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POSITIVE SOLUTIONS TO $\Delta u - Vu + Wu^p = 0$ AND ITS PARABOLIC COUNTERPART IN NONCOMPACT MANIFOLDS

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We consider the equation $\Delta u - V(x)u + W(x)u^p = 0$ and its parabolic counterpart in noncompact manifolds. Under some natural conditions on the positive functions V and W, which may only have 'slow' or no decay near infinity, we establish existence of positive solutions in both the critical and the subcritical case. This leads to the solutions, in the difficult positive curvature case, of many scalar curvature equation in noncompact manifolds. The result is new even in the Euclidean space.

In the subcritical, parabolic case, we also prove the convergence of some global solutions to nontrivial stationary solutions.

1. Introduction.

In this paper we establish positive solutions to some semilinear elliptic equations in noncompact manifolds of dimension $n(\geq 3)$, which involve both the subcritical and critical exponent $(n+2)/(n-2)$. We will also prove the convergence of global solutions to nontrivial stationary solutions for some parabolic equations. For the sake of clarity we present these results in three subsections.

1.1. Results on the scalar curvature equations in noncompact complete manifolds. The scalar curvature equations

$$\Delta u - \frac{n-2}{4(n-1)} R(x) u + W u^{(n+2)/(n-2)} = 0, \tag{1.1}$$

are the targets of intensive study over the last decades. Here $\mathbf{M}$ is a Riemannian manifold of dimension $n(\geq 3)$, R is the scalar curvature, and W is a function of x. In the case of compact manifolds, great progress have been made. For the Yamabe problem (W is a constant), we refer to the survey paper [**LP**] and the book [**Au2**] for an account of this matter. In the 80's, Yau [**Yau**] and Kazdan [**Kaz**] suggested the study of (1.1) in the noncompact setting. In the recent book [**Au2**], this study was proposed again by Aubin.

However the understanding in the noncompact case is still rather limited when the scalar curvature is nonnegative. In the negative scalar curvature case, we refer the reader to [**AM**], [**LTY**] and the references there. Some nonexistence and existence results in the positive scalar curvature case can be found in [**Jin**] ($W = 1$), [**Zh3**] ($W = 1$), [**Ki2**] ($W = 1$) and [**Ni**], [**KN**], [**Ho**] (R, W decay rapidly). In Theorem 1.1 below, we establish a general existence result on scalar curvature equations in the most difficult case, i.e., when the scalar curvature is positive and not necessarily decaying.

In order to state the result precisely, it is necessary to recall some well-known objects. We use $\mathbf{M}$ to denote a complete noncompact manifold with dimension $n \ge 3$. We use 0 to denote a fixed point in $\mathbf{M}$ and write $d(x) = d(x, 0)$, the distance from 0 to $x \in \mathbf{M}$.

(1) Let $R = R(x)$ be the scalar curvature of $\mathbf{M}$, then the Yamabe invariant is

$$Y(\mathbf{M}) \equiv \inf_{u \in C_0^\infty(\mathbf{M})} \int_{\mathbf{M}} \left(|\nabla u|^2 + \frac{n-2}{4(n-1)} R u^2 \right) dx / \|u\|^2_{L^{2n/(n-2)}(\mathbf{M})}.$$

(2) Given a function $W = W(x)$ and a domain $D \subset \mathbf{M}$, define

$$Q(W, D) = \inf_{u \in C_0^\infty(D)} \int_D \left(|\nabla u|^2 + \frac{n-2}{4(n-1)} R u^2 \right) dx \Big/ \left(\int_D W u^{2n/(n-2)} dx \right)^{(n-2)/n}.$$

The quantities $Y(\mathbf{M})$ and $Q(W, D)$ have been used widely in the study of conformal properties of both compact and noncompact manifolds. For further properties see the papers [**Au**], [**S**], [**ES**], [**E**] and [**Ki**]. For instance, Condition (a) below is exactly the noncompact version of the main assumption in [**ES**]. Note also $Q(1, \mathbf{M}) = Y(\mathbf{M})$.

We also point out that the solutions in Theorem 1.1 below have finite energy in the sense that $\int_{\mathbf{M}} |\nabla u|^2 dx < \infty$ and $\int_{\mathbf{M}} u^{n/(n-2)} dx < \infty$. This will be clear from the construction of the solution.

We hope to find solutions with infinite energy in a future study.

Theorem 1.1. *Suppose:*

(a) $\mathbf{M}$ *is a* $n(\ge 3)$ *dimensional complete noncompact manifold with nonnegative scalar curvature and* $|B(x,r)| \le C r^n$ *for all* $x \in \mathbf{M}$ *and* $r \ge 1$*; the Yamabe invariant* $Y(\mathbf{M}) > 0$*;* $W(x) \ge 0$*,* $0 \ne W \in L^\infty(\mathbf{M}) \cap C^1(\mathbf{M})$*;*

(b) *there is a compact exhaustion* $\{D_j\}$ *of* $\mathbf{M}$ *such that*

$$\sup_j \left[\left(\max_{x \in D_j} W(x) \right)^{(n-2)/n} Q(W, D_j) \right] < Q_0 = \frac{n(n-2)}{4} (\mathrm{Vol}\,(S^n))^{2/n};$$

here Q_0 *is the constant in the sharp Sobolev inequality in* $\mathbf{R}^n$*;*

(c) *there is a compact domain D such that*

$$\left(\sup_{D^c} W\right)^{(n-2)/n} Q(W, M) < Q(1, D^c).$$

(i) *Then* (1.1) *has a positive solution $u \in L^{\frac{2n}{n-2}}(\mathbf{M})$ such that $u(x) \leq C/(1+d(x)^{\frac{n-2}{2}})$.*

(ii) *If, in addition to* (a) *only, one assumes $\mathbf{M}$ is nonparabolic,* $\mathrm{Ric} \geq 0$ *and $R(x) \geq \frac{c}{1+d(x)^b} > 0$ with $b < 2$. Let $u \geq 0$ be any solution to* (1.1) *such that $\int_{\mathbf{M}} u^{2n/(n-2)} dx < \infty$. Then there exist $c_1, c_2 > 0$ and $a = a(b) > 0$ such that, for all $x \in \mathbf{M}$,*

$$u(x) \leq c_1 e^{-c_2 d(x)^a}.$$

Remark 1.1. Here we show that the assumptions in the theorem are quite natural and encompass large classes of manifolds.

Since $Q(1, B(0,r)^c) \geq Y(\mathbf{M}) > 0$, Condition (c) is satisfied if $\lim_{d(x)\to\infty} W(x) = 0$. There are noncompact manifolds satisfying Condition (c) even if $W \equiv 1$ (see [**Ki**]).

In case $W = W(x)$ reaches absolute maximum at $x_0 \in \mathbf{M}$, then $Q(W, D) \leq Q(W, D_1)$ if $x_0 \in D_1 \subset D$. Hence Condition (b) is satisfied if one can find just one compact domain D_0 containing x_0 so that

$$\left(\max_{x\in D_0} W(x)\right)^{(n-2)/n} Q(W, D_0) < Q_0.$$

The latter is the basic existence condition obtained in the compact case ([**ES**] (Proposition 1.1), [**E**]). See also [**BN**]. Ample examples of the function W are provided in these two papers. Basically W is required to satisfy some flatness condition at its maximum.

Another set of examples comes from compact perturbations of $S^2 \times \mathbf{R}^{n-2}$, $n \geq 6$. Choose a perturbed metric which is not locally conformal flat in $B(x_0, 1)$. Here x_0 is a point. By [**Au**], one can find a function $\phi \in C_0^\infty(B(x_0, 1))$ so that the Yamabe quotient involving ϕ is strictly less than Q_0. Therefore Condition (b) in Theorem 1.1 is satisfies if W is a constant in $B(x_0, 1)$ provided the constant is the absolute maximum and W converges to zero at infinity. In fact one only need some weaker flatness condition such as vanishing of certain derivatives at the maximum point (see [**ES**]). Other conditions are satisfied too since we can choose the perturbation so small that the scalar curvature is bounded away from zero.

Remark 1.2. In the papers [**ES**] and [**E**], Escobar and Schoen obtained important existence results concerning (1.1) in compact manifolds with and without boundaries.

In the paper [**Ki**, **Ki2**], under similar assumptions, Kim obtained interesting existence result for (1.1) with $W = 1$. However some clarification seems needed. In the last paragraph on p. 1987 [**Ki**], the quoted sharp Sobolev inequality of Aubin contains constants $C(\epsilon)$ that may depend on the domains. This is because the domains (Ω in [**Ki**]) may not be contained in a compact set even though their volume is finite. This complicates the claim that the 'approximate solutions u_i' are uniformly bounded in compact domains. In this paper we overcome the difficulty by proving a priori decay estimates for solutions, under merely an assumption on the volume of geodesic balls.

One can replace the volume assumption in (a) by assuming that the Ricci curvature is bounded from below and the injectivity radius is positive. Then the Sharp Sobolev inequality holds on the whole manifold (see [**He**] e.g.). In this case, the existence part of Theorem 1.1 (i) remains valid (see Remark 2.1 below).

Remark 1.3. Theorem 1.1 still holds if R is replaced by an ordinary function satisfying some similar assumptions. This can be proven by exploiting the results in [**E**] in the current setting. In both Theorem 1.1 and the Corollary 1 below, we can allow the function W to change sign. But we are not seeking the full generality this time.

It is well-known that the scalar curvature of 'most' manifolds with nonnegative Ricci curvature decay slower than the inverse of distance square, as in Theorem 1.1 (ii). In the case, (finite energy) solutions given in Theorem 1.1 (ii) decay exponentially to zero near infinity. Therefore they do not produce complete conformal metrics. This result has two interesting implications. First, it snugly complements the existence result of complete conformal metric ([**Ki2**]), where the opposite assumption on the scalar R was made. Second it seems to reveal the limit of the direct variational approach, which requires the solution to have finite energy. Moreover it provides a method of conformal compactification. This can be regarded as a generalization of the stereographic projection between S^n and R^n.

An immediate geometric application of the theorem is:

Corollary 0. *Suppose* $\mathbf{M}$ *and* W *satisfy all conditions in Theorem* 1.1 (i) *and* (ii). *If* $\mathbf{M}$ *has only one end and it is topologically simple at infinity (finite type), then* $\mathbf{M}$ *is conformal to a closed compact manifold minus one point, with scalar curvature* W.

If W decays as $c/d(x)^a$ with $a > 0$, we can obtain existence result on (1.1) via a simpler proof without Condition (c).

Corollary 1. *Let* $\mathbf{M}$ *be a complete noncompact manifold with bounded scalar curvature. Suppose the Yamabe invariant is positive and* $|B(x, r)| \leq cr^n$ *for any* $x \in \mathbf{M}$ *and all* $r > 0$. *Suppose also:*

(a) *There is a compact exhaustion* $\{D_j\}$ *of* $\mathbf{M}$ *such that*

$$\sup_j \left[\left(\max_{x\in D_j} W(x) \right)^{(n-2)/n} Q(W, D_j) \right] < Q_0 = \frac{n(n-2)}{4} (\mathrm{Vol}\,(S^n))^{2/n}.$$

(b) $0 \neq W \in L^\infty(\mathbf{M})$, $W(x) \geq 0$ *and* $W(x) \leq \frac{c}{1+d(x)^a}$ *with* $a > 0$.

Then (1.1) *has a positive solution.*

The conclusion remains valid if $\mathbf{M} = \mathbf{R}^n$, $n \geq 3$, *and* $R = R(x)$ *is any bounded function satisfying* (a) *that is nonnegative outside a compact set.*

Remark 1.4. Under further assumptions, one may be able to show that the metric $u^{4/(n-2)}g$ is complete, using the idea [**Ki2**]. But we are not able to construct an explicit example. See Remark 2.2 in the next section.

We construct an example of (1.1) covered by Corollary 1. Let $W(x) > 0$ be a function satisfying (b) and achieving global maximum at $x = 0 \in \mathbf{R}^4$. Let $V(x) = \frac{n-2}{4(n-1)} R(x)$ be a bounded smooth function which is nonnegative outside a compact set. Suppose that the first eigenvalue of the operator $-\Delta + V$ in $B(0,1)$ is positive, i.e.,

$$\int_{B(0,1)} (|\nabla u|^2 + V(x)u^2) dx \geq c \int_{B(0,1)} u^2 dx, \qquad u \in W_0^{1,2}(B(0,1)).$$

Suppose also $V(0) < 0$, then (1.1) in $\mathbf{R}^4$ has a positive solution.

Let us verify that the conditions of Corollary 1 are met. According to Theorem 3.2 in [**E**], for the above function V and W, one has

$$\left(\max_{x\in B(0,1)} W(x) \right)^{(n-2)/n} Q(W, B(0,1)) < Q_0.$$

Since W achieves absolute maximum at $x = 0$, the above holds when $B(0,1)$ is replaced by any domain containing it. Hence Condition (a) is met. All other conditions follow easily.

1.2. Results on elliptic equations in the subcritical case. Let us describe the elliptic results in the subcritical case. Consider the equation

$$\Delta u - V(x)u + W(x)u^p = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{1.2}$$

Here $1 < p < \frac{n+2}{n-2}$, $n \geq 3$. In what follows, unless otherwise stated, we will assume that $V = V(x)$ and $W = W(x)$ are locally Hölder continuous, and bounded function.

This equation has a rich history. When $V = W \equiv 1$, it is well-known that (1.2) has a so-called ground state solution, meaning a positive solution decaying exponentially to zero near infinity. In [**Lio1-2**], P.L. Lions obtained existence of nontrivial solutions to (1.2) when V is a suitable perturbation of a positive constant near infinity and $W = 1$. His approach is the famous concentration-compactness principle, which is variational in nature. Related results are also obtained in Ding and Ni [**DN**] (when $V = 1$ and W satisfies various conditions). See also [**NS**]. When V and W have rapid decay, important existence results were achieved in Ni [**Ni**], Kenig and Ni [**KN**]. See also [**Zh3**]. Here and later a function is said to have rapid decay if it is smaller than $C/|x|^b$ near infinity, where $b > 2$. Otherwise we say it has slow decay. In Stuart [**St**] existence result was obtained when $V = 1$ and W has slow decay. Subsequently many authors have taken up the study of the problem and produced numerous interesting results.

Despite these advances, the important middle range , i.e., when both V and W has slow decay, seems to be completely open in the nonradial case. Our next theorem largely fill this gap. Let us mention that this result is not a direct consequence of the variational approach. Since it is well-known that the concentrated compactness method requires that V converges to a positive constant at infinity in a special manner. Moreover if V has rapid decay and $W \equiv 1$, (1.2) does not have any positive solution if $1 < p < \frac{n}{n-2}$ (see [**Zh3**]). In this paper we will also introduce a dynamic approach to solve (1.2) (see Section 1.3).

The following table provides a glimpse of the current understanding on the existence of positive solutions to (1.2). It is not intended to be a complete account of the literature.

$\Delta u - V(x)u + W(x)u^p = 0$	existence results, $1 < p < \frac{n+2}{n-2}$
$W \to 1$ and $V \to 1$ at ∞	[**Lio1-2**], [**DN**], [**NS**] under more assumptions of the convergence
W and V decay rapidly	[**Ni**], [**KN**], [**Lin**], [**Ka**], [**Zh3**]
$V = 1$, W decays	[**St**], [**DN**], [**Li**]
V decays rapidly, W does not	there are nonexistence results [**Ni**], [**Li**], [**Zh3**]
W and V have slow decay	current paper, with additional condition on W

The second theorem of the paper is:

Theorem 1.2. *Suppose* $\frac{a}{1+|x|^b} \le V(x) \le C_1$ *with* $b \in [0, 2)$, $a > 0$. *Suppose* $0 < W(x) \le C_2/(1 + |x|^{2-((p-1)(n-2)/2)})$ *and* $1 < p < \frac{n+2}{n-2}$. *Then Equation* (1.2) *has a positive solution such that* $\int_{\mathbf{R}^n} |\nabla u|^2 dx$ *and* $\int_{\mathbf{R}^n} W u^{p+1} dx$ *are finite.*

The conclusion still holds if $\mathbf{R}^n$ *is replaced by a complete manifold of nonnegative Ricci curvature and maximum volume growth, i.e.,* $|B(x, r)| \ge cr^n > 0$ *for all* x *and* $r > 0$.

Remark 1.5. When $p \to \frac{n+2}{n-2}$ from below the number $2-((p-1)(n-2)/2)$ converges to 0. Since it is well-known that (1.2) may not have any finite energy positive solution when V and W are just bounded functions. This indicates that the decay rate on W is close to optimal.

Note also that both Theorem 1.2 (and 1.3 below) may fail if V is allowed to decay faster than $c/|x|^2$ near infinity (see [**Zh2**]).

Remark 1.6. Theorems 1.2 and 1.3 continue to hold if the Laplacian in (1.2) and (1.3) is replaced by an uniformly elliptic divergence operator with bounded measurable coefficients depending on x.

The strategy of the proof of Theorem 1.2 is the combined use of domain exhaustion method, Green's function estimates and certain scaling arguments.

1.3. Results in the parabolic case. Next we present the parabolic results of the paper. One of the central questions in nonlinear analysis is whether or not a global solution to evolution equations would converge to a nontrivial equilibrium solution. This problem is relatively well studied if the ambient space is compact. However this is not the case in noncompact setting. In fact the understanding in this case is rather limited. To illustrate this shortcoming let us consider a model parabolic equation:

$$\begin{cases} \Delta u - V(x)u + W(x)u^p - \partial_t u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x). \end{cases} \tag{1.3}$$

Here $1 < p < \frac{n+2}{n-2}$, $n \geq 3$. We assume that $V = V(x)$ and $W = W(x)$ are locally Hölder continuous, and bounded function.

Problems such as (1.3) also arises from many areas and are some of the central subjects in nonlinear analysis.

In the classical paper [**NST**], when $\mathbf{R}^n$ is replaced by a bounded domain, interesting result on the convergence of solutions of (1.3) to that of (1.2) are obtained. However except a few exceptions, the corresponding results for $\mathbf{R}^n$ and other noncompact domains have not been achieved in general. The papers [**CDE**] and [**BJP**] study (1.3) in the case $V = W = 1$. The paper [**SZ**] studies the case V, W and the solutions are radial. In the paper [**Zh3**] convergence results on (1.3) when V and W have rapid decay were obtained (see Section 1.2 for the meaning of rapid decay). As illustrated by many authors, the nature of the decay for V and W near infinity is a deciding factor for the existence and nonexistence of solutions to (1.2) and (1.3). Usually fundamental differences exist between the slower decay cases and the rapid decay ones.

Based on the study of global positive solutions of the linear part of the equation in (1.3) and a scaling argument, we will prove that Equation (1.3) has global positive solutions whose ω limit set contains nontrivial positive solutions to the elliptic equation (1.2). Up till now have not seen any comparable convergence results for Equation (1.3), except in the case when V and W are radial. We should mention that the types of equations we are studying require W to decay at infinity. An interesting remaining problem is to obtain a similar convergence results when both V and W do not decay to zero. We hope to address it in future.

Theorem 1.3. *Suppose* $\frac{a}{1+|x|^b} \le V(x) \le C_1$ *with* $b \in [0,2)$, $a>0$. *Suppose* $0 < W(x) \le C_2/(1+|x|^{2-((p-1)(n-2)/2)})$ *and* $1<p<1+\frac{4}{n}$. *Then for any compactly supported nonnegative* $f \ne 0$, *there exists* $\lambda > 0$ *such that the problem*

$$\begin{cases} \Delta u - V(x)u + W(x)u^p - \partial_t u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) \equiv \lambda f(x), \end{cases}$$

has a global positive solution. Moreover the $\omega-$ *limit set contains a nontrivial equilibrium solution. The result continues to hold if* $\mathbf{R}^n$ *is replaced by a complete manifold of nonnegative Ricci curvature and maximum volume growth.*

Remark 1.7. At this moment we do not know if the assumption $p < 1+\frac{4}{n}$ can be improved to $p < \frac{n+2}{n-2}$ in the parabolic case.

The rest of the paper is organized as follows: In Section 2 we prove Theorem 1.1. In Section 3 we prove Theorem 1.2. In Section 4 we establish several preliminary estimates on the global solutions to (1.3). We will prove Theorem 1.3 in Section 5.

The proofs of different theorems are related but independent of each other.

2. Proof of Theorem 1.1.

The proof is divided into 5 steps. We will use the idea of finite domain exhaustion. A crucial step is to establish certain a priori decay of solutions of (1.1) near infinity.

Step 1. We begin by solving, for each $j>0$, the variational problem

$$\inf_{u \in W_0^{1,2}(B(0,j))} \int_{B(0,j)} (|\nabla u|^2 + Vu^2) dx \tag{2.1}$$

subject to the constraint $\int_{B(0,j)} W u^{p+1} dx = 1$. Here and throughout the section $p=(n+2)/(n-2)$ and $V = \frac{(n-2)}{4(n-1)} R(x)$. Following the standard arguments in ([**Au**], [**S**] and [**ES**]), for each $j>0$, Problem (2.1) has a positive solution under our assumptions. In fact details of the proof are mirrored in Step 3 below where we will prove that these u_j are uniformly bounded in any compact set.

Let $u_j \geq 0$ be a solution to (2.1). Then there exists a $q_j > 0$ such that

$$\Delta u_j(x) - V u_j(x) + q_j W u_j^p(x) = 0, \qquad (2.2)$$
$$x \in B(0,j); \quad u(x) = 0, \quad x \in \partial B(0,j).$$

In fact

$$q_j = \inf_{u \in W_0^{1,2}(B(0,j))} \frac{\int_{B(0,j)} (|\nabla u|^2 + V u^2) dx}{\left(\int_{B(0,j)} W u^{p+1} dx \right)^{2/(p+1)}}.$$

Since the Yamabe invariant is positive and W is a bounded function. We know that, for any $j > 0$,

$$\begin{aligned}\left(\int_{B(0,j)} W u^{p+1} dx \right)^{2/(p+1)} &\leq (\sup W)^{2/(p+1)} \left(\int_{B(0,j)} u^{p+1} dx \right)^{2/(p+1)} \\ &\leq C (\sup W)^{2/(p+1)} \left[\int_{B(0,j)} (|\nabla u|^2 + V u^2) dx \right].\end{aligned}$$

This shows that there exists a $q > 0$ such that q_j decreases to q when $j \to \infty$. In fact $q = Q(W, \mathbf{M})$, defined in Section 1.

We extend u_j to the whole manifold by defining $u_j(x) = 0$ when x is outside of $B(0,j)$. The extended function, still denoted by u_j is a subsolution to Equation (2.2) in the whole manifold. Our goal is to show that a subsequence u_j converges to a positive solution to (1.1).

Step 2. In this step we will prove the following: There exists $R_0 > 0$ and $C > 0$ such that

$$u_j(x) \leq \frac{C}{1 + d(x)^{(n-2)/2}} \qquad (2.3)$$

for all j and x when $d(x) \geq R_0$. This estimate follows from an argument in [**Zh4**]. For simplicity we will drop the subscript j in this step.

For $R > 1$ and fixing x_0 such that $d(x_0) = 2R^2$. For each $R > 1$, let us introduce the scaled metric

$$g_1 = g/R^4.$$

Let $\mathbf{M}_1$ be the manifold $\mathbf{M}$ with g replaced by g_1 and d_1, Δ_1, ∇_1 be the corresponding distance, Laplace-Beltrami operator, gradient respectively. Note that $\Delta_1 = R^4 \Delta$. Let us consider $v \in C(\mathbf{M}_1)$ defined by

$$v(x) = R^{n-2} u(x).$$

Since $\Delta u - V u + q_j W u^{(n+2)/(n-2)} = 0$ and $R > 1$, direct computation shows

$$\Delta_1 v - V_1 v + q_j W v^{(n+2)/(n-2)} = R^{n+2} (\Delta u - V u + q_j W u^{(n+2)/(n-2)}) \geq 0,$$

$$\int_{d_1(x_0,x)\le 1} v^{2n/(n-2)}(x)d_1x = \int_{d(x_0,x)\le R^2} u^{2n/(n-2)}(x)dx,$$
$$\int_{d_1(x_0,x)\le 1} |\nabla_1 v(x)|^2 d_1x = \int_{d_1(x_0,x)\le 1} g_1(\nabla_1 v(x), \nabla_1 v(x))d_1x$$
$$= R^{-2}\int_{d(x_0,x)\le R^2} |\nabla u(x)|^2 dx.$$

Estimate (2.3) will be proven once we can show that v is bounded in $B_1(x_0,1) = \{x \in \mathbf{M}_1 \mid d_1(x_0,x) \le 1\}$. To this end we will use an argument inspired by that in [**Eg**] p. 44, which can be generalized to our case since the manifold $\mathbf{M}_1$ has nonnegative Ricci curvature outside a compact set.

Take $G(s) = s^\beta$ if $s > 0$, and zero otherwise, and put $F(u) = \int_0^u G'(s)^2 ds = \beta^2 u^{2\beta-1}/(2\beta-1)$. It is easy to verify that $sF(s) \le s^2 G'(s)^2 \le \beta^2 G(s)^2$ if $\beta > 1$.

Take $\phi \in C^\infty[0,\infty)$ such that

$$0 \le \phi \le 1; \quad \phi(r) = 1,\ r \in [0,1/2]; \quad \phi(r) = 0,\ r \in [1,\infty);$$
$$-C \le \phi'(r) \le 0; \quad |\phi''(r)| \le C.$$

Let $\eta = \eta(x) = \phi(d_1(x_0,x))$. Using $\eta^2 F(v)$ as a test function on the inequality

$$\Delta_1 v - V_1 v + q_j W v^{(n+2)/(n-2)} \ge 0,$$

we obtain

$$\int |\nabla_1 v|^2 G'(v)^2 \eta^2 d_1x + \int V_1 v F(v)\eta^2 d_1x + 2\int \nabla_1 v \nabla_1 \eta F(v)\eta d_1x$$
$$\le \beta^2 \int q_j W v^{(n+2)/(n-2)} v^{-1} G(v)^2 \eta^2 d_1x.$$

Using the inequality

$$2|\nabla_1 v \nabla_1 \eta F(v)\eta| \le \epsilon |\nabla_1 v|^2 \eta^2 v^{-1} F(v) + \epsilon^{-1}|\nabla_1 \eta|^2 F(v) v$$
$$\le \epsilon |\nabla_1 v|^2 \eta^2 G'(v)^2 + \epsilon^{-1}\beta^2 |\nabla_1 \eta|^2 G(v)^2$$

we obtain, for another $\epsilon > 0$,

$$\|\nabla_1(G(v)\eta)\|_2^2 + \frac{\beta^2}{2\beta-1}\int V_1 [G(v)\eta]^2$$
$$\le C\beta^2 \int |\nabla_1 \eta|^2 G(v)^2 d_1x + (\beta^2+\epsilon) q_j \int W v^{4/(n-2)} G(v)^2 \eta^2 d_1x.$$

Here ϵ can be chosen as any small positive number.

At this point we need to use Condition (c) which implies that $Q(1,B(0,r)^c) > 0$ when r is large. Indeed

(2.4)

$$\left[\int_{B_1(x_0,2)}[G(v)\eta]^{2n/(n-2)}\right]^{(n-2)/n} \le C_0\|\nabla_1(G(v)\eta)\|_2^2 + C_0\int V_1[G(v)\eta]^2.$$

Here in fact, we can choose

$$C_0 = 1/Q(1, B(0, R^2/2)^c) > 0.$$

Noting that $\frac{\beta^2}{2\beta-1} \ge 1$ if $\beta > 1$ and applying Hölder's inequality, we find that

(2.5)

$$\begin{aligned}
&\|G(v)\eta\|_{2n/(n-2)}^2 \\
&\le CC_0\beta^2\int|\nabla_1\eta|^2G(v)^2d_1x + C_0q_j(\beta^2+\epsilon)\int Wv^{4/(n-2)}G(v)^2\eta^2d_1x \\
&\le CC_0\beta^2\int|\nabla_1\eta|^2G(v)^2d_1x + C_0q_j(\beta^2+\epsilon)\ \|G(v)\eta\|_{2n/(n-2)}^2 \\
&\qquad\cdot\left(\int_{B_1(x_0,1)}W^{n/2}v^{2n/(n-2)}d_1x\right)^{2/n} \\
&\le CC_0\beta^2\int|\nabla_1\eta|^2G(v)^2d_1x \\
&\quad + C_0q_j\left(\sup_{B_1(x_0,1)}W\right)^{(n-2)/n}(\beta^2+\epsilon)\ \|G(v)\eta\|_{2n/(n-2)}^2 \\
&\qquad\cdot\left(\int_{B_1(x_0,1)}Wv^{2n/(n-2)}d_1x\right)^{2/n}.
\end{aligned}$$

Here all the norms are over the ball $B_1(x_0,1)$.

Since

$$\int_{B_1(x_0,1)}Wv^{2n/(n-2)}d_1x \le \int_{\mathbf{M}}Wu^{2n/(n-2)}dx = 1,$$

we have

$$\begin{aligned}
\|G(v)\eta\|_{2n/(n-2)}^2 &\le CC_0\beta^2\int|\nabla_1\eta|^2G(v)^2d_1x \\
&\quad + C_0q_j\left(\sup_{B_1(x_0,1)}W\right)^{(n-2)/n}(\beta^2+\epsilon)\ \|G(v)\eta\|_{2n/(n-2)}^2.
\end{aligned}$$

By Assumption (c) in the theorem, when R is sufficiently large,

$$\left(\sup_{B(0,R^2/2)^c} W\right)^{(n-2)/n} Q(W,M) < Q(1, B(0,R^2/2)^c).$$

Since $q_j \to Q(W,M)$ when $j \to \infty$, we have

$$\left(\sup_{B(0,R^2/2)^c} W\right)^{(n-2)/n} q_j < Q(1, B(0,R^2/2)^c) = \frac{1}{C_0}$$

when j is sufficiently large. Hence there exists a $\beta > 1$ such that

$$\sup_{B_1(x_0,1)} W^{(n-2)/n}\ (\beta^2 + \epsilon)(C_0 q_j) < \lambda < 1.$$

Using this and take $G(v) = v^{\beta_0}$, we see that, for a $\beta_0 > 1$ and sufficiently close to 1,

$$\|v^{\beta_0}\eta\|^2_{2n/(n-2)} \le 2C \int |\nabla_1 \eta|^2 v^{2\beta_0} d_1 x. \tag{2.6}$$

Choose η such that $\eta(x) = 1$ if $x \in B_1(x_0, 1/2)$, $|\nabla_1\eta| \le 2$ and $\eta(x) = 0$ when $x \in B_1(x_0,1)^c$. Use Hölder's inequality we see that

$$\begin{aligned} \|v^{\beta_0}|_{B_1(x_0,1/2)}\|^2_{2n/(n-2)} &\le C \int_{B_1(x_0,1)} v^{2\beta_0} d_1 x \\ &\le \|v\|^m_{L^{2n/(n-2)}(B_1(x_0,1))} \le C < \infty. \end{aligned}$$

Here $m > 0$. Note this is the place where we are using Condition (a) on volume, which implies $|B_1(x_0,1)| = |B(x_0,R^2)|/R^{2n} \le C$.

Now using standard method we immediately know that $v(x_0) \le C_1$. For completeness we give a sketch of the proof.

Given r_2, r_1 such that $0 < r_2 < r_1 < 1/2$, we choose η to be a radial function with support in $B_1(x_0, r_1)$ and such that $\eta = 1$ if $x \in B_1(x_0, r_1)$ and $|\nabla_1 \eta| \le 2/(r_2 - r_1)$. Clearly (2.5) remains valid for such η and any fixed $\beta \ge 1$. Let χ_r be the characteristic function of $B(x_0, r)$. By Hölder's inequality, there exists a $\delta < 1$ and $m > 0$ such that

$$\|G(v)\chi_{r_1}\|_2 \le C \|G(v)\chi_{r_1}\|_{2n\delta/(n-2)},$$

$$\begin{aligned} \int_{B_1(x_0,r_1)} v^{4/(n-2)} G(v)^2 d_1 x &\le C \|v\chi_{r_1}\|^m_{2n\beta_0/(n-2)} \|G(v)\chi_{r_1}\|^2_{2n\delta/(n-2)} \\ &\le C \|G(v)\chi_{r_1}\|^2_{2n\delta/(n-2)}. \end{aligned}$$

Substituting the last two inequalities to the first inequality of (2.5), we obtain

$$\|G(v)\chi_{r_2}\|_{2n/(n-2)} \le \frac{C\beta}{r_1 - r_2} \|G(v)\chi_{r_1}\|_{2n\delta/(n-2)}.$$

From this point, the standard Moser's iteration of taking $\beta = \delta^{-m}$ and $r_m = r_1(2+2^{-m})/4$ shows that

$$v(x_0) \le c < \infty$$

when R is sufficiently large. Therefore, since $d(x_0) = 2R^2$,

$$u(x_0) = \frac{v(x_0)}{R^{n-2}} \le \frac{c}{1+d(x_0)^{(n-2)/2}}.$$

This completes Step 2.

Step 3. We prove that u_j is uniformly bounded in any given finite domain. This follows from the ideas in [**Au**], [**ES**] and [**Ki**], together with the decay estimates in Step 2.

Let u_j be the subsolution to (2.2), produced in Step 1.

For any fixed $j > 0$, let

$$D = \{x \in \mathbf{M} \mid u_j(x) \ge 1\}. \tag{2.7}$$

By Step 2, D is contained in a fixed ball of sufficiently large radius. This is a crucial observation for the subsequent proof.

Writing $w_j = u_j - 1$, we see that

$$-\Delta w_j + V(1+w_j) \le q_j(1+w_j)^p. \tag{2.8}$$

For some $b > 0$ to be determined later, multiplying (2.8) by w_j^{1+2b} and integrating, we obtain

$$\begin{aligned} &\int_D \left[\frac{1+2b}{(1+b)^2}|\nabla w_j^{1+b}|^2 + V(1+w_j)w_j^{1+2b}\right] dx \\ &\le q_j \int_D W w_j^{1+2b}(w_j+1)^p dx. \end{aligned} \tag{2.9}$$

By virtue of the sharp Sobolev inequality of Aubin [**Au**], for any $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$\begin{aligned} &\left(\int_D w_j^{(1+b)2n/(n-2)} dx\right)^{(n-2)/n} \\ &\le (1+\epsilon)\frac{1}{Q_0}\int_D |\nabla w_j^{1+b}|^2 dx + C(\epsilon)\int_D w_j^{2(1+b)} dx. \end{aligned}$$

Here, as before, Q_0 is the best Sobolev constant in $\mathbf{R}^n$.

Substituting (2.9) to the left-hand side of the above, we see that

$$
\begin{aligned}
(2.10)\qquad &\left(\int_D w_j^{(1+b)2n/(n-2)}dx\right)^{(n-2)/n}\\
&\le (1+\epsilon)\frac{1}{Q_0}\frac{(1+b)^2}{(1+2b)}\\
&\qquad\cdot \int_D \left(q_j W w_j^{1+2b}(1+w_j)^{(n+2)/(n-2)} - V(w_j+1)w_j^{1+2b}\right)dx\\
&\quad + C(\epsilon)\int_D w_j^{2(1+b)}dx.
\end{aligned}
$$

As in [Ki], we write $D_1 = \{x \in D \mid w_j(x) \ge 2\}$ and $D_2 = D - D_1$. When $x \in D_1$, it is clear that

$$
(1+w_j(x))^{(n+2)/(n-2)} \le w_j^{(n+2)/(n-2)}(x) + c_1 w_j^{4/(n-2)}(x).
$$

When $x \in D_2$, $w_j(x) \le 2$. Hence

$$
\begin{aligned}
&\int_D W w_j^{1+2b}(1+w_j)^{(n+2)/(n-2)}dx\\
&= \int_{D_1} W w_j^{1+2b}(1+w_j)^{(n+2)/(n-2)}dx\\
&\quad + \int_{D_2} W w_j^{1+2b}(1+w_j)^{(n+2)/(n-2)}dx\\
&\le \int_{D_1} W w_j^{2n/(n-2)+2b}dx + c_1\int_{D_1} W w_j^{(n+2)/(n-2)+2b}dx\\
&\quad + \int_{D_2} W w_j^{1+2b}(1+w_j)^{(n+2)/(n-2)}dx\\
&\le \int_{D_1} W w_j^{2n/(n-2)+2b}dx + c_2\int_{D} W w_j^{(n+2)/(n-2)+2b}dx\\
&\le \left(\int_{D_1} W w_j^{(1+b)2n/(n-2)}dx\right)^{(n-2)/n}\left(\int_{D_1} W u^{2n/(n-2)}dx\right)^{2/n}\\
&\quad + c_2\int_D W w_j^{(n+2)/(n-2)+2b}dx\\
&\le \left(\int_{D_1} W w_j^{(1+b)2n/(n-2)}dx\right)^{(n-2)/n} + c_2\int_D W w_j^{(n+2)/(n-2)+2b}dx.
\end{aligned}
$$

Here we have used the Hölder's inequality. Substituting the above into (2.10), we have

$$
\begin{aligned}
(2.11)\qquad &\left(\int_D w_j^{(1+b)2n/(n-2)}dx\right)^{(n-2)/n}\\
&\le (1+\epsilon)\frac{q_j}{Q_0}\frac{(1+b)^2}{(1+2b)}\left(\int_D W w_j^{(1+b)2n/(n-2)}dx\right)^{(n-2)/n}\\
&\quad + C\int_{D_1} w_j^{((n+2)/(n-2))+2b}dx + C(\epsilon)\int_D w_j^{2(1+b)}dx + c.
\end{aligned}
$$

By our assumption in Theorem 1.1 ($\sup_j \sup_{x\in B(0,j)} W(x)^{(n-2)/n} q_j < Q_0$), we can choose ϵ and b sufficiently small so that

$$(1+\epsilon)\sup_{x\in B(0,j)} W(x)^{(n-2)/n}\frac{q_j}{Q_0}\frac{(1+b)^2}{(1+2b)} < \lambda < 1.$$

So (2.11) implies

$$
\begin{aligned}
(2.12)\qquad &\left(\int_D w_j^{(1+b)2n/(n-2)}dx\right)^{(n-2)/n}\\
&\le \frac{C}{1-\lambda}\left(\int_{D_1} w_j^{((n+2)/(n-2))+2b}dx + C(\epsilon)\int_D w_j^{2(1+b)}dx + c\right).
\end{aligned}
$$

Choosing b sufficiently small, we know, for some $l_i, l_i' > 0$, $i = 1, 2$,

$$\int_D \left[w_j^{2(1+b)} + w_j^{((n+2)/(n-2))+2b}\right] dx \le \Sigma_{i=1}^2 \left(\int_D w_j^{2n/(n-2)}dx\right)^{l_i} |D|^{l_i'}.$$

It follows that

$$(2.13)\qquad \int_D w_j^{(1+b)2n/(n-2)}dx \le C,$$

where C is independent of j. From here standard regularity theory shows that u_j is uniformly bounded in any compact domain of $\mathbf{M}$. Step 3 is complete.

Step 4. We will show that a subsequence of u_j converges pointwise to a positive solution to (1.1), up to a constant multiple.

Since u_j is uniformly bounded in any compact domain, the standard elliptic theory shows that u_j is uniformly bounded in $C^{2,\alpha}$ norm in any compact domain too. Hence a subsequence of u_j, still denoted by u_j, converges pointwise to a solution u to:

$$\Delta u(x) - Vu(x) + qWu^p(x) = 0,\ x\in \mathbf{M}^n.$$

Here q is a positive constant. Now using a dilation of u, we can obtain a solution to (1.2).

We need to prove that u is positive. We will use the Concentrated Compactness Principle of P. L. Lions, as suggested in [**Ki**]. To this end we write

$$J(u) \equiv \int_{\mathbf{M}} (|\nabla u|^2 + Vu^2)dx, \qquad v_j \equiv u_j - u.$$

Next

$$J(u_j) = J(u + v_j) = J(u) + J(v_j) + 2\int_{\mathbf{M}} (-\Delta u + Vu)v_j dx.$$

Clearly v_j converges weakly to zero. Hence

$$J(u_j) - J(v_j) \to J(u). \tag{2.14}$$

Moreover, for any fixed $R > 0$,

$$\begin{aligned} J(v_j) &= \int_{B(0,R)} (|\nabla v_j|^2 + Vv_j^2)dx + \int_{B(0,R)^c} (|\nabla v_j|^2 + Vv_j^2)dx \\ &\geq Q(1, B(0,R)^c) \left(\int_{B(0,R)^c} |v_j|^{2n/(n-2)} dx \right)^{(n-2)/n} + o(1) \\ &\geq Q(1, B(0,R)^c) \left(\int_{B(0,R)^c} W|v_j|^{2n/(n-2)} dx \right)^{(n-2)/n} \\ &\quad \cdot \left[\sup_{B(0,R)^c} W \right]^{-(n-2)/n} + o(1) \end{aligned} \tag{2.15}$$

as $R \to \infty$. Here we have used the fact that v_j converges to 0 pointwise in any compact domain.

By the Fatou Lemma due to Brezis and Lieb,

$$\int_{\mathbf{M}} Wu_j^{2n/(n-2)} dx - \int_{\mathbf{M}} Wv_j^{2n/(n-2)} dx \to \int_{\mathbf{M}} Wu^{2n/(n-2)} dx. \tag{2.16}$$

We claim that u is not identically zero. Otherwise (2.16) shows that

$$\lim_{j\to\infty} \int_{\mathbf{M}} W|v_j|^{2n/(n-2)} dx = 1, \tag{2.17}$$

since $\int_{\mathbf{M}} Wu_j^{2n/(n-2)} dx = 1$. From (2.14), (2.15), (2.17) and the assumption that $u = 0$, we see that

$$\liminf_{j\to\infty} J(u_j) = \liminf_{j\to\infty} J(v_j) \geq Q(1, B(0,R)^c) \left[\sup_{B(0,R)^c} W \right]^{-(n-2)/n} + o(1)$$

as $R \to \infty$. Recall that

$$\liminf_{j\to\infty} J(u_j) = Q(W, \mathbf{M}).$$

Hence

$$\left[\sup_{B(0,R)^c} W\right]^{(n-2)/n} Q(W,\mathbf{M}) \geq Q(1, B(0,R)^c) + o(1).$$

This is a contradiction to Assumption (c) when R is sufficiently large.

Hence u is not identically zero. By the unique continuation property, we see that $u(x) > 0$ everywhere. A suitable dilation yields a positive solution to (1.1). This proves (i).

Step 5. We prove part (ii) of the theorem.

We need to prove an a priori estimate under weaker assumptions than (i).

First we note that Step 2 of the proof still applies in this case, with only a slight change. Indeed, (2.5) remains true. Now using the a priori assumption that $\int_{\mathbf{M}} u^{2n/(n-2)}dx < \infty$ and the scaling invariance of this integral, we see that

$$\int_{B_1(x_0,1)} W v^{2n/(n-2)} d_1 x \to 0$$

when $x_0 \to \infty$. From this, (2.6) is an immediate consequence of (2.5). Following the rest of the proof of Step 2, we know that $u(x) \leq C/(1 + d(x)^{(n-2)/2})$.

Under the extra assumption that $R(x) \geq c/(1 + d(x)^b)$ with $b < 2$, we show that u actually has exponential decay near infinity.

Here is the proof. Since u is a solution to (1.1), we have

$$\Delta u - (V - Wu^{p-1})u \geq 0.$$

By the assumption that $V(x) \geq c/(1 + d(x)^b)$ with $b < 2$ and the estimate $u(x) \leq C/(1 + d(x)^{(n-2)/2})$, we know that

$$(V - Wu^{p-1})(x) \geq \frac{c}{1 + d(x)^b} - \frac{C}{1 + d(x)^2} \geq \frac{c}{2(1 + d(x)^b)}.$$

Here we just used the equality $p - 1 = 4/(n-2)$.

Let Γ_1 be the Green's function of the operator

$$\Delta - (V - Wu^{p-1}).$$

By the estimate of Green's functions [**Zh1**], we have, when $d(x) \geq 1$ and for some $a > 0$,

$$\Gamma_1(x, 0) \leq Ce^{-cd(x)^a}.$$

For completeness, we sketch a proof here.

From the heat kernel estimates (Theorem 1.1 in [**Zh1**]), we have

$$\Gamma_1(x,0) \leq \int_0^\infty \frac{C}{|B(x,\sqrt{t})|} e^{-c(\sqrt{t}/(1+d(x)^{b/2}))^{(2-b)/2}} e^{-cd(x)^2/t} dt.$$

By direct computation, the above shows

$$\Gamma_1(x,0) \le \int_0^\infty \frac{C}{|B(x,\sqrt{t})|} e^{-cd(x)^2/(2t)} dt \; e^{-cd(x)^a/2}$$

where $a > 0$. Hence

$$\Gamma_1(x,0) \le C \Bigg[\int_0^{d(x)^2} \frac{1}{|B(x,\sqrt{t})|} e^{-cd(x)^2/(2t)} dt \\ + \int_{d(x)^2}^\infty \frac{1}{|B(x,\sqrt{t})|} e^{-cd(x)^2/(2t)} dt \Bigg] \; e^{-cd(x)^a/2}.$$

By the doubling property of the balls, for $t \le d(x)^2$ and $d(x) \ge 1$,

$$|B(x,\sqrt{t})| \ge c(t/d(x)^2)^k |B(x,d(x))| \\ \ge c'(t/d(x)^2)^k |B(0,d(x))| \ge c''(t/d(x)^2)^k.$$

When $t \ge d(x)^2$ and $d(x) \ge 1$,

$$\int_{d(x)^2}^\infty \frac{1}{|B(x,\sqrt{t})|} e^{-cd(x)^2/(2t)} dt \le C \int_{d(x)^2}^\infty \frac{1}{|B(0,\sqrt{t})|} dt \le C,$$

by the extra assumption that $\mathbf{M}$ is nonparabolic. Hence

$$\Gamma(x,0) \le C[1 + d(x)^2] e^{-cd(x)^a/2} \le C' e^{-cd(x)^a/4}.$$

Note, for a large $k > 0$, $u(z) \le k\Gamma_1(z,0)$ when $d(z)$ is large, but fixed. By the maximum principle, we see that

$$u(x) \le c_1 e^{-c_2 d(x)^a}$$

for all x. □

Remark 2.1. If one assumes that the Ricci curvature is bounded from below and the injectivity radius is positive, then Aubin's sharp Sobolev inequality (with an extra zero order term) holds on the whole manifold. Then the existence result can be obtained by carrying out Steps 2 to 4.

Proof of Corollary 1. The existence part is similar to the Proof of Theorem 1.1. We give the proof in several steps.

Step 1. This is identical to Step 1 in the Proof of Theorem 1.1

Step 2. This is again similar to Step 2 before. Using the same notations as in Step 2 before, one has:

(2.19)

$$\begin{aligned}
&\|G(v)\eta\|^2_{2n/(n-2)} \\
&\le CC_1\beta^2 \int |\nabla_1\eta|^2 G(v)^2 d_1x + C_1 q_j(\beta^2+\epsilon) \int W v^{4/(n-2)} G(v)^2\eta^2 d_1x \\
&\le CC_1\beta^2 \int |\nabla_1\eta|^2 G(v)^2 d_1x + C_1 q_j(\beta^2+\epsilon)\ \|G(v)\eta\|^2_{2n/(n-2)} \\
&\qquad \cdot \left(\int_{B_1(x_0,1)} W^{n/2} v^{2n/(n-2)} d_1x \right)^{2/n} \\
&\le CC_1\beta^2 \int |\nabla_1\eta|^2 G(v)^2 d_1x \\
&\quad + C_1 q_j \left(\sup_{B_1(x_0,1)} W \right)^{(n-2)/n} (\beta^2+\epsilon)\ \|G(v)\eta\|^2_{2n/(n-2)} \\
&\qquad \cdot \left(\int_{B_1(x_0,1)} W v^{2n/(n-2)} d_1x \right)^{2/n}.
\end{aligned}$$

Here all the norms are over the ball $B_1(x_0,1)$. Since $\int_{B_1(x_0,1)} W v^{2n/(n-2)} d_1x \le 1$ and $W(x) \to 0$ as $d(x) \to \infty$, the above shows that

$$\|G(v)\eta\|^2_{2n/(n-2)} \le C_2\beta^2 \int |\nabla_1\eta|^2 G(v)^2 d_1x.$$

From here we just follow the previous arguments step by step to conclude that $v(x) \le C$ and hence $u_j(x) \le C/d(x)^{(n-2)/2}$.

Step 3. This is identical to the previous Step 3 since we are working in a bounded domain.

Step 4. Since $u_j(x) \le C/(1+d(x)^{(n-2)/2})$, by the decay condition on W and the volume growth assumption on $\mathbf{M}$, we see that

$$\begin{aligned}
\int_{B(0,r_0)^c} W u_j^{p+1} dx &\le \Sigma_{i=1}^{\infty} \int_{B(0,2^i r_0)-B(0,2^{i-1}r_0)} W u_j^{p+1} dx \\
&\le C\Sigma_{i=1}^{\infty} (2^{i-1}r_0)^{-a}.
\end{aligned}$$

Hence, for a sufficiently large r_0, one has

$$\int_{B(0,r_0)} W u_j^{p+1} dx = 1 - \int_{B(0,r_0)^c} W u_j^{p+1} dx \ge 1/2. \tag{2.20}$$

By the standard elliptic theory, u_j is uniformly bounded in $C^{2,\alpha}$ norm for some $\alpha > 0$. So a subsequence, still denoted by u_j, converges pointwise to a function u. By (2.20) and the unique continuation property, we know that $u(x) > 0$ for all x. This u is a positive solution to

$$\Delta u(x) - Vu(x) + qWu^p(x) = 0, \ x \in \mathbf{M}^n.$$

Here q is a positive constant. Now using an dilation of u, we can obtain a positive solution to (1.1). This proves the existence. $\square$

Remark 2.2. Let u be a solution in Corollary 1. We show that $u^{4/(n-2)}g$ is a complete metric under the extra assumption: $r\Delta r \le (n/2) - \delta$ for some $\delta > 0$ and, for a sufficiently small $\epsilon > 0$, $R(x) \le \epsilon/(1 + d(x)^2)$ when $d(x)$ is large. However, it seems hard to find an example of such manifolds.

We will follow an idea in [**Ki2**]. It suffices to prove that

$$u(x) \ge c_0/(1 + d(x)^{(n-2)/2})$$

for some $c_0 > 0$.

Suppose this is not true. For any small $c > 0$, consider the set

$$D \equiv \left\{ \ x \in \mathbf{M} \mid h(x) \equiv cr^{-(n-2)/2} - u(x) > 0 \right\}.$$

Here $r = d(x)$. The set D is clearly nonempty by the above assumption. Moreover D is outside any given compact set if c is sufficiently small. This is due to the fact that u is positive everywhere.

Let $D \subset B(0, r_0)^c$. Here we choose r_0 so large that $r\Delta r \le (n/2) - \delta$ when $x \in D$. By direct computation, the following holds in the distribution sense:

$$\begin{aligned} &\Delta(cr^{(2-n)/2} - u) \\ &\ge r^{-1-(n/2)} \left(\frac{(n-2)}{2} \left(\frac{n}{2} - r\Delta r \right) c - \frac{(n-2)}{4(n-1)} R(x) u(x) r^{1+(n/2)} \right). \end{aligned}$$

Using $u(x) \le c/d(x)^{(n-2)/2}$ and our extra assumption in the remark, we see that

$$\Delta h \ge cr^{-1-(n/2)} \left(\frac{(n-2)}{2} \delta - \frac{(n-2)}{4(n-1)} R(x) r^2 \right) > 0.$$

This shows that h is subharmonic in D. However $h(x) > 0$ in D and $h(x) = 0$ in ∂D. The contradiction implies that D is empty when c is sufficiently small.

3. Proof of Theorem 1.2.

We will only show the proof in the Euclidean case since the manifold case is similar.

We will use the method of domain exhaustion. We divide the proof into three steps.

Step 1. We prove an a priori decay estimate for certain sub solutions u solving

$$\Delta u + W u^p \geq 0$$

in the weak sense. During this step, we assume that u satisfy $\int_{\mathbf{R}^n} W u^{p+1} dx < \infty$ and $\int_{\mathbf{R}^n} |\nabla u|^2 dx < \infty$.

Pick $x \in \mathbf{R}^n$ and let $R = |x|/2$. Throughout the section we make a change of the variables

$$y = x/R.$$

Write

$$u_1(y) = R^k u(Ry)$$

with $k = (n-2)/2$, we know that u_1 satisfies

$$\Delta u_1 + R^{2-(p-1)k} w_1 u_1^p \geq 0. \tag{3.1}$$

Here and later $W_1(y) = W(Ry)$ and the Δ in front u_1 is the Laplacian in y variable.

From (3.1), direct computation shows, for any $l \geq 1$

$$\Delta u_1^l + l R^{2-(p-1)k} W_1 u_1^{l+p-1} \geq 0. \tag{3.2}$$

Given any y_0 such that $|y_0| = 1$ and $s_0 > 0$, we wish to show that $u_1(y_0)$ is uniformly bounded when R is sufficiently large. Much of the remainder of the step is to prove this claim.

Let Ψ be a suitable cut-off function, by standard arguments we know that

$$\int_{B(y_0,\sigma_1)} |\nabla u_1^l|^2 dy \leq C \tau^{-2} l^2 R^{2-(k-1)p} \int_{B(y_0,\sigma_2)} W_1 u_1^{2l+p-1} dy, \tag{3.3}$$

where $\tau = \sigma_2 - \sigma_1$ and $0 < \sigma_1 < \sigma_2 < 1$.

Using Sobolev embedding, it is easy to see that

$$\int_{B_{\sigma_1}} u_1^{2l\theta} dy \leq C \left[R^{2-(p-1)k} l^2 \tau^{-2} \int_{B_{\sigma_2}} W_1 u_1^{2l+p-1} dy \right]^{\theta}, \tag{3.4}$$

where $\theta = n/(n-2)$ here and throughout this section. We will modify (3.4) so that a Moser iteration can be carried out.

From scaling relation between u and u_1, it is easy to see that

$$\int_{\mathbf{R}^n} u_1^{2n/(n-2)}(z') dz' = \int_{\mathbf{R}^n} u^{2n/(n-2)}(z) dz < \infty. \tag{3.5}$$

Using the scaling $x = Ry$, $W_1(y) = W(Ry)$ and $u_1(y) = R^k u(Ry)$, it is clear that

$$\int_{B(y_0,\sigma_2)} W_1(y) u_1^{p+1}(y) dy = R^{k(p+1)-n} \int_{B(x_0,\sigma_2 R)} W(x) u^{p+1} dx.$$

By the assumption at the beginning of the step

$$\int_{B(y_0,\sigma_2)} W_1(y)u_1^{p+1}(y)dy \le CR^{k(p+1)-n}. \tag{3.6}$$

Since $k=(n-2)/2$, one has $k(p+1)-n=-(2-k(p-1))$.

Now let us go back to (3.4). Take

$$q_1=\frac{2n}{(n-2)(p-1)},\quad q_1'=\frac{q_1}{q_1-1}=\frac{2n}{2n-(n-2)(p-1)},$$

and use Hölder's inequality, we know that

$$\int_{B(y_0,\sigma_1)} u_1^{2l\theta}dy \le C\left[R^{2-(p-1)k}l^2\tau^{-2}\int_{B_{\sigma_2}} W_1u_1^{2l+p-1}dy\right]^\theta$$
$$\le C\left[R^{2-(p-1)k}l^2\tau^{-2}\left(\int_{B_{\sigma_2}} W_1^{q_1'}u_1^{2lq_1'}dy\right)^{1/q_1'}\left(\int_{B_{\sigma_2}} u_1^{(p-1)q_1}dy\right)^{1/q_1}\right]^\theta.$$

Since $(p-1)q_1=2n/(n-2)$, by (3.5),

$$\int_{B_{\sigma_1}} u_1^{2l\theta}dy \le C\left[R^{2-(p-1)k}l^2(\sigma_2-\sigma_1)^{-2}\left(\int_{B_{\sigma_2}} W_1^{q_1'}u_1^{2lq_1'}dy\right)^{1/q_1'}\right]^\theta. \tag{3.7}$$

Recall that $W(x)\le C/(1+|x|^m)$. Hence $W_1(y)\le CR^{-m}$ when $|y-y_0|\le\sigma_2\le 1/2$. So (3.7) implies

$$\int_{B_{\sigma_1}} u_1^{2l\theta}dy \le C\left[R^{2-(p-1)k-m}l^2(\sigma_2-\sigma_1)^{-2}\left(\int_{B_{\sigma_2}} u_1^{2lq_1'}dy\right)^{1/q_1'}\right]^\theta.$$

By choosing $m=2-(p-1)k=2-((p-1)(n-2)/2)$, we see that

$$\int_{B_{\sigma_1}} u_1^{2l\theta}dy \le C\left[(\sigma_2-\sigma_1)^{-2}\left(\int_{B_{\sigma_2}} u_1^{2lq_1'}dy\right)^{1/q_1'}\right]^\theta. \tag{3.8}$$

If $p<\frac{n+2}{n-2}$, then

$$q_1'=\frac{q_1}{q_1-1}=\frac{2n}{2n-(n-2)(p-1)}<\frac{n}{n-2}=\theta.$$

Therefore we can use Moser's iteration on (3.8) to conclude that

$$u_1(y_0)\le C.$$

This is so because

$$\|u_1\|_{L^{2q_1'}(B_1)}\le C.$$

This shows

$$u(x) \le C/\left(1+|x|^{(n-2)/2}\right). \tag{3.9}$$

Step 2. We show that u has uniform exponential decay:

$$u(x) \le c_1 e^{-c_2|x|^{((2-b)/2)^2}}$$

for all x.

From last step we know that $u(x) \le \frac{C}{1+|x|^{(n-2)/2}}$. By the assumption on W, we see that

$$W(x)u^{p-1}(x,t) \le \frac{C}{1+|x|^{2-((p-1)(n-2)/2)}}\frac{C}{1+|x|^{(p-1)(n-2)/2}} \le \frac{C}{1+|x|^2}.$$

Since $V(x) \ge \frac{a}{1+|x|^b}$ with $b<2$, we see that

$$\begin{aligned}0 &= \Delta u(x) - V(x)u(x) + W(x)u^p(x)\\ &= \Delta u(x) - (V(x) - W(x)u^{p-1}(x))u(x)\\ &\le \Delta u(x) - \frac{c_0}{1+|x|^b}u(x)\end{aligned}$$

when $|x| \ge r$ for a large $r>0$. Here c_0 is a positive number.

Let Γ_1 be the Green's function of the elliptic operator $\Delta - \frac{c_0}{1+|x|^b}$. Note that $u(x,0)=0$ when $|x|$ is large. It is also clear that $\Gamma_1(x,0) \ge c(|x|) > 0$. Applying the maximum principle on the exterior of a sufficiently large cylinder centered at the origin, we know that

$$u(x) \le C\Gamma_1(x,0)$$

when $|x|$ and C are sufficiently large. By the upper bound of Γ_1 in [**Mu**] (when the leading operator is the Laplacian), [**Zh1**] (general case), we have

$$\Gamma_1(x,y) \le Ce^{-c(|x-y|/(1+|x|^{b/2}))^{(2-b)/2}}/|x-y|^{n-2}.$$

It follows that

$$u(x) \le c_1 e^{-c_2|x|^{((2-b)/2)^2}}$$

for all x.

Step 3. We use the method of domain exhaustion.

We begin by solving, for each $j>0$, the standard variational problem

$$\inf_{u\in W_0^{1,2}(B(0,j))}\int_{B(0,j)}(|\nabla u|^2 + Vu^2)dx \tag{3.10}$$

subject to the constraint $\int_{B(0,j)} W u^{p+1} dx = 1$. Let $u_j > 0$ be a solution to (3.10). Then there exists a $\lambda_j > 0$ such that

(3.11)
$$\Delta u_j(x) - V u_j(x) + \lambda_j W u_j^p(x) = 0, \ x \in B(0,j); \ u(x) = 0, \ x \in \partial B(0,j).$$

We claim that there exists a $\lambda > 0$ such that λ_j decreases to λ when $j \to \infty$. Clearly $\lambda_j < \lambda_{j'}$ if $j' > j$. When W satisfies the assumption in Theorem 1.2, we have, for any suitable u,

$$\int_{\mathbf{R}^n} W u^{p+1} dx \le C \int_{\mathbf{R}^n} \frac{1}{1+|x|^{2-((p-1)(n-2)/2)}} u^{p+1}(x) dx$$
$$= C \int_{\mathbf{R}^n} \frac{1}{1+|x|^{2-((p-1)(n-2)/2)}} u^{2-((p-1)(n-2)/2)} u^{p-1+((p-1)(n-2)/2)}(x) dx.$$

Using the inequality $ab \le C(a^m + b^{m'})$ with the exponents $m = 2/(2-((p-1)(n-2)/2))$ and $m' = m/(m-1) = 4/((p-1)(n-2))$, we have

$$\int_{\mathbf{R}^n} W u^{p+1} dx \le C \int_{\mathbf{R}^n} \frac{1}{1+|x|^2} u^2 dx + C \int_{\mathbf{R}^n} u^{(p-1+((p-1)(n-2)/2))m'}(x) dx$$
$$= C \int_{\mathbf{R}^n} \frac{1}{1+|x|^2} u^2 dx + C \int_{\mathbf{R}^n} u^{2n/(n-2)}(x) dx$$
$$\le C \int_{\mathbf{R}^n} (|\nabla u|^2 + V u^2) dx.$$

This shows that λ_j is bounded away from zero. The claim is proven.

Let u_j be a solution to (3.10). We extend the domain of u_j to the whole space by setting $u_j(x) = 0$ when x is outside of the ball $B(0,j)$. It is easy to verify that the extended function, still denoted by u_j, is a subsolution to (1.2) in the weak sense, i.e.,

$$\Delta u_j(x) - V u_j(x) + \lambda_j W u_j^p(x) \ge 0, \ x \in \mathbf{R}^n, \tag{3.12}$$

in the weak sense. Since

$$\left(\int_{\mathbf{R}^n} u_j^{2n/(n-2)} \right)^{(n-2)/n} \le C \int_{\mathbf{R}^n} |\nabla u_j|^2 = C\lambda_j \int_{\mathbf{R}^n} W u_j^{p+1} = C\lambda_j \le C\lambda_1$$

for all j, we can use Step 2 to conclude that

$$u_j(x) \le c_1 e^{-c_2 |x|^{((2-b)/2)^2}} \tag{3.13}$$

for all $|x| \ge R_0$ and j. Here R_0 is a sufficiently large number. If $|x| \le R_0$, by [**GS1**] or [**GS2**], $u_j(x)$ is uniformly bounded. Hence (3.13) actually holds for all x.

This shows that there exists $r_0 > 0$ such that

$$\int_{B(0,r_0)} W u_j^{p+1} dx = 1 - \int_{B(0,r_0)^c} W u_j^{p+1} dx \ge 1/2. \tag{3.14}$$

By the standard elliptic theory, u_j is uniformly bounded in $C^{2,\alpha}$ norm for some $\alpha > 0$. So a subsequence, still denoted by u_j, converges pointwise to a function u. By (3.14) and the unique continuation property, we know that $u(x) > 0$ for all $x \in \mathbf{R}^n$. This u is a positive solution to

$$\Delta u(x) - V u(x) + \lambda W u^p(x) = 0, \ x \in \mathbf{R}^n.$$

Now using an dilation of u, we can obtain a positive solution to (1.2). □

4. Existence of global solutions and energy estimates.

As indicated in [**F**] and [**Zh2**], the existence or nonexistence of global positive solutions to (1.3) is both strongly influenced by the exponent p and the potentials V and W. In this section we will show that (1.3) with bounded W possesses global positive solutions under the condition that $V(x) \geq c/(1+|x|^b)$ with $b < 2$ and $c > 0$. In fact, concerning the existence of global positive solutions, the condition on V is sharp in general. This means that if $0 \leq V(x) \leq C/(1+|x|^b)$ with $b > 2$, $W = 1$, then (1.3) has no global positive solutions if $1 < p < 1 + \frac{2}{n}$ (see [**Zh2**]).

We will also prove some energy estimate for global solutions which will be useful for the proof of Theorem 1.3. Those results in this section which overlap those in [**SZ**] or [**Zh2**] are presented here for completeness.

Let us also mention that all results in this section remain valid if Δ is replaced by an uniformly elliptic divergence operator with bounded measurable coefficients depending on x. In this more general case one needs the Green's function estimates in [**Zh1**] to begin with. While in the special case some comparison methods are sufficient ([**SZ**]).

We denote by $e^{t(\Delta - V)}$ the semigroup (on L^∞) associated with the linear part of the equation

$$u_t - \Delta u + V u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty). \tag{4.1}$$

Namely, for all $\phi \in L^\infty$, $u(x,t) = (e^{t(\Delta - V)}\phi)(x)$ denotes the unique solution of (4.1) with initial data ϕ. Also, we denote by Γ the Green's function of the operator $\Delta - V$ and for all suitable f, we put

$$\Gamma \star f(x) \equiv \int_{\mathbb{R}^n} \Gamma(x,y) f(y) dy = \int_0^\infty e^{t(\Delta - V)} f \, dt.$$

Given $k > 0$, we introduce a weighted space L^∞_k defined as

$$L^\infty_k = \{u \mid u(.) \in L^\infty(\mathbf{R}^n), (1+|x|)^k u(x) < \infty\}.$$

The norm of this space is given by $\|u\|_{\infty,k} = \sup_x (1+|x|)^k |u(x)|$.

We will use $T(u_0)$ to denote the maximum time of existence of the solution to (1.3), which may also denoted by $u(., u_0)(.)$.

Proposition 4.1. *Suppose $V(x) \ge \frac{a}{1+|x|^b}$ with $b \in [0,2)$, $a > 0$ and let $k \ge 0$.*

There exists $C \ge 1$ such that for all $\phi \in L^\infty_k$,

$$\|e^{t(\Delta - V)}\phi\|_{\infty,k} \le C\|\phi\|_{\infty,k}, \quad t \ge 0. \tag{4.2}$$

Proof. By Theorem 1.1 in [**Zh1**], we have

$$G(x,t;y,0) \le c_1 \frac{e^{-c_2[t^{1/2}/(1+|x|^{b/2})]^\alpha} e^{-c_2[t^{1/2}/(1+|y|^b)]^\alpha}}{t^{n/2}} \; e^{-c_3|x-y|^2/t}$$

with $\alpha = (2-b)/2$.

Given $f = f(x)$, we write

$$\begin{aligned} G \star f(x,t) = &\int_{|y|\ge|x|/2} G(x,t;y,0)f(y)dy \\ &+ \int_{|y|\le|x|/2} G(x,t;y,0)f(y)dy \equiv J_1 + J_2. \end{aligned}$$

Clearly

$$J_1 \le \frac{C}{1+|x|^k}\int \frac{e^{-c_3|x-y|^2/t}}{t^{n/2}}dy \le \frac{C}{1+|x|^k}.$$

When $|y| \le |x|/2$, one has $|x-y| \ge |x|/2 \ge |y|$. Therefore

$$J_2 \le \frac{C}{t^{n/2}} \int_{|y|\le|x|/2} e^{-c_2[t^{1/2}/(1+|x|^{b/2})]^\alpha} e^{-c|x|^2/t} e^{-c|x-y|^2/t} dy.$$

Here $c > 0$ is chosen sufficiently small. If $|x| \le 1$, then obviously $J_2 \le C$. So we can assume that $|x| \ge 1$. Direct computation shows that

$$[t^{1/2}/(1+|x|^{b/2})]^\alpha + |x|^2/t \ge |x|^\theta$$

for some $\theta > 0$ and all $t > 0$. Hence

$$J_2 \le Ce^{-|x|^\theta}.$$

Combining this with the estimate on J_1 completes the proof. □

Proposition 4.2. *Suppose $V(x) \ge \frac{a}{1+|x|^b}$ with $b \in [0,2)$, $a > 0$. Suppose $0 \le f(x) \le 1/(1+|x|^k)$ for some $k \ge 0$, then*

$$\Gamma \star f(x) \equiv \int_{\mathbb{R}^n} \Gamma(x,y)f(y)dy \le \frac{C(1+|x|^b)}{1+|x|^k}.$$

Proof. According to [**Mu**] or Corollary 1 in [**Zh1**], under the assumptions in the proposition, there exist positive constants c_1, c_2 such that, for all x, y and $\alpha = (2-b)/2$,

$$\Gamma(x,y) \le c_1 e^{-c_2[|x-y|/(1+|x|^{b/2})]^\alpha} e^{-c_2[|x-y|/(1+|y|^{b/2})]^\alpha} \frac{1}{|x-y|^{n-2}}. \tag{4.3}$$

When $|x| \leq 1$ and $|y| \geq 2$, we have

$$\Gamma(x,y) \leq e^{-c|y|^{(1-(b/2))^2}} \frac{C}{|x-y|^{n-2}}.$$

Hence $\Gamma \star f(x) \leq C$ when $|x| \leq 1$.

In order to estimate $\Gamma \star f(x)$ when $|x| \geq 1$, let us write

(4.4)

$$\begin{aligned}\Gamma \star f(x) &= \int_{|y|\leq|x|/2} \Gamma(x,y)f(y)dy + \int_{|y|\geq|x|/2} \dots dy + \int_{|x|/2\leq|y|\leq 2|x|} \dots dy \\ &\equiv I_1 + I_2 + I_3.\end{aligned}$$

When $|y| \leq |x|/2$, one has $|x-y| \geq |x| - |y| \geq |x|/2 \geq |y|$. Hence

$$\begin{aligned} e^{-c_2[|x-y|/(1+|x|^{b/2})]^\alpha} &\leq e^{-c_2|x|^{(1-(b/2))(2-b)/2}}, \\ e^{-c_2[|x-y|/(1+|y|^{b/2})]^\alpha} &\leq e^{-c_2|y|^{(1-(b/2))(2-b)/2}}. \end{aligned}$$

It follows that

$$\begin{aligned} I_1 &\leq Ce^{-c_2|x|^{(1-(b/2))(2-b)/2}} \int_{\mathbb{R}^n} e^{-c_2|y|^{(1-(b/2))(2-b)/2}} \Big/ |x-y|^{n-2} dy \\ &\leq Ce^{-c_2|x|^{(1-(b/2))(2-b)/2}}. \end{aligned} \tag{4.5}$$

Similarly

$$I_2 \leq Ce^{-c_2|x|^{(1-(b/2))(2-b)/2}}. \tag{4.6}$$

Next let us estimate I_3. Since $|x|/2 \leq |y| \leq 2|x|$ in this case, we have $f(y) \leq C/(1+|x|^k)$. Hence

$$\begin{aligned} I_3 &\leq \frac{C}{1+|x|^k} \int_{|x|/2\leq|y|\leq 2|x|} \Gamma(x,y)dy \\ &\leq \frac{C}{1+|x|^k} \int_{|x|/2\leq|y|\leq 2|x|} \frac{e^{-c_2[|x-y|/(1+|x|^{b/2})]^\alpha}}{|x-y|^{n-2}} dy. \end{aligned}$$

Take the substitution $y' = y/|x|$, we obtain, as $|x| \geq 1$,

$$I_3 \leq \frac{C}{1+|x|^k}(1+|x|)^2 \int_{1/2\leq|y'|\leq 2} e^{-c|x|^{(1-(b/2))^2}|(x/|x|)-y'|^{(2-b)/2}} \frac{dy'}{|(x/|x|)-y'|^{n-2}}.$$

Note that $|x/|x|| = 1$, so if we let $r = |(x/|x|) - y'|$, then

$$I_3 \leq \frac{C}{1+|x|^k}(1+|x|)^2 \int_0^3 e^{-c|x|^{(1-(b/2))^2} r^{(2-b)/2}} r dr.$$

Letting $r_1 = |x|^{(1-(b/2))}r$, we have

$$I_3 \leq \frac{C}{1+|x|^k}(1+|x|)^2 \int_0^\infty e^{-cr_1^{(2-b)/2}} r_1 dr_1 \frac{1}{|x|^{2-b}},$$

i.e.,

$$I_3 \leq \frac{C(1+|x|^b)}{1+|x|^k}. \tag{4.7}$$

The lemma is proven by combining (3.3)-(3.5). □

The result in the next proposition is known. But we give a sketch of the proof for the sake of completeness.

Proposition 4.3. *Suppose* $\frac{a}{1+|x|^b} \leq V(x) \leq C_1$ *with* $b \in [0,2)$, $a > 0$. *Suppose* $W \in L^\infty(\mathbf{R}^n)$ *and* $1 < p < \frac{n+2}{n-2}$. *Then the following conclusions hold:*

(i) *For any compactly supported nonnegative* $f \neq 0$, *there exists* $\lambda_0 > 0$ *such that the problem*

$$\begin{cases} \Delta u - V(x)u + W(x)u^p - \partial_t u = 0 & in\ \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) \equiv \lambda f(x), \end{cases}$$

has a global positive solution when $\lambda \in (0, \lambda_0)$;

(ii) *moreover the* $\omega-$ *limit set contains a equilibrium solution;*

(iii) *all global positive solutions are bounded in* $D \times (c, \infty)$. *Here* $c > 0$ *and* D *is any compact domain.*

Proof. (i) The existence of global solutions for small initial data is a simple consequence of Propositions 4.1 and 4.2 together with a fixed point argument. If one is restricted to (1.3) only, then a comparison method also yields the result (see [**SZ**]). However the current method has the advantage that it covers the case when Δ in (1.3) is replaced by a uniformly elliptic operator with bounded measurable coefficients. We refer to [**Zh2**] for details of the proof.

(ii) Next us recall some well-known facts related to the existence of an energy functional for Equation (1.3).

For $u_0 \in L^\infty \cap H^1$ it is well-known that $u \in C([0, T(u_0)); H^1)$ and that the energy $E(t)$, defined as

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^n} V u^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^n} W u^{p+1}\, dx$$

satisfies the identity

$$E(0) - E(t) = \int_0^t \int_{\mathbb{R}^n} |u_t(x,s)|^2\, dx\, ds.$$

We will use the following two classical lemmas.

Lemma 4.1. *Let $u_0 \in L^\infty \cap H^1$. If $T(u_0) = \infty$, then $E(t) \geq 0$ for all $t \geq 0$, hence in particular*

$$\int_0^\infty \int |u_t(x,s)|^2 \, dx \, ds \leq E(0) \leq C\|u_0\|_{H^1}^2.$$

Proof. This is a consequence of the classical concavity argument of Levine (see [**Le1**]). □

Lemma 4.2. *Let $u_0 \in L^\infty \cap H^1$. If $T(u_0) = \infty$, then the ω–limit set $\omega(u_0)$ consists of equilibria (i.e., of solutions of the corresponding elliptic equation).*

Proof. Assume $u(t_j) \to v$ in L_k^∞ and fix $t > 0$. By continuous dependence of solutions of (1.3) over initial data in L^∞, it follows that $u(t+t_j) \to S(t)v$ in L^∞. For each $R > 0$, we have

$$\begin{aligned}\int_{|x|<R} |u(x,t+t_j) - u(x,t_j)|^2 \, dx &\leq C(R) \int_{t_j}^{t+t_j} \int_{|x|<R} |u_t(x,s)|^2 \, dx \, ds \\ &\leq C(R) \int_{t_j}^{\infty} \int_{\mathbb{R}^n} |u_t(x,s)|^2 \, dx \, ds.\end{aligned}$$

Since the RHS goes to 0 as $j \to \infty$ in view of Lemma 4.1, we deduce that

$$\int_{|x|<R} |(S(t)v)(x) - v(x)|^2 \, dx = 0,$$

hence $S(t)v \equiv v$ for all $t > 0$, which means that v is an equilibrium. This proves the lemma and Part (ii) of the proposition.

(iii) This follows from the scaling argument in [**Gi**]. □

The following result, essentially given in [**S**], is important for the proof of Theorem 1.3:

Proposition 4.4. *Let $u_0 \in L^\infty \cap H^1$ and assume that $T(u_0) = \infty$. Then the following holds:*

$$\frac{1}{T} \int_h^{T+h} \int_{\mathbb{R}^n} W u^{p+1}(x,t) dx dt \leq C(\|u_0\|_{H^1}), \quad T \geq 1, \quad h \geq 0.$$

Proof. Without loss of generality we take $h = 0$. We use an energy argument from [**S**, Theorem 2] (given there for $V \equiv 0$). Let $f(t) \equiv \int_{\mathbb{R}^n} u^2(x,t) \, dx$, then by Lemma 4.1,

$$
\begin{aligned}
f(t) - f(0) &= 2\int_0^t \int_{\mathbb{R}^n} u u_s \\
&\le 2\left(\int_0^t \int_{\mathbb{R}^n} u_s^2 \, dx\, ds\right)^{1/2} \left(\int_0^t \int_{\mathbb{R}^n} u^2 \, dx\, ds\right)^{1/2} \\
&\le 2E(0)^{1/2} \left(\int_0^t f(s)\, ds\right)^{1/2}.
\end{aligned}
$$

This easily implies

$$
f(t) \le C(E(0))(f(0)+t) \le C(\|u_0\|_{H^1})(t+1), \quad t \ge 0.
$$

Multiplying both sides of (1.3) by u and integrating, we obtain, for $T > 0$,

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} W u^{p+1}(x,t) dxdt \\
&= \int_0^T \int_{\mathbb{R}^n} (|\nabla u(x,t)|^2 + V(x) u^2(x,t)) dxdt + \frac{1}{2}\int_{\mathbb{R}^n} (u^2(x,T) - u^2(x,0)) dx \\
&= 2\int_0^T E(t) dt + \frac{2}{p+1}\int_0^T \int_{\mathbb{R}^n} W u^{p+1}(x,t) dxdt + \frac{1}{2}(f(T) - f(0)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{1}{T}\int_0^T \int_{\mathbb{R}^n} W u^{p+1}(x,t) dxdt \\
&\le \frac{p+1}{p-1}\left(\frac{f(T)-f(0)}{2T} + \frac{2}{T}\int_0^T E(t)dt\right) \\
&\le \frac{p+1}{p-1}\left(\frac{f(T)}{2T} + 2E(0)\right) \le C(\|u_0\|_{H^1}), \quad T \ge 1.
\end{aligned}
$$

□

5. Proof of Theorem 1.3.

Again we will only give a proof of the Euclidean case, which is divided into several steps.

Step 1. Let u solves

$$
\Delta u + W u^p - u_t \ge 0.
$$

Pick $x \in \mathbf{R}^n$ and let $R = |x|/2$. Throughout the section we make a change of the variables

$$
y = x/R, \quad s = t/R^2.
$$

Write

$$
u_1(y,s) = R^k u(Ry, R^2 s)
$$

with $k = (n-2)/2$, we know that u_1 satisfies

$$\Delta u_1 + R^{2-(p-1)k} w_1 u_1^p - \partial_s u_1 \geq 0. \tag{5.1}$$

Here and later $W_1(y) = w(Ry)$ and the Δ in front u_1 is the Laplacian in y variable.

From (5.1), direct computation shows, for any $l \geq 1$

$$\Delta u_1^l + lR^{2-(p-1)k} W_1 u_1^{l+p-1} - \partial_s u_1^l \geq 0. \tag{5.2}$$

Given any y_0 such that $|y_0| = 1$ and $s_0 > 0$, we wish to show that $u_1(y_0, s_0)$ is uniformly bounded when R is sufficiently large. Much of the remainder of the step is to prove this claim.

For a $\sigma \in (0,1)$, write

$$Q_\sigma = \{y \mid |y - y_0| \leq \sigma\} \times [s_0 - \sigma^2, s_0].$$

Since the support of u_0 is compact, the support of $u_1(.,0)$ is contained in a ball centered at 0 with radius of the order c/R. When $s < 0$ and y is outside the support of $u_1(.,0)$, we define $u_1(y,s) = 0$. In this way u_1 satisfies (5.2) in Q_σ when R is sufficiently large.

Let Ψ be a suitable cut-off function, by standard arguments we know that

$$\begin{aligned} \sup_{s_0 - \sigma_1^2 \leq s \leq s_0} \int_{|y-y_0| \leq \sigma_1} u_1^{2l}(y,s)dy + c \int_{Q_{\sigma_1}} |\nabla u_1^l|^2 dyds \\ \leq C\tau^{-2} l^2 R^{2-(k-1)p} \int_{Q_{\sigma_2}} W_1 u_1^{2l+p-1} dyds, \end{aligned} \tag{5.3}$$

where $\tau = \sigma_2 - \sigma_1$ and $0 < \sigma_1 < \sigma_2 < 1$.

Using Sobolev embedding and Hölder's inequality, it is easy to see that

$$\int_{B(y_0,r)} f^{2(1+(2/n))} dy \leq C \left(\int_{B(y_0,r)} f^2 dy \right)^{2/n} \left[\int_{B(y_0,r)} (|\nabla f|^2 + r^{-2} f^2) dy \right].$$

Using (5.3) and the above, we see that

$$\begin{aligned} \int_{Q_{\sigma_1}} u_1^{2l\theta} dyds \leq C \left[R^{2-(p-1)k} l^2 \tau^{-2} \int_{Q_{\sigma_2}} W_1 u^{2l+p-1} dyds \right] \\ \cdot \sup_{s_0 - \sigma_1^2 \leq s \leq s_0} \left(\int_{|y-y_0| \leq \sigma_1} u_1^{2l}(y,s) dy \right)^{2/n}. \end{aligned}$$

Here and later $\theta = 1 + (2/n)$. It follows that

$$\int_{Q_{\sigma_1}} u_1^{2l\theta} dyds \leq C \left[R^{2-(p-1)k} l^2 \tau^{-2} \int_{Q_{\sigma_2}} W_1 u_1^{2l+p-1} dyds \right]^\theta. \tag{5.4}$$

We will modify (5.4) so that a Moser iteration can be carried out.

From the energy estimate in Section 4, Proposition 4.4, we know that

$$\frac{1}{R^2}\int_{t-R^2}^{t}\int_{\mathbf{R}^n} W u^{p+1} \le C.$$

Since $E(t) = \frac{1}{2}\int_{\mathbf{R}^n} |\nabla u|^2 + \frac{1}{2}\int_{\mathbf{R}^n} V u^2 - \frac{1}{p+1}\int_{\mathbf{R}^n} W u^{p+1}$ and $E(t)$ is nonincreasing, we see that

$$\begin{aligned}(5.5)\quad \frac{1}{R^2}\int_{t-R^2}^{t}\int_{\mathbf{R}^n} (|\nabla u|^2 + V u^2) &\le \frac{2}{p+1}\frac{1}{R^2}\int_{t-R^2}^{t}\int_{\mathbf{R}^n} W u^{p+1} + 2E(t)\\ &\le C(E(0)).\end{aligned}$$

By Sobolev embedding, the above implies

$$(5.6)\qquad \frac{1}{R^2}\int_{t-R^2}^{t}\left(\int_{\mathbf{R}^n} u^{2n/(n-2)}\right)^{(n-2)/n} \le C(E(0)).$$

From scaling relation between u and u_1, it is easy to see that

$$\int_{\mathbf{R}^n} u^{2n/(n-2)}(z,t)dz = \int_{\mathbf{R}^n} u_1^{2n/(n-2)}(z', t/R^2)dz'.$$

Hence

$$(5.7)\qquad \int_{s-1}^{s}\left(\int_{\mathbf{R}^n} u_1^{2n/(n-2)}\right)^{(n-2)/n} \le C(E(0)).$$

Next

$$\begin{aligned}(5.8)\quad &\int_{Q_{\sigma_1}} u_1^{2(1+(2/n))}\\ &\le \int_{s_0-\sigma_1^2}^{s_0}\left(\int_{B(y_0,\sigma_1)} u_1^{2n/(n-2)}\right)^{(n-2)/n}\left(\int_{B(y_0,\sigma_1)} u_1^2\right)^{2/n}\\ &\le \sup_{s\in(s_0-\sigma_1^2,s_0)}\left(\int_{B(y_0,\sigma_1)} u_1^2\right)^{2/n}\int_{s_0-\sigma_1^2}^{s_0}\left(\int_{B(y_0,\sigma_1)} u_1^{2n/(n-2)}\right)^{(n-2)/n}.\end{aligned}$$

Combining (5.7) and (5.8) one obtains

$$(5.9)\qquad \int_{Q_{\sigma_1}} u_1^{2(1+(2/n))} \le C(E(0)) \sup_{s\in(s_0-\sigma_1^2,s_0)}\left(\int_{B(y_0,\sigma_1)} u_1^2\right)^{2/n}.$$

We would like to find an upper bound for the right-hand side of (5.9). To this end we take $l = 1$ in (5.3) to get

$$(5.10)\quad \sup_{s_0-\sigma_1^2\le s\le s_0}\int_{|y-y_0|\le\sigma_1} u_1^2(y,s)dy \le C\tau^{-2}R^{2-(k-1)p}\int_{Q_{\sigma_2}} W_1 u_1^{p+1}dyds$$

where $\tau = \sigma_2 - \sigma_1$. Using the scaling $t = sR^2$, $x = Ry$, $W_1(y) = W(Ry)$ and $u_1(y,s) = R^k u(Ry, R^2 s)$, it is clear that

$$\int_{Q_{\sigma_2}} W_1(y) u_1^{p+1}(y,s)dyds = R^{k(p+1)-n-2} \int_{t_0-\sigma_2^2 R^2}^{t_0} \int_{B(x_0,\sigma_2 R)} W(x) u^{p+1} dxdt.$$

By the energy estimate Proposition 4.4 again

$$\int_{Q_{\sigma_2}} W_1(y) u_1^{p+1}(y,s)dyds \le R^{k(p+1)-n} C(E(0)). \tag{5.11}$$

Since $k = (n-2)/2$, one has $k(p+1) - n = -(2 - k(p-1))$. Taking $\sigma_2 = 2$ and $\sigma_1 \le 1$, by (5.10) and (5.11), we deduce

$$\sup_{s_0-\sigma_1^2 \le s \le s_0} \int_{|y-y_0|\le\sigma_1} u_1^2(y,s)dy \le C(E(0))(2-\sigma_1)^{-2} \le C(E(0)). \tag{5.12}$$

Substituting (5.12) to (5.9), we obtain

$$\int_{Q_{\sigma_1}} u_1^{2(1+(2/n))} dyds \le C(E(0)) \tag{5.13}$$

for any $\sigma_1 \in (0,1)$.

Now let us go back to (5.4). Take

$$q_1 = \frac{2(n+2)}{n(p-1)}, \quad q_1' = \frac{q_1}{q_1-1} = \frac{2(n+2)}{3n-np+4},$$

and use Hölder's inequality, we know that

$$\int_{Q_{\sigma_1}} u_1^{2l\theta} dyds \le C\left[R^{2-(p-1)k} l^2 \tau^{-2} \int_{Q_{\sigma_2}} W_1 u_1^{2l+p-1} dyds\right]^\theta$$
$$\le C\left[R^{2-(p-1)k} l^2 \tau^{-2} \left(\int_{Q_{\sigma_2}} W_1^{q_1'} u_1^{2lq_1'} dyds\right)^{1/q_1'} \left(\int_{Q_{\sigma_2}} u_1^{(p-1)q_1} dyds\right)^{1/q_1}\right]^\theta.$$

Since $(p-1)q_1 = 2(1+\frac{2}{n})$, by (5.13),

$$\int_{Q_{\sigma_1}} u_1^{2l\theta} dyds \le C\left[R^{2-(p-1)k} l^2 (\sigma_2-\sigma_1)^{-2} \left(\int_{Q_{\sigma_2}} W_1^{q_1'} u_1^{2lq_1'}\right)^{1/q_1'}\right]^\theta. \tag{5.14}$$

Recall that $W(x) \le C/(1+|x|^m)$. Hence $W_1(y) \le CR^{-m}$ when $|y-y_0| \le \sigma_2 \le 1/2$. So (5.14) implies

$$\int_{Q_{\sigma_1}} u_1^{2l\theta} dyds \le C\left[R^{2-(p-1)k-m} l^2 (\sigma_2-\sigma_1)^{-2} \left(\int_{Q_{\sigma_2}} u_1^{2lq_1'} dyds\right)^{1/q_1'}\right]^\theta.$$

By choosing $m = 2 - (p-1)k = 2 - ((p-1)(n-2)/2)$, we see that

$$\int_{Q_{\sigma_1}} u_1^{2l\theta} dyds \le C \left[(\sigma_2 - \sigma_1)^{-2} \left(\int_{Q_{\sigma_2}} u_1^{2lq_1'} \right)^{1/q_1'} \right]^{\theta}. \tag{5.15}$$

If $p < 1 + \frac{4}{n}$, then

$$q_1' < \frac{2(n+2)}{3n-(n+4)+4} = \frac{n+2}{n} = \theta.$$

Therefore we can use Moser's iteration on (5.15) to conclude that $u_1(y_0, s) \le C$. Using (5.7), this shows

$$u(x,t) \le C/(1+|x|^{(n-2)/2}). \tag{5.16}$$

Step 2. We show that u has uniform exponential decay.

From last step we know that $u(x) \le \frac{C}{1+|x|^{(n-2)/2}}$. By the assumption on W, we see that

$$W(x)u^{p-1}(x,t) \le \frac{C}{1+|x|^{2-((p-1)(n-2)/2)}} \frac{C}{1+|x|^{(p-1)(n-2)/2}} \le \frac{C}{1+|x|^2}.$$

Since $V(x) \ge \frac{a}{1+|x|^b}$ with $b < 2$, we see that

$$\begin{aligned} 0 &= \Delta u(x,t) - V(x)u(x,t) + W(x,t)u^p(x,t) - u_t(x,t) \\ &= \Delta u(x,t) - (V(x) - W(x)u^{p-1}(x,t))u(x,t) - u_t(x,t) \\ &\le \Delta u(x,t) - \frac{c_0}{1+|x|^b} u(x,t) - u_t(x,t) \end{aligned}$$

when $|x| \ge r$ for a large $r > 0$. Here c_0 is a positive number.

Let Γ_1 be the Green's function of the elliptic operator $\Delta - \frac{c_0}{1+|x|^b}$. Note that $u(x,0) = 0$ when $|x|$ is large. It is also clear that $\Gamma_1(x,0) \ge c(|x|) > 0$. Applying the maximum principle on the exterior of a sufficiently large cylinder centered at the origin, we know that

$$u(x,t) \le C\Gamma_1(x,0)$$

when $|x|$ and C are sufficiently large. By the upper bound of Γ_1, as in Section 3, it follows that

$$u(x,t) \le c_1 e^{-c_2|x|^{((2-b)/2)^2}}$$

for all x, t.

Step 3. Define

$$D_0 = \left\{ u_0 \in L_k^\infty; \;\; T(u_0) = \infty \text{ and } u(t; u_0) \to 0 \text{ in } L_k^\infty \text{ as } t \to \infty \right\}.$$

By the exponential decaying property of the fundamental solution of $\Delta - V$ (Proposition 4.3), it follows that D_0 contains an open neighborhood W_0 of 0 in L_k^∞ and that

$$D_0 = \{u_0 \in L_k^\infty; \ \ T(u_0) = \infty \text{ and } 0 \in \omega_k(u_0)\}.$$

We claim that D_0 is open in L_k^∞. Indeed, if $u_0 \in D_0$, there exists $t > 0$ such that $u(t; u_0) \in W_0$. But by continuous dependence of solutions of (1.1) in L_k^∞, if $\|\overline{u}_0 - u_0\|_{\infty,k}$ is sufficiently small, then $u(t; \overline{u}_0) \in W_0 \subset D_0$, so that $\overline{u}_0 \in D_0$. The claim follows.

Let now

$$\lambda^* = \sup\{\lambda > 0; \ \lambda\phi \in D_0\}.$$

We have just seen that $\lambda\phi \in D_0$ when $\lambda > 0$ is small, and it is well-known that $T(\lambda\phi) < \infty$ if λ is large (see [**Le2**] for example). Therefore, $0 < \lambda^* < \infty$.

Let $\lambda_j \uparrow \lambda^*$ with $\lambda_j\phi \in D_0$. By standard scaling method, we have, for any bounded domain D,

$$\sup_{t\geq 0} \|u(t; \lambda_j\phi)|_D\|_\infty \leq C(\lambda_j(\|\phi\|_{H^1} + \|\phi\|_\infty), D) \leq C(D), \quad j = 1, 2, \dots.$$

Since by continuous dependence in L_k^∞, we have, for each $t \in [0, T(\lambda^*\phi))$,

$$\|u(t; \lambda^*\phi)|_D\|_\infty = \lim_j \|u(t; \lambda_j\phi)|_D\|_\infty \leq C(D),$$

it follows that $T(\lambda^*\phi) = \infty$.

On the other hand, by the openness of D_0, $\lambda^*\phi \notin D_0$. We claim that $\omega(\lambda^*\phi)$ contains a nontrivial nonnegative equilibrium v. Suppose the claim is false. Then by Step 2, there exists a sequence $\{t_j\}$ with $t_j \to \infty$, such that $\{u(t_j, \lambda^*\phi)\}$ is compact in L_k^∞ norm. Moreover a subsequence would converge to 0 in L_k^∞ norm. Hence $\|u(t_j, \lambda^*\phi)\|_{\infty,k}$ would be sufficiently small when j is large. But this would imply that $\lambda^*\phi \in D_0$. This contradiction validates the claim. The strong maximum principle finally implies that $v > 0$ in $\mathbb{R}^n$. The proof is complete. □

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References

[Au] T. Aubin, *The scalar curvature*, Differential geometry and relativity, Mathematical Phys. and Appl. Math., **3**, Reidel, Dordrecht, 1976, 5-18, MR 55 #6476, Zbl 0345.53029.

[Au2] ———, *A Course in Differential Geometry*, Graduate Studies in Mathematics, **27**, American Mathematical Society, Providence, RI, 2001, MR 2001m:53001, Zbl 0966.53001.

[AM] P. Aviles and R. McOwen, *Conformal deformations of complete manifolds with negative curvature*, J. Differential Geom., **21**(**2**) (1985), 269-281, MR 87e:53058, Zbl 0588.53028.

[BL] H. Berestycki and P.L. Lions, *Nonlinear scalar field equations* I,II, Arch. Rat. Mech. Anal., **82** (1983), 313-346, 347-376, MR 84h:35054a, Zbl 0533.35029, MR 84h:35054b, Zbl 0556.35046.

[BN] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., **36**(**4**) (1983), 437-477, MR 84h:35059, Zbl 0541.35029.

[BJP] J. Busca, M.A. Jendoubi and P. Polacik, *Convergence to equilibrium for semilinear parabolic problems in* $\mathbf{R}^n$. Preprint, 2000.

[CW] T. Cazenave and F.B. Weissler, *Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations*, Math. Z., **228** (1998), 83-120, MR 99d:35149, Zbl 0916.35109.

[CDE] C. Cortézar, M. del Pino and M. Elgueta, *The problem of uniqueness of the limit in a semilinear heat equation*, Comm. Partial Differential Equations, **24**(**11**-**12**) (1999), 2147-2172, MR 2000k:35125, Zbl 0940.35107.

[DL] K. Deng and H.A. Levine, *The role of critical exponents in blowup theorems, the sequel*, J. Math. Anal. Appl., **243** (2000), 85-126, MR 2001b:35031, Zbl 0942.35025.

[DN] W.Y. Ding and W.M. Ni, *On the existence of positive entire solutions of a semilinear elliptic equation*, Arch. Rat. Mech. Anal., **91** (1986), 283-308, MR 87b:35056, Zbl 0616.35029.

[Eg] H. Egnell, *Asymptotic results for finite energy solutions of semilinear elliptic equation*, J. Differential Equations, **98** (1992), 34-56, MR 93k:35101, Zbl 0778.35009.

[E] J.F. Escobar, *Positive solutions for some semilinear elliptic equations with critical Sobolev exponents*, Comm. Pure Appl. Math., **40**(**5**) (1987), 623-657, MR 88k:35013, Zbl 0635.35033.

[ES] J.F. Escobar and R.M. Schoen, *Conformal metrics with prescribed scalar curvature*, Invent. Math., **86**(**2**) (1986), 243-254, MR 88b:58030, Zbl 0628.53041.

[FP] E. Feireisl and H. Petzeltova, *Convergence to a ground state as a threshold phenomenon in nonlinear parabolic equations*, Differential Integral Equations, **10** (1997), 181-196, MR 97i:35070, Zbl 0879.35023.

[F] H. Fujita, *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. I, **13** (1966), 109-124, MR 35 #5761, Zbl 0163.34002.

[GS1] B. Gidas and J. Spruck, *Global and local behaviour of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math., **34** (1981), 525-598, MR 83f:35045, Zbl 0465.35003.

[GS2] ———, *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. Partial Differential Equations, **6**(**8**) (1981), 883-901, MR 82h:35033, Zbl 0462.35041.

[Gi] Y. Giga, *A bound for global solutions of semilinear heat equations*, Comm. Math. Phys., **103** (1986), 415-421, MR 87j:35187, Zbl 0595.35057.

[He] E. Hebey, *Optimal Sobolev inequalities on complete Riemannian manifolds with Ricci curvature bounded below and positive injectivity radius*, Amer. J. Math., **118**(**2**) (1996), 291-300, MR 97c:53058, Zbl 0863.53031.

[Ho] D. Holcman, *Prescribed scalar curvature problem on complete manifolds*, J. Math. Pures Appl. (9), **80**(**2**) (2001), 223–244, MR 2002a:53047.

[Jin] Z.R. Jin, *A counterexample to the Yamabe problem for complete noncompact manifolds*, Partial differential equations (Tianjin, 1986), 93–101, Lecture Notes in Math., **1306**, Springer, Berlin, 1988, MR 91a:53065, Zbl 0648.53022.

[Ka] N. Kawano, *On bounded entire solutions of semilinear elliptic equations*, Hiroshima Math. J., **14**(**1**) (1984), 125-158, MR 86b:35052, Zbl 0555.35043.

[Kaz] J. Kazdan, *Prescribing the Curvature of a Riemannian Manifold*, Amer. Math. Soc. Providence, RI, 1985, MR 86h:53001, Zbl 0561.53048.

[KN] C.E. Kenig and W.-M. Ni, *An exterior Dirichlet problem with applications to some nonlinear equations arising in geometry*, Amer. J. Math., **106**(**3**) (1984), 689–702, MR 85j:35072, Zbl 0559.35025.

[Ki] S. Kim, *Scalar curvature on noncompact complete Riemannian manifolds*, Nonlinear Anal., **26**(**12**) (1996), 1985-1993, MR 97a:53056, Zbl 0858.53029.

[Ki2] ______, *The Yamabe problem and applications on noncompact complete Riemannian manifolds*, Geom. Dedicata, **64**(**3**) (1997), 373-381, MR 98c:53052, Zbl 0878.53037.

[KP] I. Kuzin and S. Pohozaev, *Entire Solutions of Semilinear Elliptic Equations*, Birkhäuser, 1997, MR 99d:35050, Zbl 0882.35003.

[LP] J.M. Lee and T.H. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc., **17** (1987), 37-91, MR 88f:53001, Zbl 0633.53062.

[Le1] H.A. Levine, *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$*, Arch. Rat. Mech. Anal., **51** (1973), 371-386, MR 50 #714, Zbl 0278.35052.

[Le2] ______, *The role of critical exponents in blowup theorems*, SIAM Review, **32** (1990), 262-288, MR 91j:35135, Zbl 0706.35008.

[LTY] P. Li, L.-F. Tam and D.-G. Yang, *On the elliptic equation $\Delta u + ku - Ku^p = 0$ on complete Riemannian manifolds and their geometric applications*, Trans. Amer. Math. Soc., **350**(**3**) (1998), 1045-1078, MR 98e:58175, Zbl 0896.58059.

[Li] Y. Li, *Remarks on a semilinear elliptic equation on R^n*, J. Differential Equations, **74**(**1**) (1988), 34-49, MR 89h:35112, Zbl 0662.35038.

[Lin] F.-H. Lin, *On the elliptic equation $D_i[a_{ij}(x)D_jU] - k(x)U + K(x)U^p = 0$*, Proc. Amer. Math. Soc., **95**(**2**) (1985), 219-226, MR 86k:35041, Zbl 0584.35031.

[Lio1-2] P.L. Lions, *The concentration compactness principle in the calculus of variation. The locally compact case*, Part I, II, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, **1** (1984), 109-145, 223-283, MR 87e:49035a, Zbl 0541.49009, MR 87e:49035b, Zbl 0704.49004.

[Mu] M. Murata, *Structure of positive solutions to $(-\Delta + V)u = 0$ in R^n*, Duke Math. J., **53**(**4**) (1986), 869-943, MR 88f:35039, Zbl 0624.35023.

[Ni] W.M. Ni, *On the elliptic equation $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$, its generalizations, and applications in geometry*, Indiana Univ. Math. J., **31**(**4**) (1982), 493-529, MR 84e:35049, Zbl 0496.35036.

[NST] W.M. Ni, P. Sacks and J. Tavantzis, *On the asymptotic behavior of solutions of certain quasilinear parabolic equations*, J. Differential Equations, **54** (1984), 97-120, MR 87j:35057, Zbl 0565.35053.

[NS] E. Noussair and C.A. Swanson, *Existence theorems for generalized Klein Gordon equations*, Bull. Amer. Math. Soc., **31** (1983), 333-336, MR 84k:35068, Zbl 0524.35048.

[S] Ph. Souplet, *Sur l'asymptotique des solutions globales pour une équation de la chaleur semi-linéaire dans des domaines non bornés*, C.R. Acad. Sci. Paris, **323** (1996), 877-882, MR 97k:35017, Zbl 0860.35051.

[SZ] Ph. Souplet and Qi S. Zhang, *Existence of ground states for semilinear elliptic equations with decaying mass: A parabolic approach*, C.R. Acad. Sci. Paris, **332**(**6**) (2001), 515-520 (full paper to appear in Ann. Inst. Poincaré Non Linéaire), MR 2002c:35104, Zbl 0992.35029.

[St] C.A. Stuart, *Some bifurcation theory for k-set contractions*, Proc. London Math. Soc. (3), **27** (1973), 531-550, MR 48 #12178, Zbl 0268.47064.

[Yau] S.-T. Yau, ed., *Seminar on Differential Geometry*, Annals of Math. Studies, Princeton Univ. Press, **102**, 1982, MR 83a:53002, Zbl 0471.00020.

[Zh1] Qi S. Zhang, *Large time behavior of Schrödinger heat kernels and applications*, Comm. Math. Phys., **210** (2000), 371-398, MR 2001i:35129, Zbl 0978.35014.

[Zh2] ______, *The quantizing effect of potentials on the critical number of reaction-diffusion equations*, J. Differential Equations, **170**(**1**) (2001), 188-214, MR 2001k:35174, Zbl 0973.35035.

[Zh3] ______, *An optimal parabolic estimate and its applications in prescribing scalar curvature on some open manifolds with* Ricci ≥ 0, Math. Ann., **316**(**4**) (2000), 703-731, MR 2001b:53040, Zbl 0974.53031.

[Zh4] ______, *A Liouville type theorem for some critical semilinear elliptic equations on noncompact manifolds*, Indiana Univ. Math. J., **50**(**4**) (2001), 1915-1936, MR 2002k:35103.

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