Pacific Journal of Mathematics

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Volume 212 No. 2 December 2003

PACIFIC JOURNAL OF MATHEMATICS
Vol. 212, No. 2, 2003

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Let R and S be arbitrary associative rings. A left R-module ${}_RW$ is said to be cotilting if the class of modules cogenerated by ${}_RW$ coincides with the class of modules for which the functor $\mathrm{Ext}^1_R(-, W)$ vanishes. In this paper we characterize the cotilting modules which are pure-injective. The two notions seem to be strictly connected: Indeed all the examples of cotilting modules known in the literature are pure-injective. We observe that if ${}_RW_S$ is a *pure-injective cotilting bimodule*, both R and S are semiregular rings and we give a characterization of the reflexive modules in terms of a suitable "linear compactness" notion.

Introduction.

Cotilting modules first appeared as vector space duals of tilting modules over finite dimensional algebras [**12**, IV, 7.8]. Recently they have been introduced [**5**] in the framework of modules over arbitrary associative rings, acquiring a proper independent role. The cotilting modules generalize the notion of injective cogenerator: They are injectives with respect to short exact sequences of modules cogenerated by them.

For arbitrary rings R and S, a Morita duality between left R-modules and right S-modules is given by the contravariant Hom functors associated to a *Morita bimodule*, i.e., a faithfully balanced bimodule ${}_RW_S$ with ${}_RW$ and W_S both injective cogenerators. One of the major component in the theory of Morita dualities is Müller's theorem [**13**] which states that the reflexive modules are precisely the linearly compact modules. If ${}_RW_S$ is a Morita bimodule, both R and S are semiperfect rings [**16**, Theorem 2.7].

For arbitrary rings R and S, a cotilting duality between left R-modules and right S-modules is given by the contravariant Hom functors and the contravariant Ext functors associated to a cotilting bimodule, i.e., a faithfully balanced bimodule ${}_RW_S$ with both ${}_RW$ and W_S cotilting modules (see [**4**]).

All known examples of cotilting modules are *pure-injective*. In this paper we characterize the pure-injective cotilting modules. We observe that if ${}_RW_S$ is a *pure-injective cotilting bimodule*, both R and S are semiregular rings and we give a characterization of the reflexive modules in terms of a suitable “linear compactness” notion.

1. About the pure-injectivity of a cotilting module.

Let R be an associative ring with $1 \neq 0$. We denote by R-Mod the category of left unitary$R$-modules and their homomorphisms. Given a left R-module W, we consider the following classes:

- $\operatorname{Cogen} W$ denotes the class of all left R-modules *cogenerated* by ${}_RW$, that is all M in R-Mod such that there exist a cardinal λ and a monomorphism $M \hookrightarrow W^\lambda$;
- ${}^\perp W$ denotes the class of all left R-modules M such that $\operatorname{Ext}^1_R(M, W) = 0$.

A left R-module ${}_RW$ is said to be *cotilting* [**5**] if $\operatorname{Cogen} {}_RW = {}^\perp W$. The cotilting modules generalize injective cogenerators: Clearly ${}_RW$ is an injective cogenerator if and only if both the classes $\operatorname{Cogen} {}_RW$ and ${}^\perp W$ coincide with the whole category of left R-modules. A short exact sequence

$$0 \to K \to L \to M \to 0$$

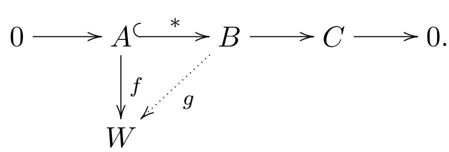
is said to be *pure* if any morphism $P \to M$, with P finitely presented, lifts to a morphism $P \to L$.

Definition 1.1. A module ${}_RW$ is *pure-injective* if it is injective with respect to any pure exact sequence.

All known examples of cotilting modules are pure-injective. It naturally arises the question how the two notions are related.

Proposition 1.2. *Let ${}_RW$ be a cotilting module. If the class* $\operatorname{Cogen} W$ *is closed under direct limits, then W is pure-injective.*

Proof. Let us show that for any pure exact sequence $0 \to A \to B \to C \to 0$ and for any map $f : A \to W$ there exists a map g making the following diagram commute:



Replacing $A \overset{*}{\hookrightarrow} B$ by $A/\operatorname{Rej}_W A \overset{*}{\hookrightarrow} B/\operatorname{Rej}_W A$,we can assume that $\operatorname{Rej}_W A = 0$. The pure exact sequence $0 \to A \to B \to C \to 0$ is a direct limit of split exact sequences $0 \to A \to B_i \to C_i \to 0$ with C_i finitely presented (cf. 34.2 [**14**]). For each index i we have the commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & \operatorname{Rej}_W B_i & \xrightarrow{\cong} & \operatorname{Rej}_W C_i & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B_i & \longrightarrow & C_i & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B_i/\operatorname{Rej}_W B_i & \longrightarrow & C_i/\operatorname{Rej}_W C_i & \longrightarrow & 0.
\end{array}$$

Applying the direct limit functor we get

$$\begin{array}{ccccccccc}
 & & & & \varinjlim \operatorname{Rej}_W B_i & \xrightarrow{\cong} & \varinjlim \operatorname{Rej}_W C_i & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & \varinjlim(B_i/\operatorname{Rej}_W B_i) & \longrightarrow & \varinjlim(C_i/\operatorname{Rej}_W C_i) & \longrightarrow & 0.
\end{array}$$

Since $\varinjlim \operatorname{Rej}_W B_i$ is in the kernel of $\operatorname{Hom}(-, W)$ and $\varinjlim(B_i/\operatorname{Rej}_W B_i)$ belongs to $\operatorname{Cogen} W$ by the assumption, we infer that $\varinjlim(B_i/\operatorname{Rej}_W B_i) \cong B/\operatorname{Rej}_W B$. Also $\varinjlim(C_i/\operatorname{Rej}_W C_i)$ belongs to $\operatorname{Cogen} W = {}^\perp W$. So f can be extended to a morphism $g' : B/\operatorname{Rej}_W B \to W$; the composition of g' with the canonical projection $B \to B/\operatorname{Rej}_W B$ yields the desired map g. □

In the cotilting case, since $\operatorname{Cogen} W = {}^\perp W$, the hypothesized closure under direct limits of the class of modules cogenerated by W is suggested by the following proposition:

Proposition 1.3. *If ${}_R W$ is a pure-injective module, then ${}^\perp_R W$ is closed under direct limits.*

Proof. Consider a direct system $\{M_i : i \in I\}$ in ${}^\perp W$. The canonical exact sequence

$$0 \to K \to \oplus_{i\in I} M_i \to \varinjlim{}_{i\in I} M_i \to 0$$

is pure (cf. [**14**, 33.9, (2)]). Applying $\operatorname{Hom}_R(-, W)$ we get the long exact sequence

$$\cdots \to \operatorname{Hom}(\oplus_{i\in I} M_i, W) \xrightarrow{f} \operatorname{Hom}(K, W) \to \operatorname{Ext}^1_R(\varinjlim{}_{i\in I} M_i, W) \to$$
$$\to \operatorname{Ext}^1_R(\oplus_{i\in I} M_i, W) = 0.$$

Since W is pure-injective, f is surjective; so $\varinjlim{}_{i\in I} M_i$ belongs to ${}^\perp W$. □

This result has been used in [**9**, Lemma 9] to prove that, if $\mathcal{C}$ is a class of pure-injective modules, every module M which has a ${}^\perp\mathcal{C}$-precover has a

${}^\perp\mathcal{C}$-cover (see [**15**] for an extensive introduction to theory of (pre)covers and (pre)envelopes of modules, including various recent results).

Corollary 1.4. *If ${}_RW$ is a cotilting module, then W is pure-injective if and only if* $\operatorname{Cogen} W$ *is closed under direct limits. In such a case any module has a* $\operatorname{Cogen} W$*-cover.*

Proof. The first claim follows by Propositions 1.2 and 1.3. The second one follows by [**1**, Corollary 2.6]. □

Open Problem 1.5. Are all cotilting (bi)modules pure-injective?[1]

It is well-known that the endomorphism rings of a Morita bimodule are semiperfect; indeed a Morita bimodule is injective and finitely cogenerated on both sides (see [**16**, Theorem 2.7, Proposition 1.19]). We are able to give an analogous result for cotilting bimodules, assuming the closure under direct limits of the classes cogenerated by them. We recall that a ring R is said to be *semiregular* if $R/J(R)$ is regular and the idempotents lift over the Jacobson radical $J(R)$.

Proposition 1.6. *Let ${}_RW_S$ be a pure-injective cotilting bimodule, i.e., faithfully balanced and cotilting and pure-injective on both sides. Then both R and S are semiregular rings.*

Proof. The notions of pure-injective and algebraically compact module coincide (cf. [**14**, 34.4]). Then, by [**17**, Theorem 9], R and S are both semiregular. □

Remark 1.7. Observe that if the ring R is regular, the classes of pure-injective and of injective R-modules coincide [**14**, 37.6]. Therefore pure-injective cotilting bimodules which are not Morita bimodules "live in the space between semiregular and regular rings".

2. Characterizing the reflexive modules.

Let ${}_RW_S$ be an arbitrary bimodule. In the sequel we denote by Δ the functors $\operatorname{Hom}_?(-, W)$, and by Γ the functors $\operatorname{Ext}^1_?(-, W)$, where ? stands for R or S. We denote by Δ^2 both the compositions $\operatorname{Hom}_R(\operatorname{Hom}_S(-, W), W)$ and $\operatorname{Hom}_S(\operatorname{Hom}_R(-, W), W)$. Given a (left R, or right S)-module M, we denote by δ_M the canonical homomorphism $M \to \Delta^2(M)$ defined by $m \mapsto [f \mapsto f(m)]$. A module M is called *reflexive* (resp. *torsionless*) if δ_M is an isomorphism (resp. monomorphism).

Clearly a torsionless module M is reflexive if and only if the evaluation map δ_M is surjective. Endowed M and $\Delta^2 M$ with any topology, the surjectivity of δ_M can be tested in a topological way asking for $\operatorname{Im}\delta_M$ to be both dense and closed in $\Delta^2 M$. As the approach of Müller [**13**] to the classical

[1] Recently Silvana Bazzoni proved that any cotilting module is pure-injective [**2**].

case of Morita dualities suggests, we introduce topological tools in order to characterize the reflexive modules.

Let us endow $\Delta^2 M$ with the *finite topology* φ: The linear topology for which the family of submodules $V(F) = \{\alpha \in \Delta^2 M : \alpha(F) = 0\}$, where F is a finite subset of ΔM, is a base for the filter of neighbourhoods of zero.

Let us endow W with the discrete topology. Given any torsionless module M, we associate with each subset A of ΔM the *weak topology* with respect to morphisms in A, denoted by τ_A. By definition τ_A is the coarsest topology on M such that all morphisms in A are continuous: It is a linear topology with a base for its filter of neighbourhoods of zero formed by finite intersections of kernels of morphisms in A. In the sequel the topology $\tau_{\Delta M}$ will be shortly denoted by τ. Note that τ is the maximum element of the set of linear topologies $\{\tau_A : A \subseteq \Delta M\}$ partially ordered by inclusion. Let L_σ be a linearly topologized module. Denote by H the σ-closure of zero in L. Note that H is equal to the intersection of all neighbourhoods of zero and, since σ is a linear topology, σ is Hausdorff if and only if $H = 0$. A σ-*Cauchy net* in L is a family X_λ, $\lambda \in \Lambda$, indexed by the upwards directed partially ordered set Λ, such that for every neighbourhood U of zero there exists an upper subset Λ' of Λ with $x_\lambda - x_{\lambda'} \in U$ for every λ, $\lambda' \in \Lambda'$. The topology σ is complete, i.e., any σ-Cauchy net in L converges in L, if and only if the topological quotient L/H is complete (see [**3**, Chap. 3, §2]). Note that a closed submodule of a complete module is complete. The completion of L/H is called the Hausdorff completion of L: Denoted by $\mathcal{J} = \{J_\lambda : \lambda \in \Lambda\}$ a base for the filter of neighbourhoods of zero in L_σ consisting of open submodules, it coincides with the inverse limit $\varprojlim L/J_\lambda$ (see [**10**, Proposition 13.7]).

Proposition 2.1. *Let M be a torsionless module.*

(i) *The topologies τ on M and φ on $\Delta^2 M$ are Hausdorff.*
(ii) *$\delta_M : M_\tau \to \Delta^2 M_\varphi$ is a topological embedding.*
(iii) *The topology φ on $\Delta^2 M$ is complete.*

Proof. (i) Since M is cogenerated by W, there exists a set X and the following maps:

$$M \overset{i}{\hookrightarrow} W^X \overset{\pi_x}{\to} W.$$

Clearly $\{0\}$ is the intersection of $\operatorname{Ker}(\pi_x \circ i)$, $x \in X$. Since any $\operatorname{Ker}(\pi_x \circ i)$ is τ-open and hence τ-closed, τ is Hausdorff. Let us consider the topology φ. The open submodules $V(F) = \{\alpha \in \Delta^2 M : \alpha(F) = 0\}$, with F finite subset of ΔM, have intersection zero: Hence φ is Hausdorff.

(ii) It follows from the fact that for any finite subset F of ΔM we have

$$V(F) \cap \delta_M(M) = \{\delta_M(m) : f(m) = 0 \ \forall\ f \in F\} = \delta_M(\cap_{f \in F} \operatorname{Ker} f).$$

(iii) The topological module $\Delta^2 M_\varphi$ is a closed submodule of $W^{\Delta M}$ endowed with the product of the discrete topologies. Since the product of complete topologies is also complete, we can conclude. $\square$

Corollary 2.2. *Each reflexive module M endowed with the topology τ is complete.*

Proof. If M is reflexive, δ_M is an isomorphism. Therefore, by Proposition 2.1, (ii), it is a topological isomorphism. We conclude by Proposition 2.1, (iii). □

Proposition 2.3. *For a torsionless module M the following statements are equivalent:*

(i) *M is reflexive.*
(ii) *τ is a complete topology and $\delta_M(M)$ is dense in $\Delta^2 M_\varphi$.*

Proof. (i) ⇒ (ii): It follows by Corollary 2.2.

(ii) ⇒ (i): Since τ is complete, by Proposition 2.1 $\delta_M(M)$ is a complete, and hence closed, topological submodule of $\Delta^2 M_\varphi$. Being $\delta_M(M)$ dense in $\Delta^2 M_\varphi$ and φ an Hausdorff topology, δ_M is an isomorphism. □

As suggested by Müller [**13**] in the case of Morita dualities, we look for a suitable notion of compactness for a module M in order to guarantee the completeness of the topology τ on M.

Note that $\delta_M(M)$ is dense in $\Delta^2 M_\varphi$ if and only if for each α in $\Delta^2 M$ and $f_1, \dots, f_n$ in ΔM there exists m in M such that $\alpha(f_i) = f_i(m)$ for each $i = 1, \dots, n$. Following [**11**], a module M satisfying the above property will be called *W-dense*.

Definition 2.4. Let M be a torsionless left R-module. A submodule K of M is called *W-closed* if M/K is torsionless (see [**11**, §2]). A linear topology on M is said to be a *W-topology* if it has a basis of neighbourhoods of zero consisting of W-closed submodules.

Each W-closed submodule K of M is closed in M_τ. Indeed, for a suitable set X, there exist the following maps:

$$M \xrightarrow{\pi} M/K \overset{i}{\hookrightarrow} W^X \xrightarrow{\pi_x} W.$$

Then, $K = \bigcap_{x \in X} \operatorname{Ker}(\pi_x \circ i \circ \pi)$ is closed, since it is an intersection of open and hence closed submodules. The converse is not true in general.

Example 2.5. Let R denote the k-algebra given by the quiver $1 \to 2 \to 3$. It is easy to verify that ${}_R R_R$ is a cotilting bimodule. Consider the projective R-module $P(2)$. The topology $\tau_{\Delta P(2)}$ is discrete since $P(2)$ embeds in ${}_R R$. Therefore each submodule of $P(2)$, in particular the simple module $S(3)$, is closed. Nevertheless $S(3)$ is not a R-closed submodule of $P(2)$, since $P(2)/S(3) \cong S(2)$ is not cogenerated by ${}_R R$.

Definition 2.6. A left R-module M is said to be *W-linearly compact* (see [**11**, §3]), briefly W-lc, (resp. *HW-linearly compact*, briefly HW-lc) if it is complete in any W-topology (resp. in any Hausdorff W-topology).

Analogously to the usual linear compactness, a left R-module M is W-linear compact if and only if any finitely solvable system of congruences $x \equiv x_\lambda \mod M_\lambda$, where $\{M_\lambda : \lambda \in \Lambda\}$ is a downwards directed collection of W-closed submodules of M, is solvable. Similarly a module M is HW-linearly compact if and only if it satisfies the previous condition restricted to downwards directed collections of W-closed submodules of M with intersection equal to zero.

Proposition 2.7. *Let ${}_RM$ be a torsionless left R-module. If M is HW-linearly compact, then M is complete in the topology τ.*

Proof. Since the intersection of the kernels of a finite number of elements of ΔM is a W-closed submodule of M, τ is a W-topology. Since M is torsionless, by Proposition 2.1 the topology τ is Hausdorff. Since M is HW-lc, τ is complete. □

Corollary 2.8. *Let ${}_RM$ be a torsionless left R-module. If M is HW-linearly compact and W-dense, then M is reflexive.*

Proof. It follows by Propositions 2.3 and 2.7. □

We can obtain a more precise result for cotilting bimodules.

Theorem 2.9. *Let ${}_RW_S$ be a cotilting bimodule. For a torsionless left R-module M the following statements are equivalent:*

(a) *M is HW-linearly compact and W-dense.*
(b) *M is reflexive and τ is the unique Hausdorff topology among those induced by subsets of ΔM.*

Proof. (a ⇒ b): By Corollary 2.8 we only have to prove that τ is the unique Hausdorff topology induced by subsets of ΔM. Since τ is the maximum element in $\{\tau_A : A \subseteq \Delta M\}$, it is sufficient to prove that if τ_A is Hausdorff, then τ is coarser than τ_A and hence $\tau_A = \tau$.

Let F be a finite subset of A. We denote by $f_F : M \to W^F$ the diagonal morphism. Let $M_F := \bigcap_{f \in F} \operatorname{Ker} f = \operatorname{Ker} f_F$ and $N_F := M/M_F$. By [**4**, Proposition 5] both the left R-modules M_F and N_F are reflexive. We call π_F the induced map $M \to N_F$. Since $\{M_F : F \subseteq A,\ F \text{ finite}\}$ is a base for the filter of τ_A-neighbourhoods of zero consisting of open submodules, $\varprojlim N_F$ is the Hausdorff completion of M endowed with the topology τ_A.

But, since τ_A is an Hausdorff W-topology and, by hypothesis, M is HW-lc, τ_A is complete. Thus $M \cong \varprojlim N_F$.

Applying the functors Δ and $\varinjlim$ to the exact sequences

$$0 \to M_F \to M \overset{\pi_F}{\to} N_F \to 0$$

we get the exact sequence of right S-modules

$$(*) \qquad 0 \longrightarrow \varinjlim \Delta N_F \xrightarrow{\varinjlim \Delta(\pi_F)} \Delta M \longrightarrow \varinjlim \Delta M_F \longrightarrow 0$$

Now $\Delta(\varinjlim \Delta(\pi_F)) \cong \varprojlim \Delta^2(\pi_F) \cong \varprojlim \pi_F$ is an isomorphism. Then, from the exact sequence

$$0 \to \Delta \varinjlim \Delta M_F \longrightarrow \Delta^2 M \xrightarrow{\Delta(\varinjlim \Delta(\pi_F))} \Delta \varinjlim \Delta N_F \longrightarrow \Gamma \varinjlim \Delta M_F \to 0,$$

we get $\varinjlim \Delta M_F$ belongs to $\operatorname{Ker}\Delta \cap \operatorname{Ker}\Gamma = 0$. Hence $\Delta M \cong \varinjlim \Delta N_F$.

Let now g be in ΔM. Since g belongs to $\Delta(\pi_F)(\Delta(N_F))$ for some finite subset F of A, there exists a morphism $h : N_F \to W$ such that $g = h \circ \pi_F$. Then since $\operatorname{Ker} g \supseteq \operatorname{Ker}\pi_F = \bigcap_{f\in F} \operatorname{Ker} f$, $\operatorname{Ker} g$ is τ_A-open. Therefore τ is coarser than τ_A.

(b $\Rightarrow$ a): We only have to prove that M is HW-lc. Let σ be a Hausdorff W-topology on M. By definition σ has a basis $\mathcal{B}$ for the filter of neighbourhoods of zero consisting of W-closed submodules; since σ is Hausdorff, the intersection of elements in $\mathcal{B}$ is equal to zero. Observe that any element V of $\mathcal{B}$ is the intersection of the kernels of a (not necessarily finite) subset A_V of ΔM. Let A the union $\cup_{V\in\mathcal{B}} A_V$. If f belongs to A_V, $\operatorname{Ker} f$ contains V and hence it is σ-open. Therefore the topology τ_A is coarser than σ. Since

$$\cap_{f\in A} \operatorname{Ker} f = \cap_{V\in\mathcal{B}} V = \{0\},$$

0 is a closed subset of M_{τ_A}, i.e., τ_A is Hausdorff. By hypothesis $\tau_A = \tau$ and, since M is reflexive, by Proposition 2.3 τ_A is complete. By [**3**, Proposition III.3.10], also the topology σ is complete. □

Lemma 2.10. *Let ${}_RW_S$ be a cotilting bimodule with* Cogen W_S *closed under direct limits. Let M be a reflexive left R-module. Then τ is the unique Hausdorff topology among those induced by subsets of ΔM.*

Proof. We can follow the first two paragraphs of the proof of Theorem 2.9, (a $\Rightarrow$ b). Applying the functors Δ and $\varinjlim$ to the exact sequence

$$0 \to M_F \to M \xrightarrow{\pi_F} N_F \to 0$$

we get the exact sequence

$$0 \to \varinjlim \Delta N_F \xrightarrow{\varinjlim \Delta(\pi_F)} \Delta M \longrightarrow \varinjlim \Delta M_F \to 0$$

of right S-modules. Observe that $\varinjlim \Delta M_F$ again belongs to $\operatorname{Ker}\Gamma = \operatorname{Cogen} W$ since, by hypothesis, $\operatorname{Cogen} W$ is closed under direct limits. Moreover

$$\Delta \varinjlim \Delta M_F \cong \varprojlim \Delta^2 M_F \cong \varprojlim M_F = \cap_{f\in A} \operatorname{Ker} f = 0.$$

Therefore, since $\operatorname{Ker}\Delta \cap \operatorname{Ker}\Gamma = 0$, we get $\varinjlim \Delta M_F = 0$. Hence $\Delta M \cong \varinjlim \Delta N_F$. We can thus conclude following the last paragraph of the proof of Theorem 2.9, (a $\Rightarrow$ b). □

Thus we obtain the characterization of reflexive modules for pure-injective cotilting bimodules.

Theorem 2.11. *Let ${}_RW_S$ be a pure-injective cotilting bimodule. For a torsionless module M the following are equivalent:*

(a) *M is reflexive.*
(b) *M is HW-linearly compact and W-dense.*
(c) *M is W-linearly compact and W-dense.*

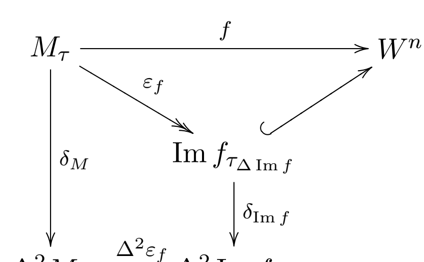
Proof. (c ⇒ b ⇒ a): They follow by Definition 2.6 and by Theorem 2.9.

(a ⇒ c): Trivially M is W-dense. Let σ be a W-topology on M and let $H \leq M$ be the τ-closure of zero. Since H is a W-closed submodule of M, M/H is reflexive (see [**4**, Proposition 5]). By Theorem 2.9 and Lemma 2.10 M/H is HW-lc and hence complete endowed with the quotient topology of τ. Therefore M_τ is complete. □

In [**13**] Müller proved that if ${}_RW_S$ is a Morita bimodule, a module M is reflexive if and only if it is linearly compact in the discrete topology if and only if it is complete in any Hausdorff linear topology. In such a case W cogenerates the whole category of modules, hence any submodule is W-closed. Therefore the notions of W-linear compactness, of HW-linear compactness and of linear compactness in the discrete topology coincide. In our setting a density condition comes out. Let us better investigate its role.

Proposition 2.12. *Let ${}_RW_S$ be a bimodule such that* $\operatorname{Cogen} W_S \subseteq \operatorname{Ker}\Gamma$. *A left R-module M is W-dense if and only if* $\operatorname{Im}(f)$ *is a reflexive left R-module for every $f \in \operatorname{Hom}_R(M, W^n)$, ($n \in \mathbb{N}$).*

Proof. Let us consider for each f in $\operatorname{Hom}(M, W^n)$, $n \in \mathbb{N}$, the following commutative diagram of linearly topologized modules and continuous morphisms:



where φ' is the finite topology on $\Delta^2 \operatorname{Im} f$. Since $\operatorname{Im} f \leq W^n$, $\tau_{\Delta \operatorname{Im} f}$ is the discrete topology; in particular it is complete.

Suppose that M is W-dense. Applying Δ to the exact sequence $0 \to \operatorname{Ker} f \to M \overset{\varepsilon_f}{\to} \operatorname{Im} f \to 0$, we get the exact sequence $0 \to \Delta \operatorname{Im} f \overset{\Delta(\varepsilon_f)}{\to} \Delta M \to C \to 0$ with C in $\operatorname{Cogen} W_S$. Since $\operatorname{Cogen} W_S \subseteq \operatorname{Ker}\Gamma$, $\Delta^2(\varepsilon_f)$ is an epimorphism. Therefore, the W-density of M implies the W-density of $\operatorname{Im} f$. By Proposition 2.3, $\operatorname{Im} f$ is reflexive.

Conversely, let $f_1, \dots, f_n$ be in ΔM. We denote by $f : M \to W^n$ their diagonal morphism, $m \mapsto (f_1(m), \dots, f_n(m))$, and by $\mu_f \circ \varepsilon_f$ the usual factorization of f through $\operatorname{Im} f$. Since $\operatorname{Im} f$ is reflexive, for each α in $\Delta^2 M$ there exists m_α in M such that

$$\Delta^2(\varepsilon_f)(\alpha) = \delta_{\operatorname{Im} f}(f(m_\alpha)).$$

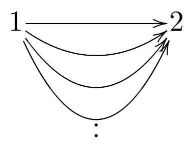
In particular, denoted by $p_i : \operatorname{Im} f \to W$ the i-th projection, we have

$$(\Delta^2(\varepsilon_f)(\alpha))(p_i) = \alpha(\Delta(\varepsilon_f))(p_i) = \alpha(p_i \circ \varepsilon_f) = \alpha(f_i)$$
$$(\delta_{\operatorname{Im} f}(f(m_\alpha))(p_i) = p_i(f(m_\alpha)) = f_i(m_\alpha);$$

therefore α and $\delta_M(m_\alpha)$ coincide on $f_1, \dots, f_n$. Therefore M is W-dense. □

If ${}_RW_S$ is a Morita bimodule, then the class of reflexive modules contains W and it is closed under submodules and finite direct sums. Therefore the W-density condition is always satisfied: Any module M is W-dense. This is not the case for cotilting bimodules.

Example 2.13. Let k be an algebraically closed field. Denote by A the *generalized Kronecker algebra* of dimension $\aleph_0$ over k given by the quiver



with a countable set of arrows from 1 to 2, i.e., the ring of lower triangular matrices

$$\begin{pmatrix} k & 0 \\ V & k \end{pmatrix} = \left\{ \begin{pmatrix} a & 0 \\ v & b \end{pmatrix} : a, b \in k, v \in V \right\}$$

where V is a k-vector space of dimension $\aleph_0$ (see [**7**, **8**]). Then, by [**8**, Lemma 2.2], A is a hereditary, coherent and perfect ring. It is easily verified that ${}_AA_A$ is a cotilting bimodule and $\operatorname{Cogen}(A)$ consists of projective modules, while $\operatorname{Ker}(\Delta)$ contains exactly the modules without projective direct summands. The reflexive modules coincide with the finitely generated projective modules [**8**, Lemma 2.3]. Denote by e_1, e_2 the primitive idempotents, i.e., $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Let $P_i = Ae_i$ and $Q_i = e_iA$, $i = 1, 2$. The socle S of P_1 is isomorphic to $P_2^{(\aleph_0)}$; therefore it is a not reflexive submodule of A. By Proposition 2.12, S is not A-dense.

In [**6**, Definition 1.2] Colpi and Fuller introduced the *W-torsionless linear compactness*, a generalization of the notion of linear compactness with respect to the torsion theories associated to a cotilting bimodule ${}_RW_S$. They prove that if a module is W-torsionless linearly compact, then it is reflexive, i.e., (see Proposition 2.3) τ is a complete topology and M is W-dense.

Our notion of W-linear compactness is strong enough to assure the completeness, but to obtain the Δ-reflexivity we need to assume explicitly the W-density. Assuming ${}_RW$ and W_S pure-injective (as in all examples known in the literature), these two notions together completely characterize the classes of reflexive modules. The notion of W-torsionless linear compactness is too strong to characterize the classes of reflexive modules in the general case; this happens if and only if the classes of reflexive left R- and right S- modules are closed under submodules [**6**, Corollary 1.9]. Observe that in this case, by Proposition 2.12, any module is W-dense. Adding the hypotheses of both the contexts we get:

Corollary 2.14. *Let ${}_RW_S$ be a pure-injective cotilting bimodule. Assume the classes of reflexive left R- and right S- modules being closed under submodules. For a module M, the following statements are equivalent:*

(a) *M is reflexive;*
(b) *M is W-torsionless linearly compact;*
(c) *M is W-linearly compact.*

Acknowledgement. The results of the first section have been obtained during an informal meeting at the University of Munchen with Lidia Angeleri Hügel and Jan Trlifaj. We wish to thank these colleagues and friends for allowing us to publish in this paper the outcome of those useful discussions. We wish also to thank Riccardo Colpi for his valuable suggestions which improved the paper.

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Received January 8, 2001 and revised January 7, 2002.

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