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Let Ω be a nonwandering, nonrecurrent Fatou component for a holomorphic self-map f of $\mathbf{P}^2$ of degree $d \geq 2$, and let h be a normal limit of the family of iterates of f. We prove that $\Sigma := h(\Omega)$ is either a fixed point of f or its normalization is a hyperbolic Riemann surface, so that the dynamics of $f|_\Sigma$ may be lifted to the unit disk. We also show that basins of attraction for holomorphic self-maps of $\mathbf{P}^k$ of degree $d \geq 2$ are taut.

1. Introduction.

Let $f : \mathbf{P}^k \to \mathbf{P}^k$ be holomorphic. By definition, therefore, there exists a homogeneous polynomial mapping $\widetilde{f} : \mathbf{C}^{k+1} \setminus \{0\} \to \mathbf{C}^{k+1} \setminus \{0\}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{C}^{k+1} \setminus \{0\} & \xrightarrow{\widetilde{f}} & \mathbf{C}^{k+1} \setminus \{0\} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathbf{P}^k & \xrightarrow[f]{} & \mathbf{P}^k. \end{array}$$

Here p denotes the standard projection from $\mathbf{C}^{k+1} \setminus \{0\}$ onto $\mathbf{P}^k$. The degree d of f is by definition the degree of $\widetilde{f}$. Throughout this paper we assume that $d > 1$.

The *Fatou set* $\mathcal{F}(f)$ is the largest open subset of $\mathbf{P}^k$ on which the family $\{f^n\}_{n \in \mathbf{N}}$ is normal. In [7], Ueda shows that $\widetilde{f}$ has a bounded basin of attraction A to the origin. Let Ω be any connected component of $\mathcal{F}(f)$. Ueda shows that there exists a set $\widetilde{\Omega} \subset \partial A$ such that the restriction of p to $\widetilde{\Omega}$ is a holomorphic covering map onto Ω. A corollary of this construction is the Kobayashi hyperbolicity of Ω. Fornæss and Sibony have exploited this fact in their classification of recurrent Fatou components for holomorphic maps on $\mathbf{P}^2$ ([4]).

Suppose now that Ω is a fixed, nonrecurrent Fatou component; that is, Ω satisfies $f(\Omega) = \Omega$ and $f^n(z) \to \partial\Omega$ for all $z \in \Omega$. Let h be a normal limit of some subsequence of $\{f^n\}$, so that $f^{n_i} \to h$ locally uniformly on Ω as

$i \to \infty$. Then $\Sigma := h(\Omega) \subset \partial\Omega$. The principal aim of this paper is to prove the following result:

Theorem 1. *Suppose that $f : \mathbf{P}^2 \to \mathbf{P}^2$ is holomorphic, and Ω a fixed, nonrecurrent Fatou component for f. Let Σ be as described above. Then either Σ is a fixed point of f, or there exists a locally injective holomorphic mapping $\sigma : \Delta \to \Sigma$, where $\Delta \subset \mathbf{C}$ is the unit disk, and a holomorphic function $F : \Delta \to \Delta$ such that the following diagram commutes:*

$$\begin{array}{ccc} \Delta & \xrightarrow{F} & \Delta \\ \sigma\big\downarrow & & \big\downarrow\sigma \\ \Sigma & \xrightarrow[f]{} & \Sigma. \end{array}$$

In the latter case, F must either be conjugate to an irrational rotation, or $F^n(z) \to \partial\Delta$ for all $z \in \Delta$.

The proof is given in Section 2.

Remark 1. A more general theorem was stated by Fornaess and Sibony in [**3**], but the proof seems incomplete.

A complex manifold M is called *taut* if the family of maps from the unit disk Δ to M is normal. Abate has asked ([**1**]) whether Fatou components for holomorphic self-maps of $\mathbf{P}^k$ are taut. In Section 3, we prove the following:

Theorem 2. *Let Ω be a Fatou component for $f : \mathbf{P}^k \to \mathbf{P}^k$ which is preperiodic to a basin of attraction. Then Ω is taut.*

2. Proof of Theorem 1.

Let f be a holomorphic self-map of $\mathbf{P}^k$, and Ω a fixed, nonrecurrent Fatou component. Choose and fix some subsequence f^{n_i} which converges locally uniformly on Ω. Let $h = \lim_{i\to\infty} f^{n_i}$, and let $\Sigma = h(\Omega)$. Then $\Sigma \subset \partial\Omega$.

Lemma 1. *Let Σ be as above. Then $f(\Sigma) = \Sigma$.*

Proof. Since $h = \lim_{i\to\infty} f^{n_i}$, h commutes with f on Ω. Let $z \in \Sigma$, $x \in h^{-1}(z)$. Let $y \in f^{-1}(x) \cap \Omega$. Then $f(z) = f(h(x)) = h(f(x)) \in \Sigma$, so $f(\Sigma) \subset \Sigma$. And $h(y) \in \Sigma$ with $f(h(y)) = h(f(y)) = h(x) = z$. Thus $f(\Sigma) = \Sigma$. $\square$

Let p be the natural projection from $\mathbf{C}^{k+1} \setminus \{0\}$ to $\mathbf{P}^k$, and $\widetilde{f} : \mathbf{C}^{k+1} \setminus \{0\} \to \mathbf{C}^{k+1} \setminus \{0\}$ the homogeneous polynomial lift of f by p. It was shown by Ueda ([**7**]) that any homogeneous polynomial self-map of $\mathbf{C}^k \setminus \{0\}$ has a bounded basin of attraction to the origin. Let A be the bounded basin of attraction to the origin for $\widetilde{f}$. Ueda showed further the existence of a set

$\widetilde{\Omega} \subset \partial A$ such that the restriction of p to $\widetilde{\Omega}$ is a holomorphic covering map onto Ω.

Lemma 2. *Let U be an open subset of Ω sufficiently small that a local inverse $q : U \to \widetilde{\Omega}$ of $p|_{\widetilde{\Omega}}$ may be defined. Then there exists $\hat{h} : U \to \partial A$ holomorphic (as a mapping into $\mathbf{C}^{k+1}$) such that $p \circ \hat{h} = h$. Furthermore, if $\hat{h}_1$ is one such lift, then $\hat{h}_2$ is another if and only if $\hat{h}_2 = e^{i\theta}\hat{h}_1$ for some real θ.*

Proof. Write $h = \lim f^{n_i}$. On U, we have $p \circ (\widetilde{f}^{n_i} \circ q) = f^{n_i}$. Since $\{\widetilde{f}^{n_i} \circ q\}$ is uniformly bounded as a family of mappings into $\mathbf{C}^{k+1}$, by passing to a subsequence, if necessary, we may assume that it has a holomorphic limit $\hat{h}$ on U. Taking limits of both sides of

$$p \circ (\widetilde{f}^{n_i} \circ q) = f^{n_i}$$

gives

$$p \circ \hat{h} = h.$$

To prove the second statement, note that $\hat{h}_1(z)$ and $\hat{h}_2(z)$ are in the same fiber of p for all $z \in U$; i.e., in the same complex line in $\mathbf{C}^{k+1}$. Thus

$$h_1(z) = \lambda(z)\hat{h}_2(z)$$

for $z \in U$, $\lambda : U \to \mathbf{C}$ holomorphic. Recall also that $h_1(z)$, $h_2(z)$ are contained in ∂A. If G is the Green's function for A, we have $\partial A = \{G = 0\}$. It is shown in [7] that for $\lambda \in \mathbf{C}$, G satisfies

$$G(\lambda z) = G(z) + \log|\lambda|.$$

Thus

$$\begin{aligned} 0 = G(\hat{h}_1(z)) &= G(\hat{h}_2(z)) \\ &= G(\lambda(z)\hat{h}_1(z)) \\ &= G(\hat{h}_1(z)) + \log|\lambda(z)|. \end{aligned}$$

Thus $|\lambda(z)| = 1$ for all $z \in U$. Since λ is holomorphic, this gives $\lambda \equiv e^{i\theta}$ for some $\theta \in \mathbf{R}$.

This shows that any two lifts $\hat{h}$ of h differ by a multiplicative constant of absolute value one. Conversely, it is easy to check that if $\hat{h} : U \to \partial A$ is a lift of h, then so is $e^{i\theta}\hat{h} : U \to \partial A$. $\square$

The next lemma is part of the classical construction of the desingularization of a Riemann surface; see [5]. We omit the proof.

Lemma 3. *Let f be a germ at 0 of a nonconstant holomorphic mapping from $\mathbf{C}$ to $\mathbf{C}^n$. Then there exists another germ g at 0 of a holomorphic mapping from $\mathbf{C}$ to $\mathbf{C}^n$ such that g is injective in a neighborhood of 0, and such that the images of f and g agree as germs.*

Lemma 4. *Given $z \in \Sigma$, let $x \in h^{-1}(z)$, and let L be a complex line through x such that $h|_L$ is not constant. Then there exists a ball U centered at x such that the restriction of p to $\hat{h}(L \cap U)$ is injective.*

Proof. Let U be sufficiently small that we may define $\hat{h}$ on U, as in Lemma 2. By shrinking U, if necessary, we may assume that x is the only critical point of both $\hat{h}$ and of $p \circ \hat{h}$ in $L \cap U$. Let $D = L \cap U$, and $D^* = D \setminus \{x\}$. By Lemma 3, shrinking U further, we may assume that both $\hat{h}(D^*)$ and $p \circ \hat{h}(D^*)$ are biholomorphic to punctured disks. Thus if $p|_{\hat{h}(D)}$ is not injective, we may assume, making the Böttcher coordinate change, that it is of the form $w \mapsto w^s$ for some $s \geq 2$.

But then we can replace $\hat{h}$ by another lift $g \circ \hat{h}$, where g, in the appropriate coordinates, is a nontrivial rotation of $\hat{h}(D)$ about $\hat{h}(x)$. In particular, $g \circ \hat{h}(x) = \hat{h}(x)$. But by Lemma 2, $g \circ \hat{h}$ must be of the form $e^{i\theta}\hat{h}$. Furthermore, $\hat{h}(x) \neq 0$, since it is in ∂A. Thus $e^{i\theta} = 1$, and g is the trivial rotation. This contradiction establishes the lemma. □

For the remainder of this section, we will assume that $k = 2$, and that $h : \Omega \to \partial\Omega$ is nonconstant. In this case, for $x \in \Omega$, there is an irreducible piece Σ_x of a Riemann surface, possibly with singularities, and a neighborhood $U(x)$ such that $h(U(x)) = \Sigma_x$. We define R to be the abstract union $\bigcup \Sigma_{x_i}$ for a covering $\{U(x_i)\}$ of Ω, with identifications of $z_i \in \Sigma_{x_i}$ to $z_j \in \Sigma_{x_j}$ if the images under h agree there as germs. R is Hausdorff, by the identity theorem. It is a one-dimensional Riemann surface, possibly with singularities. Let S be its smooth normalization. The map h factors naturally as $\pi_1 \circ h_1$, where $h_1 : \Omega \to S$ and $\pi_1 : S \to \Sigma$.

Near a regular value of h_1, h_1 has an inverse q onto some linear disk in Ω. Define f_1 locally by $f_1 = h_1 \circ f \circ q$. It is straightforward to check that f_1 is thereby well-defined and holomorphic away from critical values of h_1, and may be extended continuously to S. Thus $f : \Sigma \to \Sigma$ lifts naturally by π_1 to $f_1 : S \to S$.

Lemma 5. *The Riemann surface S described above is hyperbolic.*

Proof. Given $z_0 \in S$, let U be a neighborhood of z_0 sufficiently small that $\pi_2(U) \subset R$ contains at most one singular point, $w_0 := \pi_2(z_0)$. Assume also that U is small enough that there exists a linear disk $L \subset \Omega$ such that p maps $\hat{h}(L)$ injectively onto some set containing $\pi_1 \circ \pi_2(U)$, as in Lemma 4.

Let $z_1 \in U \setminus \{z_0\}$. Then there exists a neighborhood V of z_1 and an open subset $W \subset L$ such that $g := h|_W$ is a biholomorphism onto $\pi_1 \circ \pi_2(V)$. Consider

$$\begin{aligned} \phi : V &\to \partial A \\ z &\mapsto \hat{h} \circ g^{-1} \circ \pi_1 \circ \pi_2(z). \end{aligned}$$

Then ϕ is holomorphic, and $p \circ \phi = \pi_1 \circ \pi_2$. Any other choice of ϕ (obtained by choosing a different subset $W \subset L$) must therefore differ from the first by a multiplicative constant of absolute value one. Since z_1 was arbitrary, ϕ may therefore be extended along any path in $U \setminus \{z_0\}$. Since $p|_{\hat{h}(L)}$ is injective, this extension gives rise to a single-valued holomorphic mapping, of which z_0 is a removable singularity. Thus ϕ is holomorphic on U, with $p \circ \phi = \pi_1 \circ \pi_2$. Again, any other choice of ϕ must differ from this one by a multiplicative constant of absolute value one; and since z_0 was arbitrary, ϕ may therefore be extended along any path in S. But this defines a covering surface $\widetilde{S} \subset \partial A$ of S. Since S is covered by a bounded subset of $\mathbf{C}^3$, it is hyperbolic. □

There are four a priori possibilities for $f_2 : S \to S$ (see [6]):

1. Some iterate of f_2 is the identity.
2. There exists $a \in R$ such that $f_2^n(z) \to a$ for all $z \in S$.
3. $f_2^n(z)$ diverges to infinity with respect to the Poincaré metric on S for all $z \in S$.
4. S is conformally a disk, punctured disk, or annulus, and the action of f_2 on S is conjugate to irrational rotation.

In our case, (1) is impossible, since then some iterate of f would fix Σ. But by Bezout's theorem the number of fixed points of a holomorphic self-map of complex projective space is finite. In Case (2), the point a would be an attractive or semi-attractive fixed point of f. But then the topological dynamics in a neighborhood U of a are well understood. In both cases, if U is sufficiently small, points in $\mathcal{F}(f) \cap U$ cannot converge to $\Sigma \setminus \{a\}$. But this contradicts our assumption that h is nonconstant. Thus (2) is also impossible.

Now, we note that f_2 can in turn be lifted to a holomorphic self-map F of the unit disk, Δ. Cases (3) and (4) above give the following possibilities for F:

1. $F^n(z) \to \partial\Delta$ locally uniformly on Δ.
2. F is an irrational rotation of Δ.

Collecting the preceding lemmas gives us the following theorem:

Theorem 1. *If h is a limit of some subsequence f^{n_i} on Ω and $\Sigma := h(\Omega)$, then either Σ is a fixed point of f or there exists a surjective, locally injective holomorphic mapping $\sigma : \Delta \to \Sigma := h(\Omega)$, and a holomorphic self-map F of Δ satisfying (1) or (2) above, such that the following diagram commutes:*

$$\begin{array}{ccc} \Delta & \xrightarrow{F} & \Delta \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \Sigma & \xrightarrow[f]{} & \Sigma. \end{array}$$

Since Σ does not contain an entire curve of singularities, Case (2) gives that Σ is a disk, punctured disk, or annulus, with at most one singularity, at the fixed point. An example of this type of behavior is the following: Take

$$f : \mathbf{P}^2 \to \mathbf{P}^2$$
$$[z : w : t] \mapsto [zt + z^2 : \lambda wt + w^2 : t^2],$$

where $\lambda = e^{2\pi i\theta}$ and θ satisfies a Diophantine condition. Let S be the Siegel disk centered at 0 for the mapping $w \mapsto \lambda w + w^2$. Then $\{f^n\}$ is compactly divergent on the Fatou component containing the point $[-1 : 0 : 1]$, any uniform limit h satisfies

$$\Sigma = h(\Omega) = \{[0, w, 1] : w \in S\}$$

(note that Σ is conformally a disk), and $f|_\Sigma$ is conjugate to multiplication by λ.

In Case (1) above, the mapping σ may be very complicated. I have no example of this type of behavior, nor a proof that it cannot occur.

3. Proof of Theorem 2.

Theorem 2. *Let Ω be a Fatou component for $f : \mathbf{P}^k \to \mathbf{P}^k$ which is preperiodic to a basin of attraction. Then Ω is taut.*

Proof. Replacing f by an iterate, we may assume that Ω is an invariant basin of attraction to $q \in \Omega$. Assume, to get a contradiction, that there exists a sequence of holomorphic mappings $\{g_i : \Delta \to \Omega\}$ with no convergent subsequence. Since Ω is covered by a bounded set in $\mathbf{C}^{k+1}$, the family $\{g_i\}$ is normal as a family of maps from Δ into $\mathbf{P}^k$. Thus, passing to a subsequence if necessary, we may assume that

$$g_i \to g : \Delta \to \overline{\Omega}.$$

But, by assumption, $g(\Delta) \not\subset \Omega$ and $g(\Delta) \not\subset \partial\Omega$.

For each i, let $\widetilde{g}_i : \Delta \to \partial A$ be a lift of g_i Then $\{\widetilde{g}_i\}$ is uniformly bounded as a family of maps into $\mathbf{C}^{k+1}$, so it is normal. By passing to a subsequence if necessary, we may assume that

$$\widetilde{g}_i \to \widetilde{g} : \Delta \to \partial A.$$

Taking limits of both sides of

$$p \circ \widetilde{g}_i = g_i$$

gives

$$p \circ \widetilde{g} = g.$$

Now,

$$p \circ \widetilde{f}^n \circ \widetilde{g}_i = f^n \circ p \circ \widetilde{g}_i = f^n \circ g_i.$$

Thus, for each n and each i, $\widetilde{f^n} \circ \widetilde{g}_i$ is a lift of $f^n \circ g_i$. Taking limits with respect to i gives

$$p \circ \widetilde{f^n} \circ \widetilde{g} = f^n \circ g.$$

But $\{\widetilde{f^n} \circ \widetilde{g}\}$ is uniformly bounded as a family of mappings into $\mathbf{C}^{k+1}$. Thus it is normal, and so therefore is $\{f^n \circ g\}$. Let h be a normal limit of $\{f^n \circ g\}$. Then $h \equiv q$ on $g^{-1}(g(\Delta) \cap \Omega)$, so $h \equiv q$ on Δ. But this is impossible, since $f^n \circ g(z) \in \partial\Omega$ for all $z \in g^{-1}(g(\Delta) \cap \partial\Omega)$. $\square$

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