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An artinian ring R is square-free in case none of its inde-composable projective modules has a repeated composition factor. Let $\mathcal{Q}$ be the quiver of such a square-free ring R. In this paper we show that if R is indecomposable and transitive on the cyclic components of $\mathcal{Q}$ and if $\mathcal{Q}$ contains no n-crown, then $R \cong D \otimes_K A$ where D is the natural division ring of R, $K = \operatorname{Cen} D$, and A is a square-free K-algebra; that is, $\dim_K(eAf) \leq 1$ for every pair $e, f \in A$ of primitive idempotents.

1. Introduction.

Let R be an artinian ring with basic ring $\overline{R}$, and suppose that $\overline{R}/\overline{J} \cong D^n$ (as rings) for some division ring D and some $n \in \mathbb{N}$. Then R is said to be **split** with division ring D. If $K = \operatorname{Cen} D$, then R is a D-**algebra** in case also $R \cong D \otimes_K A$ for some (necessarily split) K-algebra A. Much of the study of such a D-algebra reduces to that of the usually more tractable K-algebra A. Of course, it is not common for an artinian ring R to be split and even less common for it to be a D-algebra. Nevertheless, it is of some interest to identify classes of artinian rings that can be realized as D-algebras. Fortunately, as we show in Theorem 3.2, the property of being a D-algebra is Morita invariant, and so we may focus on basic rings.

An artinian ring R is **square-free** in case no indecomposable projective R-module has a repeated composition factor. In [6] D'Ambrosia has shown that every indecomposable square-free ring is split. Moreover, there it is shown that there exist substantial classes of indecomposable square-free rings that are not D-algebras for any division ring D. In this paper we find a large class of square-free rings that *are* necessarily D-algebras.

Each artinian ring R determines two digraphs, the **quivers** of R. (See, for example, [7].) If R is square-free, then these two quivers are order-theoretic duals (see [6] Lemma 2.1[1]), so we work with the right quiver $\mathcal{Q}$ of R. We say that $\mathcal{Q}$ is **solvable** if it contains no n-crown (see Section 2). So, for

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[1] In [6], D'Ambrosia inadvertently reversed the roles of left and right quivers. The reader should replace "left quiver" with "right quiver" and vice versa when referring to that paper.

example, if $\mathcal{Q}$ is a tree or has a unique source or a unique sink, then $\mathcal{Q}$ is solvable.

Fuller and Haack ([**7**]) studied rings R whose quivers are trees. Such rings are necessarily square-free and solvable. The main result of [**7**] is that, in effect, if the quivers of R are trees, then R is a D-algebra and its basic ring is isomorphic to a factor ring of the D-incidence ring of a poset.

In this paper we consider the more general square-free rings with solvable quivers. Let $\mathcal{Q}$ be the quiver of a square-free ring R. If all cut edges of $\mathcal{Q}$, viewed as an undirected graph, are removed, the connected components of the resulting graph are the **cyclic components** of $\mathcal{Q}$. For the solvability of $\mathcal{Q}$ to be reflected in the behavior of R we need that R be transitive on each of these cyclic components; we say then that R is **locally transitive**. (See Section 4.) So if $\mathcal{Q}$ is a tree, then it is locally transitive. Our main result, Theorem 4.8, is that if R is an indecomposable, locally transitive, square-free ring with solvable quiver, then R is a D-algebra.

Finally, in Section 5 we look at a generalization of Gabriel's Theorem for split K-algebras. We show that a D-algebra is a factor of the D-path algebra of its quiver. From this it follows that, if R is basic square-free locally transitive with solvable quiver, then R is a twisted factor ring of the D-incidence algebra of its quiver.

2. Solvable quivers.

By a **quiver** we will mean a finite digraph $\mathcal{Q} = (V, A)$ with vertex set V and arrow set A. By a d-**path** (or **directed path**) in $\mathcal{Q}$ we mean a sequence of arrows

$$p = (a_1, a_2, \ldots, a_m)$$

such that for each $1 \leq i < m$ the terminal vertex of a_i is the initial vertex of a_{i+1}. A d-path with m arrows has **length** m. A single vertex v will be considered as a degenerate (trivial) d-path of length 0 with initial and terminal vertices both v. A **bi-directed cycle** in $\mathcal{Q}$ is the union $p \cup q$ of two d-paths of $\mathcal{Q}$ that intersect only in a common initial vertex and common terminal vertex.

We will also want to view the pair (V, A) as an undirected graph; that is, in this context we consider each $a \in A$ as an unoriented edge. Then by a **path** in $\mathcal{Q}$ we mean a path in the *undirected* graph (V, A). A **cycle** in $\mathcal{Q}$ is just a nontrivial path that is simply closed. Thus, every d-path of $\mathcal{Q}$ is a path of $\mathcal{Q}$, and every bi-directed cycle of $\mathcal{Q}$ is a cycle of $\mathcal{Q}$ or a union of cycles such as two loops with the same vertex. But the converses of these statements are true only in degenerate examples.

Example 2.1. Consider the following quivers:

$$\mathcal{Q}_1 \qquad\qquad \mathcal{Q}_2 \qquad\qquad \mathcal{Q}_3$$
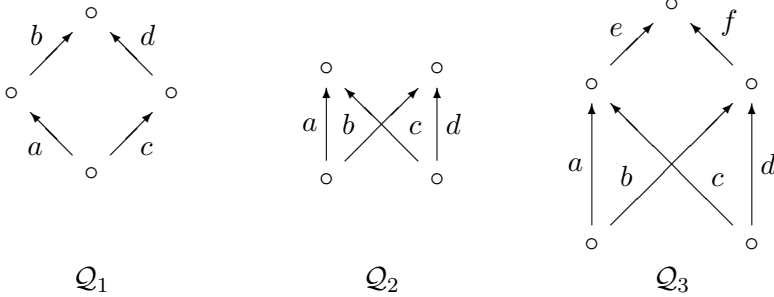
Here $\mathcal{Q}_1$ has one cycle and one bi-directed cycle, the union of the two d-paths (a, b) and (c, d). The quiver $\mathcal{Q}_2$ has one cycle and no bi-directed cycles. The graph $\mathcal{Q}_3$ has two bi-directed cycles and one other cycle.   □

Let $\mathcal{Q} = (V, A)$ be a quiver that contains no directed cycles and hence no loops. A vertex is a **source** if it is the head of no arrow and it is a **sink** if it is the tail of no arrow. Since our quiver is finite and contains no directed cycles, it must have at least one source and at least one sink. Let $U = \{u_1, \ldots, u_n\}$ be a set of sources and $W = \{w_1, \ldots, w_n\}$ a set of sinks in $\mathcal{Q}$. If $n \geq 2$, then U and W determine an n-**crown** if:

(C1) There is a d path from u_i to w_j iff $i > 1$ and $i - 1 \leq j \leq i$, or $(i, j) = (1, 1)$, or $(i, j) = (1, n)$.

(C2) If $n = 2$, no path from u_1 to w_2 shares a vertex with any path starting at u_2 and ending at w_1.

So, for example, the quiver $\mathcal{Q}_2$ of Example 2.1 is a 2-crown. For $n > 2$ an n-crown has the form
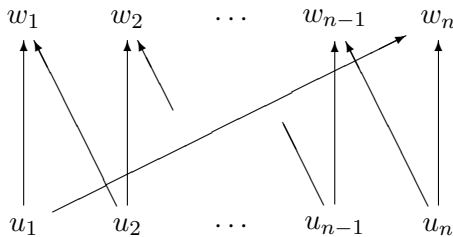
where the arrows indicate directed paths of unspecified length. We say the quiver $\mathcal{Q}$ is **solvable** in case it contains no n-crown. So note that if $\mathcal{Q}$ has either a single source or a single sink, then $\mathcal{Q}$ is solvable.

Let $U = \{u_1, \ldots, u_n\}$ be a set of sources and $W = \{w_1, \ldots, w_n\}$ a set of sinks in $\mathcal{Q}$. If there are d-paths from u_i to w_i and w_{i-1} for all $1 < i \leq n$ and from u_1 to w_1 and to w_n, then we say that U and W determine a **weak** n-**crown**. So an n-crown is a weak n-crown but not conversely.

Lemma 2.2. *Suppose that $\mathcal{Q}$ is a connected solvable quiver. Let $U = \{u_1, \ldots, u_n\}$ be a set of sources and $W = \{w_1, \ldots, w_n\}$ a set of sinks in $\mathcal{Q}$. If U and W determine a weak n-crown, then there exist $1 < i \leq n$, $1 \leq j < n$ and a vertex v with d-paths from u_1 to w_n through v and from u_i to w_j through v.*

Proof. We'll prove this by induction. For $n = 2$ this immediate from (C2). So let $n > 2$. Since $\mathcal{Q}$ has no n-crowns, there is a d-path from some u_i to some w_j where

$$0 < j - i < n - 1 \quad \text{or} \quad 1 < i - j < n.$$

In the first case

$$U' = \{u_1, \ldots, u_i, u_{j+1} \ldots, u_n\} \quad \text{and} \quad W' = \{w_1, \ldots, w_{i-1}, w_j, \ldots, w_n\}$$

form a weak $n - j + i$-crown. In the second,

$$U' = \{u_1, \ldots, u_j, u_i, \ldots, u_n\} \quad \text{and} \quad W' = \{w_1, \ldots, w_j, w_i, \ldots, w_n\}$$

form a weak $n - i + j + 1$-crown. So by induction there exist the desired vertex v and d-paths. $\square$

Now let $\mathcal{Q} = (V, A)$ be a quiver. By a **subquiver** of a quiver $\mathcal{Q}$ we mean a quiver whose vertex and arrow sets are subsets of V and A. If $H \subseteq A$, then we may view H as a subquiver with arrow set H. If H and K are two subquivers of $\mathcal{Q}$, then $H \cup K$ is the subquiver whose vertex and arrow sets are the unions of those of H and K, and $H \cap K$ is the subquiver whose vertex and arrow sets are the intersections of those of H and K. Also, if $H \subseteq A$, then the subgraph **generated by** H is the (unoriented) subgraph of $\mathcal{Q}$ consisting of all edges from H together with the vertices of these edges. The **complement** of H is the subgraph of $\mathcal{Q}$ generated by the edges of $\mathcal{Q}$ not in H.

Let T be the subgraph generated by the set of all cut edges of $\mathcal{Q}$. (A **cut edge** is an edge whose deletion separates the graph.) Let S be the complement of T, so each edge of S belongs to at least one cycle. Let $S_1, \ldots, S_r$ be the connected components of S; we call these the **cyclic components** of $\mathcal{Q}$. The subgraph T consists of a finite set of disjoint trees $T_1, \ldots, T_q$. Since each edge of T is a cut edge of $\mathcal{Q}$, each tree T_i meets any one cyclic component S_j in at most one common vertex. And if $\mathcal{Q}$ is connected with $r \geq 1$, each tree T_i meets at least one S_j in a common vertex. Note that if $\mathcal{Q}$ is connected and we shrink each cyclic component S_i to a single vertex, then the resulting graph will be a tree.

Suppose that $\mathcal{Q}$ is a finite quiver with no directed cycles. The addition or removal of a single arrow that is a cut edge of the graph will not affect the existence of an n-crown. Thus, we see that $\mathcal{Q}$ is solvable iff each of its cyclic components is solvable.

For every cyclic component S_i, its **star** is the subgraph $\mathrm{St}(S_i)$ of $\mathcal{Q}$ generated by S_i and all trees T_j in T that meet S_i. A labeling $\{S_1, \ldots, S_r\}$ of the components of S is a **proper order** in case for each $1 \leq k \leq r$ the graph

$$\mathrm{St}(S_1) \cup \mathrm{St}(S_2) \cup \cdots \cup \mathrm{St}(S_k)$$

is connected.

Lemma 2.3. *If $\mathcal{Q}$ is a connected quiver, then $\mathcal{Q}$ is a tree or there exists a proper order for the cyclic components of $\mathcal{Q}$.*

Proof. Assume that $\mathcal{Q}$ is not a tree. Let r be the number of cyclic components $S_1, \ldots, S_r$, and let $1 \leq k \leq r$. We induct on k. If $k = 1$, then the result is trivial. So let $1 \leq k < r$, and assume that cyclic components $S_1, \ldots, S_k$ have been chosen so that $H = \mathrm{St}(S_1) \cup \mathrm{St}(S_2) \cup \cdots \cup \mathrm{St}(S_k)$ is connected. Since $\mathcal{Q}$ is connected, the complement of H must have at least one vertex in common with H. If some tree T_j has a vertex in common with H, then it lies in H. Thus, there must be some cyclic component S_{k+1} sharing a vertex with H. So, clearly, $H \cup \mathrm{St}(S_{k+1})$ is connected. $\qquad\square$

3. Square-free D-algebras.

Let R be an artinian ring with radical J and with basic set $E = \{e_1, \ldots, e_n\}$; that is, the set $\{e_1, \ldots, e_n\}$ consists of pairwise orthogonal primitive idempotents, and $Re_1, \ldots, Re_n$ is a complete irredundant set of the indecomposable projective left R-modules. (See [3], page 305.) Then (see [3], Exercise 32.14, or [7]) the (right) **quiver** of R is the digraph $\mathcal{Q}(R)$ with vertex set E and with m_{ij} arrows from e_i to e_j where m_{ij} is the multiplicity of the simple module $\overline{e_j R} = e_j R / e_j J$ in the semisimple module $e_i J / e_i J^2$. The left quiver of R is defined similarly.

An artinian ring R is **square-free** in case for every primitive idempotent $e \in R$, the indecomposable projective modules Re and eR have no repeated composition factors. (See [6] for the basic facts about these rings.) If R is square-free, then ([6], Lemma 2.1) the left quiver of R is just the order-theoretic dual, $\mathcal{Q}(R)^{\mathrm{op}}$, of the right quiver $\mathcal{Q}(R)$. So, in particular, for a square-free ring the left quiver is solvable iff the right quiver is solvable. For the remainder of this paper we will assume that R is square-free. Note that this means that the quiver $\mathcal{Q}$ of R has no multiple arrows, and no arrow from a vertex to itself.

For a special case of such rings, let A be a split finite dimensional algebra over a field K. Then the K-algebra A is square free iff for each pair $e, f \in A$ of primitive idempotents

$$\dim_K(eAf) \leq 1.$$

These algebras have been described in detail in [2]. In particular, every incidence algebra of a pre-ordered set is square-free. Observe that if A is a

square-free K-algebra and if D is a division ring with $\operatorname{Cen} D = K$, then

$$R = D \otimes_K A$$

is a square-free ring. A square-free ring of this form is said to be a square-free D-**algebra**. As is shown in [**6**], there are plenty of square-free rings that are not D-algebras. Let R be an indecomposable square-free ring. Our main goal in this paper is to show that if R is locally transitive square-free (see Section 4) with solvable quiver, then it must be a D-algebra for some D. Our intermediate goal in this section is to obtain a characterization of square-free D-algebras that will provide the machinery to prove the main result.

So now let R be an indecomposable square-free ring. Then (see [**6**], Theorem 1.4) there is a division ring D, called **the division ring** of R, such that for every primitive idempotent $e \in R$, $D \cong eRe$. Moreover, for each pair $e, f \in R$ of primitive idempotents, $\dim_{eRe}(eRf) \leq 1$ and $\dim_{fRf}(eRf) \leq 1$. (See [**6**], Theorem 1.4.) It follows ([**6**], Corollary 1.5) that if $e, f \in R$ are primitive idempotents with $eRf \neq 0$, then for each $0 \neq erf \in eRf$, there is an isomorphism

$$\varphi_{erf} : eRe \longrightarrow fRf,$$

that we treat as a right operator [2] where $(exe)\varphi_{erf}$ is the unique element of fRf such that

$$exe \cdot erf = erf \cdot (exe)\varphi_{erf}.$$

This is the inverse of the isomorphism given in [**6**]. Concerning the isomorphisms φ_{erf}, we will use the following slight variation of Corollary 1.5 from [**6**]:

Lemma 3.1. *Let $e, f, g \in R$ be primitive idempotents in the square-free ring R and let $r, s, t \in R$. If $erfsg = (eae)etg \neq 0$ for some $eae \in \operatorname{Cen}(eRe)$, then*

$$\varphi_{erf}\varphi_{fsg} = \varphi_{etg}.$$

Proof. For each $exe \in eRe$,

$$
\begin{aligned}
(etg) \cdot (exe)\varphi_{erf}\varphi_{fsg} &= (eae)^{-1}erfsg \cdot (exe)\varphi_{erf}\varphi_{fsg} \\
&= (eae)^{-1}erf \cdot (exe)\varphi_{erf} \cdot fsg \\
&= (eae)^{-1} \cdot exe \cdot erfsg \\
&= (exe)(eae)^{-1}erfsg \\
&= (exe)(etg) \\
&= (etg)(exe)\varphi_{etg}.
\end{aligned}
$$

And so $\varphi_{erf}\varphi_{fsg} = \varphi_{etg}$. $\square$

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[2]In this section and the next all maps will be written as right operators unless explicitly stated otherwise.

Next, we observe that the property of being a D-algebra is Morita invariant, so that we can reduce to the basic case.

Theorem 3.2. *Let R and S be Morita equivalent rings. If $R \cong D \otimes_K A$ is a D-algebra, then there is a K-algebra A', Morita equivalent to A, such that $S \cong D \otimes_K A'$.*

Proof. Suppose that $R = D \otimes_K A$. If $e, f \in A$ are primitive idempotents with $Ae \cong Af$, then $1 \otimes e$ and $1 \otimes f$ are primitive idempotents of R that satisfy $R(1 \otimes e) \cong R(1 \otimes f)$. Thus, $\{e_1, \ldots, e_n\}$ is a basic set of primitive idempotents for A iff $\{1 \otimes e_1, \ldots, 1 \otimes e_n\}$ is a basic set for R. Assuming that those sets are basic, then

$$P' = \bigoplus_{i=1}^{n} (R(1 \otimes e_i))^{m_i} = D \otimes_K \bigoplus_{i=1}^{n} (Ae_i)^{m_i}$$

is a progenerator for R iff $m_i \geq 1$ for each i, iff $P = \oplus_{i=1}^{n}(Ae_i)^{m_i}$ is a progenerator for A. So it will suffice to prove that if $_A P$ is a finitely generated projective A-module, then (see [**3**], Corollary 22.4)

$$\operatorname{End}_R(D \otimes_K P) \cong D \otimes_K \operatorname{End}_A(P).$$

For in this case, we let $A' = \operatorname{End}_A(P)$, where P is a progenerator for A as described above.

But if $P = A^n$ is free, then

$$\begin{aligned}
\operatorname{End}_R(D \otimes_K A^n) &\cong \operatorname{End}_R(R^n) \\
&\cong \mathbb{M}_n(R) \\
&\cong D \otimes_K \mathbb{M}_n(A) \\
&\cong D \otimes_K \operatorname{End}_A(A^n).
\end{aligned}$$

Finally, if $e' \in \operatorname{End}_A(A^n)$ is an idempotent endomorphism, then there is an idempotent $e \in \operatorname{End}_R(R^n)$ with

$$\begin{aligned}
\operatorname{End}_R(D \otimes_K A^n e') &\cong \operatorname{End}_R(R^n e) \\
&\cong e \operatorname{End}_R(R^n) e \\
&\cong (1 \otimes e')(D \otimes_K \operatorname{End}_A(A^n))(1 \otimes e') \\
&\cong D \otimes_K e' \operatorname{End}_A(A^n) e' \\
&\cong D \otimes_K \operatorname{End}_A(A^n e'),
\end{aligned}$$

as claimed. $\qquad\square$

For the remainder of this section, we will assume that R is a basic indecomposable square-free ring with complete set $E = \{e_1, \ldots, e_n\}$ of orthogonal primitive idempotents. We define a binary relation $\leq$ on E by

$$e_i \leq e_j \quad \text{in case} \quad e_i R e_j \neq 0.$$

In general, this relation need not be either transitive or anti-symmetric.

For each $e_i \leq e_j$ in E, fix $0 \neq s_{ij} \in e_i R e_j$ with the proviso that $s_{ii} = e_i$ for each $i = 1, \ldots, n$. Then the set

$$B = \{s_{ij} \ : \ e_i \leq e_j\}$$

is a **basis** for R (**over** E). So if $B = \{s_{ij}\}$ is a basis for R over E, then since $\dim_{e_i R e_i}(e_i R e_j) \leq 1$ for each i, j, we have that as additive groups,

$$R = \sum_{i=1}^{n} \sum_{j=1}^{n} e_i R e_j = \sum_{e_i \leq e_j} e_i R e_i s_{ij}.$$

If B is a basis for R over E, then we will usually abbreviate the isomorphisms $\varphi_{s_{ij}}$ by

$$\varphi_{ij} = \varphi_{s_{ij}} : e_i R e_i \longrightarrow e_j R e_j$$

for each $s_{ij} \in B$.

Let B be a basis for R over E and let $B' \subseteq B$ be a subset of B. We say that B' is **solvable** in case for each $i = 1, \ldots, n$ there is an isomorphism $\mu_i : D \longrightarrow e_i R e_i$ such that

$$\varphi_{ij} = \mu_i^{-1} \mu_j$$

for all $s_{ij} \in B'$. We will also then say that $(\mu_i)_{i=1}^{n}$ is a **solution** on B'. In particular, the basis B is a **solvable basis** for R iff there is a solution $(\mu_i)_{i=1}^{n}$ on B. Note that if $(\mu_i)_{i=1}^{n}$ is a solution on B' and if $\alpha \in \operatorname{Aut} D$, then $(\alpha \mu_i)_{i=1}^{n}$ is also a solution on B'.

Lemma 3.3. *Let $B' \subseteq B$ be solvable. For all $e_i \leq e_j \leq e_k$ in E with $e_i \leq e_k$, if $s_{ij}, s_{jk}, s_{ik} \in B'$, then there is a (necessarily unique) $d = d_{ijk} \in \operatorname{Cen}(e_i R e_i)$ with*

$$s_{ij} s_{jk} = d s_{ik}.$$

Proof. Since $s_{ij} s_{jk} \in e_i R e_i s_{ik}$, there is a (necessarily unique) $d \in e_i R e_i$ with $s_{ij} s_{jk} = d s_{ik}$. We claim that $d \in \operatorname{Cen}(e_i R e_i)$. By hypothesis, there exist isomorphisms $\mu_h : D \longrightarrow e_h R e_h$ such that $\varphi_{gh} = \mu_g^{-1} \mu_h$ for all $s_{gh} \in B'$. Thus, for each $d' \in e_i R e_i$,

$$
\begin{aligned}
d' d s_{ik} &= d' s_{ij} s_{jk} \\
&= s_{ij} s_{jk} (d') \varphi_{ij} \varphi_{jk} \\
&= s_{ij} s_{jk} (d') \mu_i^{-1} \mu_k \\
&= d s_{ik} (d') \varphi_{ik} \\
&= d d' s_{ik}.
\end{aligned}
$$

Thus, $d' d = d d'$ and $d \in \operatorname{Cen}(e_i R e_i)$, as claimed. $\qquad\square$

Corollary 3.4. *Let $B' \subseteq B$ be solvable, and let $e_i \leq e_j \leq e_k$ in E with $e_i \leq e_k$. If $s_{ij}, s_{jk}, s_{ik} \in B'$, then $\varphi_{ik} = \varphi_{ij} \varphi_{jk}$.*

Proof. Apply Lemmas 3.1 and 3.3. $\qquad\square$

We now characterize square-free D-algebras in terms of solvable bases. For a different characterization, see Lemma 3.1 of [6].

Theorem 3.5. *Let R be a basic indecomposable square-free ring with division ring D. Then R is a D-algebra if and only if R has a solvable basis.*

Proof. ($\Longrightarrow$) Let $R = D \otimes_K A$ for some square-free K-algebra A where $K = \operatorname{Cen} D$. We may assume that $E = \{1 \otimes e_i : e_i \in E_A\}$ is a complete set of orthogonal primitive idempotents for R where $E_A = \{e_1, \ldots, e_n\}$ is a complete set of orthogonal primitive idempotents for A. We may choose a basis $B_A = \{s_{ij} : e_i \leq e_j\}$ for A over E_A; then $B = \{1 \otimes s_{ij} : e_i \leq e_j\}$ is a basis for R over E. For each i, let $\mu_i : d \longmapsto d \otimes e_i$. Then since $e_i A e_i \cong K$ for each i, each $\mu_i : D \longrightarrow (1 \otimes e_i)R(1 \otimes e_i)$ is an isomorphism. For each $s_{ij} \in B_A$ and $d \in D$,

$$(d \otimes e_i)(1 \otimes s_{ij}) = (d \otimes e_i s_{ij}) = (d \otimes s_{ij} e_j) = (1 \otimes s_{ij})(d \otimes e_j).$$

So $(d \otimes e_i)\varphi_{ij} = d \otimes e_j$, and $(\mu_i)_{i=1}^n$ is a solution for B. Thus, B is a solvable basis for R.

($\Longleftarrow$) Let $E = \{e_1, \ldots, e_n\}$ be a complete set of orthogonal primitive idempotents for R and let $B = \{s_{ij} : e_i \leq e_j\}$ be a solvable basis for R. Say $(\mu_i)_{i=1}^n$ is a solution for B. Define $\psi : D \longrightarrow R$ by

$$(d)\psi = \sum_{i=1}^n (d)\mu_i.$$

Observe that for all $e_i \leq e_j$ in E and all $r \in R$,

$$(d)\psi \cdot e_i r e_j = (d)\mu_i \cdot e_i r e_j \qquad \text{and} \qquad e_i r e_j \cdot (d)\psi = e_i r e_j \cdot (d)\mu_j.$$

The map $\psi : D \longrightarrow R$ is clearly additive and since $(d)\mu_i(d')\mu_j = \delta_{ij}(dd')\mu_i$, where δ_{ij} is the Kronecker delta, it follows that ψ is an isomorphism from D onto a division subring D^ψ of R.

We claim next that if $K = \operatorname{Cen} D$, then $K^\psi = (K)\psi$ is the center of R. First, we note that for each i, we have $(K)\mu_i = \operatorname{Cen}(e_i R e_i)$. Now suppose that $r \in \operatorname{Cen}(R)$. Then for each i, we have $e_i r = r e_i$, so

$$r = e_1 r e_1 + \cdots + e_n r e_n.$$

Clearly, $e_i r e_i \in \operatorname{Cen}(e_i R e_i) = (K)\mu_i$. Next, observe that if $e_i R e_j \neq 0$, then

$$(e_i r e_i)s_{ij} = s_{ij}(e_j r e_j),$$

so $e_j r e_j = (e_i r e_i)\varphi_{ij}$ or

$$(e_j r e_j)\mu_j^{-1} = (e_i r e_i)\mu_i^{-1} \in K.$$

Thus, there exists some unique $k \in K$ with $(k)\mu_i = e_i r e_i$ for all $i = 1, \ldots, n$. That is, $r = (k)\psi \in K^\psi$.

On the other hand, let $d \in K$. Then certainly, each $(d)\mu_i \in \text{Cen}(e_iRe_i)$. So it will suffice to show that for each i, j with $e_iRe_j \neq 0$, and for each $r \in R$,

$$(d)\mu_i \cdot (e_ire_j) = (e_ire_j) \cdot (d)\mu_j.$$

But if $e_ire_j \neq 0$, then $e_ire_j = e_ir'e_is_{ij}$ for some $e_ir'e_i \in e_iRe_i$. Thus, since $(d)\mu_i \in \text{Cen}(e_iRe_i)$,

$$
\begin{aligned}
(d)\mu_i \cdot (e_ire_j) &= (d)\mu_i \cdot (e_ir'e_is_{ij}) \\
&= (e_ir'e_i) \cdot (d)\mu_i \cdot s_{ij} \\
&= (e_ir'e_i)s_{ij} \cdot (d)\mu_i\varphi_{ij} \\
&= (e_ir'e_i)s_{ij} \cdot (d)\mu_i\mu_i^{-1}\mu_j \\
&= (e_ire_j) \cdot (d)\mu_j.
\end{aligned}
$$

Thus, $\text{Cen}(R) = K^\psi$ as claimed.

Now let A be the K^ψ subspace of R spanned by B. By Lemma 3.3, for each $s_{ij}, s_{jk} \in B$, if $s_{ij}s_{jk} \neq 0$, then

$$s_{ij}s_{jk} = ds_{ik}$$

for some $d \in \text{Cen}(e_iRe_i)$. But $\text{Cen}(e_iRe_i) = (K)\mu_i$, so there is a (necessarily unique) $t \in K$ with $(t)\mu_i = d$. Then $(t)\psi \in \text{Cen}(R)$ and

$$s_{ij}s_{jk} = (t)\psi s_{ik}.$$

Thus, A is actually a K^ψ-subalgebra of R. In fact, it is square-free. Indeed, E is a complete set of idempotents for A, and for each e_i, e_j, if $e_iAe_j \neq 0$, then e_iAe_j is one dimensional over K^ψ generated by s_{ij}.

Next, we claim that for each $a \in A$ and each $d \in D$,

$$a(d)\psi = (d)\psi a.$$

For this, it will clearly suffice to assume that $a = s_{ij}$ for some $e_iRe_j \neq 0$. But

$$
\begin{aligned}
((d)\psi)(s_{ij}) &= ((d)\mu_i)(s_{ij}) \\
&= (s_{ij})((d)\mu_i\varphi_{ij}) \\
&= (s_{ij})((d)\mu_j) \\
&= (s_{ij})((d)\psi),
\end{aligned}
$$

so $a(d)\psi = (d)\psi a$, as claimed.

Finally, the map $\tau : D \times A \longrightarrow R$ defined by

$$\tau : (d, a) \longmapsto ((d)\psi)a$$

is clearly K-bilinear and universal, so there is the desired ring isomorphism $\bar{\tau} : D \otimes_K A \longrightarrow R$. $\qquad\square$

4. The main result.

Let R be a basic square-free ring with basic set $E = \{e_1, \ldots, e_t\}$ and with quiver $\mathcal{Q} = \mathcal{Q}(R)$. In this section we shall show that under certain conditions a solvable quiver will imply that R is a D-algebra. Since we want the path structure of $\mathcal{Q}$ to be reflected in the algebra of R, we impose a restriction on R that will provide the desired connection between $\mathcal{Q}$ and R.

If $\mathcal{E}$ is some set of vertices in $\mathcal{Q}$, then we say that R is **transitive on $\mathcal{E}$** in case for all $e, f, g \in \mathcal{E}$

$$eJfJg = 0 \Longrightarrow eJf = 0 \text{ or } fJg = 0.$$

We say that R is **transitive** if it is transitive on the set E of all vertices of $\mathcal{Q}$ and **locally transitive** if it is transitive on the vertices of each cyclic component of $\mathcal{Q}$. If R is transitive, then (see [**6**]) the relation $\leq$ on E, defined in Section 3, is a partial order, and $\mathcal{Q}$ is the Hasse diagram for this poset. For more on the property of transitivity for square-free rings see [**6**]. Much of the significance of transitivity for us is given in the next pair of lemmas.

Lemma 4.1. *Let R be transitive. For each pair $e_i, e_j \in E$, the following are equivalent:*

   (a) $e_i \leq e_j$;
   (b) $e_i R e_j \neq 0$;
   (c) *There is a d-path from e_i to e_j in $\mathcal{Q}$.*

In particular, $e_{i_0} \leq e_{i_1} \leq \cdots \leq e_{i_k}$ in $\mathcal{Q}$ iff there is a d-path in $\mathcal{Q}$ through the sequence $(e_{i_0}, e_{i_1}, \ldots, e_{i_k})$.

Proof. The equivalence of (a) and (b) is by definition.

(b) $\Leftarrow$ (c). If there is an arrow from e_i to e_k and an arrow from e_k to e_j, then $e_i J e_k \neq 0$ and $e_k J e_j \neq 0$, so by transitivity, $e_i J e_k J e_j \neq 0$. The conclusion follows by induction on the length of the path.

(b) $\Rightarrow$ (c). We may assume $e_i \neq e_j$. If $e_i R e_j \neq 0$, then $e_i J^m e_j \neq 0$ for some maximal $m \geq 1$. If $m = 1$, then there is an arrow from e_i to e_j. If $m > 1$, then there is an e_k with $e_i J^m e_j \supseteq e_i J e_k J^{m-1} e_j \neq 0$. So there is an arrow from e_i to e_k and, by induction, a d-path from e_k to e_j. $\qquad\square$

Observe that if R is transitive, then $\mathcal{Q}$ contains no directed cycles. Let

$$B = \{s_{ij} \ : \ e_i \leq e_j\}$$

be a basis for R over E.

Lemma 4.2. *If R is transitive, and if there is a d-path in $\mathcal{Q}$ through the sequence $(e_{i_0}, e_{i_1}, \ldots, e_{i_k})$ of vertices in E, then $s_{i_0 i_1} \cdots s_{i_{k-1} i_k} \neq 0$.*

Proof. Suppose that $e_1, e_2, e_3 \in E$ and that there is a d-path through the sequence (e_1, e_2, e_3). We may assume that e_1, e_2, e_3 are distinct. Then by Lemma 4.1, $e_1 R e_2 \neq 0$ and $e_2 R e_3 \neq 0$. So

$$e_1 J e_2 = e_1 R e_2 = e_1 R e_1 s_{12} \neq 0,$$

and similarly, $e_2 J e_3 = s_{23} e_3 R e_3 \neq 0$. Thus, by transitivity, $e_1 J e_2 J e_3 \neq 0$, so that $s_{12} s_{23} \neq 0$. Now induct. □

Lemma 4.3. *If R is transitive, and if $e_i \leq e_j \leq e_k \leq e_\ell$ in E, then there exists $v \in e_j R e_k$ with*

$$s_{i\ell} = s_{ij} v s_{k\ell}.$$

Proof. This follows immediately from the fact that since R is transitive and square-free, $e_i R e_\ell = e_i R e_j R e_k R e_\ell = s_{ij} e_j R e_k s_{k\ell}$. See [**6**]. □

Our next goal is to find sufficient conditions on R for it to have a solvable basis. For this we first assume that R is transitive and indecomposable. We will prove the main result for this case, extend that to stars of cyclic components, and then piece these together for the general result. A key part of that is a lemma that gives a condition under which we can extend solvability from a subquiver to a larger subquiver.

Let $\mathcal{Q}_1$ be a subquiver of $\mathcal{Q}$. If B is a basis for R, then we set

$$B(\mathcal{Q}_1) = \{s_{ij} \in B \ : \ \text{there is a } d\text{-path from } e_i \text{ to } e_j \text{ in } \mathcal{Q}_1\}.$$

Let $V \subseteq E$. We denote by V^+ the subquiver of $\mathcal{Q}$ whose arrows are precisely those that lie on some d-path of $\mathcal{Q}$ that originates at some $e \in V$. Similarly, we denote by V^- the subquiver whose arrows lie on some d-path of $\mathcal{Q}$ that terminates at some $e \in V$. So, the vertex set of V^+ is the subposet of $\mathcal{Q}$ consisting of all upper bounds of the elements of V and the vertex set of V^- is the subposet of all lower bounds of the elements of V.

Next, we want to investigate some conditions under which a basis solvable on a subquiver can be modified to create a basis solvable on a larger subquiver.

Lemma 4.4. *Let R be an indecomposable square-free ring with quiver $\mathcal{Q}$, and suppose that B is a basis for R. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be disjoint subquivers with both $B(\mathcal{Q}_1)$ and $B(\mathcal{Q}_2)$ solvable. If there is an arrow a from a vertex e_i in $\mathcal{Q}_1$ to a vertex e_j in $\mathcal{Q}_2$, then there is a basis B' with $B'(\mathcal{Q}_1 \cup \{a\} \cup \mathcal{Q}_2)$ solvable. In particular, if a subquiver $\mathcal{Q}'$ of $\mathcal{Q}$ is a tree, then there is a basis B' for R with $B'(\mathcal{Q}')$ solvable.*

Proof. Let $(\mu_k)_{k=1}^n$ be a sequence of isomorphisms $\mu_k : \longrightarrow e_k R e_k$ that form solutions on both $B(\mathcal{Q}_1)$ and $B(\mathcal{Q}_2)$. For each $e_h \leq e_k$ in $\mathcal{Q}$ let $s'_{hk} \in e_h R e_k$ be given by

$$s'_{hk} = \begin{cases} s_{hj} s_{jk}, & \text{if } e_h \in \mathcal{Q}_1 \text{ and } e_k \in \mathcal{Q}_2; \\ s_{hk}, & \text{otherwise.} \end{cases}$$

Then $B' = \{s'_{hk} \,:\, e_h \leq e_k\}$ is a basis for R. Moreover, the sequence $(\nu_k)_{k=1}^n$ of isomorphisms given by

$$\nu_k = \begin{cases} \mu_j \varphi_{s'_{kj}}^{-1}, & \text{if } e_k \in \mathcal{Q}_1 \text{ and } e_k \leq e_j; \\ \mu_k, & \text{otherwise} \end{cases}$$

is a solution for $B'(\mathcal{Q}_1 \cup \{a\} \cup \mathcal{Q}_2)$. The final statement is now a simple induction. $\qquad\square$

Lemma 4.5. *Let R be a transitive square-free ring with quiver $\mathcal{Q}$ and basis B. Let $\mathcal{Q}_1$ be a connected subquiver of $\mathcal{Q}$, let a be an arrow of $\mathcal{Q}_1$, and let $\mathcal{Q}_2$ be the subquiver of $\mathcal{Q}_1$ obtained by deleting the arrow a. Assume that $B(\mathcal{Q}_2)$ is solvable. If a belongs to a bi-directed cycle of $\mathcal{Q}_1$, then there is a basis B' for R with $B'(\mathcal{Q}_1)$ solvable.*

Proof. Let $(\mu_i)_{i=1}^n$ be a sequence of isomorphisms $\mu_i : D \longrightarrow e_i R e_i$ that form a solution for $B(\mathcal{Q}_2)$. Suppose that a is an arrow from e_i to e_j and belongs to the bi-directed cycle $p \cup q$ where p and q are d-paths from e_ℓ to e_m. We may assume that a is an arrow on q. By Lemma 4.3 there is a $v \in e_i R e_j$ with $s_{\ell m} = s_{\ell i} v s_{j m}$. Thus,

$$\varphi_v = \varphi_{\ell i}^{-1} \varphi_{\ell m} \varphi_{jm}^{-1} = \mu_i^{-1} \mu_j.$$

For each $e_h \leq e_k$ in E define $s'_{hk} \in e_h R e_k$ by

$$s'_{hk} = \begin{cases} s_{hi} v s_{jk}, & \text{if } e_h \leq e_i \leq e_j \leq e_k; \\ s_{hk}, & \text{otherwise.} \end{cases}$$

Then $B' = \{s'_{hk} \,:\, e_h \leq e_k \text{ in } E\}$ is a basis for R and $(\mu_i)_{i=1}^n$ is a solution for $B'(\mathcal{Q}_1)$. $\qquad\square$

Let X be an arbitrary quiver and let $\mathcal{V}$ be the $\mathbb{Z}_2$-space on the set A of all arrows of X. Thus, we can also think of $\mathcal{V}$ as the Boolean algebra $\mathcal{P}(A)$ of all subsets of A. In the proof of the next lemma we will consider the $\mathbb{Z}_2$-subspace $\mathcal{B}(X)$ of $\mathcal{V}$ spanned by the bi-directed cycles of X.

Lemma 4.6. *Let H be a connected subquiver of $\mathcal{Q}$, let u be a minimal element of $\mathcal{Q}$, let w be a maximal element of $\mathcal{Q}$, let p_0 be a d-path from u to w, and let X be a subquiver of $u^+ \cap w^-$ such that p_0 lies in X and every arrow in X lies on some d-path in X from u to w. Suppose there is a basis B for R with $B(H \cup p_0)$ solvable. If $w \in H$ or if $u \in H$, then there is a basis B' for R with $B'(H \cup X)$ solvable.*

Proof. We'll prove the case with $w \in H$; the other is dual. The proof will be by induction on $b = \dim \mathcal{B}(H \cup X) - \dim \mathcal{B}(H \cup p_0)$, the difference between the dimensions of the bi-directed cycle spaces of $H \cup X$ and $H \cup p_0$. So suppose that $b = 0$. If there were some arrow a in X but not in $H \cup p_0$, then since there must be a d-path in X from u to w through a, and since a

cannot be on p_0, there is some bi-directed cycle in X containing a, contrary to $\dim \mathcal{B}(H \cup X) = \dim \mathcal{B}(H \cup p_0)$. So if $b = 0$, then $H \cup X = H \cup p_0$, and we're done.

So assume that $b > 0$. Then there must be some nontrivial d-path in X no arrow of which is in $H \cup p_0$. Let p_1 be such a d-path of maximal length. Say the initial vertex of p_1 is x and the terminal vertex is y. If there is a vertex of p_1 other than y that is the initial vertex of a bi-directed cycle in X, let x' be the greatest such vertex. Otherwise let $x' = x$. If there is a vertex different from x' on p_1 between x' and y that is the terminal vertex of a bi-directed path in X, then let y' be the smallest such vertex. Otherwise, let $y' = y$. Let p be the sub d-path of p_1 from x' to y', and let $X' = X \setminus p$. Then p_0 is in X' and every arrow in $X' \setminus H$ lies on a d-path from u to w in X'. But clearly, $\dim \mathcal{B}(H \cup X') < \dim \mathcal{B}(H \cup X)$, so by induction, there is a basis B' for R with $B'(H \cup X')$ solvable. Now every vertex of p other than x' and y' has valence 2 in $H \cup X$. Thus, by Lemma 4.4, there is a basis B'' for R with $B''(H \cup X)$ solvable. $\square$

Lemma 4.7. *Let R be an indecomposable transitive square-free ring with quiver $\mathcal{Q}$. If $\mathcal{Q}$ is solvable, then R has a solvable basis.*

Proof. Since $\mathcal{Q}$ is connected, there exist sequences $(u_1, \ldots, u_n)$ and $(w_1, \ldots, w_n)$ of minimal and maximal elements of $\mathcal{Q}$, with repetitions allowed, such that

$$u_i, u_{i+1} \leq w_i \quad \text{for all} \quad i < n \quad \text{and} \quad u_n \leq w_n,$$

and $\mathcal{Q} = \{u_1, \ldots, u_n\}^+ \cap \{w_1, \ldots, w_n\}^-$. We begin with some notation. For each m let

$$F_m = \{u_1, \ldots, u_m\}^+ \cap \{w_1, \ldots, w_m\}^-,$$

and

$$G_m = \{u_1, \ldots, u_m\}^+ \cap \{w_1, \ldots, w_{m-1}\}^-.$$

To prove this lemma, it suffices to show that for each m:

(i) For $1 \leq m \leq n$, if B is a basis for R with $B(G_m)$ solvable, then there is a basis B' for R with $B'(F_m)$ solvable.

(ii) For $1 \leq m < n$, if B is a basis for R with $B(F_m)$ solvable, then there is a basis B' for R with $B'(G_{m+1})$ solvable.

We will show (i); the arguments for (ii) are similar.

So let $m = 1$. Then there must be a d-path p_0 from u_1 to w_1. By Lemma 4.4 there is a basis B for R with $B(p_0)$ solvable. Then by Lemma 4.6, there is a basis B' for R with $B'(u_1^+ \cap w_1^-)$ solvable. Next, let $1 < m \leq n$ and assume that $B(G_m)$ is solvable. For $0 \leq k < m$, set

$$H_k = (\{u_{m-k}, \ldots, u_m\}^+ \cap w_m^-) \cup G_m.$$

We will show by induction on k that there is a basis B' for R with $B'(H_k)$ solvable. Let $k = 0$. Then there is a d-path p_0 from u_m to w_m, so by

Lemmas 4.4 and 4.5 there is a basis B'' for R with $B''(G_m \cup p_0)$ solvable. Thus, by Lemma 4.6 there is a basis B' for R with $B'(H_0)$ solvable. Now let $0 \leq k < m - 1$ and assume there is a basis B for R with $B(H_k)$ solvable. If $u_{m-k-1} \in \{u_{m-k}, \ldots, u_m\}$ or if there is no d-path from u_{m-k-1} to w_m, then $H_{k+1} = H_k$, and we're done. Otherwise, by hypothesis about the sequences $(u_1, \ldots, u_n)$ and $(w_1, \ldots, w_n)$, there must be sets $U' = \{u_{m-k-1} = u_1', u_2', \ldots, u_h'\} \subseteq \{u_{m-k-1}, \ldots, u_m\}$ and $W' = \{w_1', w_2', \ldots, w_h' = w_m\} \subseteq \{w_{m-k-1}, \ldots, w_m\}$ for which U' and W' form a weak n-crown. So since $\mathcal{Q}$ is solvable, Lemma 2.2 guarantees a pair of d-paths p_0 from u_1' to w_h' and q from some $u_i' \in U' \setminus \{u_1'\}$ to some $w_j' \in W' \setminus \{w_h'\}$ that meet in some common vertex v. But d-paths from u_1' to w_j' and from u_i' to w_h' must be in H_k, so that p_0 is in H_k. In particular, $H_k \cup p_0 = H_k$, and so by Lemma 4.6, there is a basis B' for R such that $B'(H_{k+1})$ is solvable. By induction on k, then, there is a basis B' for R with $B'(H_{m-1})$ solvable. But $H_{m-1} = F_m$, so we are done with (i). $\qquad\square$

Theorem 4.8. *If R is a locally transitive indecomposable square-free ring with solvable quiver, then R has a solvable basis and hence is a D-algebra.*

Proof. By Theorem 3.2 we may assume that R is basic. By Lemma 2.3 we may assume that the cyclic components of the quiver $\mathcal{Q}$ of R are labeled in a proper order $\{S_1, \ldots, S_n\}$. By Lemma 4.7 for each $i = 1, \ldots, n$ there is a basis B_i for R with $B_i(S_i)$ solvable. For each tree T_j in the subquiver of cut edges of $\mathcal{Q}$ there is a basis solvable on T_j. So by Lemma 4.4, for each $i = 1, \ldots, n$, we may assume that $B_i(\mathrm{St}(S_i))$ is solvable. Finally, use Lemma 4.4 and induction on n to find a basis B that is solvable on $\mathrm{St}(S_1) \cup \mathrm{St}(S_2) \cup \cdots \cup \mathrm{St}(S_n) = \mathcal{Q}$. By Theorem 3.5, R is a D-algebra. $\qquad\square$

5. Matrix representations.

If $R = D \otimes_K A$ is a D-algebra, then R acts essentially like the split K-algebra A. For example, a D-**automorphism** of R is just a ring automorphism α of R such that for some K-automorphism $\overline{\alpha}$ of A (with α and $\overline{\alpha}$ now viewed as left operators)

$$\alpha(d \otimes a) = d \otimes \overline{\alpha}(a),$$

for all $d \in D$ and $a \in A$. Thus, the D-automorphism groups of R are completely determined by the K-automorphism groups of A. For square-free algebras these groups have been studied extensively. (See, for example, [**2**].) So if R is a square-free ring satisfying the hypotheses of Theorem 4.8, then the entire analysis of the K-automorphism groups of the square-free algebra A given in [**2**] translates to that of the D-automorphism groups of R.

In this section, however, we want to describe how the D-algebras R that are locally transitive square-free rings with solvable quivers can be realized as twisted matrix rings. Thanks to Theorem 3.2 we can restrict ourselves to

basic rings. The extension of these representations to the more general case is straightforward.

First, though, let $R = D \otimes_K A$ be an arbitrary basic square-free D-algebra. Then the K-algebra A is split, basic, and square-free. Moreover, if $E = \{e_1, \ldots, e_n\}$ is a basic set of primitive idempotents for A, then $\{1 \otimes e_1, \ldots, 1 \otimes e_n\}$ is a basic set for R, and if J is the Jacobson radical for A, then $D \otimes J$ is the Jacobson radical for R. In particular, both A and R have the same quiver $\mathcal{Q}$, and R is transitive (locally transitive) iff A is.

Now let $R = D \otimes_K A$ be a basic, locally transitive, indecomposable square-free ring with quiver $\mathcal{Q}$ and division ring D, and assume that $E = \{e_1, \ldots, e_n\}$ is a basic set of primitive idempotents for A. Define a relation $\preceq$ on E by

$$e_i \preceq e_j \iff \text{there is a } d\text{-path in } \mathcal{Q} \text{ from } e_i \text{ to } e_j.$$

Let e_i and e_j belong to the same cyclic component of $\mathcal{Q}$. By Lemma 4.1, $e_i \preceq e_j$ iff $e_i A e_j \neq 0$. Thus, by Theorem 2.3 of [6], restricted to each cyclic component, this relation is a partial order with Hasse diagram the restriction of $\mathcal{Q}$ to that component. However, shrinking each cyclic component of $\mathcal{Q}$ to a point yields a tree in which there is at most one d-path from any one vertex to another. Thus, the relation $\preceq$ is a partial order on all of E and the Hasse diagram of this poset, called the **regular poset** of A and of R, is $\mathcal{Q}$. Therefore, we can assume that the labels for the elements of E have been chosen so that $i \leq j$ if $e_i \preceq e_j$, for all i, j.

If Λ is an arbitrary ring, then the Λ incidence ring ΛE of the finite poset $(E, \preceq)$ can be realized as the ring of all $n \times n$ matrices $[\![a_{ij}]\!]$ over Λ with $a_{ij} = 0$ whenever $e_i \not\preceq e_j$. If K is a field, then the K-incidence algebra KE of $(E, \preceq)$ is a subalgebra of the algebra of all $n \times n$ upper triangular matrices over K. For each $1 \leq i, j \leq n$ let e_{ij} be the (i, j) matrix unit in $\mathbb{M}_n(K)$. Then the K-incidence algebra KE has K-basis

$$B(E) = \{e_{ij} \in \mathbb{M}_n(K) \mid e_i \preceq e_j \text{ in } E\}.$$

If A is transitive with regular poset E, then in fact, A is isomorphic to the subalgebra $K_\xi E$ of upper triangular matrices with K-basis $B(E)$ and with multiplication twisted by a two-dimensional cocycle ξ of the poset E with coefficients in K^*. (See [5], [1] Theorem 2.4, or [2] Theorem 1.14.)

More generally, suppose that A is locally transitive. Consider the K-path algebra $K\mathcal{Q}$ of the quiver $\mathcal{Q}$ of A. By Gabriel's Theorem (see [8], Section 4; or [4] Theorem III.1.9) there is a surjective K-algebra homomorphism, called a **Gabriel homomorphism** for A, $\gamma : K\mathcal{Q} \longrightarrow A$ from $K\mathcal{Q}$ onto A with $\gamma(e_i) = e_i$ for each $e_i \in E$. Now suppose that p and q are two d-paths in $\mathcal{Q}$ from e to f for which $p \cup q$ forms a bi-directed cycle in $\mathcal{Q}$. Then p and q must belong to the same cyclic component, and so since A is locally transitive,

by Lemma 4.2, $eAf \neq 0$. Thus, $\dim_K(eAf) = 1$ and so

$$eAf = K\gamma(p) = K\gamma(q).$$

Thus, for each such pair (p, q) of d-paths in $\mathcal{Q}$ there is a nonzero $\rho(p, q) \in K^*$ with

$$\gamma(p) = \rho(p, q)\gamma(q).$$

Next, let I be the ideal

$$I = \langle p - \rho(p, q)q \; : p \cup q \; \text{ is a bidirected cycle in } \mathcal{Q} \rangle$$

of the K-path algebra $K\mathcal{Q}$. Let $\overline{A} = K\mathcal{Q}/I$, and let $\overline{\gamma} : K\mathcal{Q} \longrightarrow \overline{A}$ be the natural surjective algebra homomorphism. Since $I \leq \ker(\gamma)$, it follows that γ factors through $\overline{\gamma}$, say $\varphi : \overline{A} \longrightarrow A$ with $\gamma = \varphi \circ \overline{\gamma}$. Also, if p and q are two d-paths from e to f, then p and q factor as

$$p = p_1 p_2 \cdots p_t \qquad \text{and} \qquad q = q_1 q_2 \cdots q_t,$$

where for each $1 \leq k \leq t$, either $p_k = q_k$ or $p_k \cup q_k$ is a bi-directed cycle. Letting $\rho(p_k, q_k) = 1$ if $p_k = q_k$, we can define $\rho(p, q) \in K^*$ by

$$\rho(p, q) = \rho(p_1, q_1)\rho(p_2, q_2) \cdots \rho(p_t, q_t).$$

Then

$$\gamma(p) = \rho(p, q)\gamma(q).$$

For each $e_i \in E$, let $\overline{e_i} = e_i + I \in \overline{A}$. Note that $\{\overline{e_1}, \ldots, \overline{e_n}\}$ is a basic set of primitive idempotents for $\overline{A}$. Suppose $e, f \in E$ with $\overline{e}\,\overline{A}\,\overline{f} \neq 0$. A routine check shows that $\dim_K(\overline{e}\,\overline{A}\,\overline{f}) = 1$ and thus, $\overline{A}$ is square-free. If $e \neq f$ and $e(K\mathcal{Q})f \neq 0$, then there is a path from e to f in $\mathcal{Q}$. Since I contains no directed paths, we can conclude that $\overline{A}$ is transitive.

Thus, $\overline{A}$ is a transitive square-free K-algebra and hence $\overline{A} \cong K_\xi E$ for some two-dimensional cocycle ξ of E. Combining these observations with Theorem 4.8, we have the following matrix realization of locally transitive square-free rings with solvable quivers:

Theorem 5.1. *Let R be a basic, locally transitive, indecomposable square-free ring with solvable quiver $\mathcal{Q}$ on the basic set $E = \{e_1, \ldots, e_n\}$, and with division ring D. The relation defined on E by*

$$e_i \preceq e_j \iff \text{ there is a d-path in } \mathcal{Q} \text{ from } e_i \text{ to } e_j$$

is a partial order and the Hasse diagram of the poset $(E, \preceq)$ is isomorphic to the digraph $\mathcal{Q}$. Then there is a two-dimensional cocycle ξ of the poset E with coefficients in $K = \operatorname{Cen} D$ such that R is isomorphic to a factor ring of the D incidence ring $D_\xi E = D \otimes_K (K_\xi E)$ of E twisted by ξ. Finally, R is transitive iff R is isomorphic to the twisted D-incidence ring $D_\xi E$.

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