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Dedicated to Professor Hidenobu Yoshida on the occasion of his sixtieth birthday.

Our aim in this paper is to deal with growth properties at infinity for modified Poisson integrals (of fractional power) in the half space of $\mathbf{R}^n$. We also discuss weighted boundary limits for the modified Poisson integrals.

1. Introduction and statement of results.

Let $\mathbf{R}^n$ $(n \geqq 2)$ denote the n-dimensional Euclidean space with points $x = (x_1, \ldots, x_{n-1}, x_n)$. Let $D = \{x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbf{R}^n; x_n > 0\}$, whose boundary is usually identified with $\mathbf{R}^{n-1}$.

For $\lambda > 0$ and $x \in \mathbf{R}^n$, consider the kernel function

$$K_\lambda(x) = |x|^{-\lambda}.$$

The Poisson integral is defined by

$$P[f](x) = \alpha_n x_n \int_{\mathbf{R}^{n-1}} K_n(x-y) f(y) dy, \tag{1.1}$$

where f is a locally integrable function on $\mathbf{R}^{n-1}$ and $\alpha_n = 2/(n\sigma_n)$ with $\sigma_n = \pi^{n/2}/\Gamma(1+n/2)$ being the volume of the unit n-ball. The Poisson integrals are used to solve the Dirichlet problem in the half space D. Further, Sjögren ([**15**] and [**16**]), Rönning [**11**] and Brundin [**3**] treated fractional Poisson integrals with respect to the fractional power of the Poisson kernel

$$P_\theta[f](x) = \int_{\mathbf{R}^{n-1}} \{\alpha_n x_n K_n(x-y)\}^\theta f(y) dy; \tag{1.2}$$

if $n = 2$, then it defines a solution of the hyperbolic Laplacian

$$x_2^2 \Delta u = \theta(\theta - 1) u.$$

The Poisson integral $P[f]$ will be harmonic in D if

$$\int_{\mathbf{R}^{n-1}} |f(y)|(1+|y|)^{-n} dy < \infty \tag{1.3}$$

(see [2] and [5]). In this paper, we consider functions f satisfying

$$\int_{\mathbf{R}^{n-1}} |f(y)|^p (1+|y|)^{-\gamma} dy < \infty \tag{1.4}$$

for $1 \leqq p < \infty$ and a real number γ. To obtain the Dirichlet solution for the boundary data f, as in [13, 14] and [19], we use the following modified kernel function defined by

$$K_{\lambda,m}(x,y) = \begin{cases} K_\lambda(x-y) & \text{when } |y|<1, \\ K_\lambda(x-y) - \sum_{|j|\leqq m-1} \frac{x^j}{j!} \left[(\partial/\partial x)^j K_\lambda(x-y)\Big|_{x=0} \right] & \text{when } |y| \geqq 1 \end{cases}$$

for a nonnegative integer m and a point $x=(x_1,\dots,x_n)$, where $j=(j_1,\dots,j_n)$ is a multiindex with length $|j| = j_1+\cdots+j_n$, $j! = j_1!\cdots j_n!$, $x^j = x_1^{j_1}\cdots x_n^{j_n}$ and $(\partial/\partial x)^j = (\partial/\partial x_1)^{j_1}\cdots(\partial/\partial x_n)^{j_n}$. In the papers mentioned above, it is expressed by use of Gegenbauer polynomials ([18]). Write

$$K_{\lambda,m} f(x) = \int_{\mathbf{R}^{n-1}} K_{\lambda,m}(x,y) f(y) dy$$

and

$$U_{\lambda,m} f(x) = \alpha_n x_n K_{\lambda,m} f(x).$$

Here note that $U_{n,0}f$ is nothing but the Poisson integral $P[f]$.

Recently Siegel-Talvila ([14, Theorem 2.1 and Corollary 2.1]) proved the following:

Theorem A. *Let f be a continuous function on $\mathbf{R}^{n-1}$ satisfying (1.4) with $p=1$ and $\gamma = n+m$. Then the function $U_{n,m}f(x)$ satisfies*

$$U_{n,m} f \in C^2(D) \cap C^0(\overline{D}),$$
$$\Delta U_{n,m} f = 0 \qquad \text{in } D,$$
$$U_{n,m} f = f \qquad \text{on } \partial D,$$
$$U_{n,m} f(x) = o(x_n^{1-n}|x|^{n+m}) \qquad \text{as } |x|\to\infty,\ x\in D.$$

Our first aim in this paper is to establish the following theorem (cf. [14, Theorem 2.1], [13, Theorem 5.1]):

Theorem 1. *Let $1 \leqq p < \infty$, $\lambda > 0$, $\gamma > -(n-1)(p-1)$ and*

$$\begin{aligned} n-\lambda-1-(n-\gamma-1)/p < m \leqq n-\lambda-(n-\gamma-1)/p \quad & \text{in case } p>1, \\ -\lambda+\gamma \leqq m < -\lambda+\gamma+1 \quad & \text{in case } p=1. \end{aligned}$$

If f is a measurable function on $\mathbf{R}^{n-1}$ satisfying (1.4), then

$$\lim_{|x|\to\infty, x\in D} x_n^\lambda |x|^{1-n+(n-\gamma-1)/p} K_{\lambda,m} f(x) = 0$$

when $m < n - \lambda - (n-\gamma-1)/p$, *and*

$$\lim_{|x|\to\infty, x\in D} x_n^{\lambda}|x|^{1-n+(n-\gamma-1)/p}(\log|x|)^{-1/p'}K_{\lambda,m}f(x) = 0$$

when $m = n - \lambda - (n-\gamma-1)/p$, $p > 1$ *and* $p' = p/(p-1)$.

Remark 1. Siegel-Talvila [**14**, Theorem 2.1] treated the case $p = 1$ and $m = -\lambda + \gamma$ (see also [**13**, Theorem 5.1]).

Corollary 1. *Let* $p > 1$, $\gamma > -(n-1)(p-1)$ *and*

$$-1 - (n-\gamma-1)/p < m \leqq -(n-\gamma-1)/p.$$

If f *is a measurable function on* $\mathbf{R}^{n-1}$ *satisfying* (1.4), *then*

$$\lim_{|x|\to\infty, x\in D} x_n^{n-1}|x|^{1-n+(n-\gamma-1)/p}U_{n,m}f(x) = 0$$

when $m < -(n-\gamma-1)/p$,

$$\lim_{|x|\to\infty, x\in D} x_n^{n-1}|x|^{1-n+(n-\gamma-1)/p}(\log|x|)^{-1/p'}U_{n,m}f(x) = 0$$

when $m = -(n-\gamma-1)/p$.

Next we are concerned with minimally fine limits at infinity for $U_{\lambda,m}f$, as an extension of Lelong-Ferrand [**7**]. For related results, we refer the reader to the papers by Aikawa [**1**], Essén-Jackson [**4**], Miyamoto-Yoshida [**8**] and the first author [**9**]. For this purpose, consider the kernel function

$$k_{\beta,\lambda}(x,y) = x_n^{1-\beta}|x-y|^{-\lambda}.$$

To evaluate the size of exceptional sets, for a set $E \subset D$ and an open set $G \subset \mathbf{R}^{n-1}$, we consider the capacity

$$C_{k_{\beta,\lambda},p}(E;G) = \inf \int_{\mathbf{R}^{n-1}} g(y)^p \, dy,$$

where the infimum is taken over all nonnegative measurable functions g such that $g = 0$ outside G and

$$\int_{\mathbf{R}^{n-1}} k_{\beta,\lambda}(x,y)g(y)dy \geqq 1 \qquad \text{for all } x \in E.$$

We say that $E \subset D$ is (minimally) $(k_{\beta,\lambda}, p)$-thin at infinity if

$$\sum_{i=1}^{\infty} 2^{-i\{(\beta+\lambda-n)p+n-1\}}C_{k_{\beta,\lambda},p}(E_i;D_i) < \infty, \tag{1.5}$$

where $E_i = \{x \in E : 2^i \leqq |x| < 2^{i+1}\}$ and $D_i = \{x \in \mathbf{R}^{n-1} : 2^{i-1} < |x| < 2^{i+2}\}$.

Theorem 2 (cf. Aikawa [**1**] and the first author [**9**]). *Let p, λ and γ be as in Theorem 1. If f is a measurable function on $\mathbf{R}^{n-1}$ satisfying (1.4) and $\beta \leqq 1$, then there exists a set $E \subset D$ such that E is $(k_{\beta,\lambda}, p)$-thin at infinity and*

$$\lim_{|x|\to\infty, x\in D-E} x_n^{-\beta}|x|^{\beta+\lambda-n+(n-\gamma-1)/p} U_{\lambda,m}f(x) = 0.$$

It is well-known that the Poisson integral $U_{n,0}f = P[f]$ has nontangential boundary limits f at almost all boundary points. Our final goal is to show that $U_{\lambda,m}f$ has weighted boundary limits. For this purpose, we discuss the existence of boundary limits for

$$\mathcal{P}_\lambda f(x) = \frac{K_\lambda f(x)}{K_\lambda \chi_G(x)},$$

where $\lambda \geqq n-1$, G is a bounded open set in $\mathbf{R}^{n-1}$, $1 \leqq p < \infty$, $f \in L^p(G)$, χ_G denotes the characteristic function of G and

$$K_\lambda f(x) = \int_G K_\lambda(x-y) f(y)dy.$$

For a nonnegative function h on the interval $\mathbf{R}^+ = [0, \infty)$, let

$$\mathcal{A}_h(\xi) = \{x \in D : |x-\xi| < h(x_n)\}.$$

Theorem 3. *Let $1 \leqq p < \infty$ and $f \in L^p(G)$. For a.e. $\xi \in G$, $\mathcal{P}_\lambda f(x) \to f(\xi)$ as $x \to \xi$ along $\mathcal{A}_h(\xi)$, where*

$$h(t) = C \begin{cases} t & (\lambda > n-1), \\ t\left(\log \frac{1}{t}\right)^{p/(n-1)} & (\lambda = n-1) \end{cases}$$

for fixed $C > 0$.

In the unit disc, this result was proved for $\lambda = 1$ by Sjögren [**15**] and [**16**], Rönning [**11**] and Brundin [**3**].

2. Proof of Theorem 1.

Throughout this paper, let M denote various constants independent of the variables in question.

First we note the following properties for the kernel functions $K_{\lambda,m}(x, y)$:

Lemma 1. *For $t > 0$, set*

$$f(t) = f(t, x, y) = t x_n |tx - y|^{-\lambda}$$

and

$$g(t) = g(t, x, y) = |tx - y|^{-\lambda}.$$

Then $f^{(\ell)}(0) = \ell x_n g^{(\ell-1)}(0)$ *for* $\ell = 1, 2, \ldots, m,$ *and*

$$f(1) - \left(f(0) + f'(0) + \frac{1}{2!}f''(0) + \cdots + \frac{1}{m!}f^{(m)}(0)\right)$$
$$= x_n \left\{g(1) - \left(g(0) + g'(0) + \frac{1}{2!}g''(0) + \cdots + \frac{1}{(m-1)!}g^{(m-1)}(0)\right)\right\}$$
$$= x_n K_{\lambda,m}(x, y)$$

when $|y| \geqq 1$.

Corollary 2. $U_{n,m}(x, y) = \alpha_n x_n K_{n,m}(x, y)$ *is harmonic in* D *for each fixed* $y \in \mathbf{R}^{n-1}$.

In our discussions, the following estimates for the kernel functions $K_{\lambda,m}$ are fundamental (see [**6**, Lemma 4.2] and [**12**, Section 3]):

Lemma 2. *Let* m *be a nonnegative integer and* $\lambda > 0$.

(1) *If* $1 \leqq |y| \leqq |x|/2$, *then* $|K_{\lambda,m}(x, y)| \leqq M|x|^{m-1}|y|^{-\lambda-m+1}$.
(2) *If* $|x|/2 \leqq |y| \leqq 2|x|$, *then* $|K_{\lambda,m}(x, y)| \leqq M|x - y|^{-\lambda} \leqq M x_n^{-\lambda}$.
(3) *If* $|y| \geqq 2|x|$ *and* $|y| \geqq 1$, *then* $|K_{\lambda,m}(x, y)| \leqq M|x|^m|y|^{-\lambda-m}$.

Proof of Theorem 1. We prove only the case $p > 1$; the proof of the case $p = 1$ is similar. For fixed $x \in D$, $|x| > 2$, we write

$$\begin{aligned} K_{\lambda,m}f(x) &= \int_{G_1} K_{\lambda,m}(x, y)f(y)\,dy + \int_{G_2} K_{\lambda,m}(x, y)f(y)\,dy \\ &\quad + \int_{G_3} K_{\lambda,m}(x, y)f(y)\,dy + \int_{B(0,1)} K_{\lambda,m}(x, y)f(y)\,dy \\ &= U_1(x) + U_2(x) + U_3(x) + U_4(x), \end{aligned}$$

where $B(x, r)$ denotes the open ball centered at x with radius $r > 0$, and

$$G_1 = \{y \in \mathbf{R}^{n-1} : |y| \geqq 2|x|\},$$
$$G_2 = \{y \in \mathbf{R}^{n-1} : 1 \leqq |y| < |x|/2\},$$
$$G_3 = \{y \in \mathbf{R}^{n-1} : |x|/2 \leqq |y| < 2|x|\}.$$

First note that

$$|U_4(x)| \leqq (|x|/2)^{-\lambda} \int_{B(0,1)} |f(y)|dy,$$

so that

$$\lim_{|x|\to\infty, x\in D} |x|^{\lambda-n+1+(n-\gamma-1)/p} U_4(x) = 0 \tag{2.1}$$

since $\gamma > -(n-1)(p-1)$.

Lemma 3. *If $m > n - \lambda - 1 - (n - \gamma - 1)/p$, then*

$$|U_1(x)| \leqq M|x|^{-\lambda+n-1-(n-\gamma-1)/p} \left(\int_{G_1} |f(y)|^p |y|^{-\gamma} dy \right)^{1/p}.$$

Proof. If $m > n - \lambda - 1 - (n - \gamma - 1)/p$, then $(-\lambda - m + \gamma/p)p' + n - 1 < 0$, so that we obtain by Lemma 2 (3) and Hölder's inequality

$$\begin{aligned} |U_1(x)| &\leqq M|x|^m \int_{G_1} |y|^{-\lambda-m} |f(y)| \, dy \\ &\leqq M|x|^m \left(\int_{G_1} |y|^{(-\lambda-m+\gamma/p)p'} dy \right)^{1/p'} \left(\int_{G_1} |f(y)|^p |y|^{-\gamma} dy \right)^{1/p} \\ &\leqq M|x|^{-\lambda+n-1-(n-\gamma-1)/p} \left(\int_{G_1} |f(y)|^p |y|^{-\gamma} dy \right)^{1/p}, \end{aligned}$$

where $1/p + 1/p' = 1$. This proves the lemma.

By Lemma 3, we have

$$\lim_{|x|\to\infty, x\in D} |x|^{\lambda-n+1+(n-\gamma-1)/p} U_1(x) = 0. \tag{2.2}$$

Lemma 4. *If $m < n - \lambda - (n - \gamma - 1)/p$, then*

$$|U_2(x)| \leqq M|x|^{-\lambda+n-1-(n-\gamma-1)/p} \left(\int_{G_2} |f(y)|^p |y|^{-\gamma} dy \right)^{1/p};$$

if $m = n - \lambda - (n - \gamma - 1)/p$, then

$$|U_2(x)| \leqq M|x|^{-\lambda+n-1-(n-\gamma-1)/p} (\log |x|)^{1/p'} \left(\int_{G_2} |f(y)|^p |y|^{-\gamma} dy \right)^{1/p}.$$

Proof. If $m < n - \lambda - (n - \gamma - 1)/p$, then $(-\lambda - m + 1 + \gamma/p)p' + n - 1 > 0$, so that we obtain by Lemma 2 (1) and Hölder's inequality

$$\begin{aligned} |U_2(x)| &\leqq M|x|^{m-1} \int_{G_2} |y|^{-\lambda-m+1} |f(y)| \, dy \\ &\leqq M|x|^{m-1} \left(\int_{G_2} |y|^{(-\lambda-m+1+\gamma/p)p'} dy \right)^{1/p'} \left(\int_{G_2} |f(y)|^p |y|^{-\gamma} dy \right)^{1/p} \\ &\leqq M|x|^{-\lambda+n-1-(n-\gamma-1)/p} \left(\int_{G_2} |f(y)|^p |y|^{-\gamma} dy \right)^{1/p}, \end{aligned}$$

as required.

The remaining case can be proved similarly.

For $r > 1$, we have

$$\begin{aligned} U_2(x) &= \int_{G_2} K_{\lambda,m}(x,y)f(y)\,dy \\ &= \int_{G_2\cap B(0,r)} K_{\lambda,m}(x,y)f(y)\,dy + \int_{G_2-B(0,r)} K_{\lambda,m}(x,y)f(y)\,dy \\ &= U_{21}(x) + U_{22}(x). \end{aligned}$$

If $|x| > 2r$ and $m < n - \lambda - (n-\gamma-1)/p$, then

$$|U_{21}(x)| \leqq M|x|^{m-1} \int_{B(0,r)-B(0,1)} |y|^{-\lambda-m+1}|f(y)|dy,$$

so that

$$\lim_{|x|\to\infty, x\in D} |x|^{\lambda-n+1+(n-\gamma-1)/p} U_{21}(x) = 0.$$

Moreover, we have by Lemma 4

$$|U_{22}(x)| \leqq M|x|^{-\lambda+n-1-(n-\gamma-1)/p} \left(\int_{\mathbf{R}^{n-1}-B(0,r)} |f(y)|^p |y|^{-\gamma} dy \right)^{1/p}.$$

Hence, in case $m < n - \lambda - (n-\gamma-1)/p$, we find

$$\begin{aligned} &\limsup_{|x|\to\infty, x\in D} |x|^{\lambda-n+1+(n-\gamma-1)/p} |U_2(x)| \\ &\leqq M \left(\int_{\mathbf{R}^{n-1}-B(0,r)} |f(y)|^p |y|^{-\gamma} dy \right)^{1/p}, \end{aligned}$$

which implies by arbitrariness of r that

$$\lim_{|x|\to\infty, x\in D} |x|^{\lambda-n+1+(n-\gamma-1)/p} U_2(x) = 0. \tag{2.3}$$

Similarly, in case $m = n - \lambda - (n-\gamma-1)/p$, we find

$$\lim_{|x|\to\infty, x\in D} |x|^{\lambda-n+1+(n-\gamma-1)/p} (\log|x|)^{-1/p'} U_2(x) = 0. \tag{2.4}$$

Finally, by Lemma 2 (2) and Hölder's inequality, we obtain

$$\begin{aligned} |U_3(x)| &\leqq M x_n^{-\lambda} \int_{G_3} |f(y)|dy \\ &\leqq M x_n^{-\lambda} |x|^{n-1-(n-\gamma-1)/p} \left(\int_{G_3} |f(y)|^p |y|^{-\gamma} dy \right)^{1/p}. \end{aligned}$$

Hence we have

$$\lim_{|x|\to\infty, x\in D} x_n^{\lambda} |x|^{-n+1+(n-\gamma-1)/p} U_3(x) = 0. \tag{2.5}$$

Thus, collecting (2.1)-(2.5), we complete the Proof of Theorem 1. □

Corollary 3 (cf. [14, Corollary 2.1]). *Let f be a continuous function on $\mathbf{R}^{n-1}$ satisfying (1.4) with $\gamma > -(n-1)(p-1)$. Let*

$$-1-(n-\gamma-1)/p < m < -(n-\gamma-1)/p.$$

Then the function $U_{n,m}f(x)$ satisfies

(i) $U_{n,m}f \in C^2(D) \cap C^0(\overline{D})$,
(ii) $\Delta U_{n,m}f = 0$ *in* D,
(iii) $U_{n,m}f = f$ *on* ∂D,
(iv) $U_{n,m}f(x) = o(x_n^{1-n}|x|^{n-1-(n-\gamma-1)/p})$ *as* $|x| \to \infty$, $x \in D$.

Proof. We show only (iii). For $r > 2$ and $x \in B(0,r) \cap D$, we write

$$\begin{aligned} U_{n,m}f(x) &= \alpha_n x_n \int_{\mathbf{R}^{n-1}\cap B(0,2r)} K_{n,m}(x,y)f(y)dy \\ &\quad + \alpha_n x_n \int_{\mathbf{R}^{n-1}-B(0,2r)} K_{n,m}(x,y)f(y)dy = u_1(x) + u_2(x). \end{aligned}$$

In view of Lemma 2 (3), we find

$$\lim_{x\to\xi, x\in D} u_2(x) = 0$$

for every $\xi \in \mathbf{R}^{n-1} \cap B(0,r)$. Further,

$$\lim_{x\to\xi, x\in D} u_1(x) = \lim_{x\to\xi, x\in D} \alpha_n x_n \int_{\mathbf{R}^{n-1}\cap B(0,2r)} K_n(x-y)f(y)dy = f(\xi)$$

for every $\xi \in \mathbf{R}^{n-1} \cap B(0,r)$ (see [17]), so that (iii) follows. □

3. Proof of Theorem 2.

As in the Proof of Theorem 1 we write

$$U_{\lambda,m}f(x) = \alpha_n x_n \{U_1(x) + U_2(x) + U_3(x) + U_4(x)\}.$$

By (2.1) we see that

$$\lim_{|x|\to\infty, x\in D} x_n^{1-\beta}|x|^{\beta+\lambda-n+(n-\gamma-1)/p} U_4(x) = 0$$

since $1-\beta \geqq 0$. Moreover, by (2.2) and (2.3) we have

$$\lim_{|x|\to\infty, x\in D} x_n^{1-\beta}|x|^{\beta+\lambda-n+(n-\gamma-1)/p} \{U_1(x) + U_2(x)\} = 0.$$

Note that by Lemma 2 (2)

$$\begin{aligned} x_n|U_3(x)| &\leqq M x_n \int_{G_3} |x-y|^{-\lambda}|f(y)|\, dy \\ &= M x_n^{\beta} \int_{G_3} k_{\beta,\lambda}(x,y)|f(y)|\, dy. \end{aligned}$$

In view of (1.4), we can find a sequence $\{a_i\}$ of positive numbers such that $\lim_{i\to\infty} a_i = \infty$ and

$$\sum_{i=1}^{\infty} a_i \int_{D_i} |f(y)|^p |y|^{-\gamma} dy < \infty;$$

recall $D_i = \{y \in \mathbf{R}^{n-1} : 2^{i-1} < |y| < 2^{i+2}\}$. Consider the sets

$$E_i = \left\{x \in D : 2^i \leqq |x| < 2^{i+1}, x_n^{1-\beta}|U_3(x)| \geqq a_i^{-1/p} 2^{-i\{\beta+\lambda-n+(n-\gamma-1)/p\}}\right\}$$

for $i = 1, 2, \ldots$. If $x \in E_i$, then

$$\begin{aligned} a_i^{-1/p} &\leqq 2^{i\{\beta+\lambda-n+(n-\gamma-1)/p\}} x_n^{1-\beta} |U_3(x)| \\ &\leqq M 2^{i\{\beta+\lambda-n+(n-\gamma-1)/p\}} \int_{D_i} k_{\beta,\lambda}(x,y)|f(y)|dy, \end{aligned}$$

so that it follows from the definition of $C_{k_{\beta,\lambda},p}$ that

$$\begin{aligned} C_{k_{\beta,\lambda},p}(E_i; D_i) &\leqq M a_i 2^{i\{\beta+\lambda-n+(n-\gamma-1)/p\}p} \int_{D_i} |f(y)|^p dy \\ &\leqq M a_i 2^{i\{(\beta+\lambda-n)p+n-1\}} \int_{D_i} |f(y)|^p |y|^{-\gamma} dy. \end{aligned}$$

Define $E = \bigcup_{i=1}^{\infty} E_i$. Then $E \cap B(0, 2^{i+1}) - B(0, 2^i) = E_i$ and

$$\sum_{i=1}^{\infty} 2^{-i\{(\beta+\lambda-n)p+n-1\}} C_{k_{\beta,\lambda},p}(E_i; D_i) < \infty.$$

Clearly,

$$\lim_{|x|\to\infty, x\in D-E} x_n^{1-\beta} |x|^{\beta+\lambda-n+(n-\gamma-1)/p} U_3(x) = 0.$$

Thus the proof of Theorem 2 is completed. □

Remark 2. Suppose $\lambda > 1-\beta+(n-1)/p'$. Then we can find a measurable function f on $\mathbf{R}^{n-1}$ satisfying (1.4) such that

$$\limsup_{|x|\to\infty, x\in D} x_n^{-\beta} |x|^{\beta+\lambda-n+(n-\gamma-1)/p} U_{\lambda,m} f(x) = \infty. \tag{3.1}$$

To show this, take a positive number δ such that $n-\lambda-\beta < \delta < (n-1)/p$. Letting $e_j = (2^j, 0, \ldots, 0)$ and $r_j = 2^{j-1}$, we consider

$$f(y) = \sum_{j=1}^{\infty} 2^{-j(n-\gamma-1)/p} |e_j - y|^{-\delta} \chi_{B(e_j,r_j)\cap\mathbf{R}^{n-1}}(y),$$

where χ_E denotes the characteristic function of E. Then

$$\int_{\mathbf{R}^{n-1}} f(y)^p (1+|y|)^{-\gamma} dy \leqq M \sum_j 2^{-j(n-1)} r_j^{-\delta p+n-1} < \infty. \tag{3.2}$$

Moreover, if $x \in B(e_j, r_j) \cap D$, then

$$x_n^{-\beta}|x|^{\beta+\lambda-n+(n-\gamma-1)/p} U_{\lambda,m} f(x) \geqq M x_n^{n-\lambda-\beta-\delta} 2^{j(\beta+\lambda-n)},$$

so that

$$\lim_{x \to e_j, x \in D} x_n^{-\beta}|x|^{\beta+\lambda-n+(n-\gamma-1)/p} U_{\lambda,m} f(x) = \infty.$$

This proves (3.1). Thus f has all the required conditions.

4. Proof of Theorem 3.

Recall that $\lambda \geqq n-1$, G is a bounded open set in $\mathbf{R}^{n-1}$, $1 \leqq p < \infty$, $f \in L^p(G)$ and χ_G denotes the characteristic function of G.

For a proof of Theorem 3, we need some lemmas.

Lemma 5. *Consider the function*

$$H(t) = C \begin{cases} t^{n-1-\lambda} & (\lambda > n-1), \\ \log \frac{1}{t} & (\lambda = n-1), \end{cases}$$

where $C = K_\lambda \chi_{\mathbf{R}^{n-1}}(\mathbf{e})$ *with* $\mathbf{e} = (0, \ldots, 0, 1)$ *when* $n-1 < \lambda$ *and* $C = (n-1)\sigma_{n-1}$ *when* $\lambda = n-1$. *Then*

$$K_\lambda \chi_G(x) = H(x_n) + O(1) \qquad \text{as } x \in D \text{ tends to } \xi \in G.$$

Proof. We give a proof only when $\lambda > n-1$, because the case $\lambda = n-1$ can be treated similarly. In this case, let $x = (x', x_n) \in D, \xi \in G$ and note that

$$\begin{aligned} K_\lambda \chi_G(x) &= \int_{\mathbf{R}^{n-1}} (x_n^2 + |x'-y|^2)^{-\lambda/2} dy + O(1) \qquad (\text{as } x \to \xi) \\ &= x_n^{-\lambda+n-1} \int_{\mathbf{R}^{n-1}} (1+|z|^2)^{-\lambda/2} dz + O(1), \end{aligned}$$

which proves the required case. □

For fixed $\xi \in G$ and $g \in L^p(G)$, write

$$\begin{aligned} K_\lambda g(x) &= \int_G K_\lambda(x-y) g(y)\, dy \\ &= \int_{\{y \in G: |\xi-y| \leq 2r\}} K_\lambda(x-y) g(y)\, dy \\ &\quad + \int_{\{y \in G: |\xi-y| > 2r\}} K_\lambda(x-y) g(y)\, dy \\ &= I_1(x) + I_2(x), \end{aligned}$$

where $x \in D$ and $r = |x - \xi|$.

Lemma 6. *Let $g \in L^p(G)$. For $x = (x', x_n) \in D$ and $r = |x - \xi|$, we have*

$$|I_1(x)| \leqq M x_n^{-\lambda+(n-1)/p'} \left(\int_{\{y \in G: |\xi - y| \leq 2r\}} |g(y)|^p dy \right)^{1/p}.$$

Proof. Since $-\lambda p' + n - 1 < 0$, we have by Hölder's inequality

$$\begin{aligned} &|I_1(x)| \\ &\leqq \left(\int_{G \cap B(\xi, 2r)} (x_n^2 + |x' - y|^2)^{-\lambda p'/2} dy \right)^{1/p'} \left(\int_{G \cap B(\xi, 2r)} |g(y)|^p dy \right)^{1/p} \\ &\leqq x_n^{-\lambda+(n-1)/p'} \left(\int_{\mathbf{R}^{n-1}} (1 + |z|^2)^{-\lambda p'/2} dz \right)^{1/p'} \left(\int_{G \cap B(\xi, 2r)} |g(y)|^p dy \right)^{1/p}, \end{aligned}$$

which implies the required inequality. □

Note that

$$|I_2(x)| \leqq M \int_{G - B(\xi, 2r)} |\xi - y|^{-\lambda} |g(y)| \, dy. \tag{4.1}$$

Lemma 7. *If $\lim_{t \to 0} t^{1-n} \int_{G \cap B(\xi, t)} |g(y)| dy = 0$, then*

$$\lim_{r \to 0} [H(r)]^{-1} \int_{G - B(\xi, 2r)} |\xi - y|^{-\lambda} |g(y)| \, dy = 0.$$

Proof. For $r > 0$, set

$$\varepsilon(r) = \sup_{0 < t < r} t^{1-n} \int_{G \cap B(\xi, t)} |g(y)| dy;$$

then $\lim_{r \to 0} \varepsilon(r) = 0$ by our assumption. Hence we have

$$\begin{aligned} &\limsup_{r \to 0} [H(r)]^{-1} \int_{G - B(\xi, 2r)} |\xi - y|^{-\lambda} |g(y)| \, dy \\ &= \limsup_{r \to 0} [H(r)]^{-1} \int_{B(\xi, \delta) - B(\xi, 2r)} |\xi - y|^{-\lambda} |g(y)| \, dy \\ &\leqq \limsup_{r \to 0} [H(r)]^{-1} \left(\delta^{-\lambda} \int_{G \cap B(\xi, \delta)} |g(y)| dy \right. \\ &\qquad \left. + \lambda \int_{2r}^{\delta} \left\{ \int_{G \cap B(\xi, t)} |g(y)| dy \right\} t^{-\lambda - 1} dt \right) \\ &\leqq M \limsup_{r \to 0} [H(r)]^{-1} \varepsilon(\delta) \int_{2r}^{\delta} t^{n-1-\lambda-1} dt \\ &\leqq M \varepsilon(\delta) \end{aligned}$$

for $\delta > 0$, which gives the required equality. □

Now we are ready to prove Theorem 3.

Proof of Theorem 3. Letting ξ be a point such that

$$\lim_{t\to 0} t^{1-n}\int_{G\cap B(\xi,t)} |f(y)-f(\xi)|^p\, dy = 0, \tag{4.2}$$

almost every $\xi \in G$ has this property. Note that

$$\begin{aligned}
&\mathcal{P}_\lambda f(x) - f(\xi)\\
&= \frac{K_\lambda(f-f(\xi)\chi_G)(x)}{K_\lambda\chi_G(x)}\\
&= \frac{K_\lambda((f-f(\xi))\chi_{G\cap B(\xi,2r)})(x)}{K_\lambda\chi_G(x)} + \frac{K_\lambda((f-f(\xi))\chi_{G-B(\xi,2r)})(x)}{K_\lambda\chi_G(x)}\\
&= J_1(x)+J_2(x).
\end{aligned}$$

By Lemmas 5 and 6, we have

$$|J_1(x)| \leqq M\left(r^{1-n}\int_{G\cap B(\xi,2r)} |f(y)-f(\xi)|^p\, dy\right)^{1/p}$$

for $x \in \mathcal{A}_h(\xi)$ and small $r > 0$. Hence it follows from (4.2) that

$$\lim_{x\to\xi, x\in\mathcal{A}_h(\xi)} J_1(x) = 0.$$

On the other hand, we have by Lemma 5 and (4.1)

$$|J_2(x)| \leqq M[H(r)]^{-1}\int_{G-B(\xi,2r)} |\xi-y|^{-\lambda}|f(y)-f(\xi)|\, dy$$

for $x \in \mathcal{A}_h(\xi)$ and small $r > 0$, so that we see that by (4.2) and Lemma 7

$$\lim_{x\to\xi, x\in\mathcal{A}_h(\xi)} J_2(x) = 0.$$

Thus the Proof of Theorem 3 is completed. □

Remark 3. Let $1 \leqq p < \infty$ and f be a measurable function on $\mathbf{R}^{n-1}$ satisfying (1.4) for some number γ. Then, taking m as in Theorem 1, we may consider the function $K_{\lambda,m}f(x)$ instead of $K_\lambda f(x)$, and see that

$$\lim_{x\to\xi, x\in\mathcal{A}_h(\xi)} H(|x-\xi|)^{-1}K_{\lambda,m}f(x) = f(\xi)$$

for a.e. $\xi \in \mathbf{R}^{n-1}$.

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