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We show that every finitely generated nilalgebra having nilalgebras of matrices is a homomorphic image of nilalgebras constructed by the Golod method (Golod, 1965 and 1969). By applying some elements of module theory to these results, we construct over any field non-residually finite nilalgebras and Golod groups with non-residually finite quotients. This solves Šunkov's problem (Kourovka Notebook, 1995, Problem 12.102). Also, we reduce Kaplansky's problem on the existence of a f.g. infinite p-group G such that the augmentation ideal $\omega K[G]$ over a nondenumerable field K is a nilideal (Kaplansky, 1957, Problem 9) to the study of the just-infinite quotients of Golod groups.

1. Introduction.

This paper deals with finitely generated (f.g.) infinite dimensional nilalgebras and their associated groups. Using Golod's algebras Anan'in and Puczyłowski constructed over fields of characteristic zero f.g. non-nilpotent nilalgebras which are not residually finite [**2**, **15**]. On the other hand, Rowen has proved their existence over every field [**16**]. Here we shall construct such examples over every field. This will enable us to solve in the negative Šunkov's problem [**11**, Problem 12.102] by constructing Golod groups with non-residually finite quotients. To this end we shall first start constructing Golod algebras as extensions of some nilalgebras. This is a completely different view from the classical one where Golod algebras are seen as homomorphic images. On the other hand the proofs of Theorems 2 and 3 are careful analysis of the Golod method. However, a great deal of information is extracted. For example, we prove that every f.g. nilalgebra over a nondenumerable field is a homomorphic image of a Golod algebra. As a consequence, Kaplansky's problem on the existence of a f.g. infinite p-group G such that the augmentation ideal $\omega K[G]$ over a nondenumerable field K is a nilideal [**10**, Problem 9] is reduced to the study of the just-infinite quotients of Golod groups. In the denumerable case we obtain some results, although because of the Köthe conjecture [**12**] the situation is quite complicated and we are far from understanding it.

Let K be any field and let $F^{(1)}$ be the free associative algebra of polynomials without constant terms in the non-commuting indeterminates $X_1, \ldots, X_d$ ($d \geq 2$) over K. In this work an algebra means an associative algebra unless otherwise stated.

Lemma 1 ([6, 7]). *Let I be an ideal of $F^{(1)}$ generated by a family of homogeneous polynomials $f_1, f_2, \ldots$ of non-decreasing degrees greater than or equal to 2. Let r_i be the number of polynomials of each degree $i \geq 2$ in the sequence $f_1, f_2, \ldots$. If the coefficients of the series $\left(1 - dt + \sum_{i=2}^{+\infty} r_i t^i\right)^{-1}$ are positive, then the algebra $F^{(1)}/I$ is of infinite dimension. In particular this is true if for a fixed real ϵ, $0 < \epsilon < 1/2$, $r_i \leq \epsilon^2 (d - 2\epsilon)^{i-2}$, for every $i \geq 2$.*

A Golod algebra is a f.g. non-nilpotent nilalgebra which satisfies Lemma 1 and which is constructed by the Golod method as in [6, 7].

An algebra A over a field k is absolutely nil if for every extension field $K \supset k$, $A \bigotimes K$ is a nilalgebra [1, 1_c, p. 51].

We shall use the following characterization of absolutely nilalgebras:

Lemma 2 ([1, 3_c, p. 52]). *The algebra A is absolutely nil if for every finite set $g_1, \ldots, g_n$ of elements of A, there exists an integer m such that for every partition $m = \mu_1 + \cdots + \mu_n$, $\mu_i \geq 0$, $\phi_{\mu_1 \ldots \mu_n}(g_1, \ldots, g_n) = \sum g_{i_1} \cdots g_{i_m} = 0$, where $\sum$ ranges over all the products which contain g_j, μ_j times for every j.*

The smallest such integer m is called the degree of absolute nillity of $g_1, \ldots, g_n$. It is obvious that $\phi_{\mu_1 \ldots \mu_n}(g_1, \ldots, g_n)$ is a homogeneous polynomial of degree m in the subalgebra generated by $g_1, \ldots, g_n$. $\phi_{\mu_1 \ldots \mu_n}(g_1, \ldots, g_n)$ is called a $\phi_{\mu_1, \mu_n}(g_1, \ldots, g_n)$ homogeneous polynomial. When there is no ambiguity, we speak about the ϕ_{μ_1, μ_n} homogeneous polynomials (parts, components) where $\mu_1, \ldots, \mu_n$ range over all the partitions of m.

It is well-known that every f.g. nilalgebra over a nondenumerable field and every locally nilpotent algebra are absolutely nil [1]. It is observed [1, p. 56] and is proved below (see Remark 2) that Golod algebras are examples of non-locally nilpotent absolutely nilalgebras.

2. Residually finite case.

Theorem 1. *Let $A = F^{(1)}/I$ be a nilalgebra with an absolutely nil ideal J/I such that J is a homogeneous ideal of $F^{(1)}$. Then A is a homomorphic image of a residually finite nilalgebra $B = F^{(1)}/T$ such that T is a homogeneous ideal.*

Proof. Let $g \in F^{(1)}$ and n be an integer such that $g^n \in J$. Then $g^n = \sum_i \mathcal{M}_i$, where $\mathcal{M}_j$ are homogeneous polynomials of J. Since J/I is an absolutely nilalgebra, there exists an integer $m = m(\mathcal{M}_{i_1}, \dots, \mathcal{M}_{i_k})$ such that all the homogeneous polynomials in the $\mathcal{M}_j$, $\phi_{\mu_1,\mu_k} = \sum \mathcal{M}_{j_1} \cdots \mathcal{M}_{j_m} \in I$. But every element $\mathcal{M}_j$ is homogeneous in $F^{(1)}$, so all the polynomials ϕ_{μ_1,μ_k} are homogeneous in $F^{(1)}$. From the fact that $(g^n)^m = \sum_{\mu_1+\cdots+\mu_k=m} \phi_{\mu_1,\mu_k}$ we see that $(g^n)^m$ is a sum of homogeneous elements of I. Let T be the ideal of $F^{(1)}$ generated by all the homogeneous polynomials ϕ_{μ_1,μ_k}, so constructed. It is obvious that $T \subset I$ is a homogeneous ideal and that $F^{(1)}/T$ is a residually finite nilalgebra.

In view of this theorem we ask the following natural question:

Question 1. Let A be an algebra as in the previous theorem. Is A absolutely nil?

Although this question seems to be difficult, one can observe that if J/I is an ideal of A of finite codimension then A is absolutely nil. This gives the following characterization of f.g. non-absolutely nilalgebras. Examples of this sort are the nilalgebras generated by 3 elements constructed recently by Smoktunowicz [**18**].

Corollary 1. *Let A be a f.g. non-absolutely nilalgebra. Then for every $n \geq 1$, A^n is a f.g. nilalgebra which is not absolutely nil.*

Theorem 2. *Let $A = F^{(1)}/I$ be a nilalgebra over a denumerable field such that I is a homogeneous ideal. Then A is a homomorphic image of a residually finite nilalgebra $B = F^{(1)}/J$ which satisfies Lemma 1.*

Proof. We will construct by induction a family of homogeneous polynomials $f_1, f_2, \dots$ which generate the ideal J.

We suppose that the base field K is denumerable. In this case $F^{(1)}$ is denumerable. Let us enumerate its elements as $\{y_1, y_2, \dots\}$. Choose an integer n greater than or equal to the index of nilpotency of $(y_1 + I)$. Then y_1^n is in I and since I is homogeneous, each of its homogeneous components $f_1, \dots, f_t$ (with $\deg f_j < \deg f_{j+1}$) is in I. Given any number k, there is no more than one f_i with degree k. So we have the set $\{f_1, \dots, f_t\}$ satisfying Lemma 1. In particular , there exit homogeneous polynomials $f_1, \dots, f_t$ with increasing degrees in I satisfying Lemma 1 such that y_1^n is in the ideal generated by $\{f_1, \dots, f_t\} \subseteq I$.

Suppose by induction, we have a Golod set $\{f_1, \dots, f_s\} \subseteq I$ such that $\deg f_i < \deg f_{i+1}$ and for each $i = 1, \dots, k$ there is an integer n_i with $y_i^{n_i}$ in the ideal generated by $\{f_1, \dots, f_s\}$. For y_{k+1} choose an integer m greater than both the index of nilpotency of $(y_{k+1} + I)$ and $\deg f_s$. Since A is nil and since I is a homogeneous ideal, we can write $(y_{k+1})^m$ in terms of its homogeneous components all of which are in I, and all of which have degree larger than $\deg f_s$. Label these components $f_{s+1}, \dots, f_r$. Then the set $\{f_1, \dots, f_s, f_{s+1}, \dots, f_r\} \subseteq I$ of homogeneous polynomials satisfies Lemma 1 such that for each $i = 1, \dots, k+1$, there is an integer n_i with $y_i^{n_i}$ in the ideal generated by $\{f_1, \dots, f_r\}$. Now, by the induction we have an infinite set of homogeneous polynomials $f_1, f_2, \dots$ in I satisfying Lemma 1, and which generates the ideal J, such that $F^{(1)}/J$ is a nilalgebra.

Theorem 3. *Let $A = F^{(1)}/I$ be an absolutely nilalgebra. Then A is a homomorphic image of a Golod algebra $B = F^{(1)}/J$.*

Proof. The proof is by induction on the degrees of general polynomials. Let $g_1 = c_1X_1 + \cdots + c_dX_d$ be a general polynomial of degree 1 in $F^{(1)}$ and choose an integer l greater than or equal to the degree of absolute nillity of $X_1 + I, \dots, X_d + I$. Since A is absolutely nil, by Lemma 2, for every partition $l = \mu_1 + \cdots + \mu_d$, $\mu_i \geq 0$, the $\phi_{\mu_1,\mu_d}(X_1, \dots, X_d)$ polynomials are in I. These polynomials are just the coefficients (homogeneous polynomials in $X_1, \dots, X_d$) of g_1^l when seen as a polynomial in the commuting unknowns $c_1, \dots, c_d$. Let us denote these $\phi_{\mu_1,\mu_d}(X_1, \dots, X_d)$ polynomials as $f_1, \dots, f_{l_1}$. Now, since the number r_i of polynomials of each degree i (in this case $i = l$) in $\{f_1, \dots, f_{l_1}\}$ does not exceed $(l + d - 1)^{d-1}$, for l big enough, $r_i \leq (l + d - 1)^{d-1} \leq \epsilon^2(d - 2\epsilon)^{i-2}$. Thus, the set $\{f_1, \dots, f_{l_1}\}$ satisfies Lemma 1.

Suppose that we have constructed in I a system of homogeneous polynomials $f_1, \dots, f_{l_k}$ satisfying Lemma 1 and that for every polynomial $y \in F^{(1)}$ of a degree not exceeding k there exists an integer $l' = l'(y)$ such that the homogeneous parts of $y^{l'}$ are in the ideal generated by $f_1, \dots, f_{l_k}$. Let

$$g_{k+1} = c_1^{(1)}X_1 + \cdots + c_d^{(1)}X_d + c_1^{(2)}X_1^2 + c_2^{(2)}X_1X_2 + \cdots + c_{d^2}^{(2)}X_d^2 + \cdots + c_{d^{k+1}}^{(k+1)}X_d^{k+1}$$

be a general polynomial of $F^{(1)}$ of degree $k+1$. Let n be an integer greater than $\max(\deg f_1, \dots, \deg f_{l_k}, m(X_1, \dots, X_1X_d, \dots, X_d^{k+1}))$, where, $m(X_1, \dots, X_d^{k+1})$ is the degree of absolute nillity of $X_1, \dots, X_1X_d, \dots, X_d^{k+1}$. By Lemma 2, for every partition $n = \mu_1 + \cdots + \mu_q$, $\mu_i \geq 0$, $q = d + \cdots + d^{k+1}$ the $\phi_{\mu_1,\mu_q}(X_1, \dots, X_d^{k+1})$ polynomials are in I. As in the case of g_1, by the choice of the integer n, the coefficients of g_{k+1}^n, seen as a polynomial in the commuting unknowns $c_1^{(1)}, \dots, c_{d^{k+1}}^{(k+1)}$, are the $\phi_{\mu_1,\mu_q}(X_1, \dots, X_d^{k+1}) \in I$. Let us denote them by $f_{l_k+1}, \dots, f_{l_{k+1}}$ and construct a new family of homogeneous polynomials $f_1, \dots, f_{l_k}, f_{l_k+1}, \dots, f_{l_{k+1}}$ satisfying Lemma 1. Indeed, the number r_i of polynomials of degree $i > \max(\deg f_1, \dots, \deg f_{l_k})$ does not exceed $(n + q - 1)^{q-1}$. For n big enough, we have $r_i \leq (n + q - 1)^{q-1} \leq \epsilon^2(d - 2\epsilon)^{i-2}$. For $i \leq \max(\deg f_1, \dots, \deg f_{l_k})$ this property is satisfied in the system $f_1, \ldots, f_{l_k}$. So we have constructed a family of polynomials $f_1, \ldots, f_{l_{k+1}}$ satisfying Lemma 1 and for every polynomial $z \in F^{(1)}$ of a degree not exceeding $k+1$ there exists an integer $n' = n'(z)$ such that the homogeneous parts of $z^{n'}$ are in the ideal generated by $f_1, \ldots, f_{n_{k+1}}$. The union of all these families so constructed gives an infinite system of homogeneous polynomials $f_1, f_2, \ldots$ which generate the ideal J. We have proved the theorem.

Remarks.

1. If A is such that specific elements generate a nilpotent (soluble, finite dimensional,...) subalgebra, then one can construct B with the same properties as A.
2. From the proof of Theorem 3, we see that the Golod algebras are absolutely nil. Therefore, Golod algebras have nilalgebras of matrices. This solves P.M. Cohn's question [**4**, p. 387 and Exercise 6^o, p. 395].

Having in mind that a f.g. nilalgebra over a nondenumerable field is absolutely nil [**1**], we obtain:

Corollary 2. *Every f.g. nilalgebra over a nondenumerable field is a homomorphic image of a Golod algebra.*

Let A be a Golod algebra generated by $\overline{X}_1, \ldots, \overline{X}_d$ $(d \geq 2)$. The group generated by $1 + \overline{X}_1, \ldots, 1 + \overline{X}_d$ is called the Golod group of A and the Lie algebra generated by $\overline{X}_1, \ldots, \overline{X}_d$ is the Golod-Lie algebra.

Corollary 3. *For any integer $d \geq 2$, every d-generator group arising from an absolutely nilalgebra is a homomorphic image of a d-generator Golod group. In particular, so is every finite p-group, for every prime integer p.*

In [**10**, Problem 9], Kaplansky asked whether the augmentation ideal $\omega K[G]$ of a f.g. infinite p-group G could be a nilideal. A particular case is Passman's question on the use of Golod groups to solve this problem [**13**, p. 121 and Problem 18, p. 133], [**14**, p. 415]. The following result confirms Passman's observation and reduces Kaplansky's problem to the study of the quotients of Golod groups:

Corollary 4. *Let K be a nondenumerable field of characteristic $p > 0$. Then, there exists a f.g. infinite p-group $\overline{G}$ such that the augmentation ideal $\omega K[\overline{G}]$ is nil if and only if there exists a just-infinite homomorphic image G of a Golod p-group such that $\omega K[G]$ is nil.*

Proof. Let $\overline{G}$ be as in the corollary. Since it is f.g and infinite., it has a just-infinite homomorphic image G. Hence, the augmentation ideal $\omega K[G]$ is a quotient of $\omega\ K[\overline{G}]$ and so it is a nilalgebra over a nondenumerable field K. By Corollary 2, G and $\overline{G}$ are quotients of a Golod group. The converse is obvious.

On the other hand we point out that since non-absolutely nilalgebras cannot be quotients of Golod algebras, their associated groups have non-nil augmentation ideals. The only examples of this type are the nilalgebras generated by 3 elements constructed by Smoktunowicz [**18**]. The following result is analogous to the results obtained in the case of the 2-generated Grigorchuk groups [**5**], the 3-generated Gupta-Sidki groups [**17**] and the free Burnside groups [**9**]:

Corollary 5. *Let K be a nondenumerable field of characteristic $p > 0$. Let G be a f.g. p-group associated to a non-absolutely nilalgebra. Then the augmentation ideal $\omega K[G]$ is not nil. Moreover $\omega K[G]$ has a just-infinite primitive homomorphic image.*

Question 2. Could the group algebra in the preceeding Corollary contain a free associative algebra with two non-commuting indeterminates?

Corollary 6. *For any integer $d \geq 2$, every d-generator Lie algebra arising from an absolutely nilalgebra is a homomorphic image of a d-generator Golod-Lie algebra.*

3. Non-residually finite case.

We turn now to non-residually finite quotients of nilalgebras and their associated groups. We point out that a f.g. just-infinite nilalgebra or a f.g. just-infinite Jacobson radical ring is residually finite [**9**] and that some infinite dimensional quotients of Golod algebras are also Golod algebras (the same result holds for Golod groups and Golod-Lie algebras) [**8**, **19**]. A subset E of a ring A is T-nilpotent if for every sequence $g_1, g_2, \dots$ of elements of E, there exists an integer k with $g_1 g_2 \cdots g_k = 0$. It is obvious that T-nilpotency implies local nilpotency. In our investigations, a key role is played by the following generalization of Nakayama's lemma:

Lemma 3 ([**20**, §43.5, p. 386])**.** *Let A be an algebra. Then, $AM \neq M$ for every left A-module M, if and only if A is T-nilpotent.*

The existence of f.g. non-residually finite, infinite dimensional nilalgebras over every field was first proved in [**16**]. A simple observation yields a stronger result. Indeed, let $d \geq 2$ be an integer and suppose that for any d-generator nilalgebra A, any left A-module M satisfies $\cap A^i M = \langle 0 \rangle$. So, $AM \neq M$ and by Lemma 3, A is T-nilpotent. Thus every d-generator nilalgebra is nilpotent. This contradicts the Golod construction [**6**, **7**] and proves:

Proposition. *For every integer $d \geq 2$ and over any field, there exists a non-residually finite, non-nilpotent d-generator nilalgebra.*

Theorem 4. *Over any field, any f.g. non-nilpotent nilalgebra with involution is a homomorphic image of a f.g. non-residually finite nilalgebra.*

Proof. Let A be a f.g. non-nilpotent nilalgebra with involution. Since A is not locally finite, by Lemma 3 there exists a nondegenerate left A-module M such that $AM = M$. It is well-known that every left A-module can be considered as a right module over the opposite algebra A^o of A. But the fact that A has an involution yields $A \cong A^o$ and turns M to a nondegenerate (A, A)-bimodule such that $AM = MA = M$. Let m be a nondegenerate element of M and consider the submodule $N = \langle m \rangle$. Since A has an involution and N is nondegenerate, we have $AN = NA = N$. Denote by $\overline{A}$ the trivial extension of A by N,

$$\overline{A} = \{(a, n), a \in A, n \in N\}.$$

With the usual addition and the following multiplication:

$$(a, n)(a', n') = (aa', an' + na'), \quad a, a' \in A, \quad n, n' \in N,$$

$\overline{A}$ is a non-nilpotent nilalgebra such that $\overline{A}/I = A$, where I is the ideal $\langle (0, n), n \in N \rangle$. From the fact that $AN = NA = N$, it follows that I is in $\overline{A}^k$ for every integer k; thus $\overline{A}$ is not residually finite. Since A is f.g. and $N = \langle m \rangle$, $\overline{A}$ is f.g. Therefore, we proved the theorem.

Corollary 7. *Over every field, there exists a Golod algebra with non-residually finite quotients.*

Proof. Apply Theorems 1 and 2 or 3 to the non-residually finite nilalgebras of Theorem 4.

The following corollary solves in the negative Šunkov's problem [**11**, Problem 12.102]:

Corollary 8. *For every prime p (respectively $p = 0$), there exists Golod p-groups (respectively torsion free groups) with non-residually finite quotients.*

Proof. Let $\overline{A}$ be a non-residually finite homomorphic image of a Golod algebra B and denote by $Y_1, \dots, Y_d$ its generators which are images of fixed generators of B. Since $\overline{A}$ is f.g., and $N = \langle m \rangle$ is a nondegenerate module satisfying $AN = NA = N$ (see the proof of Theorem 4), $1 + (0, m) \in \overline{G}$ where, $\overline{G} = \langle 1 + Y_1, \dots, 1 + Y_d \rangle$. Thus the Golod group of B has $\overline{G}$ as a non-residually finite quotient.

We conclude with the following question which is related to Bergman's [**3**, Question 63]:

Question 3. Anan'in and Puczyłowski constructed over fields of characteristic zero, f.g. non-residually finite, non-nilpotent nilalgebras with non-radical tensor square [**2**, **15**]. Could we construct such examples in characteristic $p > 0$?

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