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Let $f : M \to \mathbf{R}^2$ be a smooth immersion of a compact connected oriented surface with boundary M into $\mathbf{R}^2$. Kauffman defined an equivalence relation called *image homotopy* and classified the set of all orientation preserving immersions of M into $\mathbf{R}^2$ up to image homotopy. When M is of genus one and the number of boundary components is strictly greater than one, Kauffman's result requires a correction. In this paper we will study this particular case.

1. Introduction.

Let M be a compact connected oriented surface of genus g with $k+1$ boundary components $c_0, \ldots, c_k$, which are oriented as the boundary of M ($g \geq 0, k \geq 0$). We choose an orientation of M and oriented simple closed curves $a_1, b_1, \ldots, a_g, b_g$ as in Figure 1. We take the orientations of a_i and b_i so that the intersection number $b_i \cdot a_i$ in M is equal to 1.

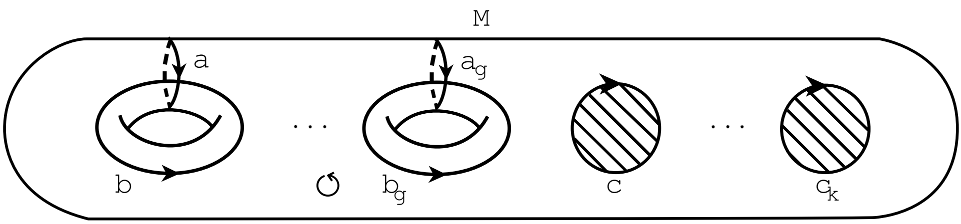

Figure 1.

Let $f : M \to \mathbf{R}^2$ be an orientation preserving immersion and $\alpha : S^1 \to M$ be an oriented simple closed curve in M. We define $D_f(\alpha) \in \mathbf{Z}$ by the rotation number of $f \circ \alpha$ (see [**W**]). Note that $\mathbf{R}^2$ and S^1 are oriented counterclockwise direction.

In [**K**], Kauffman studied the classification of all orientation preserving immersions of M into $\mathbf{R}^2$ up to regular homotopy and up to image homotopy. The definition of the image homotopy is as follows: Let $f_1, f_2 : M \to \mathbf{R}^2$ be two orientation preserving immersions. We say that f_1 and f_2 are *image homotopic* if there exists an orientation preserving diffeomorphism $h : M \xrightarrow{\cong} M$

such that f_1 is regularly homotopic to $f_2 \circ h$. But he incorrectly stated that as in the case where $g \geq 2$, the Arf-invariant and a collection of the rotation numbers of the boundary curves became a complete invariant of the image homotopy where $g = 1, k \geq 1$. This is because [**K**, Lemma 2.2] which is the formula about the sum of the rotation numbers of the boundary curves of M is incorrect. The purpose of this paper is to have a correct complete invariant of image homotopy classes where $g = 1, k \geq 1$.

We will obtain the required complete invariant in the following theorem:

Theorem 1.1. *Let M be a compact connected oriented surface of genus one with $k+1$ boundary components $c_0, c_1, \ldots, c_k$, which are oriented as the boundary of M $(k \geq 1)$. Let $f_1, f_2 : M \to \mathbf{R}^2$ be two orientation preserving immersions and $\mathfrak{S}_{k+1}$ be the symmetric group of degree $k+1$. Then f_1 is image homotopic to f_2 if and only if:*

(1) $(D_{f_1}(c_0), ..., D_{f_1}(c_k)) = (D_{f_2}(c_{\varepsilon(0)}), ..., D_{f_2}(c_{\varepsilon(k)}))$ *for some ε in $\mathfrak{S}_{k+1}$, and*

(2) $\gcd\big(D_{f_1}(a_1), D_{f_1}(b_1), D_{f_1}(c_0)+1, \ldots, D_{f_1}(c_k)+1\big)$
$= \gcd\big(D_{f_2}(a_1), D_{f_2}(b_1), D_{f_2}(c_0)+1, \ldots, D_{f_2}(c_k)+1\big)$.

Here a_1 and b_1 are the oriented curves as depicted in Figure 1, and any integers are the divisors of 0.

We have that the case where $g = 1,\ k = 0$ is contained in Theorem 1.1. This complete invariant where $g = 1$ is quite different from the complete invariants where $g \geq 2$ or $g = 0$.

This paper is organized as follows: In Section 2, we recall some basic facts about the regular homotopy and the mapping class group of M and study the changes of rotation numbers for an immersion under the action of the mapping class group. In Section 3, we prove Theorem 1.1 using the results in Section 2. This method is the same one used in [**K**]. In Section 4, we will define another invariant, *Arf-invariant*, and give an example that the Arf-invariant is not a complete invariant in general where $g = 1$. This becomes a counterexample of Theorem 4.2 (iii) in [**K**].

Throughout the paper, all surfaces and maps are differentiable of class C^∞.

2. Preliminaries.

Suppose M is a surface as in Theorem 1.1. The following proposition is well-known (see [**C**], [**K**] and [**W**]):

Proposition 2.1.

(1) *Let $\mathcal{R}(M)$ denote the set of regular homotopy classes of all orientation preserving immersions of M into $\mathbf{R}^2$ and $[f]_{\mathcal{R}} \in \mathcal{R}(M)$ denote the regular homotopy class of an orientation preserving immersion $f : M \to \mathbf{R}^2$.*

Then we have the following one-to-one mapping $\Phi : \mathcal{R}(M) \to \mathbf{Z}^{2+k}$ *where*

$$\mathcal{R}(M) \ni [f]_{\mathcal{R}} \mapsto \Phi([f]_{\mathcal{R}}) = [D_f(a_1), D_f(b_1) | D_f(c_1), \ldots, D_f(c_k)] \in \mathbf{Z}^{2+k}.$$

(2) *Let* $\mathcal{M}(M)$ *denote the mapping class group of* M *and* $\mathcal{I}(M)$ *denote the set of image homotopy classes of all orientation preserving immersions of* M *into* $\mathbf{R}^2$. *Then* $\mathcal{M}(M)$ *acts on* $\mathcal{R}(M)$ *by composition, and* $\mathcal{I}(M)$ *is in one-to-one correspondence with the orbit space* $\mathcal{R}(M)/\mathcal{M}(M)$ *as a set.*

(3) *Let* $f : M \to \mathbf{R}^2$ *be an orientation preserving immersion and* $\chi(M)$ *denote the Euler characteristic of* M. *Then* $\displaystyle\sum_{i=0}^{k} D_f(c_i) = \chi(M)$.

In order to study the set of image homotopy classes $\mathcal{I}(M)$, let us study the mapping class group $\mathcal{M}(M)$. The following proposition has been proved in [**B2**, Theorem 3]:

Proposition 2.2. *Let* a_1, b_1 *be the oriented simple closed curves as in Figure* 1 ($g = 1$). *Let* ρ_i, τ_i, *and* σ_m *be the oriented loops illustrated in Figure* 2 (a) *and Figure* 10 *in* [**B1**] ($0 \leq i \leq k, 0 \leq m \leq k-1$). *Then* $\mathcal{M}(M)$ *is generated by the isotopy classes of the following maps:*

The η*-twists* η_{a_1}, η_{b_1}, *the* ξ*-twists* ξ_{ρ_i}, ξ_{τ_i}, *and the* ζ*-twists* ζ_{σ_m} ($0 \leq i \leq k$, $0 \leq m \leq k-1$).

For the precise definitions of the *twists*, see [**B2**, Section 2]. In [**B2**], these twists are homeomorphisms. Therefore in this paper, we deform them to diffeomorphisms by smoothing.

The following lemma was implicitly stated in [**K**]:

Lemma 2.3. *Let* M *be a compact connected oriented surface of genus one with* $k+1$ *boundary components* $c_0, \ldots, c_k$, *which are oriented as the boundary of* M ($k \geq 0$). *Let* $[f]_{\mathcal{R}}$ *be the regular homotopy class of an orientation-preserving immersion* $f : M \to \mathbf{R}^2$ *and* $\Phi : \mathcal{R}(M) \to \mathbf{Z}^{2+k}$ *be the one-to-one mapping in Proposition* 2.1 (*1*). *For* $\Phi([f]_{\mathcal{R}}) = [\alpha_1, \beta_1 | \gamma_1, \ldots, \gamma_k] \in \mathbf{Z}^{2+k}$, *we have the following equations:*

$$\Phi([f \circ \eta_{a_1}^{\pm 1}]_{\mathcal{R}}) = [\alpha_1, \mp\alpha_1 + \beta_1 | \gamma_1, \ldots, \gamma_k],$$
$$\Phi([f \circ \eta_{b_1}^{\pm 1}]_{\mathcal{R}}) = [\alpha_1 \pm \beta_1, \beta_1 | \gamma_1, \ldots, \gamma_k],$$
$$\Phi([f \circ \xi_{\rho_i}^{\pm 1}]_{\mathcal{R}}) = [\alpha_1, \beta_1 \pm (\gamma_i + 1) | \gamma_1, \ldots, \gamma_k],$$
$$\Phi([f \circ \xi_{\tau_i}^{\pm 1}]_{\mathcal{R}}) = [\alpha_1 \mp (\gamma_i + 1), \beta_1 | \gamma_1, \ldots, \gamma_k] \quad (0 \leq i \leq k),$$
$$\Phi([f \circ \zeta_{\sigma_m}^{\pm 1}]_{\mathcal{R}}) = [\alpha_1, \beta_1 | \gamma_1, \ldots, \gamma_{m-1}, \gamma_{m+1}, \gamma_m, \ldots, \gamma_k] \; (1 \leq m \leq k-1), \quad \textit{and}$$
$$\Phi([f \circ \zeta_{\sigma_0}^{\pm 1}]_{\mathcal{R}}) = [\alpha_1, \beta_1 | \gamma_0, \gamma_2, \ldots, \gamma_k],$$

where $\gamma_0 = -\sum_{i=1}^{k} \gamma_i - (k+1)$ *and double signs in the same order.*

Proof. To prove these equations, we chase the change of each of the curves $a_1, b_1, c_0, \ldots, c_k$ under the action of the generators of the mapping class group $\mathcal{M}(M)$ and calculate the rotation numbers of the resulting curves. They can be easily checked. $\square$

3. Proof of Theorem 1.1.

Let $[f]_{\mathcal{R}}$ be the regular homotopy class of an orientation preserving immersion $f : M \to \mathbf{R}^2$, and $\Phi : \mathcal{R}(M) \to \mathbf{Z}^{2+k}$ be the one-to-one mapping in Proposition 2.1 (1) ($\mathcal{R}(M)$ is the set of regular homotopy classes). Set $\Phi([f]_{\mathcal{R}}) = [\alpha_1, \beta_1 | \gamma_1, \ldots, \gamma_k] \in \mathbf{Z}^{2+k}$. We define the bijection $\Psi : \mathbf{Z}^{2+k} \to \mathbf{Z}^{2+k}$ by $\Psi[\alpha_1, \beta_1 | \gamma_1, \ldots, \gamma_k] = (\alpha_1, \beta_1 | \gamma_1 + 1, \ldots, \gamma_k + 1)$, and we transfer the relations of Lemma 2.3 under this map as follows:

$$
\begin{aligned}
& (\alpha_1, \beta_1 | \gamma_1 + 1, \ldots, \gamma_k + 1) \\
\text{(i)} \quad & \approx (\alpha_1 \pm \beta_1, \beta_1 | \gamma_1 + 1, \ldots, \gamma_k + 1) \\
\text{(ii)} \quad & \approx (\alpha_1, \mp\alpha_1 + \beta_1 | \gamma_1 + 1, \ldots, \gamma_k + 1), \\
& (\alpha_1, \beta_1 | \gamma_1 + 1, \ldots, \gamma_i + 1, \ldots, \gamma_k + 1) \\
\text{(iii)} \quad & \approx (\alpha_1 \mp (\gamma_i + 1), \beta_1 | \gamma_1 + 1, \ldots, \gamma_i + 1, \ldots, \gamma_k + 1) \\
\text{(iv)} \quad & \approx (\alpha_1, \beta_1 \pm (\gamma_i + 1) | \gamma_1 + 1, \ldots, \gamma_i + 1, \ldots, \gamma_k + 1), \\
& (\alpha_1, \beta_1 | \ldots, \gamma_m + 1, \gamma_{m+1} + 1, \ldots, \gamma_k + 1) \\
\text{(v)} \quad & \approx (\alpha_1, \beta_1 | \ldots, \gamma_{m+1} + 1, \gamma_m + 1, \ldots, \gamma_k + 1), \\
& (\alpha_1, \beta_1 | \gamma_1 + 1, \gamma_2 + 1, \ldots, \gamma_k + 1) \\
\text{(vi)} \quad & \approx (\alpha_1, \beta_1 | \gamma_0 + 1, \gamma_2 + 1, \ldots, \gamma_k + 1),
\end{aligned}
$$

where $\gamma_0 = -\sum_{i=1}^{k} (\gamma_i + 1)$ and for x, y in $\mathcal{R}(M)$, $\Psi \circ \Phi(x) \approx \Psi \circ \Phi(y)$ means that x and y represent image homotopic immersions. From Proposition 2.1 (2) and Proposition 2.2, we have that two elements in $\mathcal{R}(M)$ represent the same element in the set of image homotopy classes if and only if they change each other by a finite iteration of (i)-(vi).

First, suppose that two orientation preserving immersions f_1 and f_2 are image homotopic. It is easy to see that Conditions (1) and (2) of Theorem 1.1 are the invariants of an image homotopy class.

Next, suppose that Conditions (1) and (2) of Theorem 1.1 are satisfied. Let $[f_1]_{\mathcal{R}}$ and $[f_2]_{\mathcal{R}}$ be the regular homotopy classes and set $\Phi([f_j]_{\mathcal{R}}) = [\alpha_1^j, \beta_1^j | \gamma_1^j, \ldots, \gamma_k^j]$ where $\gamma_0^j = -\sum_{i=1}^{k} (\gamma_i^j + 1)$ and $j = 1, 2$. By Theorem 1.1 (1) and Lemma 2.3, we may assume that $\gamma_i^1 = \gamma_i^2 = \gamma_i \ \ (0 \le i \le k)$.

We use the above relations to $\Psi \circ \Phi([f_j]_{\mathcal{R}})$. By (i), (ii), (iii), and (iv), we have

$$\begin{aligned}(\alpha_1^j, \beta_1^j|\gamma_1+1,\ldots,\ldots,\gamma_k+1) &\approx (\delta_1^j, 0|\gamma_1+1,\ldots,\gamma_k+1)\\ &\approx (\delta_2^j, 0|\gamma_1+1,\ldots,\gamma_k+1),\end{aligned}$$

where $\delta_1^j = \gcd(\alpha_1^j, \beta_1^j)$, $\delta_2^j = \gcd(\delta_1^j, \gamma_0+1,\ldots,\gamma_k+1) = \gcd(\alpha_1^j, \beta_1^j, \gamma_0+1,\ldots,\gamma_k+1)$ and $j=1,2$. Then by Condition (2) of Theorem 1.1, we have $\Psi\circ\Phi([f_1]_{\mathcal{R}}) \approx \Psi\circ\Phi([f_2]_{\mathcal{R}})$. This completes the proof. $\square$

Remark 3.1. If $k=0$, we have $D_f(c_0) = -1$ for any orientation preserving immersion $f : M \to \mathbf{R}^2$. Thus, when we apply the above proof, Theorem 1.1 also holds in the case where $k=0$.

4. Supplement.

We define another invariant of image homotopy classes. Let $f : M \to \mathbf{R}^2$ be an orientation preserving immersion. If all $D_f(c_i)$ are odd ($0 \le i \le k$), the definition of the *Arf-invariant* is $A(f) :\equiv (D_f(a_1)+1)(D_f(b_1)+1) \pmod 2$. On the other hand, if there exists c_i with $D_f(c_i) \equiv 0 \pmod 2$, we cannot define the Arf-invariant (see [**KB**]).

The following example shows that the Arf-invariant is not a complete invariant of image homotopy classes in general even if there is no c_i with $D_f(c_i) \equiv 0 \pmod 2$.

Example 4.1. Assume that $k=1$ and $(D_{f_j}(c_0), D_{f_j}(c_1)) = (-5, 3)$ ($j = 1, 2$). Set $\Phi([f_1]_{\mathcal{R}}) = [2, 0|3]$ and $\Phi([f_2]_{\mathcal{R}}) = [4, 0|3]$ (Φ is the map in Proposition 2.1 and see Figure 2). From Theorem 1.1, f_1 and f_2 are not image homotopic. But their Arf-invariants are equal to 1, we have that the Arf-invariant is not a complete invariant in this case.

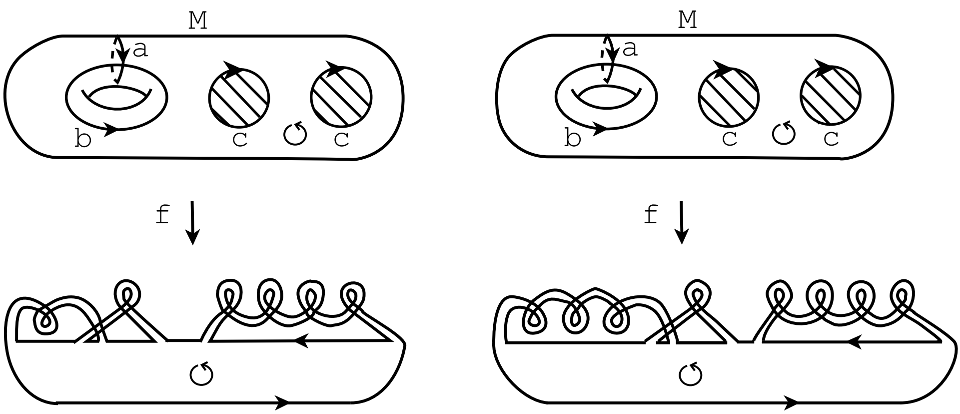

Figure 2.

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