Pacific Journal of Mathematics

ON SURFACES OF PRESCRIBED F-MEAN CURVATURE

Ulrich Clarenz and Heiko von der Mosel

Volume 213 No. 1 January 2004

PACIFIC JOURNAL OF MATHEMATICS
Vol. 213, No. 1, 2004

ON SURFACES OF PRESCRIBED F-MEAN CURVATURE

Ulrich Clarenz and Heiko von der Mosel

Hypersurfaces of prescribed weighted mean curvature, or F-mean curvature, are introduced as critical immersions of anisotropic surface energies, thus generalizing minimal surfaces and surfaces of prescribed mean curvature. We first prove enclosure theorems in $\mathbb{R}^{n+1}$ for such surfaces in cylindrical boundary configurations. Then we derive a general second variation formula for the anisotropic surface energies generalizing corresponding formulas of do Carmo for minimal surfaces, and Sauvigny for prescribed mean curvature surfaces. Finally we prove that stable surfaces of prescribed F-mean curvature in $\mathbb{R}^3$ can be represented as graphs over a planar strictly convex domain Ω, if the given boundary contour in $\mathbb{R}^3$ is a graph over $\partial\Omega$.

1. Introduction and main results.

Let $X : M \to \mathbb{R}^{n+1}$, $n \geq 2$, be an immersion of class $C^3(M, \mathbb{R}^{n+1})$ of an n-dimensional smooth manifold $M = M^n$ with boundary ∂M into $\mathbb{R}^{n+1}$. We denote the corresponding unit normal by N and the induced area element by dA, and consider general *parametric variational functionals* $\mathcal{F}$ of the form

$$\mathcal{F}(X) := \int_M F(X, N)\, dA. \tag{1.1}$$

The integrand F of class $C^0(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \backslash \{0\}))$ is a *parametric Lagrangian* characterized by the homogeneity condition

$$F(y, tz) = tF(y, z) \quad \text{for all} \quad t > 0,\, (y, z) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}. \tag{H}$$

Note that (H) implies

$$F_{zz}(y, z)z = 0 \quad \text{for all} \quad (y, z) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \backslash \{0\}); \tag{1.2}$$

hence we will identify the symmetric endomorphism $F_{zz}(y, z) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with its restriction to the space

$$z^\perp := \{\zeta \in \mathbb{R}^{n+1} : \langle \zeta, z \rangle = 0\}. \tag{1.3}$$

Important examples of parametric Lagrangians are given by the *area integrand*

$$A(z) := |z|, \tag{1.4}$$

and the integrand

$$E(y,z) := |z| + \langle Q(y), z\rangle \tag{1.5}$$

appearing in the theory of capillary surfaces. Here, Q can be chosen as a differentiable vectorfield in $\mathbb{R}^{n+1}$ with $\operatorname{div}_{\mathbb{R}^{n+1}} Q(y) = \mathcal{H}(y)$, where $\mathcal{H}(y)$ is a given function representing the prescribed mean curvature. Critical immersions of the corresponding functionals

$$\mathcal{A}(X) := \int_M A(N)\,dA = \int_M dA \tag{1.6}$$

and

$$\mathcal{E}(X) := \int_M E(X,N)\,dA \tag{1.7}$$

are *minimal surfaces* and *surfaces of prescribed mean curvature* $\mathcal{H}(X)$, respectively.

Another interesting example is

$$F(z) = \sum_{j=1}^{3} \sqrt{\delta^2|z|^2 + z_j^2}, \qquad \delta > 0, \tag{1.8}$$

which serves as a regularized version of the discrete l^1-norm used for numerical computations involving the anisotropic mean curvature flow [**7**]. Furthermore, in surface processing [**3**] such parametric functionals have become an increasingly important tool to enhance edge structures within a suitable surface evolution based on (1.6) and (1.8). For more examples of integrands and applications in numerical analysis we refer to [**8**] and [**6**].

For general parametric integrals we recall the notion of the F*-mean curvature*

$$H_F(X,N) = H_F := -\operatorname{tr}(A_F S), \tag{1.9}$$

as introduced in [**2**] and [**4**]. Here, $S \in \operatorname{End}(TM)$ is the shape operator defined by $DX \circ S := DN$ on the tangent bundle TM, and $A_F \in \operatorname{End}(TM)$ is the symmetric endomorphism field given by

$$A_F := (DX)^{-1}(F_{zz}(X,N)DX) \quad \text{on} \quad TM. \tag{1.10}$$

For the special parametric Lagrangians in (1.6) and (1.7) the F-mean curvature H_F reduces to the classical mean curvature H, since $A_F|_{T_wM} = \operatorname{Id}_{T_wM}$ for each $w \in M$ and $F(y,z) = A(z)$, or $F(y,z) = E(y,z)$, respectively. Here T_wM denotes the tangent space of M at $w \in M$.

The first author proved in [**2**] that the Euler equation for $\mathcal{F}$ can be written as

$$H_F = \sum_{i=1}^{n+1} F_{y^i z^i}(X,N). \tag{1.11}$$

Consequently, given a general parametric Lagrangian $F = F(y, z)$, critical immersions of the corresponding parametric functional $\mathcal{F}$ may be viewed as *surfaces of prescribed F-mean curvature.* In particular, we will regard critical immersions of the specific parametric functional

$$\mathcal{F}^0(X) := \int_M F(N)\, dA + \int_M \langle Q(X), N\rangle\, dA, \tag{1.12}$$

where $\operatorname{div}_{\mathbb{R}^{n+1}} Q(y) = \mathcal{H}_F(y) \in C^0(\mathbb{R}^{n+1})$ is a given function, as *surfaces of prescribed F-mean curvature* $\mathcal{H}_F(X)$. This class of surfaces yields a natural generalization of minimal surfaces if $\mathcal{H}_F(y) \equiv 0$, or of surfaces of constant mean curvature if $\mathcal{H}_F(y) \equiv \mathcal{H}_F^0 \in \mathbb{R}$. Let us point out that the parametric Lagrangian $F(z)$ in (1.12) depends on z only, and that in case $\mathcal{H}_F(y) \equiv \mathcal{H}_F^0 \in \mathbb{R}$ the second integrand in (1.12) is linear in y and z and can be interpreted as a volume term.

As a starting point for our investigations we will derive in Section 2 a differential equation for the surface normal of an arbitrary immersion in terms of the *F-Laplace-Beltrami operator* of X

$$\Delta_F := \operatorname{div}(A_F \operatorname{grad}(.)), \tag{1.13}$$

where the differential operators are taken with respect to the induced metric

$$g_w(V, W) = g(V, W) := \langle DX(V), DX(W)\rangle \quad \text{for} \quad V, W \in T_w M,\ w \in M, \tag{1.14}$$

i.e., $\operatorname{div} = \operatorname{div}_M$ and $\operatorname{grad} = \operatorname{grad}_M$.

Theorem 1.1. *Let N be the normal of an arbitrary immersion X of class $C^3(M, \mathbb{R}^{n+1})$ and let $F \in C^0(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \backslash \{0\}))$ be a parametric Lagrangian. Then*

$$\Delta_F N + \operatorname{tr}(A_F S^2) N = DX(\operatorname{div}(SA_F)). \tag{1.15}$$

Here, $\operatorname{div}(SA_F)$ denotes the divergence of the endomorphism field SA_F; see Section 2 for details.

In Section 3 we consider hypersurfaces with bounded F-mean curvature spanning[1] a given Jordan curve $\Gamma \subset \mathbb{R}^{n+1}$, i.e., we take an immersion $X : M \to \mathbb{R}^{n+1}$ mapping the boundary ∂M topologically onto Γ.

A parametric Lagrangian $F(y, z)$ is said to be (*uniformly*) *elliptic*, if there exists a constant $M_1 > 0$ such that

$$|z|\langle \zeta, F_{zz}(y, z)\zeta\rangle \geq M_1 |\zeta^{\tan}|^2 \tag{E}$$

[1]The existence of conformally parametrized $\mathcal{F}$-minimizing surfaces under Plateau type boundary conditions was proven in [**14**] and [**15**] for $n = 2$ and arbitrary co-dimension, but these solutions might have branch points. For the restricted class of boundary contours considered in Theorem 1.2, White [**24**] has constructed an embedded $\mathcal{F}$-minimizing disk in $\mathbb{R}^3$.

for all $(y,z) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1}\backslash\{0\})$, $\zeta \in \mathbb{R}^{n+1}$, where $\zeta^{\tan} := \zeta - \langle \zeta, z\rangle z/|z|^2$. Notice that $A(z)$ and $E(y,z)$ as defined in (1.4) and (1.5) are elliptic satisfying (E) with $M_1 = 1$.

Surfaces of vanishing F-mean curvature, where F is elliptic, have the *convex hull property* as proven in [**2**, Thm. 2.3]:

Theorem 1.2. *Let $F = F(z) \in C^0(\mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1}\backslash\{0\})$ be a parametric Lagrangian satisfying* (E). *Suppose $X \in C^0(\overline{M}, \mathbb{R}^{n+1}) \cap C^2(M, \mathbb{R}^{n+1})$ is an immersion of vanishing F-mean curvature, i.e., with $H_F(X,N) = \mathcal{H}_F(X) \equiv 0$. If X spans a Jordan curve $\Gamma \subset \mathbb{R}^{n+1}$ contained in the boundary of a closed convex set $K \subset \mathbb{R}^{n+1}$, then $X(\overline{M}) \subset K$.*

For surfaces of bounded (but not necessarily vanishing) F-mean curvature spanning a given Jordan curve Γ within the infinite cylinder

(1.16)
$$\mathcal{Z}_h := \left\{ (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : h\sqrt{(x^1)^2 + \cdots + (x^n)^2} \le 1 \right\}, \; h \ge 0,$$

we restrict our attention to Jordan curves $\Gamma \subset \mathbb{R}^{n+1}$ with an orthogonal projection onto an *h-convex domain* $\overline{\Omega} \subset B_{h^{-1}}(0) \subset \mathbb{R}^n$. Following Sauvigny [**21**] we call a bounded convex domain $\Omega \subset \mathbb{R}^n$ *κ-convex* for some $\kappa > 0$, if for every $w_0 \in \partial\Omega$ there is a point $\xi_0 = \xi_0(w_0) \in \mathbb{R}^n$ such that the ball $B_{1/\kappa}(\xi_0) \subset \mathbb{R}^n$ contains Ω and such that $w_0 \in \partial B_{1/\kappa}(\xi_0)$.

Theorem 1.3. *Let $F = F(y,z) \in C^0(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1} \times (\mathbb{R}^{n+1}\backslash\{0\}))$ be a parametric Lagrangian satisfying* (E). *Suppose $X : \overline{M} \to \mathcal{Z}_h$ of class $C^0(\overline{M}, \mathbb{R}^{n+1}) \cap C^2(M, \mathbb{R}^{n+1})$ is an immersion of prescribed F-mean curvature $\mathcal{H}_F \in C^0(\mathbb{R}^{n+1})$, where $\mathcal{H}_F(y)$ satisfies*

$$\|\mathcal{H}_F\|_{C^0(\mathbb{R}^{n+1})} \le M_1 h(n-1), \tag{1.17}$$

and X spans a curve $\Gamma \subset \mathcal{Z}_h$, whose orthogonal projection onto $\mathbb{R}^n$ lies in an h-convex domain $\Omega \subset B_{h^{-1}}(0) \subset \mathbb{R}^n$. Then

$$X(B) \subset \mathcal{Z}_\Omega := \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : (x^1, \ldots, x^n) \in \Omega\}. \tag{1.18}$$

In general one cannot expect that surfaces of bounded F-mean curvature satisfying the conditions of Theorem 1.3 can be represented as a graph over the h-convex domain $\Omega \subset \mathbb{R}^n$. For $n = 2$ and *stable* surfaces of bounded mean curvature $\mathcal{H}(y) \in C^{1,\alpha}(\mathbb{R}^3)$, however, Sauvigny was able to prove such a result [**21**] under a sign condition on $\frac{\partial}{\partial y^3}\mathcal{H}$, and it turns out that the same is true for stable surfaces of prescribed F-mean curvature in $\mathbb{R}^3$; see Theorem 1.4 below.

Before defining stability in Section 4 we generalize do Carmo's [**1**] second variation formula for the area functional (1.6) to the parametric functional (1.12). That is, we derive a general formula for the second variation $\delta^2\mathcal{F}^0(X, \Xi)$ of the functional (1.12) at critical immersions $X : M \to$

$\mathbb{R}^{n+1}$ in the direction of an arbitrary compactly supported vector field $\Xi \in C_0^2(M, \mathbb{R}^{n+1})$ containing normal and tangential terms[2] ; see Theorem 4.1. For immersions $X : M \to \mathbb{R}^{n+1}$ of prescribed F-mean curvature $\mathcal{H}_F$, however, the tangential term drops out (see Corollary 4.2), which additionally implies a simplified differential equation for the normal N of such surfaces derived in Corollary 4.3:

$$\Delta_F N + \Big[\mathrm{tr}\,(A_F S^2) - \langle \nabla_{\mathbb{R}^{n+1}} \mathcal{H}_F(X), N\rangle\Big] N = -\nabla_{\mathbb{R}^{n+1}} \mathcal{H}_F(X). \tag{1.19}$$

By means of this equation we are able to generalize Sauvigny's result [**21**] for surfaces of bounded mean curvature mentioned above to stable surfaces of prescribed F-mean curvature in $\mathbb{R}^3$:

Theorem 1.4. *Let $F = F(z) \in C^0(\mathbb{R}^3) \cap C^3(\mathbb{R}^3 \backslash \{0\})$ be an elliptic parametric Lagrangian satisfying* (E). *Suppose $X : \overline{B} \to \mathcal{Z}_h$ is of class $C^3(B, \mathbb{R}^3)$ $\cap C^{1,\alpha}(\overline{B}, \mathbb{R}^3)$ for some $\alpha \in (0,1)$ and a stable immersion of prescribed F-mean curvature $\mathcal{H}_F \in C^{1,\alpha}(\mathbb{R}^3)$, where $\mathcal{H}_F$ satisfies*

$$\|\mathcal{H}_F\|_{C^0(\mathbb{R}^3)} \le M_1 h. \tag{1.20}$$

We assume that X spans $\Gamma \subset \mathcal{Z}_h$, where Γ is a Jordan curve given as a graph over the boundary $\partial\Omega$ of an h-convex domain $\Omega \subset \mathbb{R}^2$. Then $X(B) \subset \mathcal{Z}_\Omega$, and $X(\overline{B})$ can be represented as a graph over Ω, if $\frac{\partial}{\partial y^3}\mathcal{H}_F(y) \ge 0$ for all $y = (y^1, y^2, y^3) \in \mathbb{R}^3$.

The proof of this result can be found in Section 5. For minimal surfaces this result is due to Radó [**19**]. Gulliver and Spruck [**13**] generalized Radó's theorem to surfaces of constant mean curvature.

Remark. For simplicity of presentation we have assumed throughout this paper that the surfaces are immersed up to the boundary. The strong smoothness hypotheses of Theorem 1.4, however, allow us to exclude *boundary branch points* for the specific boundary configurations considered in Theorems 1.2 and 1.4 with $n = 2$; see the corresponding remarks in Sections 3 and 5. That is, a *conformally parametrized surface of class $C^{1,\alpha}(\overline{B}, \mathbb{R}^3)$ without interior branch points does not have boundary branch points if it either has vanishing F-mean curvature with boundary contour $\Gamma \subset \partial K$ for some convex set $K \subset \mathbb{R}^3$, or if it has prescribed F-mean curvature $\mathcal{H}_F$ satisfying* (1.20)*, with boundary contour $\Gamma \subset \mathcal{Z}_h$ as in Theorem* 1.4.

A general boundary regularity result, however, guaranteeing $C^{1,\alpha}$-smoothness up to the boundary is currently only available for $\mathcal{F}$-minimizers; see [**16**], but not for $\mathcal{F}$-critical points.

[2] Theorem 4.1 contains as special cases the corresponding second variation formulas of Sauvigny [**21**] and Räwer [**20**] who consider only normal, or F-normal variations, respectively.

2. Preliminaries and a differential equation for the normal.

In terms of the induced metric $g : T_wM \times T_wM \to \mathbb{R}$ defined in (1.14) we can express an arbitrary tangent vector $V \in T_wM$ as

$$V = g^{kj} g(V, \partial_j)\partial_k, \tag{2.1}$$

and its image under the isomorphism $DX : T_wM \to T_{X(w)}\mathcal{M}$, where $\mathcal{M} := X(\overline{M}) \subset \mathbb{R}^{n+1}$, as

$$DX(V) = g^{kj} g(V, \partial_j)\partial_k X. \tag{2.2}$$

Here g^{kj} are the coefficients of the inverse of the metric tensor g_{kj} and

$$\{\partial_1, \ldots, \partial_n\} := \left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}$$

is the coordinate basis spanning T_wM. Let $\chi(M)$ be the space of vector fields of class C^2 on M and denote by ∇_V the covariant derivative in the direction of $V \in \chi(M)$. We set $\nabla_i := \nabla_{\partial_i}$, $i = 1, \ldots, n$. We will frequently use the following versions of the product rule:

$$U(g(V, W)) = g(\nabla_U V, W) + g(V, \nabla_U W), \tag{2.3}$$

$$\nabla_U(AV) = (\nabla_U A)V + A(\nabla_U V), \tag{2.4}$$

for all $U, V, W \in \chi(M)$ and all differentiable endomorphism fields $A \in \mathrm{End}\,(TM)$. As a consequence of (2.3) we obtain for symmetric $A \in \mathrm{End}(TM)$ and $\phi \in C^2(M)$

$$U(d\phi(AV)) = g(\nabla_U(A \operatorname{grad}\phi), V) + d\phi(A\nabla_U V), \tag{2.5}$$

where $g(\operatorname{grad}\phi, V) := d\phi(V)$, $V \in T_wM$, defines the gradient of the function ϕ on M as usual. Using the fact that $\langle DX(V), N\rangle = 0$ one can show that

$$U(DX(V)) = DX(\nabla_U V) - \langle DX(V), DX \circ S(U)\rangle N \tag{2.6}$$

for all $U, V \in \chi(M)$. The trace of an endomorphism $A \in \mathrm{End}\,(TM)$ in local coordinates is given by

$$\operatorname{tr} A = g^{ik} g(A\partial_i, \partial_k). \tag{2.7}$$

In particular, we will denote

$$\operatorname{tr}(A\nabla_\bullet V) = g^{ik} g(A\nabla_i V, \partial_k) \;\text{ for }\; A \in \mathrm{End}\,(TM), V \in \chi(M). \tag{2.8}$$

For $A := \mathrm{Id}$ we obtain the usual *divergence of a vector field* $W \in \chi(M)$

$$\operatorname{div} W = \operatorname{div}_M W := \operatorname{tr}(\nabla_\bullet W) = g^{ik} g(\nabla_i W, \partial_k). \tag{2.9}$$

The *divergence* $\mathrm{Div}\, Z$ *of a* (*not necessarily tangential*) *vector field* $Z : M \to \mathbb{R}^{n+1}$ is given in local coordinates by

$$\mathrm{Div}\, Z = g^{ik}\langle DZ(\partial_i), DX(\partial_k)\rangle. \tag{2.10}$$

For $Z := DX(W)$, $W \in \chi(M)$, we get $\operatorname{Div} Z = \operatorname{div} W$ by (2.9) and (2.10). We will also use the notion of the *divergence of an endomorphism field* $\operatorname{div} A$, $A \in \operatorname{End}(TM)$, with adjoint A^*, given by

$$g(\operatorname{div} A, V) := \operatorname{tr}(\nabla_\bullet A^* V) = g^{ik} g((\nabla_i A^*)V, \partial_k). \tag{2.11}$$

In local coordinates we can write

$$\operatorname{div} A = g^{ik}(\nabla_i A)\partial_k, \tag{2.12}$$

where $\nabla_i A$ denotes the covariant derivative of the tensor A; see [**9**, Def. 2.60].

If we denote the coefficients of the second fundamental form of (M, g) with $h_{ij} := -g(\partial_i, S(\partial_j))$, and, correspondingly, the coefficients of the F-*second fundamental form* by

$$h_{Fij} := -g(\partial_i, A_F S(\partial_j)) = -\langle F_{zz}\partial_i X, \partial_j N\rangle, \tag{2.13}$$

then the F-mean curvature H_F defined in (1.9) can be written as

$$H_F = -\operatorname{tr}(A_F S) = -g^{ij} g(\partial_i, A_F S(\partial_j)) = g^{ij} h_{Fij}. \tag{2.14}$$

Introducing the second order differential operator

$$\Theta_F := \Delta_F - \operatorname{div} A_F, \tag{2.15}$$

where Δ_F is given by (1.13), the first author could prove in [**2**] that

$$\Theta_F X = H_F N \tag{2.16}$$

holds for any immersion $X \in C^2(M, \mathbb{R}^{n+1})$. This equation reduces to the classical identity $\Delta X = \operatorname{div} \operatorname{grad} X = HN$, if $F(y, z) = A(z)$, or $F(y, z) = E(y, z)$, respectively; see (1.4), (1.5). Moreover, Θ_F is uniformly elliptic if F satisfies the ellipticity condition (E), which leads to the enclosure theorems proven in [**2**], and which will be used in the proofs of Sections 3 and 5.

Now we will conclude this section with:

Proof of Theorem 1.1. Apply (2.5) to $\phi := N^i$, $i = 1, \ldots, n+1$, and $A := A_F \in \operatorname{End}(TM)$ to obtain by (2.6) and (2.4)

$$\begin{aligned}
g(\nabla_U(A_F \operatorname{grad} N), V) \underset{(2.5)}{=} {} & U(DN(A_F V)) - DN(A_F \nabla_U V) \\
= {} & U(DX(SA_F V)) - DX(SA_F \nabla_U V) \\
\underset{(2.6)}{=} {} & DX(\nabla_U(SA_F V)) \\
& -\langle DX(SA_F V), DX \circ S(U)\rangle N \\
& -DX(SA_F \nabla_U V) \\
\underset{(2.4)}{=} {} & DX(\nabla_U(SA_F)V) - g(SA_F(V), S(U))N.
\end{aligned}$$

Choosing $U = \partial_i$, $V = \partial_k$ we obtain by (1.13), (2.9), (2.12) and (2.7)

$$\begin{aligned}
\Delta_F N &\underset{(1.13)}{=} \operatorname{div}(A_F \operatorname{grad} N) \\
&\underset{(2.9)}{=} g^{ik} g(\nabla_i (A_F \operatorname{grad} N), \partial_k) \\
&= g^{ik} DX(\nabla_i (SA_F)\partial_k) - g^{ik} g(SA_F(\partial_k), S(\partial_i))N \\
&\underset{(2.12),(2.7)}{=} DX(\operatorname{div}(SA_F)) - \operatorname{tr}(A_F S^2)N,
\end{aligned}$$

where we have used the symmetry of A_F and S to obtain the last term. $\square$

Using the Codazzi equation (cf. [**18**, p. 30])

$$(\nabla_V S)W = (\nabla_W S)V \tag{2.17}$$

one can show that

$$\operatorname{div} S = -\operatorname{grad} H. \tag{2.18}$$

Thus in case of the functionals (1.6) or (1.7), where A_F is the identity, we obtain

$$\Delta N + \operatorname{tr}(S^2)N = -DX(\operatorname{grad} H).$$

Therefore (1.15) is a generalization of [**21**, Hilfssatz 1].

3. Proofs of the enclosure theorems.

For the convenience of the reader we recall the Proof of Theorem 1.2 from [**2**].

Proof of Theorem 1.2. Since $H_F(X, N) = \mathcal{H}_F(X) = 0$, we infer from (2.16) that

$$\Theta_F(t(X)) = 0 \tag{3.1}$$

for all affine linear functions

$$t(y) := \langle a, y\rangle + b, \quad a \in \mathbb{R}^{n+1}, b \in \mathbb{R}. \tag{3.2}$$

Taking an arbitrary supporting half plane of the convex body K characterized by an affine linear function t_K, we have $t_K(X) \le 0$ on ∂M, and hence by (3.1) and the maximum principle [**11**, p. 32], $t_K(X) \le 0$ on $\overline{M}$, i.e., $X(\overline{M}) \subset K$. $\square$

Remark. For $n = 2$, $M := B = B_1(0) \subset \mathbb{R}^2$, and X immersed only in the interior of B but given in conformal parameters, i.e., with

$$|X_u|^2 = |X_v|^2 \quad \text{and} \quad \langle X_u, X_v\rangle = 0 \text{ on } \overline{B}, \tag{3.3}$$

we can exclude boundary branch points. In fact, introducing polar coordinates (r,ϑ) in B and fixing $w_0 \in \partial B$ we can apply Hopf's boundary point lemma [**11**, p. 34] together with (3.1) to obtain for $t:\mathbb{R}^3 \to \mathbb{R}$ as in (3.2)

$$\frac{\partial}{\partial \nu}[t(X(w))]\big|_{w=w_0} = \langle a, X_r(w_0)\rangle > 0.$$

Therefore we have $|X_r(w_0)| > 0$. Rewriting (3.3) in polar coordinates we conclude $|X_\vartheta(w_0)| > 0$ which shows that w_0 is not a branch point.

Proof of Theorem 1.3. For the function $R(x) := (x^1)^2 + \cdots + (x^n)^2$ we compute similarly as in [**5**, p. 7] using (2.1), (2.15), (2.16), (2.12) and (E)

$$\begin{aligned}
\frac{1}{2}\Theta_F(R(X)) \quad &\underset{\substack{(2.1)\\(2.15)\\(2.12)}}{=} \quad \sum_{i=1}^n X^i \mathrm{div}\,(A_F\,\mathrm{grad}\,(X^i)) \\
&\qquad\quad + \sum_{i=1}^n g(\mathrm{grad}\,(X^i), A_F\,\mathrm{grad}\,(X^i)) \\
&\qquad\quad - \sum_{i=1}^n X^i(\mathrm{div}\,A_F)(X^i) \\
&\underset{(2.15)}{=} \quad \sum_{i=1}^n X^i \Theta_F(X^i) + \sum_{i=1}^n g(\mathrm{grad}\,(X^i), A_F\,\mathrm{grad}\,(X^i)) \\
&\underset{\substack{(2.16)\\(\mathrm{E})}}{\geq} \quad \sum_{i=1}^n H_F(X,N)X^iN^i + M_1 \sum_{i=1}^n g(\mathrm{grad}\,(X^i), \mathrm{grad}\,(X^i)) \\
&\geq \quad -|\mathcal{H}_F(X)|\sqrt{R(X)} + M_1(n-1) \\
&\geq \quad M_1(n-1) - \|\mathcal{H}_F\|_{C^0(\mathbb{R}^3)}h^{-1}
\end{aligned}$$

on M, since $X(\overline{M}) \subset \mathcal{Z}_h$; see (1.16). Notice that we have used the relation $p^{ii} = g(\mathrm{grad}\,(X^i), \mathrm{grad}\,(X^i))$, where $P = P(w) = (p^{ij})(w) : \mathbb{R}^{n+1} \to T_{X(w)}\mathcal{M}$ is the orthogonal projection onto the n-dimensional tangent plane of $\mathcal{M} = X(\overline{M})$, with $\sum_{i=1}^{n+1} p^{ii} = n$, so that

$$\sum_{i=1}^n g(\mathrm{grad}\,(X^i), \mathrm{grad}\,(X^i)) = n - g(\mathrm{grad}\,(X^{n+1}), \mathrm{grad}\,(X^{n+1})) \geq n-1.$$

Thus $\Theta_F(R(X)) \geq 0$ on M due to (1.17), and the maximum principle implies $R(X(w)) < h^{-2}$ for all $w \in M$, since $R(X) \not\equiv h^{-2}$ in M. Following Sauvigny [**21**] we now argue as follows: Assuming that there is some point $w^* \in M$ with $X(w^*) \notin \mathcal{Z}_\Omega$ we infer that $x^* := (X^1(w^*), X^2(w^*)) \notin \Omega$. Let $y^* \in \partial\Omega$ be a point with $|y^* - x^*| = \mathrm{dist}\,(x^*, \Omega) \geq 0$. (If $x^* \in \partial\Omega$ take $y^* := x^*$.) Since Ω is h-convex there is a point $\eta_* \in \mathbb{R}^n$ such that $\Omega \subset B_{1/h}(\eta_*)$ and

$y^* \in \partial B_{1/h}(\eta_*)$. Thus $X(w^*) \notin \mathcal{Z}_{B_{1/h}(\eta_*)}$ and we can look at the 1-parameter family of cylinders $\{\mathcal{Z}(\lambda) := \mathcal{Z}_{B_{1/h}(\lambda\eta_*)}\}_{0\le\lambda\le 1}$, for which

$$X(\overline{B}) \subset \mathcal{Z}(0) = \mathcal{Z}_h,$$

and

$$X(B) \cap \partial\mathcal{Z}(1) = X(B) \cap \partial\mathcal{Z}_{B_{1/h}(\eta_*)} \neq \emptyset.$$

By continuity we find $\lambda_0 \in [0,1]$ with

$$X(\overline{B}) \subset \mathcal{Z}(\lambda_0) \quad \text{and} \quad X(B) \cap \partial\mathcal{Z}(\lambda_0) \neq \emptyset. \tag{3.4}$$

With the same computation as before we deduce for $R_0(x) := (x^1 - \lambda_0\eta_*^1)^2 + \cdots + (x^n - \lambda_0\eta_*^n)^2$ the inequality $\Theta_F(R_0(X)) \ge 0$ on B; hence by (3.4) and the maximum principle $R_0(X(w)) \equiv h^{-2}$, which is absurd. Thus we have shown (1.18). □

4. A general second variation formula and stability.

In this section we consider C^3-perturbations $X(.,\epsilon) : M \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}^{n+1}$ of an immersed hypersurface $X \in C^3(M, \mathbb{R}^{n+1})$ with

$$X(.,0) = X \quad \text{and} \tag{4.1}$$

$$\frac{\partial}{\partial\epsilon} X(.,\epsilon)\Big|_{\epsilon=0} = \varphi N + DX(V) =: \Xi, \tag{4.2}$$

where $\varphi \in C_0^2(M)$, $V \in \chi(M)$ with compact support. Notice that we admit a non-vanishing tangential component in the variational field $\Xi \in C_0^2(M, \mathbb{R}^{n+1})$ as in [**1**] but in contrast to [**21**, p. 64]. The *second variation* $\delta^2\mathcal{F}^0(X, \Xi)$ of the functional $\mathcal{F}^0$ defined in (1.12) at X in the direction of Ξ is defined as

$$\delta^2\mathcal{F}^0(X,\Xi) := \frac{d^2}{d\epsilon^2}\mathcal{F}^0(X(.,\epsilon))\Big|_{\epsilon=0}. \tag{4.3}$$

Theorem 4.1. *Let $F = F(z) \in C^0(\mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1}\backslash\{0\})$ be a parametric Lagrangian. Suppose $X \in C^3(M, \mathbb{R}^{n+1})$ is a critical immersion for the functional* (1.12) *and $\Xi \in C_0^2(M, \mathbb{R}^{n+1})$ is a variational field of the form* (4.2). *Then*

$$\begin{aligned}
\delta^2\mathcal{F}^0&(X,\Xi) \\
&= \int_M \Big\{ g(A_F \operatorname{grad}\varphi, \operatorname{grad}\varphi) - \varphi^2(\operatorname{tr}(A_F S^2) \\
&\quad - \langle \nabla_{\mathbb{R}^{n+1}}\mathcal{H}_F(X), N\rangle) + \varphi g(\operatorname{div}(SA_F) + \operatorname{grad}\mathcal{H}_F(X), V)\Big\}\, dA.
\end{aligned} \tag{4.4}$$

Note that only first order derivatives of $X(.,\epsilon)$ with respect to ϵ, i.e., merely Ξ defined in (4.2) enters the formula for the second variation which justifies the notation on the left-hand side of (4.3).

Proof of Theorem 4.1. Using the identity

$$\frac{\partial}{\partial\eta}X(.,\epsilon+\eta)_{|_{\eta=0}} = \frac{\partial}{\partial\bar\epsilon}X(.,\bar\epsilon)_{|_{\bar\epsilon=\epsilon}}$$

we obtain

$$\begin{aligned}\delta^2\mathcal{F}^0(X,\Xi) &= \frac{d^2}{d\epsilon^2}\mathcal{F}^0(X(.,\epsilon))_{|_{\epsilon=0}}\\ &= \frac{d}{d\epsilon}\left(\frac{d}{d\eta}\mathcal{F}^0(X(.,\epsilon+\eta))_{|_{\eta=0}}\right)_{|_{\epsilon=0}}\\ &= \frac{d}{d\epsilon}\left(\delta\mathcal{F}^0\left(X(.,\epsilon),\frac{\partial}{\partial\bar\epsilon}X(.,\bar\epsilon)_{|_{\bar\epsilon=\epsilon}}\right)\right)_{|_{\epsilon=0}}.\end{aligned}$$

Hence, by the first variation formula proved in [**2**, pp. 5,6] applied to (1.12) and evaluated at $X(.,\epsilon)$ in the direction $\frac{\partial}{\partial\bar\epsilon}X(.,\bar\epsilon)_{|_{\bar\epsilon=\epsilon}}$,

$$\begin{aligned}\delta^2\mathcal{F}^0(X,\Xi) = \frac{d}{d\epsilon}\Bigg(\int_M\Bigg\{&\left\langle\frac{\partial}{\partial\bar\epsilon}X(.,\bar\epsilon)_{|_{\bar\epsilon=\epsilon}},N(.,\epsilon)\right\rangle\\ &\cdot[\mathcal{H}_F(X(.,\epsilon))-H_F(X(.,\epsilon),N(.,\epsilon))]\Bigg\}\,dA\Bigg)_{|_{\epsilon=0}},\end{aligned}\tag{4.5}$$

where $N(.,\epsilon)$ is the unit normal and $H_F(X(.,\epsilon),N(.,\epsilon))$ the F-mean curvature of the perturbed immersion $X(.,\epsilon)\in C^3(M,\mathbb{R}^{n+1})$.

According to [**2**, Lemma 1.1] one has

$$\frac{\partial}{\partial\epsilon}N(.,\epsilon)_{|_{\epsilon=0}} = -DX(\operatorname{grad}\varphi)+DN(V),\tag{4.6}$$

where $\varphi\in C^2_0(M)$ and $V\in\chi(M)$ with compact support determine the normal and tangential component of Ξ defined in (4.2). From (2.14), on the other hand, we infer

$$\begin{aligned}&\frac{\partial}{\partial\epsilon}H_F(X(.,\epsilon),N(.,\epsilon))_{|_{\epsilon=0}}\\ &\underset{(2.14)}{=}\left[\left(\frac{\partial}{\partial\epsilon}g^{ij}(\epsilon)\right)h_{Fij}+g^{ij}\left(\frac{\partial}{\partial\epsilon}h_{Fij}(\epsilon)\right)\right]_{|_{\epsilon=0}}\\ &=: \mathrm{I}+\mathrm{II},\end{aligned}\tag{4.7}$$

where the argument ϵ indicates that the corresponding quantity belongs to the perturbed immersion $X(.,\epsilon)$. In particular, we write, e.g., $g^{ij}(0)=g^{ij}$, $\partial_lX(.,0)=\partial_lX$, etc. On account of $g^{ij}(\epsilon)g_{js}(\epsilon)=\delta^i_s$ for all $\epsilon\in(-\epsilon_0,\epsilon_0)$ one has

$$\frac{\partial}{\partial\epsilon}g^{ij}(\epsilon) = -g^{ik}(\epsilon)\left(\frac{\partial}{\partial\epsilon}g_{kl}(\epsilon)\right)g^{lj}(\epsilon),$$

and therefore by (4.2) and (2.6) for $U := \partial_k, \partial_l$, respectively,

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon} g^{ij}(\epsilon)|_{\epsilon=0} \\
&= -g^{ik}\left\{\left\langle \frac{\partial}{\partial\epsilon}(\partial_k X(.,\epsilon)), \partial_l X\right\rangle + \left\langle \partial_k X, \frac{\partial}{\partial\epsilon}(\partial_l X(.,\epsilon))\right\rangle\right\}|_{\epsilon=0} g^{lj} \\
&\underset{(4.2)}{=} -g^{ik}\left\{\langle \varphi\partial_k N + \partial_k(DX(V)), \partial_l X\rangle + \langle \partial_k X, \varphi\partial_l N + \partial_l(DX(V))\rangle\right\} g^{lj} \\
&\underset{(2.6)}{=} 2\varphi g^{ik} h_{kl} g^{lj} - g^{ik}\langle DX(\nabla_k V), \partial_l X\rangle g^{lj} - g^{ik}\langle DX(\nabla_l V), \partial_k X\rangle g^{lj}.
\end{aligned}$$

Thus we obtain for the expression I in (4.7) by the symmetry of the mappings A_F and S

$$\begin{aligned}
\mathrm{I} &= \left(\frac{\partial}{\partial\epsilon} g^{ij}(\epsilon)\right)|_{\epsilon=0} h_{Fij} \\
&= 2\varphi g^{ik} h_{kl} g^{lj} h_{Fij} - g^{ik}\langle DX(\nabla_k V), \partial_l X\rangle g^{lj} h_{Fij} \\
&\quad - g^{ik}\langle DX(\nabla_l V), \partial_k X\rangle g^{lj} h_{Fij} \\
&\underset{(2.13)}{=} 2\varphi g^{ik} g(\partial_k, S(\partial_l)) g^{lj} g(\partial_i, A_F S(\partial_j)) \\
&\quad + g^{ik} g(\nabla_k V, \partial_l) g^{lj} g(\partial_i, A_F S(\partial_j)) \\
&\quad + g^{ik} g(\nabla_l V, \partial_k) g^{lj} g(\partial_i, A_F S(\partial_j)) \\
&= 2\varphi g^{ik} g(S(\partial_k), g^{lj} g(S A_F(\partial_i), \partial_j)\partial_l) \\
&\quad + g^{ik} g(S A_F(\partial_i), g^{lj} g(\nabla_k V, \partial_l)\partial_j) \\
&\quad + g^{lj} g(g^{ik} g(\nabla_l V, \partial_k)\partial_i, A_F S(\partial_j)) \\
&\underset{(2.1)}{=} 2\varphi g^{ik} g(A_F S^2(\partial_k), \partial_i) + g^{ik} g(S A_F(\partial_i), \nabla_k V) \\
&\quad + g^{lj} g(\nabla_l V, A_F S(\partial_j)) \\
&\underset{\substack{(2.7)\\(2.8)}}{=} 2\varphi \mathrm{tr}\,(A_F S^2) + \mathrm{tr}\,((S A_F + A_F S)\nabla_\bullet V).
\end{aligned} \tag{4.8}$$

Furthermore we need to compute

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon} h_{Fij}(\epsilon)|_{\epsilon=0} \\
&\underset{(2.13)}{=} -\left\langle \frac{\partial}{\partial\epsilon}(F_{zz}(N(.,\epsilon))|_{\epsilon=0}\partial_i X, \partial_j N\right\rangle \\
&\quad - \left\langle F_{zz}(N)\partial_i\left[\frac{\partial}{\partial\epsilon}X(.,\epsilon)|_{\epsilon=0}\right], \partial_j N\right\rangle \\
&\quad - \left\langle F_{zz}(N)\partial_i X, \partial_j\left[\frac{\partial}{\partial\epsilon}N(.,\epsilon)|_{\epsilon=0}\right]\right\rangle
\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
\underset{(4.2),(4.6)}{=} & -\left\langle \frac{\partial}{\partial\epsilon}(F_{zz}(N(.,\epsilon))|_{\epsilon=0}\partial_i X, \partial_j N\right\rangle \\
& - \langle F_{zz}(N)\partial_i\left[\varphi N + DX(V)\right], \partial_j N\rangle \\
& - \langle F_{zz}(N)\partial_i X, \partial_j\left[-DX(\operatorname{grad}\varphi) + DN(V)\right]\rangle .
\end{aligned}
$$

Since $\langle \partial_j N, N\rangle = 0$ we have by (2.6)

$$
\begin{aligned}
\text{(4.10)} \quad \langle F_{zz}(N)\partial_i[\varphi N + DX(V)], \partial_j N\rangle = {} & \varphi\langle F_{zz}(N)\partial_i N, \partial_j N\rangle \\
& + \langle F_{zz}(N)DX(\nabla_i V), \partial_j N\rangle,
\end{aligned}
$$

and also by (2.6) and on account of $DN = DX \circ S$

(4.11)

$$
\begin{aligned}
& \langle F_{zz}(N)\partial_i X, \partial_j[-DX(\operatorname{grad}\varphi) + DN(V)]\rangle \\
& \quad = -\langle F_{zz}(N)\partial_i X, \partial_j DX(\operatorname{grad}\varphi)\rangle + \langle F_{zz}(N)\partial_i X, \partial_j DX(S(V))\rangle \\
& \quad \underset{(2.6)}{=} -\langle F_{zz}(N)\partial_i X, \partial_j DX(\operatorname{grad}\varphi)\rangle + \langle F_{zz}(N)\partial_i X, DX(\nabla_j(S(V))\rangle.
\end{aligned}
$$

Inserting (4.9)-(4.11) into the expression for II in (4.7) leads to

$$
\begin{aligned}
\mathrm{II} \;=\; & -g^{ij}\left\langle \frac{\partial}{\partial\epsilon}(F_{zz}(N(.,\epsilon)))|_{\epsilon=0}\partial_i X, \partial_j N\right\rangle - \varphi g^{ij}\langle F_{zz}(N)\partial_i N, \partial_j N\rangle \\
& - g^{ij}\langle F_{zz}(N)DX(\nabla_i V), \partial_j N\rangle + g^{ij}\langle F_{zz}(N)\partial_i X, \partial_j DX(\operatorname{grad}\varphi)\rangle \\
& - g^{ij}\langle F_{zz}(N)\partial_i X, DX(\nabla_j(S(V)))\rangle \\
\underset{(1.10)}{=} & -g^{ij}\left\langle \frac{\partial}{\partial\epsilon}(F_{zz}(N(.,\epsilon)))|_{\epsilon=0}\partial_i X, \partial_j N\right\rangle - \varphi g^{ij} g(A_F S(\partial_i), S(\partial_j)) \\
& - g^{ij} g(A_F(\nabla_i V), S(\partial_j)) + g^{ij}\langle F_{zz}(N)\partial_i X, \partial_j DX(\operatorname{grad}\varphi)\rangle \\
& - g^{ij} g(A_F(\partial_i), \nabla_j(S(V))).
\end{aligned}
$$

By the symmetry of A_F and S and by (2.7) (and (2.4) for the last term) we may rewrite this as

$$
\begin{aligned}
\text{(4.12)} \qquad \mathrm{II} = {} & -g^{ij}\left\langle \frac{\partial}{\partial\epsilon}(F_{zz}(N(.,\epsilon)))|_{\epsilon=0}\partial_i X, \partial_j N\right\rangle - \varphi\operatorname{tr}(A_F S^2) \\
& - \operatorname{tr}(S A_F \nabla_\bullet V) + g^{ij}\langle F_{zz}(N)\partial_i X, \partial_j DX(\operatorname{grad}\varphi)\rangle \\
& - \operatorname{tr}(A_F S \nabla_\bullet V) - \operatorname{tr}(A_F \circ [(\nabla_\bullet S)V]).
\end{aligned}
$$

Adding (4.8) and (4.12) in (4.7) and using the symmetry of F_{zz} we arrive at

$$
\begin{aligned}
\text{(4.13)} \quad & \frac{\partial}{\partial\epsilon} H_F(X(.,\epsilon), N(.,\epsilon))|_{\epsilon=0} \\
& \underset{(4.8),(4.12)}{=} \varphi\operatorname{tr}(A_F S^2) + g^{ij}\langle \partial_i X, F_{zz}(N)\partial_j DX(\operatorname{grad}\varphi)\rangle \\
& \qquad - g^{ij}\left\langle \frac{\partial}{\partial\epsilon}(F_{zz}(N(.,\epsilon)))|_{\epsilon=0}\partial_i X, \partial_j N\right\rangle - \operatorname{tr}(A_F \circ [(\nabla_\bullet S)V]).
\end{aligned}
$$

By virtue of (1.10), (2.6), (2.9) and (1.13) we may rewrite the second term on the right-hand side as

$$
\begin{aligned}
(4.14)\quad & g^{ij}\langle \partial_i X, F_{zz}(N)\partial_j DX(\operatorname{grad}\varphi)\rangle \\
&= g^{ij}\langle \partial_i X, \partial_j\{F_{zz}(N)DX(\operatorname{grad}\varphi)\}\rangle \\
&\quad - g^{ij}\langle \partial_i X, \partial_j(F_{zz}(N))DX(\operatorname{grad}\varphi)\rangle \\
&\underset{\substack{(1.10)\\(2.6)}}{=} g^{ij}g(\partial_i, \nabla_j(A_F(\operatorname{grad}\varphi))) - g^{ij}\langle \partial_i X, \partial_j(F_{zz}(N))DX(\operatorname{grad}\varphi)\rangle \\
&\underset{\substack{(2.9)\\(1.13)}}{=} \Delta_F\varphi - g^{ij}\langle \partial_i X, \partial_j(F_{zz}(N))DX(\operatorname{grad}\varphi)\rangle.
\end{aligned}
$$

Moreover, by the symmetry of F_{zz} we have

(4.15)

$$
g^{ij}\langle \partial_i X, \partial_j(F_{zz}(N))DX(\operatorname{grad}\varphi)\rangle = g^{ij}\langle \partial_j(F_{zz}(N))\partial_i X, DX(\operatorname{grad}\varphi)\rangle,
$$

and by (2.6), (1.10), (2.4) and (2.11) for general $W \in T_wM$

$$
\begin{aligned}
(4.16)\quad & g^{ij}\langle \partial_i X, \partial_j(F_{zz}(N))DX(W)\rangle \\
&\underset{(2.6)}{=} g^{ij}\langle \partial_j(F_{zz}(N)DX(\partial_i)) - F_{zz}(N)DX(\nabla_j\partial_i), DX(W)\rangle \\
&\underset{(1.10)}{=} g^{ij}\langle \partial_j(DX(A_F(\partial_i))) - DX(A_F(\nabla_j\partial_i)), DX(W)\rangle \\
&\underset{\substack{(2.4)\\(2.6)}}{=} g^{ij}\langle DX((\nabla_j A_F)\partial_i), DX(W)\rangle \\
&= g^{ij}g((\nabla_j A_F)\partial_i, W) \underset{(2.12)}{=} g(\operatorname{div} A_F, W).
\end{aligned}
$$

Summarizing (4.13), (4.14) and (4.16) for $W := \operatorname{grad}\varphi$ we arrive at

$$
\begin{aligned}
(4.17)\quad & \frac{\partial}{\partial\epsilon} H_F(X(.,\epsilon), N(.,\epsilon))\big|_{\epsilon=0} \\
&= \Delta_F\varphi + \varphi\operatorname{tr}(A_F S^2) - g(\operatorname{div} A_F, \operatorname{grad}\varphi) \\
&\quad - g^{ij}\left\langle \frac{\partial}{\partial\epsilon}(F_{zz}(N(.,\epsilon)))\big|_{\epsilon=0}\partial_i X, \partial_j N\right\rangle - \operatorname{tr}(A_F \circ [(\nabla_\bullet S)V]).
\end{aligned}
$$

Now writing out components one calculates

$$
\begin{aligned}
& \frac{\partial}{\partial\epsilon}(F_{z^l z^k}(N(.,\epsilon)))\big|_{\epsilon=0}\partial_i X^k \partial_j N^l \\
&= F_{z^l z^k z^s}(N)\left(\frac{\partial}{\partial\epsilon}N^s(.,\epsilon)\big|_{\epsilon=0}\right)\partial_i X^k \partial_j N^l \\
&= \partial_j(F_{z^k z^s}(N))\left(\frac{\partial}{\partial\epsilon}N^s(.,\epsilon)\big|_{\epsilon=0}\right)\partial_i X^k,
\end{aligned}
$$

whence by (4.6), (4.15) and (4.16) for $W := S(V) - \operatorname{grad}\varphi$,

$$
\begin{aligned}
(4.18)\qquad & g^{ij}\left\langle \frac{\partial}{\partial\epsilon}(F_{zz}(N(.,\epsilon)))\big|_{\epsilon=0}\partial_i X, \partial_j N\right\rangle \\
&= g^{ij}\langle \partial_j(F_{zz}(N))\partial_i X, DX\circ S(V) - DX(\operatorname{grad}\varphi)\rangle \\
&= g(\operatorname{div} A_F, S(V) - \operatorname{grad}\varphi).
\end{aligned}
$$

Next we claim that for any $V \in T_w M$

$$
(4.19)\qquad g(\operatorname{div}(SA_F), V) = \operatorname{tr}(A_F \circ [(\nabla_\bullet S)V]) + g(\operatorname{div} A_F, S(V)).
$$

This together with (4.18) and (4.17) leads to

$$
(4.20)\quad \frac{\partial}{\partial\epsilon} H_F(X(.,\epsilon), N(.,\epsilon))\big|_{\epsilon=0} = \Delta_F\varphi + \varphi\operatorname{tr}(A_F S^2) - g(\operatorname{div}(SA_F), V).
$$

By (4.5) we then conclude using (4.2)

$$
\begin{aligned}
(4.21)\quad \delta^2\mathcal{F}^0(X,\Xi) = -\int_M \varphi\{&\Delta_F\varphi + \varphi\operatorname{tr}(A_F S^2) - \varphi\langle\nabla_{\mathbb{R}^{n+1}}\mathcal{H}_F(X), N\rangle \\
&- g(\operatorname{div}(SA_F) + \operatorname{grad}\mathcal{H}_F(X), V)\}\, dA,
\end{aligned}
$$

which proves Theorem 4.1. Notice that the other terms obtained by carrying out the differentiation with respect to ϵ in (4.5) and evaluating at $\epsilon = 0$ vanish, since

$$
H_F(X(.,0), N(.,0)) = H_F(X,N) \equiv \mathcal{H}_F(X)
$$

because X is a critical immersion for (1.12).

It remains to show (4.19). By (2.11) and the symmetry of A_F and S

$$
\begin{aligned}
g(\operatorname{div}(SA_F), V) &\underset{(2.11)}{=} g^{ik} g(\nabla_i(SA_F)^* V, \partial_k) \\
&= g^{ik} g((\nabla_i A_F^*)S^* V, \partial_k) + g^{ik} g(A_F^*(\nabla_i S^*)V, \partial_k) \\
&\underset{(2.11)}{=} g(\operatorname{div} A_F, S^* V) + g^{ik} g((\nabla_i S)^* V, A_F(\partial_k)) \\
&= g(S\operatorname{div} A_F, V) + g^{ik} g(V, (\nabla_i S)A_F(\partial_k)).
\end{aligned}
$$

The Codazzi Equation (2.17) and the symmetry of S and A_F imply now

$$
\begin{aligned}
g^{ik} g(V, (\nabla_i S)A_F(\partial_k)) &= g^{ik} g(A_F \circ (\nabla_i S)V, \partial_k) \\
&\underset{(2.8)}{=} \operatorname{tr}(A_F \circ [(\nabla_\bullet S)V]),
\end{aligned}
$$

which proves the claim. □

As a consequence of Theorem 4.1 we can state:

Corollary 4.2. *Let $X \in C^3(M, \mathbb{R}^{n+1})$ be a immersion of prescribed F-mean curvature $\mathcal{H}_F \in C^1(\mathbb{R}^{n+1})$, where*

$$F = F(z) \in C^0(\mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1}\setminus\{0\})$$

is a parametric Lagrangian. Then

$$\operatorname{div}(SA_F) = -\operatorname{grad}\mathcal{H}_F(X), \qquad \textit{and} \tag{4.22}$$

$$\begin{aligned}\delta^2\mathcal{F}^0(X,\Xi) = \int_M \Big\{ & g(A_F \operatorname{grad}\varphi, \operatorname{grad}\varphi) \\ & - \Big[\operatorname{tr}(A_F S^2) - \langle \nabla_{\mathbb{R}^{n+1}}\mathcal{H}_F(X), N\rangle\Big]\varphi^2\Big\}\, dA,\end{aligned} \tag{4.23}$$

where $\Xi = \varphi N + DX(V)$, $\varphi \in C_0^2(M)$, and $V \in \chi(M)$ with compact support. In particular, the second variation of a parametric integrand depends on normal variations only.

Proof. The symmetry argument we use here is due to White [**23**]. Consider the surfaces

$$X(\,\cdot\,, \epsilon, \eta) = X + \epsilon(\varphi N + DX(V)) + \eta(\psi N + DX(W)),$$

where $\varphi, \psi \in C_0^\infty(M)$ and $V, W \in \chi(M)$ with compact support. Similarly as in (4.5) we have

$$\begin{aligned}&\frac{d}{d\epsilon}\Big|_{\epsilon=0}\left[\frac{d}{d\eta}\Big|_{\eta=0}\mathcal{F}^0(X(.,\epsilon,\eta))\right] \\ &= \frac{d}{d\epsilon}\Bigg(\int_M\Bigg\{\Bigg\langle \frac{\partial}{\partial\eta}X(.,\epsilon,\eta)\Big|_{\eta=0}, N(.,\epsilon,0)\Bigg\rangle \\ &\qquad [\mathcal{H}_F(X(.,\epsilon,0)) - H_F(X(.,\epsilon,0), N(.,\epsilon,0))]\Bigg\}\, dA\Bigg)\Big|_{\epsilon=0}.\end{aligned} \tag{4.24}$$

Hence by (4.20) we obtain

$$\begin{aligned}&\frac{d}{d\epsilon}\Big|_{\epsilon=0}\left[\frac{d}{d\eta}\Big|_{\eta=0}\mathcal{F}^0(X(.,\epsilon,\eta))\right] \\ &= \int_M \psi\{-\Delta_F\varphi - \varphi\operatorname{tr}(A_F S^2) + g(\operatorname{div}(SA_F), V)\} \\ &\quad + \psi\langle\nabla_{\mathbb{R}^{n+1}}\mathcal{H}_F(X), \varphi N + DX(V)\rangle\, dA.\end{aligned}$$

Since

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\left[\frac{d}{d\eta}\Big|_{\eta=0}\mathcal{F}^0(X(.,\epsilon,\eta))\right] = \frac{d}{d\eta}\Big|_{\eta=0}\left[\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{F}^0(X(.,\epsilon,\eta))\right],$$

we arrive at

$$\begin{aligned}(4.25)\qquad &\int_M \psi g(\mathrm{div}\,(SA_F), V) + \psi g(V, \mathrm{grad}\,\mathcal{H}_F(X))\, dA\\ &= \int_M \varphi g(\mathrm{div}\,(SA_F), W) + \varphi g(W, \mathrm{grad}\,\mathcal{H}_F(X))\, dA\end{aligned}$$

for all $\varphi, \psi \in C_0^\infty(M)$ and $V, W \in \chi(M)$ with compact support, where we used that

$$\int_M \psi(-\Delta_F\varphi - \varphi\mathrm{tr}\,(A_F S^2))\, dA = \int_M \varphi(-\Delta_F\psi - \psi\mathrm{tr}\,(A_F S^2))\, dA.$$

Equation (4.25) is only possible if $\mathrm{div}\,(SA_F) = -\mathrm{grad}\,\mathcal{H}_F(X)$, for if not, we could choose $W \equiv 0$ to have a vanishing right-hand side in (4.25), and ψ and V appropriately to obtain a positive left-hand side and thus a contradiction. □

Inserting (4.22) into formula (1.15) of Theorem 1.1 for the normal of an $\mathcal{F}^0$-critical immersion we obtain:

Corollary 4.3. *Let N be the normal of an immersion $X \in C^3(M, \mathbb{R}^{n+1})$ of prescribed F-mean curvature $\mathcal{H}_F(y) \in C^1(\mathbb{R}^{n+1})$, where $F = F(z)$ is a parametric Lagrangian of class $C^0(\mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1}\backslash\{0\})$. Then*

$$(4.26)\qquad \Delta_F N + \Big[\mathrm{tr}\,(A_F S^2) - \langle\nabla_{\mathbb{R}^{n+1}}\mathcal{H}_F(X), N\rangle\Big] N = -\nabla_{\mathbb{R}^{n+1}}\mathcal{H}_F(X).$$

The above corollary generalizes [**21**, Satz 1].

The notion of stability is defined as follows:

Definition 4.4. Let $X \in C^3(M, \mathbb{R}^{n+1})$ be an $\mathcal{F}^0$-critical immersion, where $\mathcal{F}^0$ is defined in (1.12) with a parametric Lagrangian $F = F(z) \in C^0(\mathbb{R}^{n+1})\cap C^3(\mathbb{R}^{n+1}\backslash\{0\})$. Then X is called *stable* if $\delta^2\mathcal{F}^0(X, \Xi) \geq 0$ for all $\Xi \in C_0^2(M, \mathbb{R}^{n+1})$. If $\delta^2\mathcal{F}^0(X, \Xi) > 0$ we say X is *strictly stable.*

5. Graph representation of prescribed F-mean curvature surfaces.

The Proof of Theorem 1.4 is based on a maximum principle for elliptic equations of the form $Lu = (\alpha^{ij}u_{x^j})_{x^i} + \beta^i u_{x^i} + cu$. Usually it is required that the coefficient c be nonpositive. As was carried out in [**21**] for the Laplace operator this condition may be replaced by assuming that the first eigenvalue of L is nonnegative. Our proof of the corresponding lemma for general elliptic equations is related to [**12**, Lemma 1], but we assume less regularity of the coefficients:

Lemma 5.1. *Let $Lu = (\alpha^{ij}u_{x^j})_{x^i} + \beta^i u_{x^i} + cu \leq 0$ be a linear elliptic equation in a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where α^{ij}, β^i, $c \in C^{0,\mu}(\overline{\Omega})$*

and with $\alpha^{ij} = \alpha^{ji}$ for $i, j = 1, \dots, n$. Assume that the first eigenvalue of L is nonnegative on Ω. If for $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ we have $Lu \le 0$ and $u_{|\partial\Omega} > 0$, then $\inf_\Omega u > 0$.

Proof. On account of the continuity of u we can assume that there is a smoothly bounded domain $\Omega_2 \subset\subset \Omega$ with $u_{|\partial\Omega_2} > 0$. The first eigenvalue λ of L in a domain Ω_1 with $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega$ is simple and therefore strictly positive, since otherwise we could apply [**11**, Thm. 8.38] to find a positive eigenfunction $\xi \in \mathring{H}^{1,2}(\Omega_1)$ on Ω_1 with $\xi = 0$ on $\partial\Omega_1$. Extending ξ by zero outside of Ω_1 we would obtain $\widetilde{\xi} \in \mathring{H}^{1,2}(\Omega)$ with $L\widetilde{\xi} + \lambda\widetilde{\xi} = 0$, $\widetilde{\xi} \not\equiv 0$ on Ω with $\widetilde{\xi} \equiv 0$ on $\Omega\backslash\Omega_1$, contradicting [**11**, Thm. 8.38].

The regularity of the coefficients α^{ij}, β^i, c, leads to $\xi \in C^{1,\mu}(\Omega_1)$; see e.g., [**10**, Theorem 3.2].

Thus in Ω_2 we can write $u = \xi\, v$, and due to the regularity of u and ξ we obtain a.e. on Ω_2:

$$\begin{aligned} L(\xi v) &= v(\alpha^{ij}\xi_{x^j})_{x^i} + \xi(\alpha^{ij}v_{x^j})_{x^i} + 2\alpha^{ij}v_{x^i}\xi_{x^j} + v\beta^i\xi_{x^i} + \xi\beta^i v_{x^i} + cv\xi \\ &= vL\xi + \xi[(\alpha^{ij}v_{x^j})_{x^i} + \beta^i v_{x^i}] + 2\alpha^{ij}v_{x^i}\xi_{x^j} \\ &= \xi[\ (\alpha^{ij}v_{x^j})_{x^i} + \beta^i v_{x^i} + (2/\xi)\ \alpha^{ij}\xi_{x^j}\ v_{x^i} - \lambda v]. \end{aligned}$$

Thus we obtain an elliptic differential inequality for v:

$$0 \le \int_{\Omega_2} [\alpha^{ij}v_{x^j}\varphi_{x^i} - (\widetilde{\beta}^i v_{x^i} - \lambda v)\varphi]\, dx$$

for all nonnegative $\varphi \in C_0^1(\Omega_2)$, where $\widetilde{\beta}^i := \beta^i + (2/\xi)a^{ij}\xi_{x^j} \in L^\infty(\Omega_2)$ for $i = 1, \dots, n$. Thus the weak maximum principle [**11**, Thm 8.1] holds for v and we have $\inf_{\Omega_2} v \ge 0$. By the strong minimum principle [**11**, Thm. 8.19] we obtain the strict relation $\inf_{\Omega_2} v > 0$. $\square$

Proof of Theorem 1.4. According to our assumptions on Γ and Ω in Theorem 1.3 there is a function $f \in C^2(\partial\Omega)$, such that $\Gamma = \{(x, f(x)) : x \in \partial\Omega\}$ is (positively) oriented by setting

$$P_k := (x_k, f(x_k)), \quad k = 1, 2, 3, \tag{5.1}$$

where $x_k \in \partial\Omega$, $k = 1, 2, 3$, are chosen in positive orientation with respect to $\mathbb{R}^2$.

Since X is immersed on $\overline{B}$ we may assume without loss of generality that X is conformally parametrized, i.e., satisfies the conformality relations (3.3). (Otherwise we can perform a diffeomorphism $w : \overline{B} \to \overline{B}$ of class $C^{2,\alpha}(B, \mathbb{R}^2) \cap C^{1,\alpha}(\overline{B}, \mathbb{R}^2)$ such that $\widetilde{X} := X \circ w^{-1}$ is conformally parametrized; see e.g., [**17**, Corollary 3.1.2].) Performing a suitable Möbius transformation on $\overline{B}$ we may assume that X satisfies the three-point condition

$$X(w_k) = P_k \quad k = 1, 2, 3, \tag{5.2}$$

where w_1, w_2, w_3 are fixed distinct points on ∂B.

Fix some point $w_0 = e^{i\vartheta_0} \in \partial B$. Since Ω is h-convex there is a point $\eta_0 \in \mathbb{R}^2$ such that $\Omega \subset B_{1/h}(\eta_0)$ and such that $y_0 := (X^1(w_0), X^2(w_0)) \in \partial\Omega$ is contained in $\partial B_{1/h}(\eta_0)$. Without loss of generality we may assume that $\eta_0 = 0$. By Hopf's boundary point lemma we then obtain for the function $R(x) := (x^1)^2 + (x^2)^2$

$$\frac{\partial}{\partial\nu} R(X(w))\big|_{w=w_0} > 0,$$

i.e., in polar coordinates (r, ϑ)

$$2\sum_{i=1}^{2} X^i(w)\frac{\partial}{\partial r}X^i(w)\big|_{w=w_0} > 0, \tag{5.3}$$

which implies $|X_r(w_0)| > 0$, and by conformality also

$$|X_\vartheta(w_0)| > 0. \tag{5.4}$$

Since $R(X(w_0)) = h^{-2} \geq R(X(w))$ for all $w = e^{i\vartheta} \in \partial B$ we obtain

$$0 = \frac{\partial}{\partial\vartheta}\big|_{\vartheta=\vartheta_0}(R(X(e^{i\vartheta}))) = \sum_{i=1}^{2} X^i(e^{i\vartheta})\frac{\partial}{\partial\vartheta}X^i(e^{i\vartheta})\big|_{\vartheta=\vartheta_0}. \tag{5.5}$$

Since $\Gamma = \{(x, f(x)) : x \in \partial\Omega\}$, $f \in C^2(\partial\Omega)$, we have

$$|X^3_\vartheta| = |f_{x^1}X^1_\vartheta + f_{x^2}X^2_\vartheta| \leq \|\nabla\widetilde{f}\|_{C^0(\mathbb{R}^2)}\sqrt{(X^1_\vartheta)^2 + (X^2_\vartheta)^2},$$

where $\widetilde{f} \in C^2(\mathbb{R}^2)$ is an extension of f onto $\mathbb{R}^2$ with controlled C^2-norm; see [**11**, p. 137]. Hence, by (5.4),

$$0 < |X_\vartheta(w)|^2\big|_{w=w_0} \leq (1 + \|\nabla\widetilde{f}\|^2_{C^0(\mathbb{R}^2)})\Big[(X^1_\vartheta(w))^2 + (X^2_\vartheta(w))^2\Big]\big|_{w=w_0}. \tag{5.6}$$

By (5.2) the mapping $(X^1, X^2) : \partial B \to \partial\Omega$ respects the positive orientation, thus we infer from (5.5) and (5.6) that there is a constant $\sigma > 0$ such that

$$X^1_\vartheta(w_0) = -\sigma X^2(w_0), \quad X^2_\vartheta(w_0) = \sigma X^1(w_0).$$

Therefore, by (5.3)

$$\begin{aligned} &X^1_r(w_0)X^2_\vartheta(w_0) - X^2_r(w_0)X^1_\vartheta(w_0) \\ &= \sigma(X^1(w_0)X^1_r(w_0) + X^2(w_0)X^2_r(w_0)) \underset{(5.3)}{>} 0, \end{aligned}$$

which means that $X^1_u(w_0)X^2_v(w_0) - X^1_v(w_0)X^2_u(w_0) > 0$, i.e., the third component of N^3 of the normal N is positive on ∂B. Moreover, by Corollary 4.3, N^3 satisfies the elliptic differential equation

$$\Delta_F N^3 + (\mathrm{tr}\,(A_F S^2) - \langle \nabla_{\mathbb{R}^3}\mathcal{H}_F(X), N\rangle)N^3 = -\frac{\partial \mathcal{H}_F}{\partial y^3}(X).$$

Using the assumption on $\mathcal{H}_F(X)$ this relation is given in coordinates by

$$\mathcal{L}N^3 := \partial_i(\sqrt{g}g^{ij}a_{jk}g^{kl}\partial_l N^3) + \sqrt{g}(\mathrm{tr}\,(A_F S^2) - \langle \nabla_{\mathbb{R}^3}\mathcal{H}_F(X), N\rangle)N^3 \le 0\,,$$

which we regard as a linear elliptic equation for N^3 with the differential operator $\mathcal{L}$ associated to the second variation formula (4.23). Here, $g = \det(g_{ij})$, $a_{jk} = \langle F_{zz}(N)\partial_j X, \partial_k X\rangle$ and the remaining coefficients are of class $C^{0,\alpha}(\overline{B})$, and the leading coefficients of $\mathcal{L}$ are symmetric. Since X is stable we have $\delta^2\mathcal{F}^0(X,\Xi) \ge 0$; hence the first eigenvalue of $\mathcal{L}$ is nonnegative. Thus Lemma 5.1 is applicable and we have $N^3 > 0$ on B.

Since $X : \partial B \to \Gamma$ is a topological mapping, we can apply Sauvigny's reasoning involving degree theory as in [**21**, pp. 53,54] to conclude the proof. □

Remark. We have seen in (5.3) and (5.4) that there are no branch points on the boundary by the simple Hopf maximum principle argument, which is applicable because of our regularity assumptions up to the boundary. Consequently, it would suffice to assume that X is conformal and has no *interior* branch points and maps the boundary ∂B only weakly monotonically onto Γ, but at this point it is an open question if one can relax the smoothness assumptions to $X \in C^0(\overline{B},\mathbb{R}^3) \cap C^3(B,\mathbb{R}^3)$.

References

[1] J. Barbosa and M. do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z., **185** (1984), 339-353, MR 85k:58021c, Zbl 0513.53002.

[2] U. Clarenz, *Enclosure theorems for extremals of elliptic parametric functionals*, Calc. Var. Partial Differential Equations, **15**(**3**) (2002), 313-324, CMP 1 938 817.

[3] U. Clarenz, G. Dziuk and M. Rumpf, *On generalized mean curvature in surface processing*, in 'Geometric analysis and nonlinear partial differential equations' (eds. S. Hildebrandt, H. Karcher), Springer, 2003.

[4] U. Clarenz and H. von der Mosel, *Compactness theorems and an isoperimetric inequality for critical points of elliptic parametric functionals*, Calc. Var. Partial Differential Equations, **12** (2001) 85-107, MR 2001k:35074, Zbl 0968.35039.

[5] ———, *Isoperimetric inequalities for parametric variational problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **19** (2002), 617-629, MR 2003e:49005, Zbl 1014.53009.

[6] K. Deckelnick and G. Dziuk, *Discrete anisotropic curvature flow of graphs*, Math. Modelling Numer. Anal. (RAIRO), **33** (1999), 1203-1222, MR 2000k:65169, Zbl 0948.65138.

[7] ———, *A finite element level set method for anisotropic mean curvature flow with space dependent weight*, in 'Geometric analysis and nonlinear partial differential equations' (eds. S. Hildebrandt, H. Karcher), Springer, 2003.

[8] G. Dziuk, *Discrete anisotropic curve shortening flow*, SIAM J. Numer. Anal., **36** (1999), 1808-1830, MR 2000j:65024, Zbl 0942.65112.

[9] S. Gallot, D. Hulin and J. Lafontaine, *Riemannian Geometry*, Springer, Berlin-Heidelberg-New York, 1987, MR 88k:53001, Zbl 0636.53001.

[10] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Annals of Math. Studies, **105**, Princeton Univ. Press, 1983, MR 86b:49003, Zbl 0516.49003.

[11] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Grundlehren Math. Wiss., **224**, Springer, Berlin-Heidelberg-New York, 1977, Second Edition 1983, MR 57 #13109, MR 86c:35035, Zbl 0361.35003, Zbl 0562.35001.

[12] K. Grosse-Brauckmann, *Stable constant mean curvature surfaces minimize area*, Pacific J. Math., **175**(**2**) (1996), 527-534, MR 98e:53007, Zbl 0878.49026.

[13] R. Gulliver and J. Spruck, *Surfaces of constant mean curvature which have a simple projection*, Math. Z., **129** (1972), 95-107, MR 47 #965c, Zbl 0233.53003.

[14] S. Hildebrandt and H. von der Mosel, *On two-dimensional parametric variational problems*, Calc. Var. Partial Differential Equations, **9** (1999), 249-267, MR 2000h:49006, Zbl 0934.49022.

[15] ———, *Plateau's problem for parametric double integrals: Part* I. *Existence and regularity in the interior*, Comm. Pure Appl. Math., **56** (2003), 926-955.

[16] ———, *Plateau's problem for parametric double integrals: Part* II. *Boundary regularity*, to appear in J. Reine Angew. Math.

[17] J. Jost, *Two Dimensional Geometric Variational Problems*, Wiley-Interscience Chichester, New York, 1991, MR 92h:58045, Zbl 0729.49001.

[18] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. II, Interscience, New York-London-Sydney, 1969, MR 38 #6501, Zbl 0175.48504.

[19] T. Radó, *Some remarks on the problem of Plateau*, Proc. Natl. Acad. Sci. USA, **16** (1930), 242-248, Zbl 56.0437.01.

[20] K. Räwer, *Stabile Extremalen Parametrischer Doppelintegrale in* $\mathbb{R}^3$, Dissertation, Bonn, 1993.

[21] F. Sauvigny, *Flächen vorgeschriebener mittlerer Krümmung mit eineindeutiger Projektion auf eine Ebene*, Math. Z., **180** (1982), 41-67, MR 83j:53004, Zbl 0465.53003.

[22] L. Simon, *Lectures on Geometric Measure Theory*, Proc. Centre Math. Analysis, Australian National University, Canberra, Australia, **3**, 1983 (publ. 1984), MR 87a:49001, Zbl 0546.49019.

[23] B. White, *The space of m-dimensional surfaces that are stationary for a parametric elliptic functional*, Indiana Univ. Math. J., **36** (1987), 567-602, MR 88k:58027, Zbl 0770.58005.

[24] ———, *Existence and regularity of smooth embedded surfaces of prescribed genus that minimize parametric even elliptic functionals on* 3*-manifolds*, J. Differential Geom., **33** (1991), 413-443, MR 92e:58048, Zbl 0737.53009.

Received September 30, 2002. The second author was supported by the Max-Planck Institute for Mathematics in the Sciences Leipzig and the Sonderforschungsbereich 611 at the University of Bonn.

Fachbereich Mathematik
Universität Duisburg-Essen
Lotharstrasse 65
47047 Duisburg
Germany
E-mail address: clarenz@math.uni-duisburg.de

Mathematisches Institut
Universität Bonn
Beringstrasse 1
53115 Bonn
Germany
E-mail address: heiko@math.uni-bonn.de