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CHARACTERIZATION OF THE SIMPLE $L^1(G)$-MODULES FOR EXPONENTIAL LIE GROUPS

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Let $\mathcal{G} = \exp \mathfrak{g}$ be a connected, simply connected, solvable exponential Lie group. Let $l \in \mathfrak{g}^*$ and let $\mathfrak{p}$ be an appropriate Pukanszky polarization for l in $\mathfrak{g}$. For every $\bar{p} = (p_1, \dots, p_m) \in [1, \infty]^m$ we define a representation $\pi_{l,\mathfrak{p},\overline{p}}$ by induction on an $L^{\overline{p}}$-space, where the norm $\|\cdot\|_{\overline{p}}$ of this space is in fact obtained by successive L^{p_j}-norms, with distinct p_j's in different directions. These representations are topologically irreducible and their restrictions to the subspaces generated by the vectors of the form $\pi_{l,\mathfrak{p},\overline{p}}(f)\xi$ with $f \in L^1(\mathcal{G})$, $\pi_{l,\mathfrak{p},\overline{p}}(f)$ of finite rank and $\xi \in \mathfrak{H}_{l,\mathfrak{p},\overline{p}}$ are algebraically irreducible. All the simple $L^1(\mathcal{G})$-modules are of that form, up to equivalence. We show that these representations may in fact be characterized (up to equivalence) by the $\mathcal{G}$-orbits of couples (l, ν), where $l \in \mathfrak{g}^*$ and ν is a real linear form on $\mathfrak{g}(l)/\mathfrak{g}(l) \cap \mathfrak{n}$ satisfying a certain growth condition and where $\mathfrak{g}(l)$ is the stabilizer of l in $\mathfrak{g}$.

1. Introduction.

The aim of the present paper is to give an explicit description of the algebraically irreducible representations of $L^1(\mathcal{G})$, where $\mathcal{G}$ is a connected, simply connected, exponential, solvable Lie group. These representations have first been studied by D. Poguntke in 1983 ([**Po2**]). The method of Poguntke which has been adapted and used in ([**LuMo2**]), is an important ingredient in the present paper, as we shall see later with more details. But first we have to recall the following definitions: We say that $(T, \mathfrak{V})$ is a representation of $L^1(\mathcal{G})$, if $\mathfrak{V}$ is a vector space, $\mathcal{L}(\mathfrak{V})$ the space of all linear operators on $\mathfrak{V}$ and

$$T : L^1(\mathcal{G}) \to \mathcal{L}(\mathfrak{V})$$

an algebra homomorphism. Moreover $(T, \mathfrak{V})$ is said to be algebraically irreducible if $\mathfrak{V}$ has no nontrivial invariant subspaces for the action of $L^1(\mathcal{G})$ under T. In that case we also say that $\mathfrak{V}$ is a simple $L^1(\mathcal{G})$-module. If $\mathfrak{V}$ is a topological vector space, we require moreover the action of $L^1(\mathcal{G})$ on $\mathfrak{V}$ to be strongly continuous. In that case we say that $(T, \mathfrak{V})$ is a topologically irreducible representation of $L^1(\mathcal{G})$, if $\mathfrak{V}$ has no nontrivial closed invariant subspaces. As in the general theory, we can always assume for any representation $(T, \mathfrak{V})$ of $L^1(\mathcal{G})$ that $\mathfrak{V}$ is a Banach space and that the representation $(T, \mathfrak{V})$ is bounded (see [**BoDu**]).

Assume that $(T, \mathfrak{V})$ is a topologically irreducible representation on a Banach space and that there exists $f \in L^1(\mathcal{G})$ such that $T(f)$ is a nonzero operator of finite rank. Consider

$$\mathfrak{V}^0 = \text{span } \{T(f)\xi \mid f \in L^1(\mathcal{G}), T(f) \text{ of finite rank}, \ \xi \in \mathfrak{V}\}.$$

Then $\mathfrak{V}^0 \neq \{0\}$ and the restriction of T to $\mathfrak{V}^0$, $(T|_{\mathfrak{V}^0}, \mathfrak{V}^0)$, is a simple $L^1(\mathcal{G})$-module ([**Wa**]). We shall see that in our situation all the simple $L^1(\mathcal{G})$-modules are obtained in that way (up to equivalence) and we shall give a precise description of the representations $(T, \mathfrak{V})$ to consider.

The previous definitions and results may of course be given for an arbitrary Banach algebra $\mathcal{A}$ instead of $L^1(\mathcal{G})$. Moreover the representations of $L^1(\mathcal{G})$ may be considered as the integrated forms of bounded representations of the group $\mathcal{G}$. In fact, recall that $(T, \mathfrak{V})$ is said to be a representation of the group $\mathcal{G}$ if T is a group homomorphism of $\mathcal{G}$ into the general linear group of $\mathfrak{V}$. This representation is said to be bounded if $\sup_{x\in\mathcal{G}} \|T(x)\| < \infty$, where $\|T(x)\|$ is the operator norm of $T(x)$. For such a representation of $\mathcal{G}$, we get a representation of $L^1(\mathcal{G})$ by $T(f) = \int_{\mathcal{G}} f(x)T(x)dx, \forall f \in L^1(\mathcal{G})$.

A representation π of $\mathcal{G}$, resp. $L^1(\mathcal{G})$ on a Hilbert space $\mathfrak{H}_\pi$ is said to be unitary, if $\pi(x^{-1}) = \pi(x)^*$, resp. $\pi(f^*) = \pi(f)^*$ for all $x \in \mathcal{G}$, resp. $f \in L^1(\mathcal{G})$. Recall that the unitary topologically irreducible representations π of a solvable exponential Lie group $\mathcal{G} = \exp \mathfrak{g}$ may be described as induced representations. There exist $l \in \mathfrak{g}^*$ and a Pukanszky polarization $\mathfrak{p} \subset \mathfrak{g}$ at l such that $\pi = \text{ ind }_{\mathcal{P}}^{\mathcal{G}} \chi_l$ (up to unitary equivalence), where $\mathcal{P} = \exp \mathfrak{p}$ and $\chi_l(\exp X) = e^{-i\langle l,X\rangle}$ for all $X \in \mathfrak{p}$ ([**LeLu**]). The set of equivalence classes of topologically irreducible unitary representations of $\mathcal{G}$ is noted by $\hat{\mathcal{G}}$.

If $\mathcal{G}$ is a connected, simply connected nilpotent Lie group, then all the simple $L^1(\mathcal{G})$-modules are equivalent to a module of the form $(\pi|_{\mathfrak{H}_\pi^0}, \mathfrak{H}_\pi^0)$, where $\pi \in \hat{\mathcal{G}}$ and

$$\mathfrak{H}_\pi^0 = \text{ span } \{\pi(f)\xi \mid f \in L^1(\mathcal{G}), \pi(f) \text{ of finite rank}, \xi \in \mathfrak{H}_\pi\}.$$

The same remains true for $L^1(\mathcal{G}, \omega)$, where $\mathcal{G}$ is a connected, simply connected, nilpotent Lie group and ω is a polynomial weight on $\mathcal{G}$ ([**MiMo**]). In this paper these results are generalized in the following way: If $\mathcal{G}$ is a connected, simply connected, solvable exponential Lie group, we define representations $\pi_{l,\mathfrak{p},\bar{p}}$ by induction on $L^{\bar{p}}$-spaces, where $\bar{p} = (p_1, \ldots, p_m) \in [1, \infty]^m$ is a multi-index. The norm $\| \cdot \|_{\bar{p}}$ of such an $L^{\bar{p}}$-space is obtained by successive L^{p_j}-norms with distinct p_j's in different directions. To do this, we have to introduce a precise decomposition of the Lie algebra $\mathfrak{g}$ of the group $\mathcal{G}$. These representations are topologically irreducible and admit nontrivial operators of finite rank. Hence, if we write $\mathfrak{H}_{l,\mathfrak{p},\bar{p}}$ for the space of such a representation and

$$\mathfrak{H}^0_{l,\mathfrak{p},\bar{p}} = \text{ span } \{\pi_{l,\mathfrak{p},\bar{p}}(f)\xi \mid f \in L^1(\mathcal{G}), \pi_{l,\mathfrak{p},\bar{p}}(f) \text{ of finite rank}, \xi \in \mathfrak{H}_{l,\mathfrak{p},\bar{p}}\},$$

then $\left(\pi_{l,\mathfrak{p},\bar{p}}|_{\mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}}, \mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}\right) = \left(\pi^0_{l,\mathfrak{p},\bar{p}}, \mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}\right)$ is a simple $L^1(\mathcal{G})$-module. We show that all the simple $L^1(\mathcal{G})$-modules $(T, \mathfrak{V})$ are of this type (up to equivalence). To do this we rely on the work of Poguntke ([**Po1**], [**Po2**]). In his paper ([**Po2**]) Poguntke gives a first description of simple $L^1(\mathcal{G})$-modules. Let's notice first that a representation $(T, \mathfrak{V})$ of $L^1(\mathcal{G})$ defines unique representations of $\mathcal{G}$, of $\mathcal{N}$ (by restriction) and of $L^1(\mathcal{N})$, where $\mathcal{N} = \exp \mathfrak{n}$ and $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. We shall write $\ker_{L^1(\mathcal{N})} T$ for the corresponding kernel in $L^1(\mathcal{N})$. This kernel is of the form $\ker(\mathcal{G} \cdot \tau)$, where $\tau \in \hat{\mathcal{N}}$ is the representation induced from a character χ_q defined by a linear form $q \in \mathfrak{n}^*$. Let $l \in \mathfrak{g}^*$ such that $l|_{\mathfrak{n}} = q$. The method of Poguntke ([**Po2**]) which has been adapted and used for the description of topologically irreducible representations in ([**LuMo2**]) consists in constructing an algebra of the type $L^1(\mathbb{R}^n, \omega)$, where ω is an exponential weight in general, uniquely determined by the given simple module $(T, \mathfrak{V})$ and where $\mathbb{R}^n \equiv \mathcal{G}(l)/\mathcal{G}(l) \cap \mathcal{N}$, with $\mathcal{G}(l) = \exp \mathfrak{g}(l)$ and $\mathfrak{g}(l)$ is the stabilizer of l in $\mathfrak{g}$. Then one shows that the simple $L^1(\mathcal{G})$-module $(T, \mathfrak{V})$ with given $\ker_{L^1(\mathcal{N})} T$ is completely characterized by a continuous character on $L^1(\mathbb{R}^n, \omega)$. Conversely every such character on $L^1(\mathbb{R}^n, \omega)$ leads to a unique simple $L^1(\mathcal{G})$-module (up to equivalence) with given $\ker_{L^1(\mathcal{N})} T$. In order to show that every simple $L^1(\mathcal{G})$-module is equivalent to a module of the form $\left(\pi_{l,\mathfrak{p},\bar{p}}|_{\mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}}, \mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}\right)$, it is then enough to show that every (continuous) character on $L^1(\mathbb{R}^n, \omega)$ is associated to such a representation. To do this we have to give an estimation of the weight ω using a method developed by Poguntke in ([**Po2**]). The equivalence classes of simple $L^1(\mathcal{G})$-modules are then completely characterized by the $\mathcal{G}$-orbits of the couple (l, ν), where $l \in \mathfrak{g}^*$ and ν is a real linear form on $\mathfrak{g}(l)/\mathfrak{g}(l) \cap \mathfrak{n}$ satisfying a certain growth condition.

2. Construction of special irreducible representations.

2.1. . For the rest of this paper $\mathcal{G} = \exp \mathfrak{g}$ will be a connected, simply connected, solvable exponential Lie group with Lie algebra $\mathfrak{g}$. The nil-radical of $\mathfrak{g}$ will be denoted by $\mathfrak{n}$ and $\mathcal{N} = \exp\mathfrak{n}$ will be the corresponding subgroup of $\mathcal{G}$. Take $l \in \mathfrak{g}^*$ and write $q = l|_{\mathfrak{n}} \in \mathfrak{n}^*$. We define the following stabilizers:

$$\mathfrak{g}(l) = \{X \in \mathfrak{g} \mid \langle l, [X, \mathfrak{g}]\rangle \equiv 0\},$$
$$\mathfrak{g}(q) = \{X \in \mathfrak{g} \mid \langle q, [X, \mathfrak{n}]\rangle = \langle l, [X, \mathfrak{n}]\rangle \equiv 0\},$$
$$\mathfrak{n}(q) = \{X \in \mathfrak{n} \mid \langle q, [X, \mathfrak{n}]\rangle = \langle l, [X, \mathfrak{n}]\rangle \equiv 0\} = \mathfrak{g}(q) \cap \mathfrak{n}.$$

Then we decompose the Lie algebra as follows:

$$\begin{aligned}\mathfrak{g}(l)+\mathfrak{n} &= \mathfrak{u}\oplus\mathfrak{n} && \text{with } \mathfrak{u}\subset\mathfrak{g}(l)\subset\mathfrak{g}(q),\\ \mathfrak{g}(q)+\mathfrak{n} &= \mathfrak{w}\oplus(\mathfrak{g}(l)+\mathfrak{n})=\mathfrak{w}\oplus\mathfrak{u}\oplus\mathfrak{n} && \text{with } \mathfrak{w}\subset\mathfrak{g}(q),\\ \mathfrak{g} &= \mathfrak{v}\oplus(\mathfrak{g}(q)+\mathfrak{n})=\mathfrak{v}\oplus\mathfrak{w}\oplus\mathfrak{u}\oplus\mathfrak{n}.\end{aligned}$$

2.2. . Now we choose $\mathfrak{Y}\subset\mathfrak{w}\subset\mathfrak{g}(q)$ a maximal l-isotropic subspace of $\mathfrak{w}$, i.e., a maximal subspace of $\mathfrak{w}$ such that $\langle l,[\mathfrak{Y},\mathfrak{Y}]\rangle\equiv 0$. Then there exist a subspace $\mathfrak{X}$ in $\mathfrak{w}$ and bases $\{X_1,\dots,X_c\}$ of $\mathfrak{X}$, resp. $\{Y_1,\dots,Y_c\}$ of $\mathfrak{Y}$ such that $\mathfrak{w}=\mathfrak{X}\oplus\mathfrak{Y}$ with

$$\langle l,[X_i,X_j]\rangle=0,\ \langle l,[Y_i,Y_j]\rangle=0,\ \langle l,[X_i,Y_j]\rangle=\delta_{ij},$$

i.e., $\mathfrak{X}$ is a dual space of $\mathfrak{Y}$ with respect to l. This is possible because $\{Z\in\mathfrak{w}\mid\langle l,[Z,\mathfrak{w}]\rangle\equiv 0\}=\{0\}$. As a matter of fact, $\mathfrak{w}\oplus\mathfrak{n}(q)$ modulo $\ker(q|_{\mathfrak{n}(q)})$ is a Heisenberg algebra. We write $\mathcal{U}=\exp\mathfrak{u}$, $\mathcal{V}=\exp\mathfrak{v}$, $\mathcal{W}=\exp\mathfrak{w}$, $\mathcal{X}=\exp\mathfrak{X}$, $\mathcal{Y}=\exp\mathfrak{Y}$.

2.3. Polarizations. First let us choose $\mathfrak{p}^0$ a $\mathfrak{g}(q)$-invariant polarization of q in $\mathfrak{n}$ (for example a Vergne polarization). Then $\mathfrak{p}=\mathfrak{Y}\oplus\mathfrak{p}^0\oplus\mathfrak{u}$ is a Pukanszky polarization of l in $\mathfrak{g}$. Moreover $\mathfrak{p}^0=\mathfrak{p}\cap\mathfrak{n}$. For the rest of this paper we shall stick to these polarizations. We write $\mathcal{P}^0=\exp\mathfrak{p}^0$, $\mathcal{P}=\exp\mathfrak{p}$.

2.4. Jordan-Hölder decomposition. Let

$$\mathfrak{n}=\mathfrak{n}_0\supset\mathfrak{n}_1\supset\cdots\supset\mathfrak{n}_k\supset\mathfrak{n}_{k+1}=\{0\}$$

be a Jordan-Hölder sequence for the action of $\mathfrak{g}(q)+\mathfrak{n}$ on $\mathfrak{n}$. Let

$$\begin{aligned}Y&=\{i\mid\mathfrak{p}^0+\mathfrak{n}_i\neq\mathfrak{p}^0+\mathfrak{n}_{i+1}, i=0,\dots,k\}\\ &=\{i_j\mid 1\le j\le m, 0\le i_1\le\cdots\le i_m\le k\}.\end{aligned}$$

We write $\mathfrak{p}_j=\mathfrak{p}^0+\mathfrak{n}_{i_j}$, for $j=1,\dots m$, and $\mathfrak{p}_{m+1}=\mathfrak{p}^0$. Obviously $\mathfrak{p}_1=\mathfrak{n}$. For each $j\in\{1,\dots,m\}$ we choose a subspace $\mathfrak{v}_j\subset\mathfrak{n}_{i_j}\subset\mathfrak{p}_j$ such that $\mathfrak{v}_j\oplus\mathfrak{p}_{j+1}=\mathfrak{p}_j$. Then $\sum_{j=1}^{m}{}^{\oplus}\mathfrak{v}_j\oplus\mathfrak{p}^0=\mathfrak{n}$ and

$$\Phi:\sum_{j=1}^{m}{}^{\oplus}\mathfrak{v}_j\longrightarrow\mathcal{N}/\mathcal{P}^0$$

$$V_1+\cdots+V_m\equiv(V_1,\dots,V_m)\longmapsto\exp(V_1)\dots\exp(V_m)\cdot\mathcal{P}^0$$

is a diffeomorphism.

2.5. Special representations. Let's write

$$\widetilde{\mathfrak{n}}=\sum_{j=1}^{m}{}^{\oplus}\mathfrak{v}_j,\ \mathcal{V}_j=\exp\mathfrak{v}_j \text{ and } \widetilde{\mathcal{N}}=\prod_{j=1}^{m}\mathcal{V}_j=\prod_{j=1}^{m}\exp\mathfrak{v}_j.$$

Consider the following decomposition of $\mathcal{G}$: $\mathcal{G}=\mathcal{V}\cdot\mathcal{X}\cdot\widetilde{\mathcal{N}}\cdot\mathcal{P}$. Take $\bar{p}=(p_1,\dots,p_m)\in[1,\infty]^m$. The representation space $L^{\bar{p}}(\mathcal{G}/\mathcal{P},\chi_l)$ is then defined to be the completion, for the norm $\|\cdot\|_{\bar{p}}$ given below, of the space of all functions $\xi : \mathcal{V}\cdot\mathcal{X}\cdot\widetilde{\mathcal{N}}\cdot\mathcal{P}\to\mathbb{C}$ continuous with compact support mod $\mathcal{P}$, such that $\xi(x\cdot p)=\overline{\chi_l(p)}\xi(x),\forall x\in\mathcal{G},\forall p\in\mathcal{P}$, and

$$\|\xi\|_{\bar{p}}=\Bigg(\int_{\mathcal{V}}\int_{\mathcal{X}}\Bigg(\Bigg(\int_{\mathcal{V}_1}\Bigg(\dots\Bigg(\int_{\mathcal{V}_m}|\xi(vxv_1\dots v_m)|^{p_m}dv_m\Bigg)^{\frac{1}{p_m}}\dots\Bigg)^{p_1}dv_1\Bigg)^{\frac{1}{p_1}}\Bigg)^2dxdv\Bigg)^{\frac{1}{2}}<\infty,$$

the different measures being the Lebesgue measures on $\mathfrak{v},\mathfrak{X},\mathfrak{v}_1,\dots,\mathfrak{v}_m$. If $p_j=\infty$, then $\big(\int_{\mathcal{V}_j}|\dots|^{p_j}dv_j\big)^{\frac{1}{p_j}}$ is replaced by the corresponding sup-norm. Let $L^{\bar{p}}(\mathcal{G}/\mathcal{P},\chi_l)=\mathfrak{H}_{l,\mathfrak{p},\bar{p}}$ be the space we get by completion. On this space we want to define a representation by isometric operators given essentially by left translation. This representation will be of the form

$$(\pi_{l,\mathfrak{p},\bar{p}}(s)\xi)(y)=\Delta^{-\frac{1}{\bar{p}}}(s)\xi(s^{-1}y),\quad\forall s,y\in\mathcal{G},$$

where the modular function $\Delta^{-\frac{1}{\bar{p}}}$ has to be defined in order to get isometric operators on $\mathfrak{H}_{l,\mathfrak{p},\bar{p}}$. It is easy to check that

$$\Delta^{\frac{1}{\bar{p}}}(v\cdot x\cdot n\cdot p)=e^{\sum_{j=1}^m\frac{1}{p_j}\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log p)}=e^{\sum_{j=1}^m\frac{1}{p_j}\mathrm{tr}\lambda_j(\log p)},$$

if we use the notation $\lambda_j(\cdot)=\mathrm{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\cdot)$. For $\bar{p}=\overline{2}=(2,\dots,2)$ we have

$$\Delta^{\frac{1}{2}}(s)=e^{\frac{1}{2}\mathrm{tr}\ \mathrm{ad}_{\mathfrak{n}/\mathfrak{p}^0}(\log s)}=e^{\frac{1}{2}\mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}/\mathfrak{p}}(\log s)},$$

as $\mathfrak{n}/\mathfrak{p}^0=(\mathfrak{u}\oplus\mathfrak{Y}\oplus\mathfrak{n})/\mathfrak{p}$ and as $\mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}/(\mathfrak{u}\oplus\mathfrak{Y}\oplus\mathfrak{n})}=0$. The representation $\pi_l=\pi_{l,\mathfrak{p},\overline{2}}$ is the usual induced unitary representation $\mathrm{ind}_{\mathcal{P}}^{\mathcal{G}}(\chi_l,2)$. Notice that

$$\pi_{l,\mathfrak{p},\bar{p}}(s)=\Delta^{\frac{1}{2}-\frac{1}{\bar{p}}}(s)\pi_l(s)$$

on the dense subspace of all continuous functions of $L^{\bar{p}}(\mathcal{G}/\mathcal{P},\chi_l)$ with compact support in $\mathcal{G}/\mathcal{P}$, or, more generally, on the generalized Schwartz space $\mathcal{ES}(\mathcal{G}/\mathcal{P},\chi_l)$ (see (2.7.) for the precise definition of this space).

2.6. Remarks.

a) As $\mathcal{G}(l)\subset\mathcal{P}$ and as $\Delta^{\frac{1}{\bar{p}}}\equiv 1$ on $\mathcal{N}\cap\mathcal{G}(l)$, $\Delta^{\frac{1}{\bar{p}}}$ may be considered as a character on $\mathcal{G}(l)/\mathcal{G}(l)\cap\mathcal{N}\equiv\mathcal{G}/\mathcal{H}$ given by

$$\Delta^{\frac{1}{\bar{p}}}(\dot{s})=e^{\sum_{j=1}^m\frac{1}{p_j}\ \mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log s)}$$

for all $\dot{s}\in\mathcal{G}(l)/\mathcal{G}(l)\cap\mathcal{N}$.

b) There is a relation between the Haar measures on $\mathcal{G}$, $\mathcal{P}$ and the measure on $\mathcal{G}/\mathcal{P}\equiv\mathcal{V}\cdot\mathcal{X}\cdot\widetilde{\mathcal{N}}=\mathcal{V}\cdot\mathcal{X}\cdot\prod_{j=1}^m\mathcal{V}_j$: If the Lie algebra $\mathfrak{g}$ is decomposed by

$$\mathfrak{g}=\mathfrak{v}\oplus\mathfrak{X}\oplus\mathfrak{Y}\oplus\mathfrak{u}\oplus(\widetilde{\mathfrak{n}}\oplus\mathfrak{p}^0),$$

we get a Haar measure on $\mathcal{G}$ by

$$\int_{\mathcal{G}} f(g)dg = \int_{\mathfrak{v}} \int_{\mathfrak{X}} \int_{\mathfrak{Y}} \int_{\mathfrak{u}} \int_{\mathfrak{v}_1} \cdots \int_{\mathfrak{v}_m} \int_{\mathcal{P}^0} f\big(\exp V \cdot \exp X \cdot \exp Y \cdot \exp U \cdot \exp V_1 \ldots \exp V_m \cdot p^0\big) dp^0 dV_m \ldots dV_1 dU dY dX dV,$$

where we use the Haar measure on $\mathcal{P}^0$ and the Lebesgue measures on $\mathfrak{v}, \mathfrak{X}, \mathfrak{Y}, \mathfrak{u}, \mathfrak{v}_j$. The Haar measure on $\mathcal{P}$ is given by

$$\int_{\mathcal{P}} f(p)dp = \int_{\mathfrak{Y}} \int_{\mathfrak{u}} \int_{\mathcal{P}^0} f\big(\exp Y \cdot \exp U \cdot p^0\big) dp^0 dU dY.$$

We check that

$$\int_{\mathcal{G}} f(g)dg = \int_{\mathcal{G}/\mathcal{P}} \int_{\mathcal{P}} f(gp)\Delta^{-1}(gp) dp d\dot{g}.$$

2.7. The $\mathcal{ES}$-spaces. Let the polarizations be chosen as in (2.3.). Let $\mathfrak{B}_1 = \{A_1, \ldots, A_j\}$ be a coexponential basis for $\mathfrak{p}_0$ in $\mathfrak{n}$, which has for instance been chosen in the subspaces $\mathfrak{v}_j$. Let $\mathfrak{B}_2 = \{B_1, \ldots, B_k\}$ be a coexponential basis for $\mathfrak{n} + \mathfrak{p}$ in $\mathfrak{g}$. Then $\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2$ is a coexponential basis for $\mathfrak{p}$ in $\mathfrak{g}$. Given a function F on $\mathcal{G}/\mathcal{P} \times \mathcal{G}/\mathcal{P}$, we define a function $\widetilde{F}$ on $(\mathbb{R}^k \times \mathbb{R}^j) \times (\mathbb{R}^k \times \mathbb{R}^j)$ by

$$\begin{aligned} &\widetilde{F}\left(b_1, \ldots, b_k, a_1, \ldots a_j; b_1', \ldots, b_k', a_1', \ldots a_j'\right) \\ &= F\Big(\exp b_1 B_1 \ldots \exp b_k B_k \exp a_1 A_1 \ldots \exp a_j A_j; \\ &\qquad \exp b_1' B_1 \ldots \exp b_k' B_k \exp a_1' A_1 \ldots \exp a_j' A_j \Big). \end{aligned}$$

We proceed similarly for a function defined on $\mathcal{G}/\mathcal{P}$. This allows us to give the following definition:

Definition 2.7.1. a) The space $\mathcal{E}S(\mathcal{G}/\mathcal{P} \times \mathcal{G}/\mathcal{P}, \chi_l)$ is the space of all C^∞-functions $F : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ such that:

(1) $F(xs, x's') = \overline{\chi_l(s)}\chi_l(s')F(x, x'), \quad \forall x, x' \in \mathcal{G}, \forall s, s' \in \mathcal{P}$.

(2) $$\|F\|_{\partial,\alpha,\alpha',R,R'} = \sup_{a,a' \in \mathbb{R}^j, b,b' \in \mathbb{R}^k} \left(e^{\alpha|b|} e^{\alpha'|b'|} |R(a)R'(a')\partial_a \partial_b \partial_{a'} \partial_{b'} \widetilde{F}(b, a; b', a')| \right) < \infty$$

for all $\alpha, \alpha' \geq 0$, for all polynomials R and R', for all derivation operators ∂, if $|b|$ and $|b'|$ denote the euclidean norm on $\mathbb{R}^k$.

(3) The same conditions as in (2) are required for all partial Fourier transforms of $\widetilde{F}$ in b and b'.

b) The space $\mathcal{E}S(\mathcal{G}/\mathcal{P}, \chi_l)$ is defined similarly (see [**Lu**]).

Remark. The previous spaces are independent of the choice of the coexponential bases. They also contain real analytic functions which, therefore, may be extended to functions with complex variables ([**LeLu**]).

Let $\mathfrak{B}_3 = \{C_1, \ldots, C_i\}$ be a coexponential basis for $\mathfrak{n}$ in $\mathfrak{g}$. We may choose the elements of $\mathfrak{B}_3$ in a nilpotent subalgebra $\mathfrak{Q}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{Q} + \mathfrak{n}$. Let $\mathfrak{B}_4 = \{D_1, \ldots, D_g\}$ be a Jordan-Hölder basis for $\mathfrak{n}$. For a function f defined on $\mathcal{G}$, we define $\widetilde{f}$ on $\mathbb{R}^i \times \mathbb{R}^g$ as previously. We then define:

Definition 2.7.2. The space $\mathcal{ES}(\mathcal{G})$ is the space of all C^∞-functions $f : \mathcal{G} \to \mathbb{C}$ such that

$$\|f\|_{\partial,\alpha,R} = \sup_{c\in\mathbb{R}^i, d\in\mathbb{R}^g} \left(e^{\alpha|c|} \left| R(d)\partial_c\partial_d \widetilde{f}(c,d) \right| \right) < \infty$$

for every $\alpha \geq 0$, for every polynomial R, for all derivation operators ∂, if $|c|$ denotes the euclidean norm on $\mathbb{R}^i$.

Remarks.

a) The space $\mathcal{ES}(\mathcal{G})$ is independent of the choice of the bases. It is dense in $L^1(\mathcal{G})$ ([**Lu**]). Similarly for $\mathcal{ES}(\mathcal{G}/\mathcal{P}, \chi_l)$ and $L^{\bar{p}}(\mathcal{G}/\mathcal{P}, \chi_l)$.

b) The space $\mathcal{ES}(\mathcal{G}/\mathcal{P} \times \mathcal{G}/\mathcal{P}, \chi_l)$ is in the image of the map that sends every $f \in L^1(\mathcal{G})$ to the kernel function of the operator $\pi_l(f)$ ([**LeLu**], [**Lu**]). Similarly for $\pi_{l,\mathfrak{p},\bar{p}}(f)$ instead of π_l thanks to the following observation: For $f \in \mathcal{ES}(\mathcal{G}) \subset L^1(\mathcal{G})$, we have $\pi_{l,\mathfrak{p},\bar{p}}(f) = \pi_l(\Delta^{\frac{1}{2}-\frac{1}{\bar{p}}} \cdot f)$ and $\pi_l(f) = \pi_{l,\mathfrak{p},\bar{p}}(\Delta^{\frac{1}{\bar{p}}-\frac{1}{2}} \cdot f)$, where $\frac{1}{2} - \frac{1}{\bar{p}} = \left(\frac{1}{2} - \frac{1}{p_1}, \ldots, \frac{1}{2} - \frac{1}{p_m}\right)$.

c) Put $\mathfrak{H}^0_{l,\mathfrak{p},\bar{p}} = \operatorname{span}\{\pi_{l,\mathfrak{p},\bar{p}}(f)\xi \mid \xi \in \mathfrak{H}_{l,\mathfrak{p},\bar{p}}, f \in L^1(\mathcal{G})$ such that $\pi_{l,\mathfrak{p},\bar{p}}(f)$ of finite rank$\}$. Hence $\mathcal{ES}(\mathcal{G}/\mathcal{P}, \chi_l) \subset \mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}$, by b).

As in ([**Wa**]) we can prove the following theorem, using c):

Theorem 2.7.3. *The representation* $\left(\pi_{l,\mathfrak{p},\bar{p}}, \mathfrak{H}_{l,\mathfrak{p},\bar{p}}\right)$ *is topologically irreducible and the sub-representation* $\left(\pi_{l,\mathfrak{p},\bar{p}}|_{\mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}}, \mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}\right) = \left(\pi^0_{l,\mathfrak{p},\bar{p}}, \mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}\right)$ *is algebraically irreducible.*

3. Analysis of an arbitrary simple $L^1(\mathcal{G})$-module.

3.1. . In this chapter we shall use the methods of Poguntke ([**Po1**], [**Po2**]) which have been used and modified in ([**LuMo2**]) in order to study the topologically irreducible representations. As a matter of fact most of the analysis of ([**LuMo2**]) remains true in the situation of simple $L^1(\mathcal{G})$-modules. Therefore we shall give no proofs in this chapter and just recall the main results of ([**LuMo2**]) and ([**Po2**]).

Proposition 3.2. *Let* $(T, \mathfrak{U})$ *be an algebraically irreducible representation of* $L^1(\mathcal{G})$. *Let's write* $\ker_{L^1(\mathcal{N})} T$ *for the kernel of the corresponding representation of* $L^1(\mathcal{N})$. *Then there exist* $\tau \in \hat{\mathcal{N}}$ *and* $q \in \mathfrak{n}^*$, $\mathfrak{p}_0$ *a polarization* *of q in $\mathfrak{n}$ and $\mathcal{P}_0 = \exp \mathfrak{p}_0$, such that*

$$\tau = \mathrm{ind}_{\mathcal{P}^0}^{\mathcal{N}} \chi_q \text{ and } \ker_{L^1(\mathcal{N})} T = \ker(\mathcal{G} \cdot \tau) = \bigcap_{g \in \mathcal{G}} \ker({}^g\tau).$$

The kernel $\ker_{L^1(\mathcal{N})} T$ is completely determined by the $\mathcal{G}$-orbit $\mathcal{G} \cdot \tau$.

3.3. Corresponding unitary representations. The aim of this section is to introduce the largest subgroup $\mathcal{H}$ on which it is possible, in a certain sense, to work with a unitary representation. Let $l \in \mathfrak{g}^*$ be such that $l|_{\mathfrak{n}} = q$. Using the same decompositions as in (2.1.), we define $\mathfrak{h} = \mathfrak{v} \oplus \mathfrak{w} \oplus \mathfrak{n}$, $\mathcal{H} = \exp \mathfrak{h}$, $r = l|_{\mathfrak{h}}$. Then $\mathfrak{p}^1 = \mathfrak{Y} \oplus \mathfrak{p}^0$ is a Pukanszky polarization of r in $\mathfrak{h}$. Moreover, $\mathfrak{p}^1 = \mathfrak{p} \cap \mathfrak{h}$. Let $\mathcal{P}^1 = \exp \mathfrak{p}^1$. As in (2.5.) we get a decomposition of $\mathcal{H}$ by writing $\mathcal{H} = \mathcal{V} \cdot \mathcal{X} \cdot \widetilde{\mathcal{N}} \cdot \mathcal{P}^1$. Imitating the definition of $\pi_{l,\mathfrak{p},\bar{p}}$, we similarly define representations $\gamma_{\bar{p}}$ of $\mathcal{H}$ and $L^1(\mathcal{H})$ on the representation space $\mathfrak{H}_{\gamma_{\bar{p}}} = L^{\bar{p}}(\mathcal{H}/\mathcal{P}^1, \chi_r)$. Notice that the corresponding character $\Delta^{\frac{1}{\bar{p}}}$ is the same as for $\pi_{l,\mathfrak{p},\bar{p}}$. For $\bar{p} = \overline{2} = (2, \ldots, 2)$ we simply write $\gamma = \gamma_{\overline{2}} = \gamma_2$. For every extension l of r to $\mathfrak{g}$, the representation $\gamma_{\bar{p}}$ may be extended to a representation $\gamma_{l,\bar{p}}$ of $\mathcal{G}$ in the following way:

(1) $\mathfrak{H}_{\gamma_{l,\bar{p}}} = \mathfrak{H}_{\gamma_{\bar{p}}}$

(2) $\gamma_{l,\bar{p}}(h) = \gamma_{\bar{p}}(h), \quad \forall h \in \mathcal{H}$

(3) $\Big(\gamma_{l,\bar{p}}(t)\xi\Big)(x) = \Delta^{-\frac{1}{\bar{p}}}(t)\chi_l(t)\xi(t^{-1}xt), \quad \forall \xi \in \mathfrak{H}_{\gamma_{\bar{p}}}, \forall t \in \mathcal{U}, \forall x \in \mathcal{H}$

(4) $\gamma_{l,\bar{p}}(th) = \gamma_{l,\bar{p}}(t)\gamma_{l,\bar{p}}(h), \quad \forall t \in \mathcal{U}, \forall h \in \mathcal{H}.$

For $\bar{p} = \overline{2}$ we simply write γ_l instead of $\gamma_{l,\overline{2}}$. It is easy to check that $\gamma_{l,\bar{p}}$ is a well-defined representation that is equivalent to $\pi_{l,\mathfrak{p},\bar{p}}$. Hence $\gamma_{\bar{p}}$ may also be viewed as the restriction of $\pi_{l,\mathfrak{p},\bar{p}}$ to the subgroup $\mathcal{H}$. One may check that different extensions r and r' of $q \in \mathfrak{n}^*$ to $\mathfrak{h}$ give the same representation $\gamma_{\bar{p}}$ (up to equivalence), whereas different extensions l and l' of $r \in \mathfrak{h}^*$ to $\mathfrak{g}$ lead to representations $\gamma_{l,\bar{p}}$ and $\gamma_{l',\bar{p}}$ that differ by the unitary character $\chi_{l-l'}$ on $\mathcal{U}$. One defines of course the spaces $\mathcal{ES}(\mathcal{H})$, $\mathcal{ES}(\mathcal{H}/\mathcal{P}^1, \chi_r)$, $\mathcal{ES}(\mathcal{H}/\mathcal{P}^1 \times \mathcal{H}/\mathcal{P}^1, \chi_r)$ and one has the equivalent of (2.7.3.) for the representations $\gamma_{\bar{p}}$.

Take $\lambda \in \mathcal{ES}(\mathcal{H}/\mathcal{P}^1, \chi_r)$ such that $\langle \lambda, \lambda \rangle = 1$ and let $p_\lambda \in L^1(\mathcal{H})$ be an element such that the kernel of the operator $\gamma(p_\lambda)$ is the projector $P_{\lambda,\lambda}$, i.e., such that

$$\Big(\gamma(p_\lambda)\xi\Big)(x) = \int_{\mathcal{H}/\mathcal{P}^1} \lambda(x)\overline{\lambda(y)}\xi(y)d\dot{y}.$$

Put $p = p_\lambda \mod \ker \gamma$. Then p is an idempotent element of $L^1(\mathcal{H})/\ker \gamma$. We have that

$$\ker \gamma = \Big(L^1(\mathcal{H}) * \ker(\mathcal{G} \cdot \tau)\Big)^{-L^1(\mathcal{H})} = \ker_{L^1(\mathcal{H})} T,$$

where $\ker_{L^1(\mathcal{H})} T$ stands for the kernel of the corresponding representation of $L^1(\mathcal{H})$ (obtained by $T|_{\mathcal{H}}$) and

$$\left(L^1(\mathcal{G}) * \ker\gamma\right)^{-L^1(\mathcal{G})} \subset \ker T.$$

In particular,

$$T(p_\lambda) \neq 0 \text{ and } \mathfrak{W} = T(p_\lambda)\mathfrak{U} \neq \{0\}.$$

3.4. Some quotient algebras. Thanks to the decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{h}$ with $\mathfrak{u} \subset \mathfrak{g}(l)$, we put $\mathcal{U} = \exp\mathfrak{u}$ and we may identify the sets $\mathcal{U}$ and $\mathcal{G}(l)/\mathcal{G}(l) \cap \mathcal{N}$. As in ([**Po2**]) and ([**LuMo2**]) we introduce generalized convolution and involution formulas in $L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma)$. It is then easy to check that the algebras $L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma)$ and $L^1(\mathcal{G})/(L^1(\mathcal{G}) * \ker\gamma)^{-L^1(\mathcal{G})}$ $= L^1(\mathcal{G})/(L^1(\mathcal{G}) * \ker_{L^1(\mathcal{N})} T)^{-L^1(\mathcal{G})}$ are isomorphic and isometric (see [**Po2**] and [**LuMo2**]). Notice that the latter algebra is completely determined by the initial representation $(T, \mathcal{U})$.

3.5. A special subalgebra. Take p_λ as in (3.3.). For any $f \in L^1(\mathcal{G})$, let's define $\widetilde{f} \in L^1(\mathcal{U}, L^1(\mathcal{H}))$ by $\widetilde{f}(u)(h) = f(u \cdot h)$ for almost all $u \in \mathcal{U}$ and almost all $h \in \mathcal{H}$. It is then easy to check that

$$(p_\lambda * f * p_\lambda)\widetilde{}(x) = p_\lambda^x *_{L^1(\mathcal{H})} \widetilde{f}(x) *_{L^1(\mathcal{H})} p_\lambda$$

for every $f \in L^1(\mathcal{G})$ and every $x \in \mathcal{G}$, where p_λ^x is the function of $L^1(\mathcal{H})$ obtained by the action of x on p_λ:

$$p_\lambda^x(y) = \Delta_{\mathcal{G}}(x) p_\lambda(xyx^{-1}), \quad \forall y \in \mathcal{H}.$$

We recall that $\pi = \pi_{l,\mathfrak{p},\overline{2}} = \operatorname{ind}_{\mathcal{P}}^{\mathcal{G}} \chi_l$, that $\gamma = \operatorname{ind}_{\mathcal{P}^1}^{\mathcal{H}} \chi_r$ and that the extension γ_l is equivalent to π. One has the following formulas:

$$\gamma(p_\lambda^x) = P_{\gamma_l(x)^*\lambda, \gamma_l(x)^*\lambda},$$
$$\gamma(p_\lambda^x * g * p_\lambda) = \langle \gamma(g)\lambda, \gamma_l(x)^*\lambda \rangle P_{\gamma_l(x)^*\lambda, \lambda},$$

for every $g \in L^1(\mathcal{H})$. By ([**LeLu**], [**Lu**]) there exists $v_{\lambda,l}(x) \in L^1(\mathcal{H})$ such that $\gamma(v_{\lambda,l}(x)) = P_{\gamma_l(x)^*\lambda,\lambda}$ and the map $x \to v_{\lambda,l}(x)$ from $\mathcal{G}$ to $L^1(\mathcal{H})$ is continuous. Hence, for every $g \in L^1(\mathcal{H})/\ker\gamma$ and every $x \in \mathcal{G}$, there is a constant $c(x,g) = \langle \gamma(g)\lambda, \gamma_l(x)^*\lambda \rangle$ such that

$$p_\lambda^x * g * p_\lambda = c(x,g) v_{\lambda,l}(x) \mod \ker\gamma.$$

Moreover

$$v_{\lambda,l}(x) = p_\lambda^x * v_{\lambda,l}(x) * p_\lambda \mod \ker\gamma.$$

Let's write

$$p = p_\lambda \mod \ker\gamma, \quad v_l(x) = v_{\lambda,l}(x) \mod \ker\gamma$$

in the quotient space $L^1(\mathcal{H})/\ker\gamma$. Then the space

$$p^x * (L^1(\mathcal{H})/\ker\gamma) * p = (p_\lambda^x * L^1(\mathcal{H}) * p_\lambda)/\ker\gamma$$

is one dimensional for every fixed $x \in \mathcal{G}$ and it has $v_l(x)$ as a basis.

On the other hand, it is easy to check that

$$\gamma(p_\lambda^x * p_\lambda) = \langle \gamma_l(x)\lambda, \lambda\rangle P_{\gamma_l(x)^*\lambda,\lambda} = \langle \gamma_l(x)\lambda, \lambda\rangle \gamma(v_{\lambda,l}(x)),$$

i.e., that

$$p_\lambda^x * p_\lambda = \langle \gamma_l(x)\lambda, \lambda\rangle v_{\lambda,l}(x) \quad \mod \ker\gamma.$$

If we apply the representation $\gamma_{\bar{p}}$ instead of γ, the formulas are more complicated. As $\ker_{L^1(\mathcal{H})} \pi_{l,\mathfrak{p},\bar{p}} = \ker_{L^1(\mathcal{H})} \gamma_{l,\bar{p}} = \ker\gamma$ (by (3.3.)),

$$\begin{aligned}\gamma_{l,\bar{p}}(v_{\lambda,l}(x)) &= \frac{1}{\langle \gamma_l(x)\lambda, \lambda\rangle}\gamma_{l,\bar{p}}(p_\lambda^x * p_\lambda) \\ &= \frac{1}{\langle \gamma_l(x)\lambda, \lambda\rangle}\gamma_{l,\bar{p}}(x^{-1})\gamma_{l,\bar{p}}(p_\lambda)\gamma_{l,\bar{p}}(x)\gamma_{l,\bar{p}}(p_\lambda).\end{aligned}$$

In order to compute the exact value of $\gamma_{l,\bar{p}}(v_{\lambda,l}(x))$, we have to introduce a more precise decomposition of the Lie algebra $\mathfrak{g}$ (see (5.)).

Definition of $v(x)$**.** The previous definition of $v_l(x)$ is the one used in ([**Po2**]) and ([**LuMo2**]). It depends on the extension l of q we have chosen. If l and l' are two different extensions such that $l|_{\mathfrak{h}} = l'|_{\mathfrak{h}} = r$, then γ_l and $\gamma_{l'}$ differ only by the unitary character $\chi_{l-l'}$ on $\mathcal{U}$. Hence, if the corresponding functions are named $v_{\lambda,l}$ and $v_{\lambda,l'}$, then

$$v_{\lambda,l'}(x) = \chi_{l-l'}(x)v_{\lambda,l}(x) \mod \ker\gamma, \forall x \in \mathcal{U}.$$

Let $l_0 \in \mathfrak{g}^*$ be a fixed extension of r. We have

$$v_{\lambda,l_0}(x) = \chi_{l-l_0}(x)v_{\lambda,l}(x) \mod \ker\gamma$$

and we define $v(x)$ to be $v_{l_0}(x) = v_{\lambda,l_0}(x) \mod \ker\gamma$.

Let's put $\omega(x) = \|v(x)\|_{L^1(\mathcal{H})/\ker\gamma}$. By ([**Po2**], [**LuMo2**]) the function ω is a symmetric weight function on $\mathcal{G}$, which is constant on the classes modulo $\mathcal{H}$. Notice that ω is independent of the choice of the fixed linear form l_0 used to define v. Moreover, ω may be considered as a function on $\mathcal{G}(l)/\mathcal{G}(l)\cap\mathcal{N} = \mathcal{G}(l_0)/\mathcal{G}(l_0)\cap\mathcal{N}$.

Recall that p_λ acts on $L^1(\mathcal{G})$ and $p = p_\lambda \mod \ker\gamma$ acts on $L^1(\mathcal{G})/(L^1(\mathcal{G})*\ker\gamma)^{-L^1(\mathcal{G})}$ by convolution. Moreover $f \mod (L^1(\mathcal{G})*\ker\gamma)^{-L^1(\mathcal{G})} \mapsto \widetilde{f} \mod \ker\gamma$ is an isometric isomorphism between $L^1(\mathcal{G})/(L^1(\mathcal{G})*\ker\gamma)^{-L^1(\mathcal{G})}$ and $L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma)$. As

$$(p_\lambda * f * p_\lambda)\widetilde{\ }(x) = p_\lambda^x *_{L^1(\mathcal{H})} \widetilde{f}(x) *_{L^1(\mathcal{H})} p_\lambda$$

for every $f \in L^1(\mathcal{G})$ and every $x \in \mathcal{G}$, we may consider a similar action on $L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma)$ by

$$\begin{aligned}(p * \widetilde{f} * p)(x) &= p^x *_{L^1(\mathcal{H})/\ker\gamma} \widetilde{f}(x) *_{L^1(\mathcal{H})/\ker\gamma} p \in p^x * (L^1(\mathcal{H})/\ker\gamma) * p \\ &= \mathbb{C}\cdot v(x)\end{aligned}$$

for every $\widetilde{f} \in L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma)$. As a matter of fact,

$$(p * \widetilde{f} * p)(x) = \langle \gamma(\widetilde{f}(x))\lambda, \gamma_l(x)^*\lambda\rangle \overline{\chi_l(x)}\chi_{l_0}(x) \cdot v(x) \mod \ker\gamma$$
$$= h(x) \cdot v(x) \mod \ker\gamma,$$

if we define the function $h : \mathcal{U} \to \mathbb{C}$ by $h(x) = \langle \gamma(\widetilde{f}(x))\lambda, \gamma_l(x)^*\lambda\rangle \overline{\chi_l(x)}\chi_{l_0}(x)$. Of course the same argument is valid for every function $f \in L^1(\mathcal{G})$ and every $x \in \mathcal{G}$, if we define $\widetilde{f}(x) \in L^1(\mathcal{H})/\ker\gamma$ by $\widetilde{f}(x)(h) = f(xh) \mathrm{mod} \ker\gamma$. As shown in ([**LuMo2**]), the map $\Lambda : p * \widetilde{f} * p = h \cdot v \mapsto h$ is an isometric isomorphism from $p * L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma) * p$ onto $L^1(\mathcal{U}, \omega)$.

Remarks.

a) Notice that the function h given by

$$h(x) = \langle \gamma(\widetilde{f}(x))\lambda, \gamma_l(x)^*\lambda\rangle \overline{\chi_l(x)}\chi_{l_0}(x) = \langle \gamma(\widetilde{f}(x))\lambda, \gamma_{l_0}(x)^*\lambda\rangle$$

is independent of the choice of l such that $l|_{\mathfrak{h}} = r$ is fixed.

b) For a given f in $L^1(\mathcal{G})/(L^1(\mathcal{G}) * \ker\gamma)^{-L^1(\mathcal{G})}$, the function h defined by the previous formulas may be considered as a function on all of $\mathcal{G}(l)$. It is then constant on the classes of $\mathcal{G}(l)$ modulo $\mathcal{G}(l) \cap \mathcal{N}$. Hence we may consider h as a function in $L^1(\mathcal{G}(l)/\mathcal{G}(l) \cap \mathcal{N}, \omega)$, where $\mathcal{G}(l)$ just depends on $l|_{\mathfrak{n}} = q$. In particular, h is independent of the choice of the supplementary space $\mathfrak{u}$ in $\mathfrak{g}(l)$.

c) If we take another $l_0 \in \mathfrak{g}^*$ having the same restriction to $\mathfrak{h}$ and another $v(x) = v_{\lambda, l_0}(x) \mod \ker\gamma$, then the h functions are all multiplied by the same unitary character χ such that $\chi|_{\mathcal{H}} \equiv 1$.

d) Let's take $\lambda, \mu \in \mathcal{ES}(\mathcal{H}/\mathcal{P}^1, \chi_r)$ such that $\langle\lambda, \lambda\rangle = \langle\mu, \mu\rangle = 1$. If $p_\lambda, p_\mu \in L^1(\mathcal{H})$ are such that $\gamma(p_\lambda) = P_{\lambda,\lambda}$ and $\gamma(p_\mu) = P_{\mu,\mu}$, then the algebras

$$(p_\lambda \mod \ker\gamma) * L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma) * (p_\lambda \mod \ker\gamma)$$

and

$$(p_\mu \mod \ker\gamma) * L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma) * (p_\mu \mod \ker\gamma)$$

are $*$-isomorphic. The resulting weights are equivalent. In fact, take $s_{\lambda,\mu} \in L^1(\mathcal{H})$ and $s = s_{\lambda,\mu} \mod \ker\gamma$ such that $\gamma(s_{\lambda,\mu}) = P_{\lambda,\mu}$. Then the map

$$\Phi : (p_\lambda \mathrm{mod} \ker\gamma) * \widetilde{f} * (p_\lambda \mathrm{mod} \ker\gamma)$$
$$\mapsto s^* * ((p_\lambda \mathrm{mod} \ker\gamma) * \widetilde{f} * (p_\lambda \mathrm{mod} \ker\gamma)) * s$$

is the corresponding $*$-isomorphism. Moreover the different $\lambda, \mu \in \mathcal{ES}(\mathcal{H}/\mathcal{P}^1, \chi_r)$ together with the corresponding $*$-isomorphism lead to the same function h, for given functions f and $\widetilde{f}$. The algebra $L^1(\mathcal{G}(l)/\mathcal{G}(l) \cap \mathcal{N}, \omega)$ is hence independent of the choice of λ.

e) The algebra $L^1(\mathcal{U},\omega) \equiv L^1(\mathcal{G}(l)/\mathcal{G}(l)\cap\mathcal{N},\omega)$ is abelian (see [**Po2**], [**LuMo2**]).

3.6. Relation with the simple $L^1(\mathcal{G})$-module. Let's recall that if $(T,\mathfrak{U})$ is a simple $L^1(\mathcal{G})$-module, there is a unique orbit $G\cdot\tau\subset\hat{\mathcal{N}}, \tau\in\hat{\mathcal{N}}$, such that $\ker_{L^1(\mathcal{N})} T = \ker(\mathcal{G}\cdot\tau)$. Then we construct $\mathcal{H}\subset\mathcal{G}$ and $\gamma\in\hat{\mathcal{H}}$ as explained previously. To characterize completely $(T,\mathfrak{U})$ with a given $\ker T$, it is of course enough to study the algebraically irreducible representations of

$$L^1(\mathcal{G})/\left(L^1(\mathcal{G})*\ker_{L^1(\mathcal{N})} T\right)^{-L^1(\mathcal{G})} = L^1(\mathcal{G})/(L^1(\mathcal{G})*\ker\gamma)^{-L^1(\mathcal{G})} \simeq L^1(\mathcal{U}, L^1(\mathcal{H})/\ker\gamma),$$

as $\left(L^1(\mathcal{G})*\ker_{L^1(\mathcal{N})} T\right)^{-L^1(\mathcal{G})}\subset\ker T$. By ([**Po2**], Theorem 1) these are determined by the simple $(p*L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)*p)$-modules. But $\mathcal{B} = p*L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)*p \simeq L^1(\mathbb{R}^n,\omega)$ is abelian and its simple modules coincide with the characters of $L^1(\mathbb{R}^n,\omega)$. Hence, if we put $\mathcal{A} = L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)$ and if $(S,\mathfrak{U})$ is a simple $\mathcal{A}$-module, this means that the subspace $\mathfrak{V} = S(p)\mathfrak{U}$ is one-dimensional. So there exists a character χ on $L^1(\mathbb{R}^n,\omega)\simeq p*L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)*p$ such that for every $v\in\mathfrak{V}$ and $f\in\mathcal{B}$ we have $S(f)v=\chi(f)v$. Hence the maximal modular left ideal M of $\mathcal{A}$ consisting of all f in $\mathcal{A}$ for which $S(f)v=0, v\in\mathfrak{V}$, is given by $M=\{f\in\mathcal{A}\mid \chi(p*\mathcal{A}*f*p)\equiv 0\}$. The given simple $L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)$-module is then isomorphic to $(L,\mathcal{A}/M)$ where L is the left multiplication on $\mathcal{A}/M$.

On the other hand, for a given $(T,\mathfrak{U})$, let $q\in\mathfrak{n}^*$ be as in (3.2.). We want to show that $(T,\mathfrak{U})$ is equivalent to $\pi^0_{l,\mathfrak{p},\bar{p}}$ for some $l,\mathfrak{p},\bar{p}$ such that $l|_{\mathfrak{n}} = q$. But $\ker_{L^1(\mathcal{N})} T = \ker(\mathcal{G}\cdot\tau) = \ker_{L^1(\mathcal{N})}\pi_{l,\mathfrak{p},\bar{p}}$ for every $l\in\mathfrak{g}^*$ such that $l|_{\mathfrak{n}}=q$, for every multi-index $\bar{p}$, τ being given by $\tau = \mathrm{ind}^{\mathcal{N}}_{\mathcal{P}^0}\chi_q$. Hence the algebraically irreducible representations $\left(\pi_{l,\mathfrak{p},\bar{p}}|_{\mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}}, \mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}\right)$ give rise to the same algebra $L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)$ as $(T,\mathfrak{U})$ does (if we make the same choices for $\mathcal{H},\mathcal{U},\mathfrak{p},\dots$). To show that $(T,\mathfrak{U})$ is equivalent to such a $\left(\pi_{l,\mathfrak{p},\bar{p}}|_{\mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}}, \mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}\right)$ with $l|_{\mathfrak{n}}=q$ it is therefore enough to show that the corresponding characters on $L^1(\mathbb{R}^n,\omega)$ coincide for some $\bar{p}$. To do this we first have to study the weight ω.

Example 3.7. Let $\gamma_l\equiv\pi_{l,\mathfrak{p},2}\in\hat{G}$ such that $\gamma_l|_{\mathcal{H}}=\gamma$ and consider the simple module $(\gamma_l|_{\mathfrak{H}^0_\gamma},\mathfrak{H}^0_\gamma)$. Let's compute the character of $L^1(\mathbb{R}^n,\omega)\equiv p*L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)*p$ associated to $\gamma_l|_{\mathfrak{H}^0_\gamma}$. Recall that this is done by considering the action of $p*L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)*p$ on $\gamma(p)\mathfrak{H}^0_\gamma=\gamma(p_\lambda)\mathfrak{H}^0_\gamma$. Take $h\cdot v\in p*L^1(\mathcal{U},L^1(\mathcal{H})/\ker\gamma)*p$ corresponding to $h\in L^1(\mathbb{R}^n,\omega)$. Then one checks that

$$\gamma_l(h\cdot v)(\gamma(p_\lambda)\xi)=\int_{\mathcal{U}}\int_{\mathcal{H}} h(t)v(t)(s)\gamma_l(t)\gamma(s)\gamma(p_\lambda)\xi ds dt=\hat{h}(l-l_0)\gamma(p_\lambda)\xi.$$

Hence the character of $L^1(\mathbb{R}^n,\omega)\equiv L^1(\mathcal{U},\omega)$ corresponding to $\gamma_l\equiv\pi_{l,\mathfrak{p},2}$ is χ_{l-l_0}. Similarly, we may compute $\gamma_{l,\bar{p}}(h\cdot v)$:

$$\gamma_{l,\bar{p}}(h\cdot v)(\gamma_{\bar{p}}(p_\lambda)\xi)=\int_{\mathcal{U}} h(t)\chi_{l-l_0}(t)\frac{1}{\langle\gamma_l(t)\lambda,\lambda\rangle}\gamma_{\bar{p}}(p_\lambda)\gamma_{l,\bar{p}}(t)\gamma_{\bar{p}}(p_\lambda)\xi dt,$$

by (3.5.). In order to conclude, we need to know $\gamma_{\bar{p}}(p_\lambda)$. This computation requires a more precise decomposition of the Lie algebra $\mathfrak{g}$ and will be done in (5.6.1.). We shall see that the character corresponding to $\gamma_{l,\bar{p}}\equiv\pi_{l,\mathfrak{p},\bar{p}}$ is $\chi_{l,\bar{p}}=\Delta^{\frac{1}{2}-\frac{1}{\bar{p}}}\cdot\chi_{l-l_0}$.

4. Characters of $L^1(\mathbb{R}^n,\omega)$.

4.1. . Let's fix $x=\exp X\in\mathcal{U}\subset\mathcal{G}(l)$ and let's study the growth of $\omega(\exp tX)$ for $t\in\mathbb{R}$ and X fixed. Take $\lambda,p_\lambda,v_\lambda,v$ as in (3.3.) and (3.5.). Recall that

$$\omega(\exp tX)=\|v(\exp tX)\|_{L^1(\mathcal{H})/\ker\gamma}$$

where $v(\exp tX)=v_{\lambda,l_0}(\exp tX)\mod\ker\gamma$. Moreover let's choose for λ the Gaussian function. This is possible because different choices of λ give equivalent weights. Put $\sigma(g)=e^{\frac{1}{2}\sum_{j=1}^m|\operatorname{tr}\lambda_j(\log g)|}$ for $g\in\mathcal{G}(l)$, where $\lambda_j(\cdot)=\mathrm{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\cdot)$. Using a method developed by Poguntke ([**Po2**]), one checks that there are constants C and C' (depending on the choice of X but not on t) such that

$$\omega(\exp tX)\le C'\cdot(1+|t|)^C\cdot e^{\frac{|t|}{2}\sum_{j=1}^m|\operatorname{tr}\lambda_j(X)|}=C'\cdot(1+|t|)^C\cdot\sigma(\exp tX).$$

Proposition 4.2. *Let χ be a continuous character on $L^1(\mathcal{G}(l)/\mathcal{G}(l)\cap\mathcal{N},\omega)$ $\equiv L^1(\mathcal{U},\omega)\equiv L^1(\mathbb{R}^n,\omega)$. Then*

$$|\chi(\exp X)|\le\prod_{i=1}^m e^{\frac{1}{2}|\operatorname{tr}\lambda_i(X)|}=\sigma(\exp X)$$

for all $X\in\mathfrak{u}\equiv\mathfrak{g}(l)/\mathfrak{g}(l)\cap\mathfrak{n}$.

Proof. As χ is a continuous character on $L^1(\mathcal{U},\omega)$, $|\chi(\exp X)|\le\omega(\exp X)$, $X\in\mathcal{U}$. Let's write $\chi(\exp X)=e^{\rho(X)}$ with ρ a complex linear form on $\mathcal{U}\equiv\mathbb{R}^n$. Then $|\chi(\exp X)|=e^{\mathfrak{Re}\rho(X)}$, $\forall X\in\mathcal{U}$. Assume that there is $X_0\in\mathcal{U}$ such that $\mathfrak{Re}\rho(X_0)>0$ (otherwise, change X_0 to $-X_0$) and such that

$$|\chi(\exp X_0)|=e^{\mathfrak{Re}\rho(X_0)}=e^{|\mathfrak{Re}\rho(X_0)|}>\prod_{i=1}^m e^{\frac{1}{2}|\operatorname{tr}\lambda_i(X_0)|}.$$

Hence

$$\prod_{i=1}^{m} e^{\frac{|t|}{2}|\mathrm{tr}\,\lambda_i(X_0)|} < |\chi(\exp|t|X_0)|$$

$$\leq \omega(\exp|t|X_0) \leq C'(1+|t|)^C \prod_{i=1}^{m} e^{\frac{|t|}{2}|\mathrm{tr}\,\lambda_i(X_0)|}$$

and

$$1 < e^{|t|(|\mathfrak{Re}\rho(X_0)|-\frac{1}{2}\sum_{i=1}^{m}|\mathrm{tr}\,\lambda_i(X_0)|)} \leq C'(1+|t|)^C$$

for all $t \in \mathbb{R}^*$. As this is impossible, we have that

$$|\chi(\exp X)| \leq \prod_{i=1}^{m} e^{\frac{1}{2}|\mathrm{tr}\,\lambda_i(X)|} = \sigma(\exp X),$$

for all $X \in \mathcal{U}$. □

5. Characterization of all the simple modules.

We proceed now as written in (3.6.).

5.1. Identification of $\mathfrak{H}_{\gamma_{\bar{p}}} = L^{\bar{p}}(\mathcal{H}/\mathcal{P}^1, \chi_r)$ and $L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X})$.

a) We use the decompositions and notations introduced in (2.1.) to (2.5.) and in (3.3.). Recall in particular that $\mathfrak{h} = \mathfrak{v} \oplus \mathfrak{n} \oplus \mathfrak{w} = \mathfrak{v} \oplus \mathfrak{n} \oplus \mathfrak{Y} \oplus \mathfrak{X}$. Let's define $\mathfrak{k} = \mathfrak{v} \oplus \mathfrak{n}$. Hence $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{Y} \oplus \mathfrak{X}$. In order to get an isometry between $\mathfrak{H}_{\gamma_{\bar{p}}} = L^{\bar{p}}(\mathcal{H}/\mathcal{P}^1, \chi_r)$ and $L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X})$, let's define

$$\widetilde{\xi}(k,x) = \xi(k\cdot x), \quad \forall k \in \mathcal{K}, \forall x \in \mathcal{X}, \forall \xi \in \mathfrak{H}_{\gamma_{\bar{p}}},$$

and

$$(S_{\bar{p}}\xi)(k,x) = e^{-\sum_{j=1}^{m}\frac{1}{p_j}\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log x)}\widetilde{\xi}(k,x)$$

$$= e^{-\sum_{j=1}^{m}\frac{1}{p_j}\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log x)}\xi(k\cdot x).$$

Let's write $\delta^{-\frac{1}{\bar{p}}}(x) = e^{-\sum_{j=1}^{m}\frac{1}{p_j}\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log x)}$, $\forall x \in \mathcal{X}$. Then it is easy to see that the map $S_{\bar{p}} : \xi \mapsto \delta^{-\frac{1}{\bar{p}}}\cdot\widetilde{\xi}$ is an isometry between $L^{\bar{p}}(\mathcal{H}/\mathcal{P}^1, \chi_r)$ and $L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X})$ if the norm on $L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X})$ is given by

$$\|\widetilde{f}\| = \left(\int_{\mathcal{X}} \|\widetilde{f}(\cdot,x)\|_{\bar{p}}^2 dx\right)^{\frac{1}{2}}$$

$$= \left(\int_{\mathcal{X}}\int_{\mathcal{V}}\left(\left(\int_{\mathcal{V}_1}\left(\dots\left(\int_{\mathcal{V}_m}|\widetilde{f}(vv_1\dots v_m,x)|^{p_m}dv_m\right)^{\frac{1}{p_m}}\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\dots\right)^{p_1}dv_1\right)^{\frac{1}{p_1}}\right)^2 dv dx\right)^{\frac{1}{2}}.$$

In particular, for $\xi \in \mathfrak{H}_{\gamma_{\bar{p}}}$,

$$\begin{aligned}(S_{\bar{p}}\xi)(k\cdot p_0, x) &= \delta^{-\frac{1}{\bar{p}}}(x)\xi(k\cdot p_0\cdot x)\\ &= \delta^{-\frac{1}{\bar{p}}}(x)\overline{\chi_r(x^{-1}p_0x)}\xi(k\cdot x) = \overline{\chi_r(p_0)}(S_{\bar{p}}\xi)(k,x),\end{aligned}$$

because $x \in \mathcal{X} \subset \mathcal{G}(q), p_0 \in \mathcal{P}^0$, $\mathcal{P}^0$ is $\mathcal{G}(q)$-invariant and $\langle r, [\log x, \log p_0]\rangle = 0$.

b) Notice that $L^2(\mathcal{X})$ may be identified with $L^2(\mathfrak{X})$. In fact, if $x, x' \in \mathcal{X} \subset \mathcal{G}(q)$, then

$$x\cdot x' = q(x,x')\cdot \exp(\log x + \log x')$$

with $q(x,x') \in \mathcal{N}(q) \subset \mathcal{P}^0$ and $\chi_r(q(x,x')) = 1$. Hence

$$\xi(k\cdot x\cdot x') = \xi(k\cdot q(x,x')\cdot \exp(\log x + \log x')) = \widetilde{\xi}(k, \exp(\log x + \log x'))$$

and we may identify $L^2(\mathcal{X})$ with $L^2(\mathfrak{X})$ where $\mathcal{X} = \exp\mathfrak{X}$ as before. Moreover

$$(S_{\bar{p}}\xi)(k, x\cdot x') = e^{-\sum_{j=1}^m \frac{1}{p_j}\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log x + \log x')}\cdot \widetilde{\xi}(k, \exp(\log x + \log x'))$$

and we may consider $S_{\bar{p}}\xi$ as a function on $(\mathcal{K}/\mathcal{P}^0)\times\mathfrak{X}$. Similarly we shall consider

$$\mathcal{E}S(\mathcal{X}) \equiv \mathcal{E}S(\mathfrak{X}) \subset L^2(\mathcal{X}) \equiv L^2(\mathfrak{X}),$$

an $\mathcal{E}S$-space with decay conditions as in (2.7.1.).

5.2. Equivalent representations. Let $\gamma_{\bar{p}}$ be the representation defined on $\mathfrak{H}_{\gamma_{\bar{p}}} = L^{\bar{p}}(\mathcal{H}/\mathcal{P}^1, \chi_r)$. If we define $\kappa_{\bar{p}}$ on $L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X})$ by

$$(\kappa_{\bar{p}}(h)(S_{\bar{p}}\xi))(k,x) = S_{\bar{p}}(\gamma_{\bar{p}}(h)\xi)(k,x) = \delta^{-\frac{1}{\bar{p}}}(x)\Delta^{-\frac{1}{\bar{p}}}(h)\xi(h^{-1}kx)$$

$\forall h \in \mathcal{H}$, then the representations $(\kappa_{\bar{p}}, L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X}))$ and $(\gamma_{\bar{p}}, L^{\bar{p}}(\mathcal{H}/\mathcal{P}^1, \chi_r))$ are equivalent. Similarly, we define $\kappa_{l,\bar{p}}$ on $L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X})$ by

$$\begin{aligned}(\kappa_{l,\bar{p}}(t)(S_{\bar{p}}\xi))(k,x) &= S_{\bar{p}}(\gamma_{l,\bar{p}}(t)\xi)(k,x)\\ &= \delta^{-\frac{1}{\bar{p}}}(x)\Delta^{-\frac{1}{\bar{p}}}(t)\chi_l(t)\xi(k^{t^{-1}}\cdot(t^{-1}xtx^{-1})x)\end{aligned}$$

for $t \in \mathcal{U} \subset \mathcal{G}(l)$ and $\kappa_{l,\bar{p}}(h) = \kappa_{\bar{p}}(h)$ for $h \in \mathcal{H}$. As $t^{-1}xtx^{-1} \in \mathcal{G}(q)\cap\mathcal{N} \subset \mathcal{P}^0$ and as $\chi_r(t^{-1}xtx^{-1}) = 1$, we have that

$$\begin{aligned}(\kappa_{l,\bar{p}}(t)(S_{\bar{p}}\xi))(k,x) &= \delta^{-\frac{1}{\bar{p}}}(x)\Delta^{-\frac{1}{\bar{p}}}(t)\chi_l(t)\widetilde{\xi}(k^{t^{-1}}, x)\\ &= \Delta^{-\frac{1}{\bar{p}}}(t)\chi_l(t)(S_{\bar{p}}\xi)(k^{t^{-1}}, x),\end{aligned}$$

i.e., $t \in \mathcal{G}(l)$ acts only on $\mathcal{K}$. The representations $(\kappa_{l,\bar{p}}, L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X}))$ and $(\gamma_{l,\bar{p}}, L^{\bar{p}}(\mathcal{H}/\mathcal{P}^1, \chi_r))$ are equivalent by construction.

5.3. Kernel of $\kappa_{\overline{p}}(f)$. The kernel of the operator $\kappa_{\overline{p}}(f)$, $f \in \mathcal{ES}(\mathcal{H}) \subset L^1(\mathcal{H})$, is given by the following computations:

$$\begin{aligned}
&(\kappa_{\overline{p}}(f)(S_{\overline{p}}\xi))(k',x') \\
&= \delta^{-\frac{1}{\overline{p}}}(x') \int_{\mathcal{Y}}\int_{\mathcal{X}}\int_{\mathcal{K}} f((xy)k)\Delta^{-\frac{1}{\overline{p}}}(xyk)\xi(k^{-1}(xy)^{-1}k'x')dkdxdy \\
&= \delta^{-\frac{1}{\overline{p}}}(x') \int_{\mathcal{Y}}\int_{\mathcal{X}}\int_{\mathcal{K}} f((xy)((k')^{(xy)^{-1}})k^{-1}) \cdot \Delta^{-\frac{1}{\overline{p}}}(k'(xy)k^{-1}) \\
&\qquad \cdot \Delta_{\mathcal{K}}(k)^{-1}\xi(k(xy)^{-1}x')dkdxdy \\
&= \int_{\mathcal{Y}}\int_{\mathcal{X}}\int_{\mathcal{K}} f((xy)((k')^{(xy)^{-1}})k^{-1})\Delta^{-\frac{1}{\overline{p}}}(k'(xy)k^{-1}) \\
&\qquad \cdot \Delta_{\mathcal{K}}(k)^{-1} \cdot e^{-i\langle r,\log(y)\rangle}e^{-i\langle r,[\log(y),\log(x^{-1}x')]\rangle} \\
&\qquad \cdot \delta^{-\frac{1}{\overline{p}}}(x)(S_{\overline{p}}\xi)(k,x^{-1}x')dkdxdy \\
&= \int_{\mathcal{Y}}\int_{\mathcal{X}}\int_{\mathcal{K}} f((x'x^{-1}y)((k')^{(x'x^{-1}y)^{-1}})k^{-1})\Delta^{-\frac{1}{\overline{p}}}(k'(x'x^{-1}y)k^{-1}) \\
&\qquad \cdot \Delta_{\mathcal{K}}(k)^{-1} \cdot e^{-i\langle r,\log(y)\rangle}e^{-i\langle r,[\log(y),\log(x)]\rangle} \\
&\qquad \cdot \delta^{-\frac{1}{\overline{p}}}(x'x^{-1})(S_{\overline{p}}\xi)(k,x)dkdxdy \\
&= \int_{\mathcal{Y}}\int_{\mathcal{X}}\int_{\mathcal{K}/\mathcal{P}^0}\int_{\mathcal{P}^0} f(((x'x^{-1})y)((k')^{((x'x^{-1})y)^{-1}}) \cdot p_0k^{-1}) \\
&\qquad \cdot \Delta_{\mathcal{K}}(k)^{-1} \cdot \Delta^{-\frac{1}{\overline{p}}}(y) \cdot e^{-i\langle r,\log(p_0)\rangle} \cdot e^{-i\langle r,\log(y)\rangle} \\
&\qquad \cdot e^{-i\langle r,[\log(y),\log(x)]\rangle}\delta^{-\frac{1}{\overline{p}}}(x'x^{-1})(S_{\overline{p}}\xi)(k,x)dp_0d\dot{k}dxdy,
\end{aligned}$$

as $\Delta \equiv 1$ on $\mathcal{K}$. Consider $\rho_{\overline{p}} = \mathrm{ind}_{\mathcal{P}^0}^{\mathcal{K}}(\chi_q,\overline{p})$ and $\rho_{\overline{2}} = \mathrm{ind}_{\mathcal{P}^0}^{\mathcal{K}}(\chi_q,2)$. Let's write $f(x,y)(k) = f(x\cdot y\cdot k)$, $k \in \mathcal{K}$. Then the kernel of $\kappa_{\overline{p}}(f)$ may be written

$$\begin{aligned}
&(f_{\kappa_{\overline{p}}})((k',x'),(k,x)) \\
&= \int_{\mathcal{Y}} f((x'x^{-1}),y)_{\rho_{\overline{p}}}((k')^{((x'x^{-1})\cdot y)^{-1}},k) \\
&\qquad \cdot \Delta^{-\frac{1}{\overline{p}}}(y) \cdot e^{-i\langle r,\log(y)\rangle} \cdot e^{-i\langle r,[\log(y),\log(x)]\rangle}\delta^{-\frac{1}{\overline{p}}}(x'x^{-1})dy,
\end{aligned}$$

where the kernel of $\rho_{\overline{p}}(g), g \in L^1(\mathcal{K})$, is given by

$$(\rho_{\overline{p}}(g))(k',k) = \int_{\mathcal{P}^0} g(k'p_0k^{-1}) \cdot e^{-i\langle r,\log p_0\rangle}\Delta_{\mathcal{K}}(k)^{-1}dp_0.$$

In particular, for $g \in \mathcal{ES}(\mathcal{K})$, $g_{\rho_{\overline{p}}}(k',k) = g_{\rho_{\overline{2}}}(k',k)$ for every multi-index $\overline{p}$. Hence the representations $\rho_{\overline{p}}$ and $\rho_{\overline{2}}$ are given by the same formulas (but act on different spaces).

5.4. Behavior of projectors. For $\bar{p} = (p_1, \ldots, p_m)$ we define $\bar{q} = (q_1, \ldots, q_m)$ such that $\frac{1}{\bar{q}} = 1 - \frac{1}{\bar{p}}$, which means that $\frac{1}{q_i} = 1 - \frac{1}{p_i}$ for each i. Take $\alpha, \beta \in \mathcal{ES}(\mathcal{H}/\mathcal{P}^1, \chi_r) \subset L^{\bar{p}}(\mathcal{H}/\mathcal{P}^1, \chi_r) \cap L^{\bar{q}}(\mathcal{H}/\mathcal{P}^1, \chi_r)$. One checks that $\langle \alpha, \beta \rangle = \langle S_{\bar{p}}\alpha, S_{\bar{q}}\beta \rangle$. Hence, if $\gamma_{\bar{p}}(f)$ is the projector $P_{\alpha,\beta}$, then $\kappa_{\bar{p}}(f)$ is the projector $P_{S_{\bar{p}}\alpha, S_{\bar{q}}\beta}$. Conversely, every projector $\kappa_{\bar{p}}(f) = P_{\alpha',\beta'}$, with $\alpha', \beta' \in \mathcal{ES}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}\mathcal{ES}(\mathcal{X})$, comes from a projector $\gamma_{\bar{p}}(f) = P_{\alpha,\beta}$ where $\alpha(vxv_1 \ldots v_m) = \delta^{-\frac{1}{\bar{p}}}(x^{-1})\alpha'(v \cdot (xv_1x^{-1}) \ldots (xv_mx^{-1}), x)$. Similarly for β and β'.

5.5. Choice of a particular p_λ. Let μ be an arbitrary function in $\mathcal{ES}(\mathcal{K}/\mathcal{P}^0, \chi_r)$ such that $\langle \mu, \mu \rangle = 1$. For every $s \in \mathcal{G}$ there is a function $g(s) \in \mathcal{ES}(\mathcal{K})$ such that the kernel of $\rho_{\bar{2}}(g(s))$ is given by $g(s)_{\rho_{\bar{2}}}(k', k) = \mu((k')^s)\overline{\mu(k)}$. Moreover, as $\Delta|_{\mathcal{P}^0} \equiv 1$, the kernels $g(s)_{\rho_{\bar{p}}}$ and $g(s)_{\rho_{\bar{2}}}$ coincide. Let's choose a real-valued analytic function $\nu \in \mathcal{ES}(\mathcal{X}) \equiv \mathcal{ES}(\mathfrak{X}) \subset L^2(\mathcal{X})$ such that $\langle \nu, \nu \rangle = 1$. Put

$$\alpha(x) = e^{\frac{1}{2}\sum_{j=1}^m \mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log x)}\nu(x) = \delta^{\frac{1}{2}}(x)\nu(x)$$

and define $\lambda \in \mathcal{ES}(\mathcal{H}/\mathcal{P}^1, \chi_r)$ by the formulas

$$\begin{aligned}\lambda(vxv_1 \ldots v_m) &= \widetilde{\lambda}(v(xv_1x^{-1}) \ldots (xv_mx^{-1}), x) \\ &= \mu(v(xv_1x^{-1}) \ldots (xv_mx^{-1})) \cdot \alpha(x).\end{aligned}$$

One checks that

$$\langle \lambda, \lambda \rangle = \langle \mu, \mu \rangle \langle \nu, \nu \rangle = 1 = \langle S_{\bar{p}}\lambda, S_{\bar{q}}\lambda \rangle = \langle S_2\lambda, S_2\lambda \rangle,$$

where the last equalities are due to (5.4.). Moreover, for $k = vv_1 \ldots v_m \in \mathcal{K}$,

$$\begin{aligned}&(S_{\bar{2}}\lambda)(k, x) = \mu(k)\nu(x), \\ &(S_{\bar{p}}\lambda)(k, x) = e^{\sum_{j=1}^m (\frac{1}{2} - \frac{1}{p_j})\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log x)}\mu(k)\nu(x) = \delta^{\frac{1}{2} - \frac{1}{\bar{p}}}(x)\mu(k)\nu(x).\end{aligned}$$

In order to construct the function $p_\lambda \in L^1(\mathcal{H})$ that will give us the projectors associated to λ, let's put

$$a(x, y) = \int_{\mathcal{X}} \alpha(xu)\alpha(u)e^{i\langle r, [\log y, \log u]\rangle} \cdot e^{i\langle r, \log y\rangle} \cdot \Delta^{\frac{1}{2}}(y)du, \quad \forall x \in \mathcal{X}, \forall y \in \mathcal{Y},$$

and define $p_\lambda \in \mathcal{ES}(\mathcal{H}) \subset L^1(\mathcal{H})$ by $p_\lambda(xyk) = p_\lambda(x, y)(k) = a(x, y)g(xy)(k)$.

Proposition 5.5.1. *For every $\bar{p} \in [1, \infty]^m$, the operator $\gamma_{\bar{p}}(p_\lambda)$ is a rank one operator. This is in particular true for the operator $\gamma(p_\lambda) = \gamma_{\bar{2}}(p_\lambda)$.*

Proof. Using the previous computations and the observation (5.1.b) we get

$$\begin{aligned} & p_\lambda(x'x^{-1},y)_{\rho_{\overline{p}}}((k')^{(x'x^{-1}y)^{-1}},k) \\ &= a(x'x^{-1},y)\mu(k')\overline{\mu(k)} \\ &= \mu(k')\overline{\mu(k)}\int_{\mathcal{X}}\alpha(x'x^{-1}u)\alpha(u) \\ &\qquad \cdot e^{i\langle r,[\log y,\log u]\rangle}e^{i\langle r,\log y\rangle}\Delta^{\frac{1}{2}}(y)\delta^{-1}(u)du, \end{aligned}$$

where $\delta^{-1}(u)=e^{-\sum_{j=1}^m \operatorname{tr}\,\operatorname{ad}_{\mathfrak{p}_j/\mathfrak{p}_{j+1}}(\log u)}$. Hence

$$\begin{aligned} & (p_\lambda)_{\kappa_{\overline{p}}}((k',x'),(k,x)) \\ &= \mu(k')\overline{\mu(k)}\int_{\mathcal{Y}}\int_{\mathcal{X}}\alpha(x'x^{-1}u)\alpha(u) \\ &\qquad \cdot e^{i\langle r,[\log y,\log u]\rangle}e^{-i\langle r,[\log y,\log x]\rangle}\Delta^{\frac{1}{2}-\frac{1}{\overline{p}}}(y)\delta^{-1}(u)\delta^{-\frac{1}{\overline{p}}}(x'x^{-1})du, \\ &= \mu(k')\overline{\mu(k)}\int_{\mathcal{Y}}\int_{\mathcal{X}}\nu(x'x^{-1}u)\nu(u) \\ &\qquad \cdot e^{i\langle r,[\log y,\log u]\rangle}e^{-i\langle r,[\log y,\log x]\rangle}\Delta^{\frac{1}{2}-\frac{1}{\overline{p}}}(y)\delta^{\frac{1}{2}-\frac{1}{\overline{p}}}(x'x^{-1})du. \end{aligned}$$

In particular, for $\overline{p}=\overline{2}$ and $\kappa=\kappa_{\overline{2}}$, we have

$$\begin{aligned} (p_\lambda)_\kappa((k',x'),(k,x)) &= \mu(k')\overline{\mu(k)}\int_{\mathcal{Y}}\int_{\mathcal{X}}\nu(x'x^{-1}u)\nu(u) \\ &\qquad \cdot e^{i\langle r,[\log y,\log u]\rangle}e^{-i\langle r,[\log y,\log x]\rangle}dudy \\ &= \mu(k')\overline{\mu(k)}\nu(x')\overline{\nu(x)} \\ &= (S_{\overline{2}}\lambda)(k',x')\overline{(S_{\overline{2}}\lambda)(k,x)}, \end{aligned}$$

as ν is in fact a real-valued function. Hence $\kappa(p_\lambda)$ is a projector, i.e.,

$$\kappa(p_\lambda)=P_{S_{\overline{2}}\lambda,S_{\overline{2}}\lambda} \text{ and } \gamma(p_\lambda)=P_{\lambda,\lambda}.$$

In order to characterize the kernel of $\kappa_{\overline{p}}(p_\lambda)$, let's recall that

$$\Delta^{(\frac{1}{2}-\frac{1}{\overline{p}})}(y)=e^{\sum_{j=1}^m\left(\frac{1}{2}-\frac{1}{p_j}\right)\operatorname{tr}\lambda_j(\log y)}$$

where $\sum_{j=1}^m\left(\frac{1}{p_j}-\frac{1}{2}\right)\operatorname{tr}\lambda_j(\cdot)$ is a linear form on $\mathfrak{Y}$ and may hence be identified with an element of $\mathfrak{X}$, because of the duality between $\mathfrak{X}$ and $\mathfrak{Y}$ (see (2.2.)). Let's write $\frac{1}{\overline{p}}-\frac{1}{2}$ for this element of $\mathfrak{X}$, i.e.,

$$\left\langle r,\left[\frac{1}{\overline{p}}-\frac{1}{2},\log y\right]\right\rangle=\sum_{j=1}^m\left(\frac{1}{p_j}-\frac{1}{2}\right)\operatorname{tr}\lambda_j(\log y),\quad \forall y\in\mathcal{Y},$$

by definition. The function ν has been chosen analytic in $\mathcal{E}S(\mathcal{X}) \equiv \mathcal{E}S(\mathfrak{X})$. Hence it admits an extension to a complex-valued analytic function on $\mathfrak{X}_{\mathbb{C}}$ which we shall also denote by ν. We compute

$$\begin{aligned}
&(p_\lambda)_{\kappa_{\bar{p}}}((k',x'),(k,x))\\
&= \mu(k')\overline{\mu(k)}\delta^{\frac{1}{2}-\frac{1}{\bar{p}}}(x'x^{-1})\int_{\mathcal{Y}}\int_{\mathcal{X}}\nu((x'x^{-1})\cdot u)\nu(u)\\
&\qquad \cdot e^{i\langle r,[\log y,\log u]\rangle}e^{-i\langle r,[\log y,\log x-i(\frac{1}{\bar{p}}-\frac{1}{2})]\rangle}dudy\\
&= \mu(k')\overline{\mu(k)}\delta^{-\frac{1}{\bar{p}}}(x')\delta^{\frac{1}{2}}(x')\nu\left(\log x'-i\left(\frac{1}{\bar{p}}-\frac{1}{2}\right)\right)\\
&\qquad \cdot\delta^{-\frac{1}{\bar{q}}}(x)\delta^{\frac{1}{2}}(x)\nu\left(\log x-i\left(\frac{1}{\bar{p}}-\frac{1}{2}\right)\right),
\end{aligned}$$

if we identify ν with a function on the complex vector space $\mathfrak{X}_{\mathbb{C}}$ and if $\frac{1}{\bar{q}} = 1-\frac{1}{\bar{p}}$.

Let's define new functions $\nu_{\bar{p}} \in \mathcal{E}S(\mathcal{X}) \equiv \mathcal{E}S(\mathfrak{X})$, $\zeta_1,\zeta_2 \in L^{\bar{p}}(\mathcal{K}/\mathcal{P}^0, \chi_r)\hat{\otimes}L^2(\mathcal{X})$ and $\lambda_{\bar{p}},\lambda'_{\bar{p}} \in \mathcal{E}S(\mathcal{H}) \subset L^1(\mathcal{H})$ by

$$\nu_{\bar{p}}(x) = \delta^{\frac{1}{2}}(x)\nu\left(\log x - i\left(\frac{1}{\bar{p}}-\frac{1}{2}\right)\right), \quad \forall x \in \mathcal{X},$$

$$\begin{aligned}
(S_{\bar{p}}\lambda_{\bar{p}})(k,x) &= \delta^{-\frac{1}{\bar{p}}}(x)\mu(k)\nu_{\bar{p}}(x) = \zeta_1(k,x)\\
(S_{\bar{q}}\lambda'_{\bar{p}})(k,x) &= \delta^{-\frac{1}{\bar{q}}}(x)\mu(k)\overline{\nu_{\bar{p}}(x)} = \zeta_2(k,x),
\end{aligned}$$

i.e., $\lambda_{\bar{p}} = S_{\bar{p}}^{-1}(\zeta_1)$ and $\lambda'_{\bar{p}} = S_{\bar{q}}^{-1}(\zeta_2)$. Hence

$$(p_\lambda)_{\kappa_{\bar{p}}}((k',x'),(k,x)) = (S_{\bar{p}}\lambda_{\bar{p}})(k',x')\overline{(S_{\bar{q}}\lambda'_{\bar{p}})(k,x)},$$

i.e., $\kappa_{\bar{p}}(p_\lambda)$ is the projector $P_{S_{\bar{p}}\lambda_{\bar{p}},S_{\bar{q}}\lambda'_{\bar{p}}}$. So $\gamma_{\bar{p}}(p_\lambda)$ is also a projector, more precisely $\gamma_{\bar{p}}(p_\lambda) = P_{\lambda_{\bar{p}},\lambda'_{\bar{p}}}$. Both projectors are idempotent because

$$\begin{aligned}
\langle\lambda_{\bar{p}},\lambda'_{\bar{p}}\rangle &= \langle S_{\bar{p}}\lambda_{\bar{p}}, S_{\bar{q}}\lambda'_{\bar{p}}\rangle\\
&= \int_{\mathcal{X}}\int_{\mathcal{K}}\delta^{-\frac{1}{\bar{p}}}(x)\mu(k)\nu_{\bar{p}}(x)\delta^{-\frac{1}{\bar{q}}}(x)\overline{\mu(k)}\nu_{\bar{p}}(x)dkdx\\
&= \langle\mu,\mu\rangle\cdot\int_{\mathcal{X}}\delta^{-1}(x)(\nu_{\bar{p}}(x))^2dx\\
&= \int_{\mathcal{X}}\left(\nu\left(\log x - i\left(\frac{1}{\bar{p}}-\frac{1}{2}\right)\right)\right)^2dx = 1,
\end{aligned}$$

as $\frac{1}{\bar{p}}+\frac{1}{\bar{q}} = 1$, $\langle\mu,\mu\rangle = 1$ and

$$\int_{\mathcal{X}}\left(\nu\left(\log x - i\left(\frac{1}{\bar{p}}-\frac{1}{2}\right)\right)\right)^2dx = \int_{\mathcal{X}}(\nu(x))^2dx = \langle\nu,\nu\rangle = 1$$

by Cauchy's theorem. □

5.5.2. . The following computation will be important in the characterization of the character associated to $\gamma_{l,\bar{p}} \equiv \pi_{l,\mathfrak{p},\bar{p}}$:

$$\begin{aligned}
&\langle \gamma_{l,\bar{p}}(t)\lambda_{\bar{p}}, \lambda'_{\bar{p}}\rangle \\
&= \langle \kappa_{l,\bar{p}}(t)(S_{\bar{p}}\lambda_{\bar{p}}), S_{\bar{q}}\lambda'_{\bar{p}}\rangle \\
&= \Delta^{-\frac{1}{\bar{p}}}(t)\chi_l(t) \int_{\mathcal{X}}\int_{\mathcal{K}} (S_{\bar{p}}\lambda_{\bar{p}})(k^{t^{-1}}, x)\overline{(S_{\bar{q}}\lambda'_{\bar{p}})(k,x)}dkdx \\
&= \Delta^{-\frac{1}{\bar{p}}}(t)\chi_l(t) \int_{\mathcal{X}}\int_{\mathcal{K}} \delta^{-1}(x)\mu(k^{t^{-1}})\nu_{\bar{p}}(x)\overline{\mu(k)}\nu_{\bar{p}}(x)dkdx \\
&= \Delta^{-\frac{1}{\bar{p}}}(t)\chi_l(t)\langle \mu^{t^{-1}}, \mu\rangle.
\end{aligned}$$

In particular, for $\bar{p} = \overline{2} = (2, \dots, 2)$, $\lambda_{\overline{2}} = \lambda'_{\overline{2}} = \lambda$ (as ν is real-valued) and

$$\frac{\langle \widetilde{\gamma}_{l,\bar{p}}(t)\lambda_{\bar{p}}, \lambda'_{\bar{p}}\rangle}{\langle \widetilde{\gamma}_l(t)\lambda, \lambda\rangle} = \Delta^{\frac{1}{2}-\frac{1}{\bar{p}}}(t).$$

5.6. Character of $L^1(\mathbb{R}^n, \omega)$ corresponding to $\gamma_{l,\bar{p}}$.

5.6.1. . Using the computations of (3.5.) and (3.7.) we get

$$\begin{aligned}
\gamma_{l,\bar{p}}(v_{\lambda,l}(t)) &= \frac{1}{\langle \gamma_l(t)\lambda, \lambda\rangle}\gamma_{l,\bar{p}}(t^{-1})P_{\lambda_{\bar{p}},\lambda'_{\bar{p}}}\gamma_{l,\bar{p}}(t)P_{\lambda_{\bar{p}}\lambda'_{\bar{p}}} \\
&= \Delta^{\frac{1}{2}-\frac{1}{\bar{p}}}(t)\gamma_{l,\bar{p}}(t^{-1})\gamma_{\bar{p}}(p_\lambda)
\end{aligned}$$

and

$$(\gamma_{l,\bar{p}}(h\cdot v))(\gamma_{\bar{p}}(p_\lambda)\xi) = \left(\int_{\mathcal{U}} h(t)\chi_{l-l_0}(t)\Delta^{\frac{1}{2}-\frac{1}{\bar{p}}}(t)dt\right)(\gamma_{\bar{p}}(p_\lambda)\xi).$$

Hence $\chi_{l,\bar{p}}(t) = \Delta^{\frac{1}{2}-\frac{1}{\bar{p}}}(t)\chi_{l-l_0}(t)$ is the character of $L^1(\mathcal{U},\omega) \equiv L^1(\mathcal{G}(l)/\mathcal{G}(l)\cap \mathcal{N},\omega)$ associated to the representation $\gamma_{l,\bar{p}}$.

Remark. If l and l' are two different extensions of $r \in \mathfrak{h}^*$ to $\mathfrak{g}$, then $\chi_{l',\bar{p}}(t) = \chi_{l'-l}(t)\cdot\chi_{l,\bar{p}}(t)$, $\forall t \in \mathcal{U} \subset \mathcal{G}(l)$. Hence, if l and l' are in the same $\mathcal{G}$-orbit, then $\chi_{l',\bar{p}}(t) = \chi_{l,\bar{p}}(t)$, $\quad \forall t \in \mathcal{U} \subset \mathcal{G}(l)$.

Corollary 5.6.2. *The weight ω satisfies the inequality*

$$\Delta^{(\frac{1}{2}-\frac{1}{\bar{p}})}(x) = e^{\sum_{j=1}^m\left(\frac{1}{2}-\frac{1}{p_j}\right)\mathrm{tr}\lambda_j(\log x)} \le \omega(x), \quad \forall x \in \mathcal{G}(l),$$

for all $\bar{p}$. Hence

$$\begin{aligned}
e^{\frac{|t|}{2}\sum_{j=1}^m |\mathrm{tr}\lambda_j(X)|} &\le \omega(\exp tX) \\
&\le C'(1+|t|)^C e^{\frac{|t|}{2}\sum_{j=1}^m |\mathrm{tr}\lambda_j(X)|}, \quad \forall t \in \mathbb{R}, \forall X \in \mathfrak{g}(l).
\end{aligned}$$

Proof. If $t \in \mathcal{G}(l) \cap \mathcal{N}$, then $\Delta^{(\frac{1}{2}-\frac{1}{\bar{p}})}(t) = 1$ and $\omega \geq 1$. For $t \in \mathcal{U}$, for every multi-index $\bar{p} \in [1,\infty]^m$, $\Delta^{(\frac{1}{2}-\frac{1}{\bar{p}})}(t) = |\chi_{l,\bar{p}}(t)| \leq \omega(t)$ as $\chi_{l,\bar{p}}$ is a continuous character on $L^1(\mathbb{R}^n,\omega) \equiv L^1(\mathcal{G}(l)/\mathcal{G}(l)\cap\mathcal{N},\omega)$ (see proof of (4.2.)). See (4.1.) for the last assertion. □

5.7. Characterization of an arbitrary simple $L^1(\mathcal{G})$-module.

Proposition 5.7.1. *Let S_1 and S_2 be the following subsets of $(\mathbb{R}^n)^* \equiv \mathfrak{u}^* \equiv (\mathfrak{g}(l)/\mathfrak{g}(l)\cap\mathfrak{n})^*$:*

$$S_1 = \left\{ \sum_{i=1}^m \left(\frac{1}{2} - \frac{1}{p_i}\right) \mathrm{tr}\lambda_i(\cdot) \mid 1 \leq p_i \leq \infty \right\}$$

$$= \left\{ \sum_{i=1}^m C_i \mathrm{tr}\lambda_i(\cdot) \mid |C_i| \leq \frac{1}{2}, C_i \in \mathbb{R} \right\}$$

$$S_2 = \left\{ \rho \in \mathfrak{u}^* \mid |\rho(X)| \leq \sum_{i=1}^m \frac{1}{2}|\mathrm{tr}\lambda_i(X)|,\ \forall X \in \mathfrak{u} \equiv \mathbb{R}^n \right\}.$$

Then $S_1 = S_2$.

Proof. Notice first that the linear form $\nu(\cdot) = \sum_{i=1}^m (\frac{1}{2} - \frac{1}{p_i})\mathrm{tr}\lambda_i(\cdot)$ of $\mathfrak{g}(l)$ is constant on the classes modulo $\mathfrak{g}(l)\cap\mathfrak{n}$ and may hence be considered as a linear form on $\mathfrak{g}(l)/\mathfrak{g}(l)\cap\mathfrak{n}$. The sets S_1 and S_2 are closed convex subsets of $(\mathbb{R}^n)^*$ such that $S_1 \subset S_2$. Assume there exists $\rho \in S_2 \backslash S_1$. By the Hahn-Banach theorem there is $X_0 \in \mathbb{R}^n \equiv \mathfrak{u}$ and $\alpha \in \mathbb{R}$ such that $s_1(X_0) < \alpha < \rho(X_0)$, $\forall s_1 \in S_1$. Let's then choose $s_1 \in S_1$ by $s_1(X) = \sum_{i=1}^m \frac{1}{2}\varepsilon_i \mathrm{tr}\lambda_i(X)$, $\forall X \in \mathfrak{u}$, where $\varepsilon_i = 1$ if $\mathrm{tr}\lambda_i(X_0) \geq 0$ and $\varepsilon_i = -1$ if $\mathrm{tr}\lambda_i(X_0) < 0$. Hence

$$\sum_{i=1}^m \frac{1}{2}|\mathrm{tr}\lambda_i(X_0)| = s_1(X_0) < \rho(X_0),$$

which contradicts the fact that $\rho \in S_2$. □

Corollary 5.7.2. *Let χ be a continuous character on $L^1(\mathbb{R}^n,\omega) \equiv L^1(\mathcal{U},\omega)$. Then there is a multi-index $\bar{p} = (p_1,\dots,p_m)$ and $l' \in \mathfrak{g}^*$ with $l'|_{\mathfrak{h}} = l|_{\mathfrak{h}}$ such that*

$$|\chi(\exp X)| = e^{\sum_{i=1}^m \left(\frac{1}{2}-\frac{1}{p_i}\right)\mathrm{tr}\lambda_i(X)}, \quad \forall X \in \mathfrak{u}$$

and such that

$$\chi(\exp X) = \chi_{l'-l_0}(\exp X) \cdot e^{\sum_{i=1}^m \left(\frac{1}{2}-\frac{1}{p_i}\right)\mathrm{tr}\lambda_i(X)} = \chi_{l',\bar{p}}(\exp X), \quad \forall X \in \mathfrak{u},$$

i.e., every continuous character on $L^1(\mathcal{G}(l)/\mathcal{G}(l)\cap\mathcal{N},\omega)$ is of the form

$$\chi(\exp X) = \chi_{l'-l_0}(\exp X) e^{\nu(X)},$$

where $l' - l_0 \in \mathfrak{h}^\perp$ *and* $\nu \in (\mathfrak{g}(l)/\mathfrak{g}(l) \cap \mathfrak{n})^*$ *such that*

$$|\nu(X)| \leq \frac{1}{2} \sum_{j=1}^{m} |\mathrm{tr}\lambda_j(X)|.$$

Proof. We may write $\chi(\exp X) = e^{-i\rho_1(X)} \cdot e^{\rho_2(X)}$ with $\rho_1, \rho_2 \in (\mathbb{R}^n)^* \equiv \mathfrak{u}^* \equiv (\mathfrak{g}(l)/\mathfrak{g}(l) \cap \mathfrak{n})^*$. By (4.2.) and (5.6.1.) $\rho_2 \in S_2 = S_1$ and hence there is a multi-index $\bar{p} = (p_1, \ldots, p_m)$, $p_i \in [1, \infty]$ for all i, such that

$$|\chi(\exp X)| = e^{\rho_2(X)} = e^{\sum_{i=1}^m \left(\frac{1}{2} - \frac{1}{p_i}\right)\mathrm{tr}\lambda_i(X)}, \quad \forall X \in \mathfrak{u} \equiv (\mathfrak{g}(l)/\mathfrak{g}(l) \cap \mathfrak{n})^*.$$

We may then choose $l' \in \mathfrak{g}^*$ such that $l'|_\mathfrak{h} = l|_\mathfrak{h}$ and such that $l' - l_0 = \rho_1$ on $\mathfrak{u}$. □

Theorem 5.7.3.

a) *Let* $(T, \mathfrak{U})$ *be a simple* $L^1(\mathcal{G})$*-module. Then there exists* $l \in \mathfrak{g}^*$*, a polarization* $\mathfrak{p}$ *for* l *in* $\mathfrak{g}$ *and a multi-index* $\bar{p} \in [1, \infty]^m$*, such that* $(T, \mathfrak{U})$ *is equivalent to the simple module* $\left(\pi^0_{l,\mathfrak{p},\bar{p}}, \mathfrak{H}^0_{l,\mathfrak{p},\bar{p}}\right)$.

b) *Let* $\bar{p}$, $\bar{q} \in [1, \infty]^m$ *be two multi-indices. Then* $\left(\pi^0_{l,\mathfrak{p},\bar{q}}, \mathfrak{H}^0_{\pi_{l,\mathfrak{p},\bar{q}}}\right) \simeq \left(\pi^0_{l,\mathfrak{p},\bar{p}}, \mathfrak{H}^0_{\pi_{l,\mathfrak{p},\bar{p}}}\right)$ *if and only if*

$$\sum_{i=1}^{m} \left(\frac{1}{2} - \frac{1}{q_i}\right) \mathrm{tr}\,\lambda_i(\cdot) = \sum_{i=1}^{m} \left(\frac{1}{2} - \frac{1}{p_i}\right) \mathrm{tr}\,\lambda_i(\cdot) = \nu(\cdot)$$

on $\mathfrak{u}$ *and hence on* $\mathfrak{g}(l)$*, i.e., if the corresponding linear forms* $\nu \in (\mathfrak{g}(l)/\mathfrak{g}(l) \cap \mathfrak{n})^*$ *are the same.*

Proof. By (3.6.) and (5.7.2.). □

5.7.4. Remarks.

a) One can show that up to equivalence the representations $\pi^0_{l,\mathfrak{p},\bar{p}}$ are independent of the choice of the polarization $\mathfrak{p}$.

b) Let's write $\widetilde{\mathcal{G}}$ for the space of the equivalence classes of simple $L^1(\mathcal{G})$-modules. Let's write $\widetilde{\mathfrak{g}^*}$ for the collection of all pairs (l, ν) with $l \in \mathfrak{g}^*$, $\nu \in (\mathfrak{g}(l)/\mathfrak{g}(l) \cap \mathfrak{n})^*$ such that $|\nu(X)| \leq \frac{1}{2} \sum_{j=1}^m |\mathrm{tr}\lambda_j(X)|, \forall X \in \mathfrak{g}(l)$. The group $\mathcal{G}$ acts on $\widetilde{\mathfrak{g}^*}$ by conjugation. Let $\widetilde{\mathfrak{g}^*}/\mathcal{G}$ be the set of all equivalence classes for this action.

We then get our final theorem:

Theorem 5.7.5. *There is a bijection between* $\widetilde{\mathfrak{g}^*}/\mathcal{G}$ *and* $\widetilde{\mathcal{G}}$.

6. Final remarks.

As it was already pointed out in the introduction, the algebraically simple $L^1(\mathcal{G})$-modules for a solvable exponential Lie group are essentially obtained in the same way as in the case of the nilpotent groups, except that one has to generalize the induced representations. This is no longer true for topologically irreducible representations, as it was shown in ([**LuMo2**]). Two major differences exist. Usually there are a lot of extensions of a topologically irreducible representation of the subalgebra $p * L^1(\mathcal{G}/\mathcal{H}, L^1(\mathcal{H})/\ker\gamma) * p$ to a topologically irreducible representation of the algebra $L^1(\mathcal{G}/\mathcal{H}, L^1(\mathcal{H})/\ker\gamma)$, whereas this extension is unique in the algebraic case. These different extensions are characterized by different extension norms. But the main difference arises from the irreducible representations of $L^1(\mathbb{R}^n, \omega)$. These representations coincide with the characters in the algebraic case. In the topological case there are a lot of irreducible inifinite dimensional representations of $L^1(\mathbb{R}^n, \omega)$ if the weight ω is exponential, which happens if and only if the group $\mathcal{G}$ is nonsymmetric. The corresponding representations of $L^1(\mathcal{G})$ are fundamentally different from induced representations. The construction of such representations is linked to the invariant subspace problem, as it was shown in ([**LuMo2**]).

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