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In this paper, we studied the real vector bundle decomposition problem. We first give a general decomposition result which relates a given vector bundle to some cohomology classes with local coefficients in the homotopy group of a Grassmann manifold; it is those classes that obstruct the decomposition. Those classes are natural with respect to the induced vector bundle by a map. For some special decompositions, we gave a relationship between those classes and the well-known characteristic classes such as Stiefel-Whitney classes and Chern classes. We determined the local coefficients in the cohomology group which contain the decomposition obstruction classes. We find applications in the study of subbundles of low codimension. In particular, codimension 1 decomposition classes are investigated in which we find that one of the two decomposition classes for the universal bundle over $BO(2n+1)$ is in $H^{2n+1}(BO(2n+1), Z)$. This result gives rise to a new geometric interpretation for the order two elements in the integral cohomology group in odd dimension. We further make use of the cellular structure of the classifying space $BO(n)$ to see the 'local' structure for the restriction of the universal bundle to each cell. In this way, we can construct the obstruction classes for the codimension 1 vector bundle decomposition. We gave an example to calculate the decomposition obstruction for the tangent bundle of RP^{2n}, which turns out to be the generator in the cohomology of RP^{2n} with twisted integer coefficients. On the other hand, we exhibit a trivial summand in the tangent bundle for any odd dimensional cobordism classes.

1. Introduction.

The classification for vector bundles is a classical problem which has been studied by many mathematicians. Grothendieck proved that every algebraic vector bundle over CP^1 can be decomposed as a direct sum of complex line bundles which gives rise to a complete classification for algebraic vector bundle over CP^1. Hirzebruch [6] applied the Riemann-Rock theorem to the vector bundles over CP^n and obtained an integral condition on the Chern classes of the vector bundle. Using the Hirzebruch's results, Schwarzenberger

[12] gave a partial classification for complex algebraic 2-vector bundles over CP^2 and formulated the following conditions for the Chern classes c_1 and c_2 of a 2-vector bundle over CP^n:

$$S : \binom{\delta_1}{k} + \binom{\delta_2}{k} \in Z \qquad 2 \leq k \leq n,$$

where $\delta_1 + \delta_2 = c_1$, $\delta_1\delta_2 = c_2$.

Atiyah and Rees [1] classified the complex topological 2-vector bundles over CP^3. In [2], Barth and Van de Ven gave a decomposability criterion for complex algebraic 2-bundles over CP^n. Switzer [14] contributed in the classification of complex topological 2-vector bundle over CP^n for $n = 4, 5$ and 6.

In the real case the literature is sketchy at best. In this paper we shall begin the study of the decomposition of real vector bundles. Unlike complex vector bundles, the decomposition problem for real vector bundles will involve cohomology groups with local coefficients. We give a general decomposition result (Theorem 2.1.5) which relates a given vector bundle to some cohomology classes with local coefficients in the homotopy group of a Grassmann manifold; it is those classes that obstruct the decomposition. Those classes are natural with respect to the induced vector bundle by a map (see 2.1.7). For some special decompositions, we gave a relationship between those classes and the well-known characteristic classes such as Stiefel-Whitney classes and Chern classes (see 2.2.8, 2.2.9 and 2.2.10). We find applications in the study of subbundles of low codimension. In particular, codimension 1 decomposition classes are investigated in 2.2.6 in which we find that one of the two decomposition classes for the universal bundle over $BO(2n+1)$ is in $H^{2n+1}(BO(2n+1), Z)$. This result gives rise to a new geometric interpretation for the order two elements in the integral cohomology group in odd dimension. We further make use of the cellular structure of the classifying space $BO(n)$ to see the 'local' structure (see 2.2.11) for the restriction of the universal bundle to each cell. In this way, we can construct the obstruction classes for the codimension 1 vector bundle decomposition. In example 3.1, we calculated the decomposition obstruction for the tangent bundle of RP^{2n}, which turns out to be the generator in the cohomology of RP^{2n} with twisted integer coefficients. On the other hand, in Example 3.4, we exhibit a trivial summand in the tangent bundle for any odd dimensional cobordism classes.

Our approach is based on the following considerations. Let G be a compact Lie group and H be a closed subgroup of G. Designated by BG and BH the classifying spaces of G and H respectively, classical results on compact Lie groups and their classifying spaces give us an important fibration $BH \longrightarrow BG$ with fiber G/H [15]. The lifting problem for certain fibrations between the classifying spaces of classical Lie groups and their closed subgroups is an extensively studied problem during the past twenty years [7], [8], [11] and [9]. We will discuss the following fibration in some detail:

$$B(O(n) \times O(m)) \longrightarrow BO(n+m).$$

The fibre of the fibration is $O(n+m)/O(n) \times O(m) = G_{n,m}$, the Grassmann manifold. Let X be a CW-complex and ξ^n be an n-dimensional vector bundle over X, then by classification theorem there is a continuous map, the classifying map of ξ^n, $f : X \longrightarrow BO(n)$ such that $\xi^n \approx f^*(\eta_n)$ where η_n is the canonical n-dimensional vector bundle over $BO(n)$. A vector bundle ξ^n can be decomposed into a Whitney sum $\xi^n = \xi^k \oplus \xi^{n-k}$ of two bundles if and only if the structure group of ξ^n can be reduced into the subgroup $O(k) \times O(n-k)$ which means that the classifying map f of ξ^n can be lifted to the classifying space $B(O(k) \times O(n-k))$ up to a homotopy. So the problem of decomposing a vector bundle is equivalent to the lifting problem of its classifying map for the fibration $B(O(k) \times O(n-k)) \longrightarrow BO(n)$. We apply the obstruction theory to study the corresponding lifting problem for this fibration.

2. Main results.

2.1. The general decomposition results. In this section we will use the obstruction theory to consider some problems about vector bundles, in particular the decomposition problem of vector bundles over a CW-complex.

Lemma 2.1.1. *Let ξ^m be an m-dimensional vector bundle over a paracompact space X, then ξ^m has a Whitney sum decomposition $\xi^k \oplus \xi^{m-k}$ if and only if there exists a commutative diagram up to homotopy:*

$$\begin{array}{ccc} & & B(O(k) \times O(m-k)) \\ & \nearrow & \big\downarrow p \\ X & \xrightarrow[f]{} & BO(m) \end{array}$$

where f is the classifying map of ξ^m and p is the map between the classifying spaces induced by the inclusion $O(k) \times O(m-k) \subset O(m)$.

Proof. This result can be proved by considering the structure group of a vector bundle. □

There are different ways to construct the classifying space BG for a compact Lie-group G. Here I give a geometric way to construct $B(O(k) \times O(m-k))$ so that there is a natural fibration $p : B(O(k) \times O(m-k)) \longrightarrow BO(m)$ with fiber $G_k(R^m)$, the Grassmann. Let η_m be the universal m-dimensional vector bundle over $BO(m) = G_m(R^\infty)$. Define

$$G_k(\eta_m) = \{(X,Y)|Y \text{ is any } k\text{-dimensional subspace of } X\} \subset G_m(R^\infty) \times G_k(R^\infty)$$

with the subspace topology.

Proposition 2.1.2. *$G_k(\eta_m)$ is a classifying space for $O(k) \times O(m-k)$. The natural projection $p : G_k(\eta_m) \longrightarrow G_m(R^\infty)$ is a fibration with fiber $G_k(R^m)$. The universal bundle over $G_k(\eta_m)$ is $p^*(\eta_m)$ which is isomorphic to $\omega_k \oplus \overline{\omega}_{m-k}$, whose total spaces are as follows:*

$$E(\omega_k) = \{((X,Y),\nu)|\nu \in Y\} \subset G_k(\eta_m) \times R^\infty;$$
$$E(\overline{\omega}_{m-k}) = \{((X,Y),\nu)|\nu \in X, \nu \perp Y\} \subset G_k(\eta_m) \times R^\infty.$$

Proof. The proof is based on the uniqueness of the classifying space for compact Lie-group and the fact that the canonical inclusion $i : G_k(R^\infty) \times G_{m-k}(R^\infty) \longrightarrow G_m(R^\infty)$ factors through $G_k(\eta_m)$. □

By the local triviality of the universal bundle η_m, $G_k(\eta_m)$ is also a local trivial fiber space with fiber $G_k(R^\infty)$, therefore, the projection p is a fibration with fiber $G_k(R^m)$.

Let ξ^m be any m-dimensional vector bundle over a CW-complex X with classifying map $f : X \longrightarrow G_m(R^\infty)$. We define $G_k(\xi^m) \longrightarrow X$, called the Grassmann bundle associated with ξ^m, to be the pull-back of the fibration $p : G_k(\eta_m) \longrightarrow G_m(R^\infty)$ by f. To justify the definition, we need to prove that if f is homotopic to g, then their pull-backs must be homeomorphism. By the classification theorem, $f^*(\eta_m) \underset{h}{\approx} g^*(\eta_m) \approx \xi^m$. Now we can define a homeomorphism $h^* : f^*(G_k(\eta_m)) \approx g^*(G_k(\eta_m))$ by

$$h^*(x,(f(x),Y)) = (x,(h(f(x)),h(Y))) = (x,(g(x),h(Y)))$$

where Y is any k-dimensional subspace of $f(x)$. Geometrically, $G_k(\xi^m)$ is the space of all k-dimensional subspaces in the fibers of η_m. As a special case, $G_1(\xi^m)$ is the well-known projective bundle space $RP(\xi^m)$ associated with ξ^m.

Lemma 2.1.3. *The projection $p_\xi : G_k(\xi^m) \longrightarrow X$ is a fibration with fiber $G_k(R^m)$. ξ^m has a decomposition $\xi^m \approx \xi_1^k \oplus \xi_2^{m-k}$ if and only if there is a section for the fibration p_ξ.*

Proof. Let $f : X \longrightarrow G_m(R^\infty)$ be the classifying map for ξ^m, then we have a pull-back diagram:

$$\begin{array}{ccc} G_k(\xi^m) & \xrightarrow{f^*} & G_k(\eta_m) \\ {\scriptstyle p_\xi}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & G_m(R^\infty). \end{array}$$

By 2.1.1 and 2.1.2 $\xi^m \approx \xi_1^k \oplus \xi_2^{m-k}$ if and only if there is a lifting for the fibration $p : G_k(\eta_m) \longrightarrow G_m(R^\infty)$ which is equivalent to the existence of a section of the fibration $p_\xi : G_k(\xi^m) \longrightarrow X$. □

Let $p : E \longrightarrow B$ be a fibration with fiber F, and let E, B and F be path-connected CW-spaces. Then there exist fibrations q_n and maps h_n making the digram

$$\begin{array}{ccccccccc} E & & & & & & & & \\ {\scriptstyle p}\downarrow & \overset{h_1}{\searrow} & & \overset{h_{n-1}}{\searrow} & & \overset{h_n}{\searrow} & & & \\ B & \xleftarrow[q_1]{} & E_1 & \xleftarrow[q_2]{} & \cdots E_{n-1} & \xleftarrow[q_n]{} & E_n & \longleftarrow & \cdots \end{array}$$

commute, and such that for $n > 1$ (if $n = 1$, $\pi_1(F)$ needs to be abelian):

(1) q_n is a fibration with fiber $K(\pi_n(F), n)$, the Eilenberg-Maclane space.
(2) h_n is $(n + 1)$-connected.

In the following, we will use the above so-called Postnikov decomposition for a fibration to study $p : G_k(\eta_m) \longrightarrow G_m(R^\infty)$ or $p_\xi : G_k(\xi^m) \longrightarrow X$. We start with a typical fibration in the Postnikov decomposition.

Lemma 2.1.4. *Let $p : E \longrightarrow B$ be a fibration with fiber $K(\Pi, n)$, where Π is an abelian group, and $f : X \longrightarrow B$ be a map between connected* CW-*spaces. Let $\widehat{\Pi}_p$ be the local coefficients and $\mathrm{ob}(p) \in H^{n+1}(B; \widehat{\Pi}_p)$ be the primary cohomology obstruction with respect to the trivial section on the base point. Then $f^*(\mathrm{ob}(p)) = 0 \in H^{n+1}(X; f^*\widehat{\Pi}_p)$ if and only if f can be lifted to $\widetilde{f} : X \longrightarrow E$.*

Proof. See [**3**].

Now we can apply the above results to prove our general decomposition theorem.

Theorem 2.1.5. *Let ξ^m be an m-dimensional vector bundle over a connected* CW-*complex X, $p_\xi : G_k(\xi^m) \longrightarrow X$ be the Grassmann bundle of ξ^m with Postnikov decomposition $\{\widetilde{X}_n, \widetilde{h}_n, \widetilde{q}_n\}$. Let*

$$\mathrm{ob}_n^k(\xi^m) \in H^{n+1}(\widetilde{X}_{n-1}, \widetilde{\pi}_n(G_{m-k,k})), \qquad n = 1, 2, \ldots$$

be the n-th Postnikov invariance for p_ξ. For $n \geq 1$ and a given homomorphism $\theta : \pi_1(X) \longrightarrow \pi_1(\widetilde{X}_1)$, set

$$\mathrm{OB}_{n,\theta}^k(\xi^m) = \Big\{ s_{n-1}^*(\mathrm{ob}_n^k(\xi^m)) \,|\, \textit{for all sections}$$
$$s_{n-1} : X \longrightarrow \widetilde{X}_{n-1} \text{ s.t. } (\widetilde{q}_2 \ldots \widetilde{q}_{n-1} s_{n-1})_* = \theta \Big\}$$
$$\subset H^{n+1}(X, \theta_* \widehat{\pi}_n(G_{m-k,k}))$$

where $s_{n-1}^* : H^{n+1}(\widetilde{X}_{n-1}; \widehat{\pi}_n(G_{m-k,k})) \longrightarrow H^{n+1}(X; \theta_* \widehat{\pi}_n(G_{m-k,k}))$ *is the induced homomorphism in the cohomology groups with local coefficients. Then*

$$\{\mathrm{ob}_n^k(\xi^m) \,|\, n = 1, 2, \ldots\} \text{ and } \{\mathrm{OB}_{n,\theta}^k(\xi^m) \,|\, n = 1, 2, \ldots\}$$

have the following properties:

(1) *If* $\mathrm{ob}_n^k(\xi^m) = 0$ *for every* $n < \dim X$*, then* ξ^m *can be decomposed as a Whitney sum* $\xi^m = \xi^k \oplus \xi^{m-k}$.
(2) *If* ξ^m *can be decomposed as a Whitney sum* $\xi^m = \xi^k \oplus \xi^{m-k}$*, then there exists a homomorphism* $\theta : \pi_1(X) \longrightarrow \pi_1(\widetilde{X}_1)$ *such that* $0 \in \mathrm{OB}_{n,\theta}^k(\xi^m)$ *for* $n = 1, 2, \ldots$.
(3) *If* $N = \dim X < \infty$ *and* $0 \in \mathrm{OB}_{N-1,\theta}^k(\xi^m)$*, then* ξ^m *can be decomposed as a Whitney sum* $\xi^m = \xi^k \oplus \xi^{m-k}$.
(4) ξ^m *can be decomposed as a Whitney sum* $\xi^m = \xi^k \oplus \xi^{m-k}$ *if and only if there exists a section* $s : X \longrightarrow \varprojlim_n \widetilde{X}_n$ *for the fibration* $pr : \varprojlim_n \widetilde{X}_n \longrightarrow X$.

Proof. By Lemma 2.1.3, the problem of decomposition $\xi^m = \xi^k \oplus \xi^{m-k}$ is equivalent to the existence of a section for the fibration $p_\xi : G_k(\xi^m) \longrightarrow X$ which has fiber $G_{m-k,k}$.

If $k = 1$, and $m = 2$, then $G_{m-k,k} = G_{1,1} = S^1$ and $\pi_1(G_{1,1}) = Z$. If $m > 2$, then it is well-known that

$$\pi_1(G_{m-k,k}) \approx \pi_0(O(k)) = Z_2.$$

As a result, $\pi_1(G_{m-k,k})$ is always an abelian group so that we can apply the obstruction theory for the fibration $p_\xi : G_k(\xi^m) \longrightarrow X$.

Proof of (1). By Lemma 2.1.4, $\mathrm{ob}_n^k(\xi^m) = 0$ if and only if the fibration $\widetilde{q}_n : \widetilde{X}_n \longrightarrow \widetilde{X}_{n-1}$ has a section. So if $\mathrm{ob}_n^k(\xi^m) = 0$ for each n, then there exists a section $s : X \longrightarrow \varprojlim_n \widetilde{X}_n$ for the fibration $pr : \varprojlim_n \widetilde{X}_n \longrightarrow X$, which is the composition of all sections $\widetilde{q}_n : \widetilde{X}_n \longrightarrow \widetilde{X}_{n-1}$. From [3], $\varprojlim_n \widetilde{h}_n : G_k(\xi^m) \longrightarrow \varprojlim_n \widetilde{X}_n$ is a weak homotopy equivalence. Since X is a CW-complex, one can apply J.H.C. Whitehead theorem, which in our situation says that $h^* : [X, G_k(\xi^m)] \approx [X, \varprojlim_n \widetilde{X}_n]$, to get a section for the fibration $p_\xi : G_k(\xi^m) \longrightarrow X$. By Lemma 2.1.3, ξ^m can be decomposed as a Whitney sum: $\xi^m = \xi^k \oplus \xi^{m-k}$.

Proof of (2). By Lemma 2.1.3, $\xi^m = \xi^k \oplus \xi^{m-k}$ implies $p_\xi : G_k(\xi^m) \longrightarrow X$ has a sections $s : X \longrightarrow G_k(\xi^m)$ which gives rise to a sequence of sections $\{s_n = h_n s : X \longrightarrow \widetilde{X}_n \,|\, \text{ s.t. } \widetilde{q}_n s_n = s_{n-1},\ n = 1, 2, \ldots\}$. Take $\theta = s_{1*} : \pi_1(X) \longrightarrow \pi_1(\widetilde{X}_1)$. Then for every n, consider the pull-back diagram:

$$\begin{array}{ccc} s_{n-1}^*(\widetilde{X}_n) & \xrightarrow{s_{n-1}^*} & \widetilde{X}_n \\ \downarrow & & \downarrow \widetilde{q}_n \\ X & \xrightarrow[s_{n-1}]{} & \widetilde{X}_{n-1}. \end{array}$$

By definition, $\mathrm{ob}_n^k(\xi^m)$ is the primary cohomology obstruction for the fibration $\widetilde{q}_n : \widetilde{X}_n \longrightarrow \widetilde{X}_{n-1}$. Since s_{n-1} has a lifting s_n, by Lemma 2.1.4,

$$s_{n-1}^*(\mathrm{ob}_n^k(\xi^m)) = 0 \in H^{n+1}(X; s_{n-1*}\widehat{\pi}_n(G_{m-k,k})).$$

Now we look at the local coefficients. From the long exact sequence of the fibration $\widetilde{q}_n : \widetilde{X}_n \longrightarrow \widetilde{X}_{n-1}$:

$$\dots \xrightarrow{\partial_2} \pi_1(K(\pi_n(F), n)) \xrightarrow{i_*} \pi_1(\widetilde{X}_n) \xrightarrow{\widetilde{q}_{n^*}} \pi_1(\widetilde{X}_{n-1}) \longrightarrow 0$$

we see that $\widetilde{q}_{n^*} : \pi_1(\widetilde{X}_n) \longrightarrow \pi_1(\widetilde{X}_{n-1})$ is an isomorphism for $n \geq 2$. Noticing that

$$\widetilde{q}_2 \dots \widetilde{q}_{n-1} s_{n-1} = s_1,$$

we thus prove that $s_{n-1*}\widehat{\pi}_n(G_{k,m-k}) = \theta_*\widehat{\pi}_n(G_{k,m-k})$, and

$$s_{n-1}^*(\mathrm{ob}_n^k(\xi^m)) = 0 \ \text{ in } \ H^{n+1}(X; \theta_*\widehat{\pi}_n(G_{m-k,k})).$$

By definition, $0 \in \mathrm{OB}_{n,\theta}^k(\xi^m)$, for $n = 1, 2, \dots$.

Proof of (3). By definition, $0 \in \mathrm{OB}_{N-1,\theta}^k(\xi^m)$ implies that there exists a section s_{N-2} such that $s_{N-2}^*(\mathrm{ob}_{N-1}(\xi^m)) = 0$ in $H^N(X, \theta_*\widehat{\pi}_{N-1}(G_{m-k,k}))$, by Lemma 2.1.4, s_{N-2} has a lifting s_{N-1} such that $p_{N-1}s_{N-1} = s_{N-2}$. Since $\dim X = N$, for any local coefficients $\widetilde{G}$ on X

$$H^i(X, \widetilde{G}) = 0 \qquad \text{for } \ i > N.$$

But $s_{N-1}^*(\mathrm{ob}_N(\xi^m)) \in H^{N+1}(X, s_{N-1*}\widehat{\pi}_N(G_{k,m-k})) = 0$, so $s_{N-1}^*(\mathrm{ob}_N(\xi^m)) = 0$, and by Lemma 2.1.4, there exists a section s_N such that $p_N s_N = s_{N-1}$. By repeating this procedure, we can obtain a sequence of sections $\{s_n \mid \text{s.t. } p_n s_n = s_{n-1}\}$ which gives a section for the fibration $p_\xi : G_k(\xi^m) \longrightarrow X$ and by Lemma 2.1.3, $\xi^m = \xi^k \oplus \xi^{m-k}$.

Proof of (4). Since $\varprojlim_n \widetilde{h}_n : G_k(\xi^m) \longrightarrow \varprojlim_n \widetilde{X}_n$ is a weak homotopy equivalence and X is a CW-complex, from Lemma 2.1.1 and J.H.C. Whitehead theorem as in proof of (1) the result follows. □

Definition 2.1.6. The classes $\{\mathrm{ob}_n^k(\xi^m) \in H^{n+1}(\widetilde{X}_{n-1}, \widehat{\pi}_n(G_{m-k,k})),\ n = 1, 2 \dots\}$ in the above theorem are called the decomposition obstructions of $\xi^m = \xi^k \oplus \xi^{m-k}$.

Corollary 2.1.7. $\{\mathrm{ob}_n^k(\xi^m) \in H^{n+1}(\widetilde{X}_{n-1}, \widehat{\pi}_n(G_{m-k,k})),\ n = 1, 2 \ldots\}$ *are natural in the following sense: If* $g : Y \longrightarrow X$ *is a map, then* g *pulls the tower*

$$X \underset{\widetilde{q}_1}{\longleftarrow} \widetilde{X}_1 \longleftarrow \cdots \widetilde{X}_{n-1} \underset{\widetilde{q}_n}{\longleftarrow} \widetilde{X}_n \longleftarrow \cdots$$

back over Y *such that*

$$g_{n-1}^*(\mathrm{ob}_n^k(\xi^m)) = \mathrm{ob}_n^k(g^*(\xi^m)) \quad \textit{and} \qquad g^*(\mathrm{OB}_{n\theta}^k(\xi^m)) \subset \mathrm{OB}_{n,g_*\theta}^k(g^*(\xi^m))$$

where $g_{n-1} : g^*(\widetilde{X})_{n-1} \longrightarrow \widetilde{X}_{n-1}$ *is the induced map at* $(n-1)$ *stage, in particular,*

$$\mathrm{ob}_n^k(\xi^m) = \mathrm{ob}_n^k(f^*(\eta_m)) = f_{n-1}^*(\mathrm{ob}_n^k(\eta_m)) \quad \textit{and} \qquad f^*(\mathrm{OB}_{n,\theta}^k(\eta_m)) \subset \mathrm{OB}_{n,f_*\theta}^k(\xi^m)$$

where η_m *is the universal* m*-vector bundle, and* f *is the classifying map of* ξ^m*.*

Proof. The proof is essentially based on the naturality of the primary cohomology obstruction and that of the Postnikov decomposition. By definition $\mathrm{ob}_n^k(\xi^m)$ is the n-th Postnikov invariant in the induced Postnikov decomposition. But the Postnikov invariants are natural since they are defined to be the primary cohomology obstructions.

To prove that $g^*(\mathrm{OB}_{n\theta}^k(\xi^m)) \subset \mathrm{OB}_{n,g_*\theta}^k(g^*(\xi^m))$, let $s_{n-1} : X \longrightarrow \widetilde{X}_{n-1}$ be a section such that $(\widetilde{q}_2 \cdots \widetilde{q}_{n-1} s_{n-1})_* = \theta$, consider the commutative diagram:

$$\begin{array}{ccc} g^*(X_{n-1}) & \xrightarrow{g_{n-1}} & \widetilde{X}_{n-1} \\ \big\downarrow & & \big\downarrow \\ Y & \xrightarrow{g} & X. \end{array}$$

Since s_{n-1} induces a section $g^*(s_{n-1})$ for the induced fibration by g such that $s_{n-1}g = g_{n-1}\ g^*(s_{n-1})$, by the naturality of cohomology with local coefficients, we have

$$\begin{aligned} g^*(s_{n-1}^*(\mathrm{ob}_n^k(\xi^m))) &= (s_{n-1}g)^*(\mathrm{ob}_n^k(\xi^m)) \\ &= (g_{n-1}g^*(s_{n-1}))^*(\mathrm{ob}_n^k(\xi^m)) \\ &= (g^*(s_{n-1}))^* g_{n-1}^*(\mathrm{ob}_n^k(\xi^m)) \\ &= (g^*(s_{n-1}))^*(\mathrm{ob}_n^k(g^*(\xi^m))) \end{aligned}$$

which means that $g^*(\mathrm{OB}^{k,n,\theta}(\xi^m)) \subset \mathrm{OB}_{n,g_*\theta}^k(g^*(\xi^m))$. □

In the following we study the decomposition obstructions in some details. Consider the inclusions:

$$O(m-k) \subset O(m-k) \times O(k) \subset O(m)$$

which induce fibrations in their classifying spaces:

$$BO(m-k) \longrightarrow B(O(m-k) \times O(k)) \longrightarrow BO(m).$$

We look at the following two fibrations:

$$p : BO(m-k) \longrightarrow BO(m) \text{ and } p' : B(O(m-k) \times O(k)) \longrightarrow BO(m)$$

which have fibers $O(m)/O(m-k) = V_{m,k}$, and $O(m)/(O(m-k) \times O(k)) = G_{m-k,k}$ respectively, and view the third fibration $BO(m-k) \longrightarrow B(O(m-k) \times O(k))$ as a map between the two fibrations. By the naturality of Postnikov decomposition, there exists a commutative diagram:

$$\begin{array}{ccc}
E & \xrightarrow{\widetilde{f}} & E' \\
h_n \big\downarrow & & \big\downarrow h'_n \\
E_n & \xrightarrow{f_n} & E'_n \\
q_{n-1} \big\downarrow & & \big\downarrow q'_{n-1} \\
E_{n-1} & \xrightarrow{f_{n-1}} & E'_{n-1} \\
p_{n-1} \big\downarrow & & \big\downarrow p'_{n-1} \\
BO(m) & = & BO(m)
\end{array}$$

where $\{E_n, .q_n, h_n\}$ and $\{E'_n, .q'_n, h'_n\}$ are the Postnikov decompositions of $p : BO(m-k) \longrightarrow BO(m)$ and $p' : B(O(m-k) \times O(k)) \longrightarrow BO(m)$ respectively.

Now we study the Postnikov decomposition $\{E_n, .q_n, h_n\}$ of $p : BO(m-k) \longrightarrow BO(m)$. Since q_n is a fibration with fiber $K(\pi_n(V_{m,k}), n)$ and the Stiefel manifold $V_{m,k}$ is $(m-k-1)$-connected, the Postnikov invariant

$$k_i(p) \in H^{i+1}(E_{i-1}; \widehat{\pi}_i(V_{m,k}))$$

vanishes for $0 < i < m-k$. Applying Lemma 2.1.4 repeatedly, we have:

Proposition 2.1.8. *There exists a section: $BO(m) \longrightarrow E_{m-k-1}$ in the Postnikov decomposition $\{E_n, q_n, h_n\}$ of the fibration $p : BO(m-k) \longrightarrow BO(m)$.*

Corollary 2.1.9. *There exists a section: $BO(m) \longrightarrow E'_{m-k-1}$ in the Postnikov decomposition $\{E'_n, q'_n, h'_n\}$ of the fibration $p' : B(O(m-k) \times O(k)) \longrightarrow BO(m)$.*

Proof. This is a direct result of the Proposition 2.1.8 and the above commutative diagram. □

Corollary 2.1.10. *Let ξ^m be an m-dimensional vector bundle over a connected N-dimensional* CW*-complex X. If $N \leq m-k$, then ξ^m can be decomposed as a Whitney sum $\xi^m = \xi^k \oplus \xi^{m-k}$.*

Proof. By Corollary 2.1.9, we have a section: $s : BO(m) \longrightarrow E'_{m-k-1}$ in the Postnikov decomposition $\{E'_n, q'_n, h'_n\}$ of the fibration $p' : BO((m-k) \times O(k)) \longrightarrow BO(m)$. So there exists a section: $\widetilde{s} : X \longrightarrow \widetilde{X}_{m-k-1}$ in the induced Postnikov decomposition. By definition, $0 \in \mathrm{OB}^k_{m-k-1,\theta}(\xi^m)$, using Theorem 2.1.5 (3), one concludes that ξ^m can be decomposed as a Whitney sum $\xi^m = \xi^k \oplus \xi^{m-k}$. □

In Theorem 2.1.5, the obstruction set $\mathrm{OB}^k_{n,\theta}(\xi^m)$ depends on $\theta : \pi_1(X) \to \pi_1(\widetilde{X}_1)$. In the following, we will see that there are exactly two different θ's for the universal m-vector bundle η^m.

Theorem 2.1.11. *In the Postnikov decomposition $\{E'_n, q'_n, h'_n\}$ of the fibration $p' : BO((m-k) \times O(k)) \longrightarrow BO(m)$, if $m > 2$, then there are exactly two homomorphisms*

$$\theta_1, \theta_2 : \pi_1(BO(m)) \longrightarrow \pi_1(E'_1)$$

which are induced by some sections from $BO(m)$ to E'_1.

Proof. Recall that in the Postnikov decomposition $\{E'_n, q'_n, h'_n\}$, $q'_1 : E'_1 \longrightarrow BO(m)$ is a fibration with fiber $K(\pi_1(G_{m-k,k}), 1)$. Consider the homotopy sequence of the fibration

$$\cdots \longrightarrow \pi_2(BO(m)) \xrightarrow{\partial} \pi_1(K(\pi_1(G_{m-k,k}), 1)) \xrightarrow{i_*} \pi_1(E'_1) \xrightarrow{q'_{1*}} \pi_1(BO(m)) \longrightarrow 0$$

and the fact that $\pi_1(G_{m-k,k}) = Z_2$ for $m > 2$ and $\pi_1(BO(m)) = Z_2$. Since $B(O(m-k) \times O(k)) \sim BO(m-k) \times BO(k)$, we have

$$\pi_1(B(O(m-k) \times O(k))) \approx \pi_1(BO(m-k)) \oplus \pi_1(BO(m-k)) \approx Z_2 \oplus Z_2.$$

But $h' : B(O(m-k) \times O(k)) \longrightarrow E'_1$ is 2-connected, so

$$h'_{1*} : \pi_1(B(O(m-k) \times O(k))) \longrightarrow \pi_1(E'_1)$$

is an isomorphism and hence $\pi_1(E'_1) \approx Z_2 \oplus Z_2$. Therefore, the above exact sequence actually is the following sequence:

$$\pi_2(BO(m)) \xrightarrow{\partial} Z_2 \xrightarrow{i_*} Z_2 \oplus Z_2 \xrightarrow{q'_{1*}} Z_2 \longrightarrow 0.$$

By the exactness, the image of ∂ must be 0, and we get a short exact sequence:

$$0 \longrightarrow Z_2 \xrightarrow{i_*} Z_2 \oplus Z_2 \xrightarrow{p'_*} Z_2 \longrightarrow 0.$$

It is easy to see that there are exactly two homomorphisms

$$\theta_1, \theta_2 : Z_2 \longrightarrow Z_2 \oplus Z_2$$

which satisfy the condition: $p'_* \circ \theta_1 = p'_* \circ \theta_2 = 1$. By the 2-extendability thorem [**3**], we get two sections: $s_1, s_2 : BO(m) \longrightarrow E'_1$ such that

$$s_{i^*} = \theta_i : \pi_1(BO(m)) = Z_2 \longrightarrow \pi_1(E'_1) = Z_2 \oplus Z_2 \qquad \text{for } i = 1, 2.$$

This is what we need to prove. □

2.2. Codimension 1 decomposition. Now we consider two special cases in which there is only one decomposition obstruction. The first one is that the dimension of the vector bundle is 2. The other one is the case in which the codimension is 1.

Proposition 2.2.1. *For any two dimensional vector bundle ξ^2 over a connected* CW*-complex X, there is only one decomposition obstruction*

$$\mathrm{ob}_1(\xi^2) \in H^2(X; f * \widetilde{Z})$$

such that $\mathrm{ob}_1(\xi^2) = 0$ *if and only if ξ^2 can be decomposed as $\xi^2 = \xi^1 \oplus \eta^1$, where $f : X \longrightarrow BO(2)$ is the classifying map of ξ^2, and $\widetilde{Z}$ is the twisted integer.*

Proof. In the fibration $p : B(O(1) \times O(1)) \longrightarrow BO(2)$, the fiber is

$$O(2)/(O(1) \times O(1)) \approx G_{1,1} \approx S^1 \approx K(Z, 1),$$

so the fibration itself is the Postnikov decomposition, and

$$\mathrm{ob}_n(\xi^2) = 0 \qquad \text{for } \ n > 1.$$

Since $\mathrm{ob}_1(\xi^2)$ is the only none trivial decomposition obstruction, by Theorem 2.1.5, $\mathrm{ob}_1(\xi^2) = 0$ if and only if ξ^2 can be decomposed as $\xi^2 = \xi^1 \oplus \eta^1$. □

Corollary 2.2.2. *Let η_2 be the universal principal $O(2)$-bundle, then for any two dimensional vector bundle ξ^2 over a connected* CW*-complex X with classifying map f,*

$$f^*(\mathrm{ob}_1(\eta_2)) = 0 \ \textit{ if and only if } \xi^2 \textit{ can be deconposed as } \ \xi^2 = \xi^1 \oplus \eta^1.$$

Proof. By the naturality of the decomposition obstruction

$$f^*(\mathrm{ob}_1(\eta_2)) = \mathrm{ob}_1(f^*(\eta_2)) = \mathrm{ob}_1(\xi^2).$$

From Proposition 2.2.1, we have this corollary. □

Now we turn to consider the second special case where the dimension of the vector bundle is same as that of the base space. The decomposition is such that one of the bundles in the sum is a line bundle.

In our definition of decomposition obstructions $\mathrm{ob}_n^k(\xi^m)$, we use the Postnikov decomposition induced by the classifying map of the vector bundle ξ^m

from the fibration $p : B(O(k) \times O(m-k)) \longrightarrow BO(m)$. The advantage of this approach is that those decomposition obstruction classes $\mathrm{ob}_n^k(\xi^m)$ only depend on the vector bundle ξ^m and are natural in the sense as indicated in Corollary 2.1.7. If one has a section s to the $(n-1)$-level in the induced Postnikov decomposition, then the vanishing of $s^*(\mathrm{ob}_n^k(\xi^m))$ is equivalent to the existence of the n-level lifting for s. So the class $s^*(\mathrm{ob}_n^k(\xi^m))$ has the similar property as that of the obstruction cohomology class for a section. On the other hand, as stated in [**3**], the construction for the Postnikov decomposition is much more difficult than that of CW-decomposition. In some special cases, one may give a detail description of the Postnikov decomposition [**14**]. However, Eckmann and Hilton [**5**] showed that the cohomology obstructions for Postnikov and CW-decompositions are equivalent. To say precisely, let (X, A) be a relative CW-complex with CW-decomposition $X = \varinjlim_n X_n$, and let $p : E \longrightarrow X$ be a fibration with Postnikov decomposition $\{E_n, q_n, h_n\}$, then there is a bijection:

$$\lambda : \langle X_n, E\rangle^{u,\theta} \,|\, X_{n-1} \approx \langle X, E_{n-1}\rangle^{\theta'}$$

where $u : A \longrightarrow p^{-1}(A) = E_{-1}$ is a partial section, $\theta : \pi_1(X) \longrightarrow \pi_1(E)$ is a splitting of p (see [**3**]) and $\theta' = h_{1^*} \circ \theta : \pi_1(X) \longrightarrow \pi_1(E_1)$, and $\langle X_n, E\rangle^{u,\theta}$ denoted the section homotopy classes relative to u and compatible with θ, and $\langle X_n, E\rangle^{u,\theta} \,|\, X_{n-1}$ is the set of all the restrictions on X_{n-1}. The bijection mapping λ is given in the following way:

Let $\phi_n \in \langle X_n, E\rangle^{u,\theta}$, then $h_{n-1}\phi_n \,|\, X_{n-1} : X_{n-1} \longrightarrow E_{n-1}$ has an extension $h_{n-1}\phi_n : X_n \longrightarrow E_{n-1}$. But the fiber of the fibration: $E_{n-1} \longrightarrow X$ has no non-vanishing homotopy groups in dimension greater than $n-1$, so the section $h_{n-1}\phi_n : X_n \longrightarrow E_{n-1}$ can be extended to a section over X which is defined to be $\lambda(\phi_n|X_{n-1})$. From [**3**], this is well-defined bijection. Under this bijection, Eckmann and Hilton's result says that

$$[\mathrm{ob}_n(\phi_n)] = \phi^{n-1} * (k_n(q_n)) \in H^{n+1}(X, A; \theta * \pi_n(F))$$

where $\phi^{n-1} = \lambda(\phi_n \,|\, X_{n-1})$, and $k_n(q_n) \in H^{n+1}(E_{n-1}; \pi_n(F)_{q_n})$ is the n-th Postnikov invariant for the fibration $q_n : E_n \longrightarrow E_{n-1}$.

Using the above result, we see that from the Postnikov invariants, one can recover all the obstructions classes defined by using the CW-decomposition of the base space. It is because of this, we can compute the obstruction $\mathrm{OB}_{n\theta}^k(\xi^m)$ without knowing the Postnikov decomposition. This makes it possible to compute $\mathrm{OB}_{n\theta}^k(\xi^m)$ by using only the CW-decomposition of the base space.

In order to actually compute $\mathrm{OB}_{n\theta}^k(\xi^m) \subset H^{n+1}(X, \theta_*\widehat{\pi}_n(G_{m-k,k}))$, one still needs to know the local coefficients $\widehat{\pi}_n(G_{m-k,k})$. By the naturality 2.1.7, we know that $f^*(\mathrm{OB}_{n,\theta}^k(\eta_m)) \subset \mathrm{OB}_{n,f_*\theta}^k(\xi^m)$, where f is the classifying map for the vector bundle ξ^m, and η_m is the universal m-plane bundle over $BO(m)$. So it will be enough to determine the local coefficients for the fibration $p : B(O(k) \times O(m-k)) \longrightarrow BO(m)$ which is difficult in general because one does not know $\pi_n(G_{k,m-k})$ for each n. In order to relate this coefficients to some known coefficients, we need the following lemma:

Lemma 2.2.3. *Let $E_2 \xrightarrow{q_2} E_1 \xrightarrow{q_1} X$ be a tower of fibrations, and let F_2 and F_1 be the fibers of q_1q_2 and q_1 respectively, then for each n, the induced homomorphism $q_{2^*} : \pi_n(F_2) \longrightarrow \pi_n(F_1)$ is a homorphism $q_{2^*} : \pi_n(F_2)^{q_1q_2} \longrightarrow \pi_n(F_1)^{q_1}$ between the two systems of local coefficients determined by the fibrations q_1q_2 and q_1.*

Proof. Let $\widehat{E}_2, \widehat{E_2}$, be the mapping cylinders of q_1q_2 and q_1 respectively, then q_2 induces an obvious map $\widehat{q}_2 : (\widehat{E}_2, E_2) \longrightarrow (\widehat{E}_1, E_1)$ which induces operator homomorphisms between the two homotopy exact sequences:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \pi_q(E_2) & \xrightarrow{i_*} & \pi_q(\widehat{E_2}) & \xrightarrow{j_*} & \pi_q(\widehat{E_2}, E_2) & \xrightarrow{\partial} & \pi_{q-1}(E_2) & \longrightarrow & \cdots \\
 & & {\scriptstyle q_{2^*}}\big\downarrow & & {\scriptstyle \widehat{q}_{2^*}}\big\downarrow & & {\scriptstyle \widehat{q}_{2^*}}\big\downarrow & & {\scriptstyle q_{2^*}}\big\downarrow & & \\
\cdots & \longrightarrow & \pi_q(E_1) & \xrightarrow{i_*} & \pi_q(\widehat{E_1}) & \xrightarrow{j_*} & \pi_q(\widehat{E_1}, E_1) & \xrightarrow{\partial} & \pi_{q-1}(E_1) & \longrightarrow & \cdots .
\end{array}$$

In particular, $\widehat{q}_{2^*} : \pi_q(\widehat{E}_2, E_2) \longrightarrow \pi_q(\widehat{E}_1, E_1)$ is an operator homorphism. Recall that $\pi_n(F_1)^{q_1}$ is defined via an isomorphism: $\Delta' : \pi_q(\widehat{E_1}, E_1) \longrightarrow \pi_{q-1}(F_1)$. In fact, the isomorphism is given by the following composition:

$$\Delta' : \pi_q(\widehat{E_1}, E_1) \xrightarrow{k_1^{-1}} \pi_q(\widehat{F_1}, F_1) \xrightarrow{\partial} \pi_{q-1}(F_1)$$

where k_1 is the inclusion which induces an isomorphism [**15**]. Thus we have a commutative diagram:

$$\begin{array}{ccccc}
\pi_q(\widehat{E}_2, E_2) & \xrightarrow{k_1^{-1}} & \pi_q(\widehat{F_2}, F_2) & \xrightarrow{\partial} & \pi_{q-1}(F_2) \\
{\scriptstyle \widehat{q}_{2^*}}\big\downarrow & & {\scriptstyle \widehat{q}_{2^*}}\big\downarrow & & {\scriptstyle q_{2^*}}\big\downarrow \\
\pi_q(\widehat{E}_1, E_1) & \xrightarrow{k_1^{-1}} & \pi_q(\widehat{F_1}, F_1) & \xrightarrow{\partial} & \pi_{q-1}(F_1).
\end{array}$$

Therefore $q_{2^*} : \pi_n(F_2)^{q_1q_2} \longrightarrow \pi_n(F_1)^{q_1}$ is a homomorphism between the two local systems. □

Corollary 2.2.4. *With the above notations, for any n-skeleton section $s : X^n \longrightarrow E_2$ $(n \geq 2)$, there is an induced homomorphism:*

$$q_{2*} : H^{n+1}(X; s_*\pi_n(F_2)) \longrightarrow H^{n+1}(X; q_2 \circ s_*\pi_n(F_1))$$

such that

$$q_{2*}(\mathrm{ob}_n(s)) = \mathrm{ob}_n(q_2 \circ s).$$

Proof. By Lemma 2.2.3, $q_{2*} : \pi_n(F_2) \longrightarrow \pi_n(F_1)$ is a homomorphism $q_{2*} : \pi_n(F_2)^{q_1q_2} \longrightarrow \pi_n(F_1)^{q_1}$ between the two systems of local coefficients determined by the fibrations q_1q_2 and q_1. The sections $s : X^n \longrightarrow E_2$ and $q_2 s : X^n \longrightarrow E_1$ pull the two systems of local coefficients back on X^n so that $q_{2*} : \pi_n(F_2)^{q_1q_2} \longrightarrow \pi_n(F_1)^{q_1}$ induces a homomorphism of the systems of local coefficients:

$$q_{2*} : s_*\pi_n(F_2) \longrightarrow (q_2 s)_*\pi_n(F_1).$$

From [**15**], q_{2*} induces a homomorphism for the cohomology groups with local coefficients:

$$q_{2*} : H^{n+1}(X; s_*\pi_n(F_2)) \longrightarrow H^{n+1}(X; q_2 \circ s_*\pi_n(F_1)).$$

Now we consider the diagram:

$$\begin{array}{ccccc} \pi_{n+1}(E_2^{n+1}, E_2^n) & \xrightarrow{\partial} & \pi_n(E_2^n) & \xrightarrow{s_\#} & \pi_n(F_2) \\ {\scriptstyle q_{2*}}\downarrow & & {\scriptstyle q_{2*}}\downarrow & & {\scriptstyle q_{2*}}\downarrow \\ \pi_{n+1}(E_1^{n+1}, E_1^n) & \xrightarrow{\partial} & \pi_n(E_1^n) & \xrightarrow{(q_2 s)_\#} & \pi_n(F_1) \end{array}$$

where all the spaces and homomorphisms are the same as in the definition of obstruction cocycle (see [**3**]). We only need to check the commutativity of the right square.

Let $\alpha \in \pi_n(E_2^n)$, by definition, we have

$$\begin{aligned} i_*q_{2*}s_\#(\alpha) &= q_{2*}i_*s_\#(\alpha) = q_{2*}(\alpha - s_*(q_1q_2)_*(\alpha)) \\ &= q_{2*}(\alpha) - q_{2*}s_*(q_1q_2)_*(\alpha) \\ &= q_{2*}(\alpha) - (q_2 s)_*(q_1(q_{2*}(\alpha))) = i_*(q_2 \circ s)_\# q_{2*}(\alpha). \end{aligned}$$

Since i_* is injective, we obtain that $q_{2*}s_\# = (q_2 \circ s)_\# q_{2*}$. Therefore, we have

$$q_{2*}s_\#\partial q_{2*}^{-1} = (q_2 \circ s)_\#\partial.$$

By the definition of cocycle, and that of the induced homomorphism,

$$q_{2*}(\mathrm{ob}_n(s)) = \mathrm{ob}_n(q_2 \circ s).$$

This completes the proof. □

Corollary 2.2.5. *The inclusions $O(m) \subset O(m) \times O(n) \subset O(m+n)$ induce a tower of fibrations $BO(m) \xrightarrow{q_2} B(O(m) \times O(n)) \xrightarrow{q_1} BO(m+n)$. Let $m \geq n$ and $m + n > 2$. Then q_2 induces a homomorphism:*

$$q_{2*} : H^{q+1}(BO(m+n); \pi_q(V_{m+n,n})_{q_1q_2}) \longrightarrow H^{q+1}(BO(m+n); \theta_{1_*}\pi_q(G_{m,n}))$$

where $\theta_1 : \pi_1(BO(m+n)) = \pi_1(BO(m)) = Z_2 \xrightarrow{q_{2*}} \pi_1(B(O(m) \times O(n))) = Z_2 \oplus Z_2$. If s is a q-skeleton section with $q \geq 2$, then

$$q_{2*}(\mathrm{ob}_q(s)) = \mathrm{ob}_q(q_2 \circ s).$$

Proof. The fibers of q_1 and q_1q_2 are $G_{m,n}$ and $V_{m+n,n}$ respectively. By our assumption, $m \geq 2$, so $V_{m+n,n}$ is at least 1-connected and the fibration q_1q_2 has a unique (up to homotopy) 2-skeleton section s and

$$s_* : \pi_1(BO(m+n)) = Z_2 \longrightarrow \pi_1(BO(m)) = Z_2$$

is an isomorphism. The result follows by Corollary 2.2.4 if we simply think s_* as an identity. □

The following theorem is the main result about the codimension 1 decomposition which reveals an obstruction class in $H^{2n+1}(BO(2n+1); Z)$, the cohomology with ordinary integer coefficients.

Theorem 2.2.6. *Let η_m be the m-dimensional universal bundle over $BO(m)$ with $m \geq 3$, then there are exactly two obstruction classes in dimension m for the decomposition $\eta_m \approx \xi^{m-1} \oplus \lambda$. Furthermore, one of the two classes is the primary obstruction for the decomposition $\eta_m \approx \xi^{m-1} \oplus R$, which is in $H^m(BO(m); \widetilde{Z})$, where $\widetilde{Z}$ is the twisted integers; the other is in $H^m(BO(m); \widetilde{Z})$ if m is even and is in $H^m(BO(m); Z)$ if m is odd.*

Proof. Consider the tower of fibrations:

$$BO(m-1) \xrightarrow{q_2} B(O(m-1) \times O(1)) \xrightarrow{q_1} BO(m)$$

by Corollary 2.2.5, q_2 induces a homomorphism:

$$q_{2*} : H^{q+1}(BO(m); \pi_q(V_{m,1})_{q_1q_2}) \longrightarrow H^{q+1}(BO(m); \theta_{1*}\pi_q(G_{m-1,1}))$$

where

$$\begin{aligned}\theta_1 : \pi_1(BO(m)) &= \pi_1(BO(m-1)) \\ &= Z_2 \xrightarrow{q_{2*}} \pi_1(B(O(m-1) \times O(1))).\end{aligned}$$

Noticing that $V_{m,1} = S^{m-1}$ and $G_{m-1,1} = RP^{m-1}$, we know that q_2 induces an isomorphism $q_{2*}\pi_{m-1}(V_{m,1}) \approx Z \longrightarrow \pi_{m-1}(G_{m-1,1}) \approx Z$ between the two systems of local coefficients. As is well-known, the coefficients in the first cohomology is the twisted integer $\widetilde{Z}$ when $q = m-1$, so is the coefficients in the second one. Thus q_2 induces an isomorphism

$$q_{2*} : H^m(BO(m); \widetilde{Z}) \longrightarrow H^m(BO(m); \widetilde{Z}).$$

Let s be any $(m-1)$-skeleton section for the fibration $q_1\, q_2 : BO(m-1) \longrightarrow BO(m)$, then $q_2\, s$ is $(m-1)$-skeleton section for the fibration $q_1 : B(O(m-1) \times O(1)) \longrightarrow BO(m)$. From Corollary 2.2.5, we obtain that

$$q_{2*}(\mathrm{ob}_{m-1}(s)) = \mathrm{ob}_{m-1}(q_2 \circ s).$$

From [**15**], $\mathrm{ob}_{m-1}(s)$ is the primary obstruction for the vector bundle decomposition $\eta_m \approx \xi^{m-1} \oplus R$. From the primary obstruction theorem, the primary obstruction is independent with the choice of sections. Hence $\mathrm{ob}_{m-1}(q_2 \circ s)$ is one of the primary obstruction for the decomposition $\eta_m \approx$

$\xi^{m-1} \oplus \lambda$. But $q_2{*}$ is an isomorphism, so we may regard $\mathrm{ob}_{m-1}(s) = \mathrm{ob}_{m-1}(q_2 \circ s) \in H^m(BO(m); \widetilde{Z})$.

In the following, we will prove that the other obstruction for the decomposition $\eta_m \approx \xi^{m-1} \oplus \lambda$ is either in $H^m(BO(m); \widetilde{Z})$ if m is even, or in $H^m(BO(m); Z)$ if m is odd. As in the proof of Theorem 2.1.11, there is an exact sequence:

$$0 \longrightarrow \pi_1(G_{m-1,1}) \longrightarrow \pi_1(B(O(m-1) \times O(1))) \xrightarrow{q_{1*}} \pi_1(B(O(m))) \longrightarrow 0$$

and $\pi_1(G_{m-1,1}) \approx \pi_1(B(O(m))) \approx Z_2$. Therefore the above sequence splits and there are exactly two splitting homomorphisms:

$$\theta_1, \theta_2 : \pi_1(BO(m)) \longrightarrow \pi_1(B(O(m-1) \times O(1))) \approx \pi_1(B(O(m))) \oplus \pi_1(G_{m-1,1})$$

which are given by $\theta_1(\alpha) = (\alpha, 0)$ and $\theta_2(\alpha) = (\alpha, \beta)$, where α, β are the generators of $\pi_1(BO(m))$ and $\pi_1(G_{m-1,1})$ respectively. We have already proved that $\theta_1(\alpha) = (\alpha, 0)$ corresponds to the obstruction class of the decomposition $\eta_m \approx \xi^{m-1} \oplus R$. In order to see the local coefficients determined by $\theta_2(\alpha) = (\alpha, \beta)$, we first recall the action of $\pi_1(G_{m-1,1})$ on $\pi_{m-1}(G_{m-1,1})$. The universal covering space of $G_{m-1,1} = RP^{m-1}$ is S^{m-1}. The only nontrivial covering translation is the antipodal map h which corresponds to β. From [**15**], there is a commutative diagram

$$\begin{array}{ccc} [S^{m-1}, S^{m-1}] & \xrightarrow{p_*} & [S^{m-1}, *; RP^{m-1}, *] = \pi_{m-1}(G_{m-1,1}) \\ \Big\downarrow \underline{h} & & \Big\downarrow \tau_\beta \\ [S^{m-1}, S^{m-1}] & \xrightarrow{p_*} & [S^{m-1}, *; RP^{m-1}, *] = \pi_{m-1}(G_{m-1,1}). \end{array}$$

It is well-known that the degree of the antipodal map on S^{m-1} is $(-1)^m$, from the above diagram, we know that the action of β is given by

$$\tau_\beta(\xi) = (-1)^m \xi, \text{ for any } \xi \text{ in } \pi_{m-1}(G_{m-1,1}) = Z.$$

But the action of $(0, \beta)$ on $\pi_{m-1}(G_{m-1,1})$ is the same as that of β, and the action of $(\alpha, 0)$ on $\pi_{m-1}(G_{m-1,1})$ is the same as that of α which is the twisted action, i.e.,

$$\tau_\alpha(\xi) = -\xi, \text{ for any } \xi \text{ in } \pi_{m-1}(G_{m-1,1}) = Z.$$

Now the action induced by $\theta_2(\alpha) = (\alpha, \beta)$ is simply the product of the actions of α and β. So we have $\theta_2(\alpha)(\xi) = (-1)^{m+1}\xi$, and thus the action is either the twisted integr $\widetilde{Z}$ if m is even or the ordinary integer Z if m is odd. □

The most interesting part in the theorem is the following corollary:

Corollary 2.2.7. *There is an element o_{2n+1} in $H^{2n+1}(BO(2n+1); Z)$ such that for any map $f : X^{2n+1} \longrightarrow BO(2n+1)$, if $f^*(o_{2n+1}) = 0$, then $f^*(\eta_{2n+1}) \approx \xi^{2n} \oplus \lambda$, where X^{2n+1} is a $(2n+1)$-dimensional* CW*-complex and η_{2n+1} is the universal vector bundle over $BO(2n+1)$.*

Proof. Let o_{2n+1} be the obstruction corresponding to the decomposition $\xi^{2n} \oplus \lambda$ as in Theorem 2.2.6. By the primary obstruction theorem, there is a $(2n)$-skeleton section s for the fibration $p : B(O(2n) \times O(1)) \to BO(2n+1)$ such that $o_{2n+1} = \mathrm{ob}(s)$. By cellular approximation theorem, we may assume that f is a cellular map. By the naturality of the primary obstruction class, we have

$$f^*(o_{2n+1}) = f^*(\mathrm{ob}(s)) = \mathrm{ob}(f^*s)$$

where $f^*(s)$ is the induced section by f over the $(2n)$-skeleton of X^{2n+1}. So if $f^*(o_{2n+1}) = 0$, then $f^*(s)$ can be extended to a section over X^{2n+1}, hence $f : X^{2n+1} \longrightarrow BO(2n+1)$ has a lifting for the fibration $p : B(O(2n) \times O(1)) \longrightarrow BO(2n+1)$. By Lemma 2.1.1, $f^*(\eta_{2n+1}) \approx \xi^{2n} \oplus \lambda$. □

The following corollary gives us some relationship between the decomposition obstruction and the well-known characteristic classes such as Stiefel-Whitney classes, Euler classes and Chern classes.

Corollary 2.2.8. *If the coefficient is reduced to Z_2, then one of the two obstruction classes in Theorem 2.2.6 will be the top Stiefel-Whitney class of the universal bundle.*

Proof. By definition, the top universal Stiefel-Whitney class can be defined as the primary obstruction class for the vector bundle decomposition $\eta_m \approx \xi^{m-1} \oplus R$ reduced the coefficient to Z_2. The corollary follows by Theorem 2.2.6. □

Recall that the classifying space for oriented m-dimensional vector bundles is $BSO(m)$, and the inclusion $SO(m) \longrightarrow O(m)$ induced a universal covering map

$$\pi : BSO(m) \longrightarrow BO(m)$$

which is the classifying map for the universal oriented m-dimensional vector bundle ζ_m over $BSO(m)$.

Corollary 2.2.9. *Let $\mathrm{OB}^1_{m-1,\theta_1}(\eta_m)$ and $\mathrm{OB}^1_{m-1,\theta_2}(\eta_m)$ be the two decomposition obstruction classes as in Theorem 2.2.6, and $\pi : BSO(m) \longrightarrow BO(m)$ be the projection, then*

$$\pi^*(\mathrm{OB}^1_{m-1,\theta_1}(\eta_m)) = \pi^*(\mathrm{OB}^1_{m-1,\theta_2}(\eta_m)) \in H^m(BSO(m), Z),$$

which is the Euler class of the universal oriented m-dimensional vector bundle ζ_m over $BSO(m)$.

Proof. By the naturality of the obstruction classes, both $\pi^*(\mathrm{OB}^1_{m-1,\theta_1}(\eta_m))$ and $\pi^*(\mathrm{OB}^1_{m-1,\theta_2}(\eta_m))$ are the primary obstruction classes corresponding to the splitting homomorphism $\theta_1\pi_*$ and $\theta_2\pi_*$ for the codimension one decomposition of ζ_m. But $BSO(m)$ is 1-connected, hence $\theta_1\pi_* = \theta_2\pi_* = 0$. By the uniqueness of the primary obstruction class, we have $\pi^*(\mathrm{OB}^1_{m-1,\theta_1}(\eta_m)) = \pi^*(\mathrm{OB}^1_{m-1,\theta_2}(\eta_m)) \in H^m(BSO(m), Z)$. Noticing that any line bundle over $BSO(m)$ is trivial, we see that $\pi^*(\mathrm{OB}^1_{m-1,\theta_1}(\eta_m))$ is the obstruction for the decomposition $\zeta_m = \zeta^{m-1} \oplus R$. But Euler class can be defined to be the primary obstruction of the decomposition $\zeta_m = \zeta^{m-1} \oplus R$. □

Recall that any n-dimensional complex vector bundle ζ^n can be regarded as a $2n$-dimensional real vector bundle $\mathrm{Re}\,(\zeta^n)$. In terms of the structure groups, one has an inclusion $U(n) \subset O(2n)$ which induces a fibration $p : BU(n) \longrightarrow BO(2n)$. If γ_n is the universal n-dimensional complex vector bundle over $BU(n)$, then $\mathrm{Re}\,(\gamma_n) = p^*(\eta_{2n})$. Let $c_n(\zeta^n)$ be the n-th Chern class of a complex vector bundle ζ^n, then we have the following corollary:

Corollary 2.2.10. *With the above notations,*

$$\begin{aligned} c_m(\gamma_m) &= p^*(\mathrm{OB}^1_{2m-1,\theta_1}(\eta_{2m})) \\ &= p^*(\mathrm{OB}^1_{2m-1,\theta_2}(\eta_{2m})) \in H^{2m}(BU(m), Z). \end{aligned}$$

Proof. Since $BU(m)$ is 1-connected, $\mathrm{Re}\,(\gamma_n) = p^*(\eta_{2n})$ is oriented vector bundle, as in Corollary 2.2.7, we know that

$$p^*(\mathrm{OB}^1_{2m-1,\theta_1}(\eta_{2m})) = p^*(\mathrm{OB}^1_{2m-1,\theta_2}(\eta_{2m})),$$

which is the Euler class of $\mathrm{Re}\,(\gamma_n)$. But the top Chern class of γ_n is just the Euler class of $\mathrm{Re}\,(\gamma_n)$ (see [**10**]). □

In the following, we further consider the codimension 1 decomposition, that is the decomposition $\xi^m \approx \xi^{m-1} \oplus \lambda$. We try to begin from CW-structure of the classify space to see the restriction of the universal bundle to each cell.

Recall that a partition of $r \geq 0$ is an unordered sequence $(i_1, i_2, \ldots, i_s)$ of positive numbers such that the sum of the numbers is equal to r. For our purpose, we define a partition of $r \geq 0$ with length n is an unordered sequence $(i_1, i_2, \ldots, i_n)$ of non-negative numbers such that the sum of the numbers is equal to r. We can always assume that $0 \leq i_1 \leq i_2 \leq \cdots \leq i_n$. Then there is an one to one correspondence $(i_1, i_2, \ldots, i_n) \leftrightarrow (\sigma_1, \sigma_2, \ldots, \sigma_n)$ given by $\sigma_j = i_j + j$, for $j = 1, 2, \ldots, n$. From [**10**], the Schubert symbol $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ determines an unique r-cell $e(\sigma)$, which is the set of n-planes in R^m such that:

$$e(\sigma) = \Big\{ X \mid \dim(X \cap R^{\sigma_i}) = i,\ \dim(X \cap R^{\sigma_i - 1}) = i - 1;\ i = 1, 2, \ldots, n \Big\}.$$

From [10], we know that the Grassmann manifold $G_n(R^m)$ has the following CW-structure:

$$G_n(R^m) = \Big\{ \cup\, e(\sigma) \,|\, \sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \text{ is such that } 0 < \sigma_1 < \sigma_2 < \cdots < \sigma_n \leq m \Big\}.$$

Taking the direct limit as $m \longrightarrow \infty$, one gets the infinite Grassmann manifold $G_n(R^\infty)$, which is the classifying space $BO(n)$.

Proposition 2.2.11. *Let $e(\sigma)$ be a $(2k+1)$-dimensional cell in $G_n(R^\infty)$, then*

$$\eta_{2k+1} \,|\, \overline{e}(\sigma) \approx \xi^{2k} \oplus \lambda$$

where η_{2k+1} is the universal vector bundle over $G_n(R^\infty)$.

Proof. Let $\sigma_j = i_j + j$, for $j = 1, 2, \dots, 2k+1$, then

$$\dim e(\sigma) = \sum_{j=1}^{2k+1} i_j = 2k+1, \quad \text{and } \ 0 \leq i_1 \leq i_2 \leq \dots i_{2k+1}.$$

It is easy to see that the only $(2k+1)$-cell such that $i_1 > 0$ is the cell with partition $(1, 1, \dots, 1)$. The corresponding Schubert symbol is $\sigma = (2, 3, \dots, 2k+2)$. By definition, $e(\sigma)$ consists of all the $(2k+1)$-subspace X in R^{2k+2} such that $\dim(X \cap R^1) = 0$. It is not difficult to count all the faces of this cell, in fact, all the faces in the partition form are $(0, 0, \dots, 0, 1, \dots, 1)$. So the closure $\overline{e}(\sigma)$ for this cell is just $G_{2k+1}(R^{2k+2}) = RP^{2k+1}$. And $\eta_{2k+1}|G_{2k+1}(R^{2k+2})$ is the canonical $(2k+1)$-plane bundle of $G_{2k+1}(R^{2k+2})$ which is the tangent bundle for $k > 0$. It is well-known that there exists an nowhere 0 vector field for $\tau(S^{2k+1})$, in fact,

$$(x_0, x_1, \dots, x_{2k}, x_{2k+1}) \longrightarrow ((x_0, x_1, \dots, x_{2k}, x_{2k+1}), (x_1, -x_0, \dots, x_{2k+1}, -x_{2k}))$$

is such a vector field. It is easy to see that this vector field induces a section for the fibration $RP(\eta_{2k+1}|\overline{e}(\sigma)) \longrightarrow \overline{e}(\sigma)$, the projective space bundle. By Lemma 2.1.3

$$\eta_{2k+1} \,|\, \overline{e}(\sigma) \approx \xi^{2k} \oplus \lambda$$

where $\sigma = (2, 3, \dots, 2k+2)$.

For any other $(2k+1)$-cell $e(\sigma)$, σ_1 must be equal to 1 and each face of $e(\sigma)$ must also have the first entry 1 which means that $X \cap R^1 = R^1$ for any $X \in \overline{e}(\sigma)$. So $RP(\eta_{2k+1}|\overline{e}(\sigma)) \longrightarrow \overline{e}(\sigma)$ has a section given by $X \longrightarrow (X, R^1)$. Again by Lemma 2.1.3,

$$\eta_{2k+1} \,|\, \overline{e}(\sigma) \approx \xi^{2k} \oplus \lambda$$

where λ can be even chosen to be trivial line bundle. Thus we complete the proof of the proposition. ∎

3. Two examples.

In this section, we will give two examples. In the first example, we demonstrate a method to calculate the obstruction classes. In the second one, we try to find as many as possible the trivial lines in the tangent bundle of a manifold up to cobordism.

Example 3.1. $H^{2n}(RP^{2n}, \widetilde{Z}) \approx Z$ and the generator is the obstruction class for the tangent bundle decomposition $\tau(RP^{2n}) \approx \xi^{2n-1} \oplus \lambda$.

Let $p : S^{2n} \longrightarrow RP^{2n}$ be the covering map and $\tau : S^{2n} \longrightarrow S^{2n}$ be the antipodal map. By Eilenberg Theorem $H^{2n}(RP^{2n}, \widetilde{Z}) \approx E^{2n}(S^{2n}, \widetilde{Z})$, where $E^{2n}(S^{2n}, \widetilde{Z})$ is the equivariant cohomology group which can be determined by the complex

$$\Big\{H_p(S^p, S^{p-1}; Z) \xrightarrow{\partial} H_{p-1}(S^{p-1}, S^{p-2}; Z) \,|\, p = 1, 2, \ldots, 2n\Big\}.$$

From [15], one can choose the orientation e^p of the cell E^p_+, the upper hemisphere so that $H_p(S^p, S^{p-1}; Z) = \langle e^p, \tau e^p\rangle$ and the boundary operator is given by

$$\partial(e^p) = e^{p-1} + (-1)^p \tau e^{p-1}$$

where τe^p is the induced orientation on the cell of lower hemisphere by the antipodal map. As an equivariant group $H_p(S^p, S^{p-1}; Z)$ has one generator e^p, so the set of equivariant homomorphism $\mathrm{Hom}^{Z_2}(H_p(S^p, S^{p-1}; Z), \widetilde{Z})$ has one generator c_{e^p} which maps e^p to 1, where the antipodal map generates Z_2 and acts on integers by multliplying (-1). Now we can calculate the coboundary operator:

$$\delta(c_{e^p})(e^{p+1}) = c_{e^p}(\partial e^{p+1}) = c_{e^p}(e^p + (-1)^{p+1}\tau e^p) = 1 + (-1)^p$$

where we use the fact that $c_{e^p}(\tau e^p) = \tau(c_{e^p}(e^p)) = -1$. Now it is easy to see that $E^{2n}(S^{2n}, \widetilde{Z}) = Z$ with generator c_{e^p}. So $H^{2n}(RP^{2n}, \widetilde{Z}) \approx Z$.

In the following obstruction class for the decomposition $\tau(RP^{2n}) \approx \xi^{2n-1}\lambda$ is generator of $H^{2n}(RP^{2n}, \widetilde{Z}) \approx Z$. Consider the fibration

$$p : RP(\tau(RP^{2n})) \longrightarrow RP^{2n}$$

whose fiber is RP^{2n-1}. The obstruction for the existence of a section for p is the same as that of for the decomposition $\tau(RP^{2n}) \approx \xi^{2n-1} \oplus \lambda$. If we have a section s_{2n-1} for $p : RP(\tau(RP^{2n})) \longrightarrow RP^{2n}$ on the $(2n-1)$-skeleton of RP^{2n}. Let

$$h : (B^{2n}, S^{2n-1}) \longrightarrow (RP^{2n}, RP^{2n-1})$$

be the characteristic map of the only $2n$-cell of RP^{2n}. In the pull-back diagram

$$\begin{array}{ccc} h^*(RP(\tau(RP^{2n}))) & \xrightarrow{h^*} & RP(\tau(RP^{2n})) \\ \Big\downarrow & & \Big\downarrow p \\ (B^{2n}, S^{2n-1}) & \xrightarrow{h} & (RP^{2n}, RP^{2n-1}) \end{array}$$

$h^*(RP(\tau(RP^{2n}))) \approx B^{2n} \times RP^{2n-1}$, since B^{2n} is contractible. The section s_{2n-1} induces a section for the trivial fibration $S^{2n-1} \times RP^{2n-1} \longrightarrow S^{2n-1}$ which determines a map

$$h' : S^{2n-1} \longrightarrow RP^{2n-1}.$$

The obstruction cocycle $\mathrm{ob}(s_{2n-1})$ is just defined to be the correspondence:

$$\mathrm{ob}(s_{2n-1}(e^{2n})) = [h'] \in \pi_{2n-1}(RP^{2n-1}) \approx Z$$

which in turn determines an element

$$[\mathrm{ob}(s_{2n-1})] \text{ in } H^{2n}(RP^{2n}, \widetilde{\pi}_{2n-1}(RP^{2n-1})).$$

We claim that the local coefficients $\widetilde{\pi}_{2n-1}(RP^{2n-1})$ is the twisted integer $\widetilde{Z}$. To see this, we consider the natural inclusion:

$$i : RP^{2n} = G_{2n}(R^{2n+1}) \subset G_{2n}(R^{\infty}) = BO(2n).$$

From Theorem 2.2.6, the obstruction for the decomposition $\eta_{2n} \approx \xi^{2n-1} \oplus \lambda$ is in

$$H^{2n}(BO(2n); \widetilde{Z}).$$

But the inclusion i induces an isomorphism on the fundamental groups:

$$i_* : \pi_1(G_{2n}(R^{2n+1})) \approx \pi_1(G_{2n}(R^{\infty})) \approx Z_2.$$

By the naturality of the decomposition obstruction classes, i induces a homomorphism:

$$i^* : H^{2n}(G_{2n}(R^{\infty}), \widetilde{Z}) \longrightarrow H^{2n}(G_{2n}(R^{2n+1}), i_*\widetilde{Z})$$

which maps the decomposition obstructions of η_{2n} to that of $i*(\eta_{2n}) = \tau(RP^{2n})$. So $\widetilde{\pi}_{2n-1}(RP^{2n-1}) \approx i_*\widetilde{Z} = \widetilde{Z}$. Thus

$$[\mathrm{ob}(s_{2n-1})] \in H^{2n}(RP^{2n}, \widetilde{\pi}_{2n-1}(RP^{2n-1})) \approx H^{2n}(RP^{2n}, \widetilde{Z}).$$

In order to calculate $[h'] \in \pi_{2n-1}(RP^{2n-1}) \approx Z$, we need to choose a specific characteristic map h for the $2n$-cell, and find the trivilization of the pull-back of fibration $p : RP(\tau(RP^{2n})) \longrightarrow RP^{2n}$. Let

$$q : V_k(R^m) \longrightarrow G_k(R^m)$$

be the principal $O(k)$-bundle, where $V_k(R^m)$ is the Stiefel manifold, then the pull-back of the canonical k-bundle over $G_k(R^m)$ by q is isomorphic to the trivial bundle

$$V_k(R^m) \times R^k \longrightarrow V_k(R^m).$$

The isomorphism is given by

$$((\nu_1,\nu_2,\dots,\nu_k),(t_1,t_2,\dots,t_k)) \longrightarrow \left((\nu_1,\nu_2,\dots,\nu_k),\left(\langle V\rangle,\sum_1^k t_i\nu_i\right)\right)$$

where $\nu_1,\nu_2,\dots,\nu_k$ are k unit orthogonal vectors in R^m, and $\langle V\rangle$ is the k-dimensional subspace in R^m generated by $\nu_1,\nu_2,\dots,\nu_k$. The inverse is given by

$$((\nu_1,\nu_2,\dots,\nu_k),(\langle V\rangle,\nu)) \longrightarrow ((\nu_1,\nu_2,\dots,\nu_k),(\nu\cdot\nu_1,\nu\cdot\nu_2,\dots,\nu\cdot\nu_k)).$$

So any vector bundle factors out though $q: V_k(R^m)\longrightarrow G_k(R^m)$ is trivial and the trivilization is given by the above isomorphisms. In particular, we know the trivilization on any subspace of $V_k(R^m)$. The following theorem [**10**] states that the characteristic map for any cell in $G_k(R^m)$ can be chosen to be the restriction of $q: V_k(R^m)\longrightarrow G_k(R^m)$ to some subspace of $V_k(R^m)$.

Theorem 3.2. *In the* CW*-structure of* $G_n(R^m)$,

$$G_n(R^m)=\Big\{\cup\, e(\sigma)\,|\,\sigma=(\sigma_1,\sigma_2,\dots,\sigma_n) \quad \textit{is such that}\ \ 0<\sigma_1<\sigma_2\cdots<\sigma_n\le m\Big\}$$

the characteristic map of the cell $e(\sigma)$ *can be given by*

$$q|_{\overline{H}^{\sigma_1}\times\cdots\times\overline{H}^{\sigma_n}}: V_n(R^m)\cap\overline{H}^{\sigma_1}\times\cdots\times\overline{H}^{\sigma_n}\longrightarrow G_n(R^m)$$

where $\overline{H}^{\sigma_1}=\{(x_1,x_2,\dots,x_{\sigma_i},0,\dots,0)\in R^m\,|\,x_{\sigma_i}\ge 0\}$.

Before further considering the characteristic map for the top cell in $G_{2n}(R^{2n+1})$, we need the following lemma:

Lemma 3.3. *Let* $(\nu_1,\nu_2,\dots,\nu_k)\in V_k(R^{k+1})$ *be written in matrix form:*

$$\begin{bmatrix}\nu_1\\ \nu_2\\ \vdots\\ \nu_k\end{bmatrix}=\begin{bmatrix}\nu_{1,1} & \nu_{1,2} & \cdots & \nu_{1,k+1}\\ \nu_{2,1} & \nu_{2,2} & \cdots & \nu_{2,k+1}\\ \cdots & \cdots & \cdots & \cdots\\ \nu_{k,1} & \nu_{k,2} & \cdots & \nu_{k,k+1}\end{bmatrix}$$

then the vector $(\nu_1,\dots,\nu_k)^\perp=(A_1,A_2,A_{k+1})$, *where*

$$A_i=(-1)^{i+1}\begin{vmatrix}\nu_{1,1} & \cdots & \widehat{\nu}_{1,i} & \cdots\nu_{1,k+1}\\ \cdots & \cdots & \cdots & \cdots\\ \nu_{k,1} & \cdots & \widehat{\nu}_{k,i} & \cdots\nu_{k,k+1}\end{vmatrix}$$

is an unit vector and orthogonal to each ν_i *for* $1,2,\dots,k$.

Proof. Let $w = (w_1, w_2, \dots, w_{k+1})$ be the unique unit vector that is orthogonal to each ν_i for $i = 1, 2, \dots, k$ and such that

$$(*) \qquad \begin{vmatrix} \nu_{1,1} & \cdots & \nu_{1,i} & \cdots & \nu_{1,k+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \nu_{k,1} & \cdots & \nu_{k,i} & \cdots & \nu_{k,k+1} \\ w_1 & \cdots & w_{k,i} & \cdots & w_{k+1} \end{vmatrix} = (-1)^k.$$

From the determinant properties we know that $(\nu_1, \dots, \nu_k)^\perp = (A_1, A_2, \dots, A_{k+1})$ is orthoganal to each ν_i for $i = 1, 2, \dots, k$ and so

$$(\nu_1, \dots, \nu_k)^\perp = tw$$

hence $\langle (\nu_1, \dots, \nu_k)^\perp, w\rangle = t\langle w, w\rangle = t$. Expanding the determinant $(*)$ on the last row, we see that $(-1)^k \sum_{i=1}^{k+1} w_i A_i = (-1)^k$, that is $\langle (\nu_1, \dots, \nu_k)^\perp, w\rangle = 1$. Thus we proved that $(\nu_1, \dots, \nu_k)^\perp = w$ and hence $(\nu_1, \dots, \nu_k)^\perp$ is an unit vector and orthogonal to each ν_i for $i = 1, 2, \dots, k$.

Now we consider the $2n$-cell in $G_{2n}(R^{2n+1})$ which corresponds to the Schubert symbol $(2, 3, \dots, 2n+1)$. In this special case, we can further give a specific homeomorphism:

$$h_{2n} : I^{2n} \longrightarrow V_{2n}(R^{2n+1}) \cap \overline{H}^2 \times \cdots \times \overline{H}^{2n+1}$$

by

$$h_{2n}(x_1, x_2, \dots, x_{2n}) = (\nu_1, \nu_2 \dots, \nu_{2n})$$

where

$$\begin{aligned} \nu_1 &= x_1 e_1 + \sqrt{1 - x_1^2} e_2 \\ \nu_2 &= x_2 \left(\sqrt{1 - x_1^2} e_1 - x_1 e_2 \right) + \sqrt{1 - x_2^2} e_3 \\ &\cdots \cdots \cdots \cdots \\ \nu_k &= x_k (\nu_1, \dots \nu_{k-1})^\perp + \sqrt{1 - x_k^2} e_{k+1} \end{aligned}$$

with inverse given by

$$h_{2n}^{-1}(\nu_1, \nu_2, \dots, \nu_{2n}) = (\langle \nu_1, e_1\rangle, \langle \nu_2, (\nu_1)^\perp\rangle, \dots, \langle \nu_{2n}, (\nu_1, \dots, \nu_{2n-1})^\perp\rangle).$$

Now we consider the $(2n-1)$-skeleton section s_{2n-1} for the fibration

$$p : RP(\tau(RP^{2n})) \longrightarrow RP^{2n} = G_{2n}(R^{2n+1})$$

given by

$$s_{2n-1}(X) = (X, [e_1]).$$

In fact, for any cell $e(\sigma)$ in $BO(2n)$ and $X \in e(\sigma)$, $\dim(X \cap R^1) = 1$ if $\dim e(\sigma) < 2n$. So s_{2n-1} is a section over the $(2n-1)$-skeleton for the fibration

$$RP(\eta_{2n}) \longrightarrow BO(2n).$$

Now we can calculate the obstruction cocycle for the section s_{2n-1} as follows: In the pull-back diagram

$$\begin{array}{ccc} h^*(RP(\tau(RP^{2n}))) & \xrightarrow{h^*} & RP(\tau(RP^{2n})) \\ \downarrow & & \downarrow p \\ (B^{2n}, S^{2n-1}) & \xrightarrow{h} & (RP^{2n}, RP^{2n-1}) \end{array}$$

if we choose the characteristic map to be

$$qh_{2n} : I^{2n} \longrightarrow V_{2n}(R^{2n+1}) \cap \overline{H}^{2n+1} \times \cdots \times \overline{H}^{2n+1} \longrightarrow G_{2n}(R^{2n+1})$$

then the pull-back diagram will be equivalent to the following diagram

$$\begin{array}{ccc} I^{2n} \times RP^{2n-1} & \xrightarrow{qh^*_{2n}} & RP(\tau(RP^{2n})) \\ \downarrow & & \downarrow p \\ I^{2n} & \xrightarrow{qh_{2n}} & G_{2n}(R^{2n+1}) \end{array}$$

where $I^{2n} \times RP^{2n-1} \overset{qh^*_{2n}}{\longrightarrow} RP(\tau(RP^{2n}))$ is given by

$$qh^*_{2n}((x_1, \ldots, x_{2n}), [t_1, \ldots, x_{2n}]) = \left(qh_{2n}(x_1, \ldots, x_{2n}), \sum_i t_i \nu_i \right)$$

where $h_{2n}(x_1, x_2, \ldots, x_{2n}) = (\nu_1, \nu_2, \ldots, \nu_{2n})$. The induced section s^*_{2n-1} over $\partial(I^{2n}) \approx S^{2n-1}$ is given by

$$s^*_{2n-1}(x_1, \ldots, x_{2n}) = ((x_1, \ldots, x_{2n}), [\langle e_1, \nu_1\rangle, \ldots \langle e_1, \nu_{2n}\rangle])$$

which induces a map $h : \partial(I^{2n}) \approx S^{2n-1} \longrightarrow RP^{2n-1}$ given by

$$h(x_1, \ldots, x_{2n}) = [\langle e_1, \nu_1\rangle, \ldots, \langle e_1, \nu_{2n}\rangle].$$

Taking a close look at the formula for $h_{2n}(x_1, x_2, \ldots, x_{2n}) = (\nu_1, \nu_2, \ldots, \nu_{2n})$, we find that

$$[\langle e_1, \nu_1\rangle, \ldots, \langle e_1, \nu_{2n}\rangle] = \left[x_1, x_2\sqrt{1 - x_1^2}, \ldots, x_{2n}\sqrt{1 - x_1^2} \cdots \sqrt{1 - x_{2n-1}^2} \right].$$

Thus $h(x_1, \ldots, x_{2n}) = \left[x_1, x_2\sqrt{1 - x_1^2}, \ldots, x_{2n}\sqrt{1 - x_1^2} \cdots \sqrt{1 - x_{2n-1}^2} \right]$ which is homotopic to the map $(x_1, \ldots, x_{2n}) \longrightarrow [x_1, \ldots, x_{2n}]$ via the following homotopy

$$H_t(x_1, \ldots, x_{2n}) = \left[x_1, x_2\sqrt{1 - tx_1^2}, \ldots, x_{2n}\sqrt{1 - tx_1^2} \cdots \sqrt{1 - tx_{2n-1}^2} \right].$$

Therefore $[h]$ represents the generator in $\pi_{2n-1}(RP^{2n-1})$, hence the obstruction class $[\mathrm{ob}(s_{2n-1})]$ for the decomposition $\tau(RP^{2n}) \approx \xi^{2n-1} \oplus \lambda$ is the generator in $H^{2n}(RP^{2n}\widetilde{Z}) \approx Z$. Thus we complete our first example □

In the following example, we will see that the decomposition $\xi^{2n+1} \approx \xi^{2n} \oplus \lambda$ is often possible. Let MO_* denote the Thom cobordism ring. It is well-known that $MO_* = \sum_{n \geq 0} MO_n = Z_2[X_n \,|\, n \neq 2^k - 1]$ is a graded polynomial algebra over Z_2 with one generator in each dimension n not in the form $2^k - 1$ for all $k > 0$.

Example 3.4. Let $[M_{2k+1}] \in MO_{2k+1}$ be a $(2k+1)$-dimensional cobordism class, then we can choose M_{2k+1} such that $\tau(M_{2k+1}) \approx \xi^{2k} \oplus R$.

Proof. For any odd dimensional generator, we will choose such a representative. Consider the vector bundle $\lambda_1 \oplus \cdots \oplus \lambda_m$ over $RP(n_1) \times \cdots \times RP(n_m)$, where λ_1 is the pull-back of the canonical line bundle over i-th factor. Let $RP(n_1, \ldots, n_m)$ be the projective space bundle of $\lambda_1 \oplus \cdots \oplus \lambda_m$, then it is a n-dimensional smooth manifold, where $n = \sum_1^m n_i + m - 1$. From [13], $RP(n_1, \ldots, n_m)$ is indecomposable in MO_n if and only if $\binom{n-1}{n_1} + \cdots + \binom{n-1}{n_m} = 1 \bmod 2$.

Let n be a positive odd number that is not in the form $2^k - 1$, then n can be uniquely written as $n = 2^{p+1}q + 2^p - 1$, where p, q are positive integers. Let X_n be the manifold $RP(2^p, 1, \ldots, 1, 0)$ where the number of 1's is $2^p q - 1$ which is greater than 0. Noticing that 2^p and 1 do not appear in the binary expression of $n-1$, one can check that $RP(2^p, 1, \ldots, 1, 0)$ is indecomposable. From Borel-Hirzebruch [4], the tangent bundle of the projective space bundle $RP(\xi)$ associated with a vector bundle ξ over a smooth manifold M always splits:

$$\tau(RP(\xi)) \approx p^*(\tau(M)) \oplus \tau_2$$

where $p : RP(\xi) \longrightarrow M$ is the projection, and τ_2 is the bundle along the fiber. From this result, noticing that the tangent bundle of RP^1 is trivial, we see that the tangent bundle of $RP(2^p, 1, \ldots, 1, 0)$ has a $(2^p q - 1)$-dimensional trivial summand.

For even number n, we may just choose RP^n to be the generator. From the polynomial structure of MO_*, any $[M_{2k+1}] \in MO_{2k+1}$ has the form:

$$[M_{2k+1}] = \sum_I \varepsilon_I X_{i_1} \cdots X_{i_r}$$

where $\varepsilon \in Z_2$, and $\sum_{j=1}^r i_j = 2k+1$. So at least one of the generator has odd dimension. Therefore the tangent bundle of $\sum_I \varepsilon_I X_{i_1} \cdots X_{i_r}$ has a trivial summand. □

Since the characteristic numbers are cobordism invariants, noticing that $w_{2n+1}(\xi^{2n} \oplus R) = 0$, and using the above example, we get a different proof of the fact that $\langle w_{2n+1}[M^{2n+1}] \rangle = 0$ for any odd dimensional closed manifold M^{2n+1}.

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