, $e \in E^1$, $t \in \mathbb{T}$.

1.2. *K*-theory. The K-theory of a graph algebra $C^*(E)$ can be calculated as follows: Let V_E be the collection of all those vertices which are not sinks and emit finitely many edges, and let $\mathbb{Z}V_E$ and $\mathbb{Z}E^0$ be free abelian groups on free generators V_E and E^0, respectively. We have

$$K_0(C^*(E)) \cong \mathrm{coker}(K_E),$$
$$K_1(C^*(E)) \cong \ker(K_E),$$

where $K_E : \mathbb{Z}V_E \to \mathbb{Z}E^0$ is the map defined on generators as

$$K_E(v) = \left(\sum_{e \in E^1 \colon s(e)=v} r(e) \right) - v.$$

(See [**5**, Proposition 3.1], [**16**, Corollary 6.12], [**22**, Theorem 3.2], [**24**, Proposition 2] and [**7**, Theorem 3.1].)

1.3. Ideals. We assume that E is a row-finite graph (i.e., each vertex of E emits only finitely many edges) without sinks, since this is all we need in the present article. At first we describe closed 2-sided ideals of $C^*(E)$ invariant under the gauge action, as well as the corresponding quotients [**5**, **12**, **16**, **2**, **1**, **10**]. To this end we consider hereditary and saturated subsets of E^0. A subset $X \subseteq E^0$ is hereditary and saturated if the following two conditions are satisfied:

(HS1) If $v \in X$, $w \in E^0$ and there exists a path from v to w then $w \in X$.

(HS2) If $v \in E^0$ and for each $e \in E^1$ with $s(e) = v$ we have $r(e) \in X$, then $v \in X$.

We denote by Σ_E the collection of all hereditary and saturated subsets of E^0. Any hereditary and saturated set X gives rise to a gauge invariant ideal generated by $\{P_v \mid v \in X\}$ and denoted J_X. The quotient $C^*(E)/J_X$ is naturally isomorphic to $C^*(E/X)$, where E/X denotes the restriction of the graph E to $E^0 \setminus X$. There exists a bijection between Σ_E and the collection of all gauge invariant ideals of $C^*(E)$, given by the following two maps:

$$X \mapsto J_X, \quad J \mapsto \{v \in E^0 \mid P_v \in J\}.$$

We now turn to the description of primitive ideals of the graph algebra $C^*(E)$ corresponding to a row-finite graph E with no sinks [**12**, **16**, **2**, **1**, **10**]. Key objects used in the classification of primitive ideals of graph algebras are maximal tails, defined as follows: A nonempty subset $M \subseteq E^0$ is called a maximal tail if the following three conditions are satisfied:

(MT1) If $v \in E^0$, $w \in M$ and there is a path in E from v to w then $v \in M$.

(MT2) If $v \in M$ then there exists an edge $e \in E^1$ such that $s(e) = v$ and $r(e) \in M$.

(MT3) For any $v, w \in M$ there is a $y \in M$ such that there exist paths in E from v to y and from w to y.

The collection $\mathcal{M}(E)$ of all maximal tails is a disjoint union of its two subcollections $\mathcal{M}_\gamma(E)$ and $\mathcal{M}_\tau(E)$, defined as follows: A maximal tail M belongs to $\mathcal{M}_\gamma(E)$ if and only if every vertex simple loop $(e_1, e_2, \ldots, e_k)$ (where $e_i \in E^1$, $r(e_i) = s(e_{i+1})$, $r(e_k) = s(e_1)$ and $r(e_i) \neq r(e_j)$ for $i \neq j$) whose all vertices $s(e_i)$ belong to M has an exit $e \in E^1$ (that is, $s(e) \in \{s(e_1), \ldots s(e_k)\}$ but $e \notin \{e_1, \ldots, e_k\}$) with $r(e) \in M$. Otherwise M belongs to $\mathcal{M}_\tau(E)$. It can be shown that each maximal tail from $\mathcal{M}_\gamma(E)$ gives rise to a primitive ideal of $C^*(E)$ invariant under the gauge action, and each maximal tail from $\mathcal{M}_\tau(E)$ gives rise to a circle of primitive ideals none of which is invariant under the gauge action. Let $\mathrm{Prim}(C^*(E))$ denote the set of all primitive ideals of $C^*(E)$. If E is a finite graph with no sinks then there exists a bijection

$$\mathcal{M}_\gamma(E) \cup (\mathcal{M}_\tau(E) \times \mathbb{T}) \leftrightarrow \mathrm{Prim}(C^*(E)).$$

A complete description of the closure operation in the hull-kernel topology is also available. See [**12**, **16**, **2**, **1**, **10**] for the details.

We finish this section with the following lemma, which will be needed in the proof of Theorem 2.5. Recall that a closed 2-sided ideal J of a C^*-algebra A is essential if and only if for each nonzero element a of A we have $aJ \neq \{0\}$.

Lemma 1.1. *If E is a row-finite graph and $X \neq \emptyset$ is a hereditary and saturated subset of E^0 then J_X is an essential ideal of $C^*(E)$ if and only if for each vertex $v \in E^0 \setminus X$ there exists a path in E from v to a vertex in X.*

Proof. Suppose that for each vertex $v \in E^0 \setminus X$ there exists a path in E from v to a vertex in X. With the gauge action $\gamma : \mathbb{T} \to \mathrm{Aut}(C^*(E))$, the formula $\Gamma(b) = \int_{t \in \mathbb{T}} \gamma_t(b) dt$ (the integration with respect to the normalized Haar measure) defines a faithful conditional expectation from $C^*(E)$ onto the fixed point algebra $C^*(E)^\gamma$. Let $a \neq 0$ be an element of $C^*(E)$ and let J' be the closed 2-sided ideal of $C^*(E)$ generated by $\Gamma(a^*a)$. Since J' is a nonzero gauge invariant ideal there exists a vertex $v \in E^0$ such that $P_v \in J'$ (cf. [**2**, Theorem 4.1]). If α is a path from v to a vertex in X then $S_\alpha \in J_X$

and $P_v S_\alpha \neq 0$. Consequently, $\{0\} \neq \Gamma(a^*a)J_X = \left(\int_{t\in\mathbb{T}} \gamma_t(a^*a)dt\right) J_X$. Thus, there exists a $t \in \mathbb{T}$ such that $\gamma_t(a^*a)J_X \neq \{0\}$. Since $\gamma_t(J_X) = J_X$ this implies $aJ_X \neq \{0\}$. Therefore, the ideal J_X is essential, as required. The converse implication is trivial. $\square$

1.4. Quantum odd dimensional spheres. For $n = 1, 2, \ldots$ and $q \in (0,1)$ the C^*-algebra $C(S_q^{2n-1})$ of continuous functions on the quantum sphere S^{2n-1} is given in [**26**] as the universal C^*-algebra generated by elements $z_1, z_2, \ldots, z_n$, subject to the following relations:

$$z_j z_i = q z_i z_j \quad \text{for } i < j, \tag{1}$$

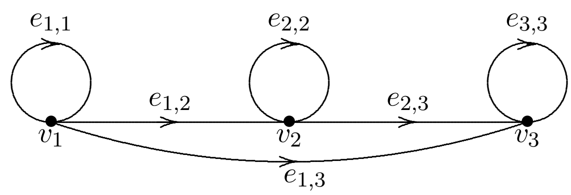
$$z_j^* z_i = q z_i z_j^* \quad \text{for } i \neq j, \tag{2}$$

$$z_i^* z_i = z_i z_i^* + (1-q^2)\sum_{j>i} z_j z_j^* \quad \text{for } i = 1, \ldots, n, \tag{3}$$

$$\sum_{i=1}^{n} z_i z_i^* = I. \tag{4}$$

It is shown in [**11**, Theorem 4.4] that the C^*-algebra $C(S_q^{2n-1})$ is isomorphic with a graph algebra $C^*(L_{2n-1})$. The graph L_{2n-1} has n vertices $\{v_1, \ldots, v_n\}$ and $n(n+1)/2$ edges $\bigcup_{i=1}^{n}\{e_{i,j} \mid j = i, \ldots, n\}$ with $s(e_{i,j}) = v_i$ and $r(e_{i,j}) = v_j$. It is a finite graph without sinks. For example, if $n = 3$ then the corresponding graph L_5 looks as follows:

L_5



The isomorphism $\phi : C(S_q^{2n-1}) \to C^*(L_{2n-1})$ is given explicitly on the generators as

$$\phi : z_n \mapsto \sum_{k_1,\ldots,k_{n-1}\in\mathbb{N}} q^{k_1+\cdots+k_{n-1}} T(k_1,\ldots,k_{n-1}) S_{e_{n,n}} T(k_1,\ldots,k_{n-1})^*, \tag{5}$$

$$\begin{aligned}\phi : z_i \mapsto \sum_{k_1,\ldots,k_i\in\mathbb{N}} & q^{k_1+\cdots+k_{i-1}} \left(\sqrt{1-q^{2(k_i+1)}} - \sqrt{1-q^{2k_i}}\right) \times \\ & \times T(k_1,\ldots,k_i)\left(\sum_{j=i}^{n} S_{e_{i,j}}\right) T(k_1,\ldots,k_i)^*,\end{aligned} \tag{6}$$

for $i = 1, \ldots, n-1$. Here for $k_1, \ldots, k_i \in \mathbb{N}$ we denoted

$$T(k_1, \ldots, k_i) = \left(\sum_{j=1}^{n} S_{e_{1,j}} \right)^{k_1} \left(\sum_{j=2}^{n} S_{e_{2,j}} \right)^{k_2} \cdots \left(\sum_{j=i}^{n} S_{e_{i,j}} \right)^{k_i}, \tag{7}$$

an element of $C^*(L_{2n-1})$.

2. Quantum lens spaces.

We begin by recalling the definition of the classical lens spaces [**13**]. Namely, for $n = 1, 2, \ldots$ let $S^{2n-1} = \{(y_1, \ldots, y_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} |y_i|^2 = 1\}$ be the sphere of dimension $2n-1$. We fix an integer $p \geq 2$ and n integers $m_1, \ldots, m_n$. If $\theta = e^{2\pi i/p}$ then

$$(y_1, \ldots, y_n) \mapsto (\theta^{m_1} y_1, \ldots, \theta^{m_n} y_n) \tag{8}$$

is a homeomorphism of S^{2n-1} which gives rise to an action of $\mathbb{Z}_p$, the cyclic group of order p, on S^{2n-1}. The (generalized) lens space $L(p; m_1, \ldots, m_n)$ of dimension $2n-1$ is defined as the orbit space of this action. It is normally assumed that each of $m_1, m_2, \ldots, m_n$ is relatively prime to p. This assumption is equivalent to freeness of the action (8).

We now turn to the quantum case. With the sole exception of Lemma 2.1, we always assume that each of $m_1, m_2, \ldots, m_n$ is relatively prime to p. The universal property of $C(S_q^{2n-1})$ implies that the assignment

$$\widetilde{\Lambda} : z_i \mapsto \theta^{m_i} z_i, \tag{9}$$

for $i = 1, \ldots, n$, gives rise to an automorphism $\widetilde{\Lambda}$ of $C(S_q^{2n-1})$ of order p. For $q \in (0,1)$ we define the C^*-algebra $C(L_q(p; m_1, \ldots, m_n))$ of continuous functions on the quantum lens space as the fixed point algebra corresponding to this automorphism, i.e.,

$$C(L_q(p; m_1, \ldots, m_n)) = C(S_q^{2n-1})^{\widetilde{\Lambda}}. \tag{10}$$

Let $\phi : C(S_q^{2n-1}) \to C^*(L_{2n-1})$ be the isomorphism given by (5)-(6). Setting $\Lambda = \phi \widetilde{\Lambda} \phi^{-1}$ we get

$$\Lambda : P_{v_i} \mapsto P_{v_i}, \tag{11}$$

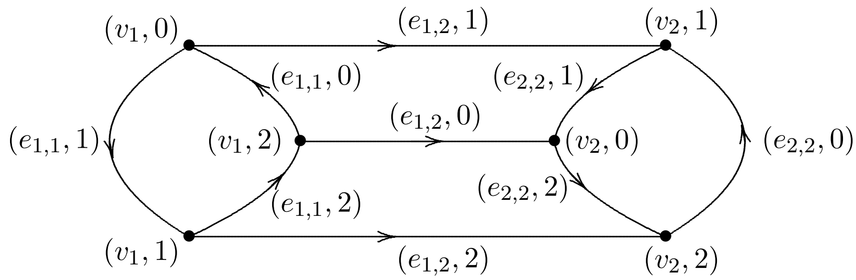
$$\Lambda : S_{e_{i,j}} \mapsto \theta^{m_i} S_{e_{i,j}}, \tag{12}$$

for $i = 1, \ldots, n$ and $j = i, \ldots, n$. This gives

$$C(L_q(p; m_1, \ldots, m_n)) = C(S_q^{2n-1})^{\widetilde{\Lambda}} \cong C^*(L_{2n-1})^{\Lambda}. \tag{13}$$

Actions of this type have been studied by Kumjian and Pask [**14**]. Let $c : L_{2n-1}^1 \to \mathbb{Z}_p$ be a labeling of the edges of L_{2n-1} such that $c(S_{e_{i,j}}) = m_i$. The skew product graph $L_{2n-1} \times_c \mathbb{Z}_p$ is defined so that its vertices are $L_{2n-1}^0 \times \mathbb{Z}_p$ and its edges are $L_{2n-1}^1 \times \mathbb{Z}_p$ with $s(e_{i,j}, m) = (v_i, m - m_i)$ and

$r(e_{i,j}, m) = (v_j, m)$, for $m \in \mathbb{Z}_p$, $i = 1, \ldots, n$ and $j = i, \ldots, n$. We note that through each vertex of this graph passes precisely one vertex simple loop (composed of p edges), and for any two vertices $(v_i, m), (v_j, k)$ there exists a path from (v_i, m) to (v_j, k) if and only if $i \leq j$. For example, if $n = 2$, $p = 3$, $m_1 = 1$ and $m_2 = 2$ then $L_3 \times_c \mathbb{Z}_3$ looks as follows:



By virtue of [**14**, Corollary 2.5] there exists a C^*-algebra isomorphism

$$C^*(L_{2n-1} \times_c \mathbb{Z}_p) \cong C^*(L_{2n-1}) \times_\Lambda \mathbb{Z}_p. \tag{14}$$

Let U be a unitary in $C^*(L_{2n-1}) \times_\Lambda \mathbb{Z}_p$ such that $U^p = I$ and $UxU^* = \Lambda(x)$ for all $x \in C^*(L_{2n-1})$. For $m = 0, 1, \ldots, p-1$ let $Q_m = \frac{1}{p}\sum_{i=0}^{p-1} \theta^{im} U^i$ be the spectral projection of U. The isomorphism (14) is given explicitly by

$$P_{(v_i,m)} \mapsto P_{v_i} Q_m, \tag{15}$$

$$S_{(e_{i,j},m)} \mapsto S_{e_{i,j}} Q_m, \tag{16}$$

for $i = 1, \ldots, n$, $j = i, \ldots, n$ and $m = 0, \ldots, p-1$.

We have

$$Q_0(C^*(L_{2n-1}) \times_\Lambda \mathbb{Z}_p)Q_0 = C^*(L_{2n-1})^\Lambda Q_0, \tag{17}$$

and the map $C^*(L_{2n-1})^\Lambda \to C^*(L_{2n-1})^\Lambda Q_0$, $x \mapsto xQ_0$, is a C^*-algebra isomorphism. On the other hand, the isomorphism (14) (cf. Formulae (15) and (16)) identifies $Q_0 = \sum_{i=1}^n P_{v_i} Q_0$ with the projection $\sum_{i=1}^n P_{(v_i,0)}$ in $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$. Consequently, there is a C^*-algebra isomorphism

$$C(L_q(p; m_1, \ldots, m_n)) \cong \left(\sum_{i=1}^n P_{(v_i,0)}\right) C^*(L_{2n-1} \times_c \mathbb{Z}_p) \left(\sum_{i=1}^n P_{(v_i,0)}\right). \tag{18}$$

In the following lemma we only require that m_1 be relatively prime to p and no assumptions on the remaining parameters $m_2, \ldots, m_n$ are made whatever. The lemma says that if m_1 is relatively prime to p then the action Λ is saturated, as expected.

Lemma 2.1. *If m_1 is relatively prime to p then for each vertex (v_k, m) there exists a path in $L_{2n-1} \times_c \mathbb{Z}_p$ from $(v_1, 0)$ to (v_k, m). Thus, Formula*

(18) *gives an isomorphism between the* C^**-algebra* $C(L_q(p; m_1, \dots, m_n))$ *and a full corner of* $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$.

Proof. Let $k \in \{1, \dots, n\}$, $m \in \mathbb{Z}_p$, and let r be a positive integer such that $rm_1 = m$ in $\mathbb{Z}_p$. Then

$$((e_{1,1}, m_1), (e_{1,1}, 2m_1), \dots, (e_{1,1}, (r-2)m_1), (e_{1,1}, (r-1)m_1), (e_{1,k}, rm_1))$$

is the desired path. Consequently,

$$S_{(e_{1,1},m_1)} S_{(e_{1,1},2m_1)} \cdots S_{(e_{1,1},(r-2)m_1)} S_{(e_{1,1},(r-1)m_1)} S_{(e_{1,k},rm_1)}$$

is a partial isometry in $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$ whose domain projection equals $P_{(v_k,m)}$ and whose range projection is majorized by $P_{(v_1,0)}$. Thus all projections $P_{(v_k,m)}$, $k = 1, \dots, n$, $m \in \mathbb{Z}_p$, belong to the ideal of $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$ generated by $\sum_{i=1}^n P_{(v_i,0)}$. Since $I = \sum_{k=1}^n \sum_{m \in \mathbb{Z}_p} P_{(v_k,m)}$ in $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$, Formula (18) implies that the C^*-algebra $C(L_q(p; m_1, \dots, m_n))$ is isomorphic to a full corner of $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$, as claimed. □

Lemma 2.1 implies that $C(L_q(p; m_1, \dots, m_n))$ and $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$ are strongly Morita equivalent [**23**, Chapter 3]. Consequently, the K-groups of these two C^*-algebras are isomorphic [**4**, **8**]. In order to calculate the K-groups of $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$ we assume that each of $m_1, \dots, m_n$ is relatively prime to p. For short, we write Φ for the map $K_{L_{2n-1} \times_c \mathbb{Z}_p}$ which determines these K-groups (cf. Section 1.2). Thus, the K_0 and K_1 groups of $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$ are isomorphic to the cokernel and kernel, respectively, of the endomorphism Φ of the free abelian group with a basis $(L_{2n-1} \times_c \mathbb{Z}_p)^0$, given by

$$\Phi : (v_i, m) \mapsto \left(\sum_{j=i}^{n} (v_j, m + m_i) \right) - (v_i, m). \tag{19}$$

Proposition 2.2. *If each of* $m_1, \dots, m_n$ *is relatively prime to* p *then*

$$K_1(C(L_q(p; m_1, \dots, m_n))) \cong \mathbb{Z}.$$

Proof. By Lemma 2.1 it sufficies to calculate the K_1-group of $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$, which is isomorphic to the kernel of the map Φ from (19). Let $\lambda_i^m \in \mathbb{Z}$, for $i = 1, \dots, n$ and $m \in \mathbb{Z}_p$, be such that $\Phi(\sum_{i=1}^n \sum_{m \in \mathbb{Z}_p} \lambda_i^m (v_i, m)) = 0$. This can only happen if $\sum_{j=1}^i \lambda_j^{m-m_j} = \lambda_i^m$ for each $i \in \{1, \dots, n\}$ and $m \in \mathbb{Z}_p$. Setting $i = 1$, we get $\lambda_1^m = \lambda_1^0$ for all $m \in \mathbb{Z}_p$, because m_1 is relatively prime to p. Then, considering $i = 2$, we get $\lambda_1^0 + \lambda_2^{m-m_2} = \lambda_2^m$ for all $m \in \mathbb{Z}_p$. Summing this identity over m we see that $\lambda_1^0 = 0$. Consequently, $\lambda_2^m = \lambda_2^0$ for all $m \in \mathbb{Z}_p$. Again, we use here the fact that m_2 is relatively prime to p. Continuing inductively in this manner we get $\lambda_i^m = 0$ for $i = 1, \dots, n-1$ and $\lambda_n^m = \lambda_n^0$ for $m \in \mathbb{Z}_p$. Thus, the kernel of Φ is isomorphic to $\mathbb{Z}$, as claimed. □

It is also possible to calculate the cokernel of the map Φ and, therefore, the K_0 group of $C(L_q(p; m_1, \ldots, m_n))$. This is a simple matter if $n = 2$, and we get

$$K_0(C(L_q(p; m_1, m_2))) \cong \mathbb{Z} \oplus \mathbb{Z}_p,$$

similarly to the result of Matsumoto and Tomiyama [**18**]. However, if $n \geq 3$ then the calculation becomes a bit more elaborate. We illustrate with a particular case.

Proposition 2.3. *If $n = 3$, $m_2 = m_3$ and both m_1 and m_2 are relatively prime to p then*

$$K_0(C(L_q(p; m_1, m_2, m_3))) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_{2p} \oplus \mathbb{Z}_{\frac{p}{2}} & \text{if } p \text{ is even,} \\ \mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p & \text{if } p \text{ is odd.} \end{cases}$$

Proof. We must determine the cokernel of Φ. It is easy to see that $\{(v_i, 0) \mid i = 1, 2, 3\}$ together with the range of Φ generate the entire group $\mathbb{Z}(L_5 \times_c \mathbb{Z}_p)^0$. Now let $d_i, \lambda_i^m \in \mathbb{Z}$, for $i = 1, 2, 3$ and $m \in \mathbb{Z}_p$, be such that $\Phi(\sum_{i=1}^3 \sum_{m \in \mathbb{Z}_p} \lambda_i^m (v_i, m)) = \sum_{i=1}^3 d_i (v_i, 0)$. This is equivalent to

$$d_i = \left(\sum_{j=1}^{i} \lambda_j^{-m_j} \right) - \lambda_i^0, \tag{20}$$

$$0 = \left(\sum_{j=1}^{i} \lambda_j^{m-m_j} \right) - \lambda_i^m, \quad \text{for} \quad m \neq 0. \tag{21}$$

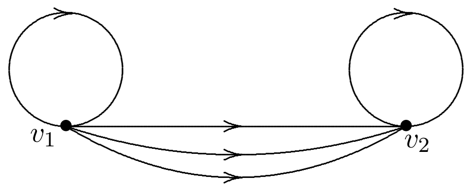
If $i = 1$ then (21) gives $\lambda_1^m = \lambda_1^0$ for all $m \in \mathbb{Z}_p$ and then $d_1 = 0$ by (20). If $i = 2$ then substituting $m = km_2$ in (21), with $k = 1, \ldots, p-1$, we get $\lambda_2^{km_2} = k\lambda_1^0 + \lambda_2^0$ for all $k = 0, \ldots, p-1$. Then (20) yields $d_2 = p\lambda_1^0$. If $i = 3$ then substituting $m = km_2$ in (21), with $k = 1, \ldots, p-1$, we get $\lambda_3^{km_2} = \frac{k(k+1)}{2}\lambda_1^0 + k\lambda_2^0 + \lambda_3^0$ for all $k = 0, \ldots, p-1$. Then (20) yields $d_3 = \frac{p(p+1)}{2}\lambda_1^0 + p\lambda_2^0$. Thus, $(v_1, 0)$ has infinite order in the cokernel. If p is even then $(v_2, 0)$ and $2(v_2, 0) + (v_3, 0)$ generate a subgroup of the cokernel isomorphic to $\mathbb{Z}_{2p} \oplus \mathbb{Z}_{\frac{p}{2}}$. If p is odd then $(v_2, 0)$ and $(v_3, 0)$ generate a subgroup of the cokernel isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$. $\square$

We now show that $C(L_q(p; m_1, \ldots, m_n))$ itself is isomorphic to a graph algebra. The following construction of the graph $L_{2n-1}^{(p;m_1,\ldots,m_n)}$ and the argument of Theorem 2.5, below, are similar to [**25**, Section 4 and Lemma 6]. Again, we assume that each of $m_1, \ldots, m_n$ is relatively prime to p.

At first we define the graph $L_{2n-1}^{(p;m_1,\ldots,m_n)}$, as follows: The vertices of the graph $L_{2n-1}^{(p;m_1,\ldots,m_n)}$ are $\{v_1, v_2, \ldots, v_n\}$. The edges of $L_{2n-1}^{(p;m_1,\ldots,m_n)}$ consist of all finite (vertex simple) paths $\alpha = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r))$ in $L_{2n-1} \times_c \mathbb{Z}_p$ such that $k_1 = m_{i_1}$, $k_a \neq 0$ for $a \neq r$, $k_r = 0$ and $(v_{j_a}, k_a) \neq (v_{j_b}, k_b)$ for

$a \neq b$. The source and range functions are defined as $s(\alpha) = v_{i_1}$ and $r(\alpha) = v_{j_r}$. We note that this is a finite graph without sinks, through each vertex there passes precisely one vertex simple loop (composed of a single edge), and for each pair of vertices v_i, v_j there exists a path from v_i to v_j if and only if $i \leq j$. For example, if $n = 2$, $p = 3$, $m_1 = 1$ and $m_2 = 2$ then $L_3^{(3;1,2)}$ looks as follows:

$L_3^{(3\,;\,1,2)}$



The following Lemma 2.4 essentially follows from [**25**, Lemma 5]. However, for the sake of completeness and reader's convenience, we give a self-contained proof.

Lemma 2.4. *If each of $m_1, \ldots, m_n$ is relatively prime to p then for any $l \in \{1, \ldots, n\}$ and any $m \in \mathbb{Z}_p$ we have*

$$P_{(v_l,m)} = \sum_{\alpha} S_{(e_{i_1,j_1},k_1)} \cdots S_{(e_{i_r,j_r},k_r)} S^*_{(e_{i_r,j_r},k_r)} \cdots S^*_{(e_{i_1,j_1},k_1)}$$

(in $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$), where the summation extends over all $\alpha = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r))$, vertex simple paths in $L_{2n-1} \times_c \mathbb{Z}_p$ such that $i_1 = l$, $k_1 - m_{i_1} = m$, $k_a \neq 0$ for $a \neq r$, $k_r = 0$ and $(v_{j_a}, k_a) \neq (v_{j_b}, k_b)$ for $a \neq b$.

Proof. For $\nu = 1, 2, \ldots$ we define A_ν to be the collection of all vertex simple paths $\alpha = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r))$ in $L_{2n-1} \times_c \mathbb{Z}_p$ such that the length of α is not greater than ν (and nonzero), $i_1 = l$, $k_1 - m_{i_1} = m$, $k_a \neq 0$ for $a \neq r$, $k_r = 0$ and $(v_{j_a}, k_a) \neq (v_{j_b}, k_b)$ for $a \neq b$, and let B_ν be the collection of all paths $\beta = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r))$ such that the length of β equals ν, $i_1 = l$, $k_1 - m_{i_1} = m$, $k_a \neq 0$ for $a = 1, \ldots, r$ and $(v_{j_a}, k_a) \neq (v_{j_b}, k_b)$ for $a \neq b$. We show, by induction on ν, that

$$P_{(v_l,m)} = \sum_{\alpha \in A_\nu} S_\alpha S_\alpha^* + \sum_{\beta \in B_\nu} S_\beta S_\beta^*. \tag{22}$$

Indeed, the collection of all edges in $L_{2n-1} \times_c \mathbb{Z}_p$ with source equal to (v_l, m) is the union of A_1 and B_1. Thus, (22) holds with $\nu = 1$ by virtue of (G5). Now suppose (22) holds for some ν. If $\beta = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r))$ in B_ν, then applying Condition (G5) at the range vertex of β, equal to (v_{j_r}, k_r), we get

$$S_\beta S_\beta^* = S_\beta P_{(v_{j_r},k_r)} S_\beta^* = \sum_{d=j_r}^{n} S_\beta S_{(e_{j_r,d},k_r+m_{j_r})} S^*_{(e_{j_r,d},k_r+m_{j_r})} S_\beta^*. \tag{23}$$

Let $\beta' = ((e_{i_1,j_1}, k_1), \dots, (e_{i_r,j_r}, k_r), (e_{j_r,d}, k_r+m_{j_r}))$. We claim that $(v_d, k_r+m_{j_r}) \neq (v_{j_a}, k_a)$ for $a = 1, \dots, r$. This is obvious if $d \neq j_r$. For $d = j_r$ let b be the smallest index such that $j_b = j_r$. Since β is a path we have $j_b = j_{b+1} = \dots = j_r$ and $k_{b+h} = k_b + hm_{j_b}$ for $h = 1, \dots, r-b$. Since m_{j_b} is relatively prime to p it follows that $k_r + m_{j_r} \notin \{k_b, \dots, k_r\}$, as claimed. Thus $\beta' \in (A_{\nu+1} \setminus A_\nu) \cup B_{\nu+1}$. Consequently, from the inductive hypothesis, Formula (23) and the above discussion we get

$$\begin{aligned} P_{(v_l,m)} &= \sum_{\alpha \in A_\nu} S_\alpha S_\alpha^* + \sum_{\beta \in B_\nu} S_\beta S_\beta^* \\ &= \sum_{\alpha \in A_\nu} S_\alpha S_\alpha^* + \sum_{\beta' \in (A_{\nu+1} \setminus A_\nu)} S_{\beta'} S_{\beta'}^* + \sum_{\beta' \in B_{\nu+1}} S_{\beta'} S_{\beta'}^* \\ &= \sum_{\alpha \in A_{\nu+1}} S_\alpha S_\alpha^* + \sum_{\beta \in B_{\nu+1}} S_\beta S_\beta^*, \end{aligned}$$

and the inductive step follows.

Since $L_{2n-1} \times_c \mathbb{Z}_p$ is a finite graph there exists a ν large enough so that $B_\nu = \emptyset$. With this ν Formula (22) gives the lemma. □

Theorem 2.5. *If each of the numbers $m_1, \dots, m_n$ is relatively prime to p then the C^*-algebra $C(L_q(p; m_1, ..., m_n))$ is isomorphic to $C^*\left(L_{2n-1}^{(p;m_1,\dots,m_n)}\right)$.*

Proof. At first we observe that there exists a C^*-algebra homomorphism

$$\psi : C^*\left(L_{2n-1}^{(p;m_1,\dots,m_n)}\right) \to \left(\sum_{i=1}^n P_{(v_i,0)}\right) C^*(L_{2n-1} \times_c \mathbb{Z}_p) \left(\sum_{i=1}^n P_{(v_i,0)}\right)$$

such that

$$\begin{aligned} \psi : P_{v_l} &\mapsto P_{(v_l,0)}, \\ \psi : S_\alpha &\mapsto S_{(e_{i_1,j_1},k_1)} S_{(e_{i_2,j_2},k_2)} \dots S_{(e_{i_r,j_r},k_r)}, \end{aligned}$$

for all $l = 1, \dots, n$ and for all $\alpha = ((e_{i_1,j_1}, k_1), \dots, (e_{i_r,j_r}, k_r))$, vertex simple paths in $L_{2n-1} \times_c \mathbb{Z}_p$ such that $k_1 = m_{i_1}$, $k_a \neq 0$ for $a \neq r$, $k_r = 0$ and $(v_{j_a}, k_a) \neq (v_{j_b}, k_b)$ for $a \neq b$. Due to the universal property of $C^*\left(L_{2n-1}^{(p;m_1,\dots,m_n)}\right)$, to this end it sufficies to verify that the elements $\{\psi(P_{v_l}), \psi(S_\alpha)\}$ of $C^*(L_{2n-1} \times_c \mathbb{Z}_p)$ satisfy Conditions (G1)–(G5) for the graph $L_{2n-1}^{(p;m_1,\dots,m_n)}$. But it is obvious that Conditions (G1)–(G4) are satisfied, and Condition (G5) is equivalent to Lemma 2.4 with $m = 0$.

For surjectivity of ψ it sufficies to show that:

(i) If α is a path in $L_{2n-1} \times_c \mathbb{Z}_p$ such that both $s(\alpha)$ and $r(\alpha)$ are in $\{(v_i, 0) \mid i = 1, \dots, n\}$ then S_α belongs to the range of ψ.

(ii) If α, β are two paths such that $r(\alpha) = r(\beta)$ and both $s(\alpha)$ and $s(\beta)$ are in $\{(v_i, 0) \mid i = 1, \dots, n\}$ then $S_\alpha S_\beta^*$ belongs to the range of ψ.

To this end we first note that any loop in $L_{2n-1} \times_c \mathbb{Z}_p$ must pass through a vertex of the form $(v_i, 0)$. Now let $\alpha = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r))$ be a path as in (i). Let $a_1 < a_2 < \cdots < a_\nu = r$ be all the indices for which $k_{a_t} = 0$. We also set $a_0 = 0$. For each $t = 1, \ldots, \nu$ the path $\alpha_t = ((e_{i_{1+a_{t-1}},j_{1+a_{t-1}}}, k_{1+a_{t-1}}), \ldots, (e_{i_{a_t},j_{a_t}}, k_{a_t}))$ is an edge of the graph $L_{2n-1}^{(p;m_1,\ldots,m_n)}$. Hence, $S_\alpha = S_{\alpha_1} \ldots S_{\alpha_\nu}$ belongs to the range of ψ, since each S_{α_t} does. Now let α and β be two paths as in (ii). Let $\alpha = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_r,j_r}, k_r))$. By virtue of Part (i) it sufficies to consider the case $k_r \neq 0$. Let μ be the greatest index such that $k_\mu = 0$, or 0 if such an index does not exist. We set $\alpha_1 = ((e_{i_1,j_1}, k_1), \ldots, (e_{i_\mu,j_\mu}, k_\mu))$ and $\alpha_2 = ((e_{i_{\mu+1},j_{\mu+1}}, k_{\mu+1}), \ldots, (e_{i_r,j_r}, k_r))$. We have $S_\alpha = S_{\alpha_1} S_{\alpha_2}$ and S_{α_1} is in the range of ψ by Part (i) (if $\mu = 0$ then $\alpha_1 = \emptyset$ and $S_{\alpha_1} = I$). Furthermore, for $\mu + 1 \leq a, b \leq r$ we have $(v_{j_a}, k_a) \neq (v_{j_b}, k_b)$ if $a \neq b$. We have an analogous factorization $S_\beta = S_{\beta_1} S_{\beta_2}$, with S_{β_1} in the range of ψ. Let $P_{(v_{j_r},k_r)} = \sum_x S_x S_x^*$ be the decomposition as in Lemma 2.4. Then we have

$$S_\alpha S_\beta = S_{\alpha_1} S_{\alpha_2} P_{(v_{j_r},k_r)} S_{\beta_2}^* S_{\beta_1}^* = \sum_x S_{\alpha_1} S_{\alpha_2} S_x S_x^* S_{\beta_2}^* S_{\beta_1}^*.$$

Consequently, $S_\alpha S_\beta$ belongs to the range of ψ, since S_{α_1}, S_{β_1} and all $S_{\alpha_2} S_x$ and $S_{\beta_2} S_x$ do. This completes the proof of surjectivity of ψ.

Now we show that the homomorphism ψ is injective. Our argument is essentially the same as in [**5**, Remark 3]. Since for each $i \in \{1, \ldots, n-1\}$ there exists a path from v_i to v_n, the ideal J of $C^*\left(L_{2n-1}^{(p;m_1,\ldots,m_n)}\right)$ generated by P_{v_n} is essential by Lemma 1.1. Thus, it sufficies to show that $J \cap \ker(\psi) = \{0\}$. To this end, we notice that in the graph $L_{2n-1}^{(p;m_1,\ldots,m_n)}$ the vertex v_n emits a unique edge, which we call e, and the range of this edge is v_n. Since there are infinitely many paths from other vertices to v_n it follows (cf. [**15**] and [**5**, Remark 3]) that

$$J \cong P_{v_n} J P_{v_n} \otimes \mathcal{K} = C^*(S_e) \otimes \mathcal{K} \cong C(\mathbb{T}) \otimes \mathcal{K}.$$

Hence, in order to prove injectivity of ψ it sufficies to show that $C^*(S_e) \cap \ker(\psi) = \{0\}$. This follows from the fact that

$$\psi(S_e) = S_{(e_{n,n},m_n)} S_{(e_{n,n},2m_n)} \cdots S_{(e_{n,n},pm_n)}$$

is a partial unitary with full spectrum (cf. [**15**]). $\square$

With help of Theorem 2.5 it is easy to determine the ideal structure of $C(L_q(p; m_1, \ldots, m_n))$. For example, we have seen in the proof of Theorem 2.5 that the ideal of $C^*\left(L_{2n-1}^{(p;m_1,\ldots,m_n)}\right)$ generated by P_{v_n} is isomorphic to $C(\mathbb{T}) \otimes \mathcal{K}$. The corresponding quotient is $C^*\left(L_{2n-3}^{(p;m_1,\ldots,m_{n-1})}\right)$, and this C^*-algebra is in turn isomorphic to $C(L_q(p; m_1, \ldots, m_{n-1}))$. Thus, there exists an exact sequence

(24)
$$0 \to C(\mathbb{T}) \otimes \mathcal{K} \to C(L_q(p; m_1, \dots, m_n)) \to C(L_q(p; m_1, \dots, m_{n-1})) \to 0.$$

Using the exact sequence (24) or the general results about graph algebras, outlined in Section 1.3, it is easy to understand the primitive spectrum of $C(L_q(p; m_1, \dots, m_n))$. Therefore, we omit the proof of the following proposition:

Proposition 2.6. *If each of $m_1, \dots, m_n$ is relatively prime to p then the primitive ideal space of $C(L_q(p; m_1, \dots, m_n))$ consists of n disjoint copies $C_1, \dots, C_n$ of the circle. The hull-kernel topology restricted to each of the circles coincides with the natural one. The closure of a point in C_k contains $C_1 \cup \dots \cup C_{k-1}$. Thus,* $\mathrm{Prim}(C(L_q(p; m_1, \dots, m_n)))$ *and* $\mathrm{Prim}(C(S_q^{2n-1}))$ *are homeomorphic* (*cf.* [**11**, Section 4.1]).

Concluding remarks. For a fixed integer $p \geq 2$ the infinite lens space $L(p; \infty)$ is defined as the inductive limit of the lens spaces $L(p; 1_n)$, corresponding to the natural imbeddings $L(p; 1_n) \hookrightarrow L(p; 1_{n+1})$. (If $m_1 = \dots = m_n = 1$ then we simply write $L(p; 1_n)$ instead of $L(p; 1, \dots, 1)$.) It turns out that $L(p; \infty)$ is identical with the Eilenberg-MacLane space of type $(\mathbb{Z}_p, 1)$ [**3**].

The results of the previous section lead to quantum versions of this classical topological setting. Namely, the inclusion $L(p; 1_n) \hookrightarrow L(p; 1_{n+1})$ corresponds to the surjective homomorphism $\widetilde{\theta}_{n+1} : C(L_q(p; 1_{n+1})) \to C(L_q(p; 1_n))$ such that the kernel of $\widetilde{\theta}_{n+1}$ is generated by $z_{n+1} z_{n+1}^*$. Consequently, the quantum infinite lens space, or the quantum Eilenberg-MacLane space of type $(\mathbb{Z}_p, 1)$, may be defined as the inverse limit

$$C(L_q(p; \infty)) = \varprojlim (C(L_q(p; 1_n)),\ \widetilde{\theta}_n). \tag{25}$$

Under the isomorphisms $C(L_q(p; 1_k)) \cong C^*\left(L_{2k-1}^{(p)}\right)$ (if $m_1 = \dots = m_n = 1$ then we simply write $L_{2n-1}^{(p)}$ instead of $L_{2n-1}^{(p;1,\dots,1)}$), described in Theorem 2.5, the homomorphism $\widetilde{\theta}_{n+1}$ is carried onto a surjective C^*-algebra homomorphism $\theta_{n+1} : C^*\left(L_{2n+1}^{(p)}\right) \to C^*\left(L_{2n-1}^{(p)}\right)$, whose kernel is generated by the projection $P_{v_{n+1}}$. Thus, we have the C^*-algebra isomorphism

$$C(L_q(p; \infty)) \cong \varprojlim \left(C^*\left(L_{2n-1}^{(p)}\right),\ \theta_n\right). \tag{26}$$

It is not difficult to see, and we omit the details, that this inverse limit itself may be realized as the graph algebra $C^*\left(L_\infty^{(p)}\right)$. The graph $L_\infty^{(p)}$ is the increasing limit of the graphs $L_{2n-1}^{(p)}$, corresponding to the natural imbeddings $L_{2n-1}^{(p)} \hookrightarrow L_{2n+1}^{(p)}$ such that the v_i vertex in $L_{2n-1}^{(p)}$ is identified with the v_i vertex in $L^{(p)}_{2n+1}$, and the edges from v_i to v_j in $L^{(p)}_{2n-1}$ are bijectively identified with the edges from v_i to v_j in $L^{(p)}_{2n+1}$. The graph $L^{(p)}_{\infty}$ has infinitely many vertices $\{v_1, v_2 \dots\}$, and for each pair $i \le j$ there exists at least one edge from v_i to v_j. These two properties imply that $C^*\left(L^{(p)}_{\infty}\right)$ is a primitive, purely infinite C^*-algebra (but not simple) [**1**]. Furthermore, $K_0\left(C^*\left(L^{(p)}_{\infty}\right)\right) \cong \bigoplus^{\infty} \mathbb{Z}$ and $K_1\left(C^*\left(L^{(p)}_{\infty}\right)\right) = 0$ [**22**, **7**].

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Applied Mathematics
Korea Maritime University
Busan 606–791
South Korea
E-mail address: hongjh@hanara.kmaritime.ac.kr

Mathematics
The University of Newcastle
Callaghan, NSW 2308
Australia
E-mail address: wojciech@frey.newcastle.edu.au