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A closed Riemann surface X which can be realised as a p-sheeted covering of the Riemann sphere is called p-gonal, and such a covering is called a p-gonal morphism. A p-gonal Riemann surface is called real p-gonal if there is an anticonformal involution (symmetry) σ of X commuting with the p-gonal morphism. If the p-gonal morphism is a cyclic regular covering the Riemann surface is called real cyclic p-gonal, otherwise it is called real generic p-gonal. The species of the symmetry σ is the number of connected components of the fixed point set Fix(σ) and the orientability of the Klein surface $X/\langle\sigma\rangle$. In this paper we find the species for the possible symmetries of real cyclic p-gonal Riemann surfaces by means of Fuchsian and NEC groups.

1. Introduction.

A closed Riemann surface X which can be realised as a p-sheeted covering of the Riemann sphere is called p-gonal, and such a covering is called a *p-gonal morphism*. The p-gonal Riemann surfaces have been extensively studied, see [1], [2], [6], [8], [9], [12] and [13]. A p-gonal Riemann surface is called *real p-gonal* if there is an anticonformal involution (symmetry) σ of X commuting with the p-gonal morphism.

Let X_g be a real p-gonal Riemann surface of genus $g \geq 2$. A *symmetry* σ of X_g is an anticonformal involution of X_g. The topological type of a symmetry is determined by the number of connected components, called *ovals*, of the fixed-point set Fix(σ) and the orientability of the Klein surface $X/\langle\sigma\rangle$. We say that σ has *species* $\Sigma_\sigma = +k$ if Fix(σ) consists of k ovals and $X/\langle\sigma\rangle$ is orientable, and $\Sigma_\sigma = -k$ if Fix(σ) consists of k ovals and $X/\langle\sigma\rangle$ is non-orientable. The set Fix(σ) corresponds to the real part of a complex algebraic curve representing X, which admits an equation with real coefficients.

If the p-gonal morphism is a cyclic regular covering, then the Riemann surface is called *real cyclic p-gonal*. When $p = 2$ the surface X_g is called hyperelliptic. A Riemann surface represented by an algebraic curve given by an equation of the form

$$y^p = \prod(x-a_i)\prod(x-b_j)^2\cdots\prod(x-m_j)^{p-1} \tag{1.1}$$

where the coefficients of the polynomial $\prod(x-a_i)\cdots\prod(x-m_j)^{p-1}$ are real is a real cyclic p-gonal Riemann surface. The complex conjugation induces a symmetry on the above curve. A natural problem is to study and classify all possible symmetries of such a Riemann surface up to conjugacy, as they will produce non-isomorphic real models of the complex algebraic curve.

In Section 2 we characterise real cyclic p-gonal Riemann surfaces, where p is an odd prime, in terms of signatures of Fuchsian and NEC groups. In Section 3 we determine all possible symmetries of a real cyclic p-gonal Riemann surface represented by an algebraic curve with equation (1.1).

2. Signatures of real cyclic p-gonal Riemann surfaces.

Let X_g be a compact Riemann surface of genus $g \geq 2$. The surface X_g can be represented as a quotient $X_g = \mathcal{H}/\Gamma$ of the upper half plane $\mathcal{H}$ under the action of a surface Fuchsian group Γ, that is, a cocompact orientation-preserving subgroup of the group $\mathcal{G} = \mathrm{Aut}(\mathcal{H})$ of conformal and anticonformal automorphisms of $\mathcal{H}$ without elliptic elements. A discrete, cocompact subgroup Γ of $\mathrm{Aut}(\mathcal{H})$ is called an NEC (*non-euclidean crystallographic*) *group*. The subgroup of Γ consisting of the orientation-preserving elements is called the *canonical Fuchsian subgroup of* Γ, it is denoted by Γ^+. The algebraic structure of an NEC group and the geometric structure of its quotient orbifold are given by the signature of Γ:

$$s(\Gamma) = (h, \pm, [m_1, \ldots, m_r], \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}). \tag{2.1}$$

The orbit space $\mathcal{H}/\Gamma$ is an orbifold with underlying surface of genus h, having r cone points and k boundary components, each with $s_j \geq 0$ corner points. The signs $''+''$ and $''-''$ correspond to orientable and non-orientable orbifolds respectively. The integers m_i are called the proper periods of Γ and they are the orders of the cone points of $\mathcal{H}/\Gamma$. The brackets $(n_{i1}, \ldots, n_{is_i})$ are the period cycles of Γ and the integers n_{ij} are the link periods of Γ and the orders of the corner points of $\mathcal{H}/\Gamma$. The group Γ is called the *fundamental group* of the orbifold $\mathcal{H}/\Gamma$.

A group Γ with signature (2.1) has a *canonical presentation* with generators:

$$x_1, \ldots, x_r, e_1, \ldots, e_k, c_{ij}, 1 \leq i \leq k, 1 \leq j \leq s_i + 1, \text{ and} \tag{2.2}$$
$$a_1, b_1, \ldots, a_h, b_h$$

if $\mathcal{H}/\Gamma$ is orientable, or

$$d_1, \ldots, d_h$$

otherwise, and relators:

$$x_i^{m_i}, \quad i = 1, \dots, r, \tag{2.3}$$

$$c_{ij}^2,\ (c_{ij-1}c_{ij})^{n_{ij}},\ c_{i0}e_i^{-1}c_{is_i}e_i, \quad i = 1, \dots, k, j = 2, \dots, s_i + 1$$

and $x_1 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h b_h a_h^{-1} b_h^{-1}$ or $x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_h^2$ according to whether $\mathcal{H}/\Gamma$ is orientable or not. This last relation is called the long relation.

The hyperbolic area of the orbifold $\mathcal{H}/\Gamma$ coincides with the hyperbolic area of an arbitrary fundamental region of Γ and equals:

$$\mu(\Gamma) = 2\pi \left(\varepsilon h - 2 + k + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right) \right), \tag{2.4}$$

where $\varepsilon = 2$ if there is a $''+''$ sign and $\varepsilon = 1$ otherwise. If Γ' is a subgroup of Γ of finite index then Γ' is an NEC group and the following Riemann-Hurwitz formula holds:

$$[\Gamma : \Gamma'] = \mu(\Gamma')/\mu(\Gamma). \tag{2.5}$$

An NEC group Γ without elliptic elements is called a *surface group* and it has signature $(h; \pm; [-], \{(-), \overset{k}{\dots}, (-)\})$. In such a case $\mathcal{H}/\Gamma$ is a *Klein surface*, i.e., a surface with a dianalytic structure of topological genus h, orientable or not according to the sign $''+''$ or $''-''$, and having k boundary components. Conversely, a Klein surface whose complex double has genus greater than one can be expressed as $\mathcal{H}/\Gamma$ for some NEC surface group Γ. Furthermore, given a Riemann (resp. Klein) surface represented as the orbit space $X = \mathcal{H}/\Gamma$, with Γ a surface group, a finite group G is a group of automorphisms of X if and only if there exists an NEC group Δ and an epimorphism $\theta : \Delta \to G$ with $\ker(\theta) = \Gamma$ (see [**5**]). The NEC group Δ is the lifting of G to the universal covering $\pi : \mathcal{H} \to \mathcal{H}/\Gamma$ and is called *the universal covering transformation group* of (X, G).

Definition 1. For a prime p, a real cyclic p-gonal Riemann surface is a triple (X, f, σ) where σ is a symmetry of X, f is a cyclic p-gonal morphism and $f \circ \sigma = c \circ f$, and c is the complex conjugation.

Notice that by Lemma 2.1 in [**1**] the condition $f \circ \sigma = c \circ f$ is automatically satisfied for genera $g \geq (p-1)2 + 1$, since the p-gonal morphism is unique. From now on, the genera will satisfy the condition above. As a consequence of the assumption $g \geq (p-1)2 + 1$ for the genera of the p-gonal surface X_g we have that the group C_p generated by the p-gonal morphism is a normal subgroup of $\mathrm{Aut}^+(X_g)$. Notice the the classification method fails for surfaces with genera in the range $2 \leq g \leq (p-1)2$. For instance, there are two 7-gonal surfaces of genus 3. One of them, X_3, is the Klein's quartic with $\mathrm{Aut}^+(X_3) \simeq PSL_2(7)$, in this case C_7 is non-normal in $PSL_2(7)$.

We give now a characterisation of real cyclic p-gonal Riemann surfaces represented by real equations via NEC groups.

Theorem 1 ([7]). *Let X be a Riemann surface with genus g. The surface X admits a symmetry σ and a meromorphic function f such that (X, f, σ) is a real cyclic p-gonal Riemann surface represented by a curve with real equation $y^p = \prod(x-a_i)\cdots\prod(x-m_j)^{p-1}$ if and only if there are an* NEC *group Δ with signature $(0,+,[p,\dots,p],\{(p,\dots,p)\})$ and an epimorphism $\theta : \Delta \to D_p$ such that X is conformally equivalent to $\mathcal{H}/\operatorname{Ker}\theta$ and $\operatorname{Ker}\theta$ is an* NEC *Fuchsian surface group.*

Let (X, f, σ) be a real cyclic p-gonal Riemann surface uniformised by a Fuchsian surface group Γ. Consider the automorphism $\varphi : X \to X$ such that $X/\langle\varphi\rangle$ is the Riemann sphere and φ is a deck-transformation of the covering f. Notice that the group Δ is the universal covering transformation group of (X, φ, σ), that $D_p = \langle\varphi, \sigma\rangle$ and that the canonical Fuchsian subgroup Δ^+ is the universal covering transformation group of (X, φ). Thus $X/\langle\varphi\rangle$ is a sphere with conic points of order p . Let $\overline{\sigma}$ be the symmetry in the Riemann sphere $X/\langle\varphi\rangle$ induced by σ. Since the triple (X, φ, σ) is represented by the equation $y^p = \prod(x-a_i)\cdots\prod(x-m_j)^{p-1}$, the symmetry σ is given by the map $\sigma : X \to X$ defined by $\sigma : (x, y) \to (\overline{x}, \overline{y})$. The set of real solutions of 1.1 is the set $\operatorname{Fix}(\sigma)$. Thus $\overline{\sigma}$ is conjugated to the complex conjugation. Then $X/\langle\varphi,\sigma\rangle = X/\langle\varphi\rangle/\langle\overline{\sigma}\rangle$ is a disc with corner(s) and conic points of order p.

With the above notation:

Theorem 2. *Let X be a real cyclic p-gonal Riemann surface such that $\langle\varphi,\sigma\rangle$ is isomorphic to D_p. If G is the group of conformal and anticonformal automorphisms of X, then X/G is uniformised by an* NEC *group Λ such that there is a surface Fuchsian subgroup $\Gamma \leq \Lambda$ uniformising X and the group Λ has one of the following signatures:*

(I) $(0,+,[\overbrace{p,\dots,p}^{r}, qp^{\epsilon}],\{(\overbrace{p,\dots,p}^{s})\})$, *where* $\epsilon = 0$ *or* 1 *and* $2r+s = \frac{2g+2(1-\epsilon)(p-1)}{q(p-1)}$. $G/\langle\varphi\rangle = C_q \times C_2$.

(II) $(0,+,[\overbrace{p,\dots,p}^{r}],\{(qp^{\epsilon_1},\overbrace{p,\dots,p}^{s_1}, qp^{\epsilon_2},\overbrace{p,\dots,p}^{s_2})\})$, *where* $\epsilon_i = 0$ *or* 1 *and* $2r+s_1+s_2 = \frac{2g+(2-\epsilon_1-\epsilon_2)(p-1)}{q(p-1)}$. $G/\langle\varphi\rangle = D_q$.

(III) $(0,+,[\overbrace{p,\dots,p}^{r}, 2p^{\epsilon_1}],\{(qp^{\epsilon_2},\overbrace{p,\dots,p}^{s})\})$, *where* $\epsilon_i = 0$ *or* 1 *and* $2r+s = \frac{g+(1-q\epsilon_1-\epsilon_2)(p-1)}{q(p-1)}$. $G/\langle\varphi\rangle = D_q \rtimes C_2$.

(IV) $(0,+,[\overbrace{p,\dots,p}^{r}],\{(2p^{\epsilon_1},\overbrace{p,\dots,p}^{s_1}, 2p^{\epsilon_2},\overbrace{p,\dots,p}^{s_2}, qp^{\epsilon_3},\overbrace{p,\dots,p}^{s_3})\})$, *where* $\epsilon_i = 0$ *or* 1 *and* $2r+s_1+s_2+s_3 = \frac{2g+(2-q\epsilon_1-q\epsilon_2-2\epsilon_3)(p-1)}{2q(p-1)}$. $G/\langle\varphi\rangle = D_q \times C_2$.

(V) $(0,\ +,\ [\overbrace{p,\ \ldots,\ p}^{r}], \{(2p^{\epsilon_1},\ \overbrace{p,\ \ldots,\ p}^{s_1}, 3p^{\epsilon_2},\ \overbrace{p,\ \ldots,\ p}^{s_2}, 3p^{\epsilon_3},\ \overbrace{p, \ldots, p}^{s_3})\})$, *where* $\epsilon_i = 0$ *or* 1 *and* $2r + s_1 + s_2 + s_3 = \frac{g+(1-3\epsilon_1-2\epsilon_2-2\epsilon_3)(p-1)}{6(p-1)}$. $G/\langle\varphi\rangle = S_4$.

(VI) $(0, +, [\overbrace{p, \ldots, p}^{r}, 3p^{\epsilon_1}], \{(2p^{\epsilon_2}, \overbrace{p, \ldots, p}^{s})\})$, *where* $\epsilon_i = 0$ *or* 1 *and* $2r + s = \frac{g+(1-4\epsilon_1-3\epsilon_2)(p-1)}{6(p-1)}$. $G/\langle\varphi\rangle = A_4 \rtimes C_2$.

(VII) $(0,\ +,\ [\overbrace{p,\ \ldots,\ p}^{r}],\ \{(2p^{\epsilon_1}, \overbrace{p,\ \ldots,\ p}^{s_1}, 3p^{\epsilon_2},\ \overbrace{p,\ \ldots,\ p}^{s_2},\ 4p^{\epsilon_3},\ \overbrace{p, \ldots, p}^{s_3})\})$, *where* $\epsilon_i = 0$ *or* 1 *and* $2r + s_1 + s_2 + s_3 = \frac{g+(1-6\epsilon_1-4\epsilon_2-3\epsilon_3)(p-1)}{12(p-1)}$. $G/\langle\varphi\rangle = S_4 \times C_2$.

(VIII) $(0,\ +,\ [\overbrace{p,\ \ldots,\ p}^{r}],\ \{(2p^{\epsilon_1},\ \overbrace{p,\ \ldots,\ p}^{s_1},\ 3p^{\epsilon_2},\ \overbrace{p,\ \ldots,\ p}^{s_2},\ 5p^{\epsilon_3},\ \overbrace{p, \ldots, p}^{s_3})\})$, *where* $\epsilon_i = 0$ *or* 1 *and* $2r + s_1 + s_2 + s_3 = \frac{g+(1-15\epsilon_1-10\epsilon_2-6\epsilon_3)(p-1)}{30(p-1)}$. $G/\langle\varphi\rangle = A_5 \times C_2$.

Notice that in cases (VII) and (VIII) the factor group C_2 of $G/\langle\varphi\rangle$ is generated by the antipodal map.

Proof. Consider the chain of coverings $X = \mathcal{H}/\Gamma \to X/\langle\varphi\rangle = \mathcal{H}/\Delta^+ \to X/G = \mathcal{H}/\Lambda$ with uniformising groups $\Gamma \leq \Delta^+ \leq \Lambda$, where $s(\Delta^+) = (0, +, [\overbrace{p, \ldots, p}^{\rho}], \{\})$ and $s(\Lambda) = (h, \pm, [m_1, \ldots, m_r], \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\})$. Furthermore by Lemma 2.1 in [**1**] the group $\langle\varphi\rangle$ is a normal subgroup of G. By Theorem 1 the factor group $\overline{G} = G/\langle\varphi\rangle$ is a finite group of conformal and anticonformal automorphisms of the Riemann sphere. (See also [**12**].)

In other words, we have an epimorphism $\theta : \Lambda \to \overline{G}$ with $\operatorname{Ker}\theta = \Delta^+$. This yields the signature of the group Λ in terms of the signature of Δ^+ and the group $\overline{G}$. Let p_i and q_{ij} be the orders in $\overline{G}$ of $\theta(x_i)$ and $\theta(c_{ij-1}c_{ij})$ respectively, where x_i, c_{ij} are generators in the canonical presentation of Λ associated to the signature (2.1). By [**3**] and [**5**] each proper period m_i induces $\frac{|\overline{G}|}{p_i}$ proper periods $\frac{m_i}{p_i}$ in $s(\Delta^+)$. Each link-period n_{ij} induces $\frac{|\overline{G}|}{2q_{ij}}$ proper periods $\frac{n_{ij}}{q_{ij}}$ in $s(\Delta^+)$. But $\frac{m_i}{p_i} = p$ or $\frac{m_i}{p_i} = 1$ and $\frac{n_{ij}}{q_{ij}} = p$ or $\frac{n_{ij}}{q_{ij}} = 1$, since Δ^+ is the group of the Riemann sphere with conic points of prime order p. We denote $K_1 = \{i \quad | \frac{m_i}{p_i} = 1\}$, $K_p = \{i \quad | \frac{m_i}{p_i} = p\}$, $H_1 = \{(i,j) \quad | \frac{n_{ij}}{q_{ij}} = 1\}$ and $H_p = \{(i,j) \quad | \frac{n_{ij}}{q_{ij}} = p\}$. Thus $\rho = \sum_{i \in K_p} \frac{|\overline{G}|}{p_i} + \sum_{(i,j) \in H_p} \frac{|\overline{G}|}{2q_{ij}}$

Using the Riemann-Hurwitz formula $|\overline{G}| = \mu(\Delta^+)/\mu(\Lambda)$ we obtain

$$\begin{aligned}(2.6)\qquad & -2 + \left(\sum_{i \in K_p} \frac{|\overline{G}|}{p_i} + \sum_{(i,j) \in H_p} \frac{|\overline{G}|}{2q_{ij}} \right) \frac{(p-1)}{p} \\ &= |\overline{G}|(\alpha h - 2 + k) + \sum_{i \in K_p} |\overline{G}| \left(1 - \frac{1}{pp_i}\right) + \sum_{i \in K_1} |\overline{G}| \left(1 - \frac{1}{p_i}\right) \\ &\quad + \sum_{(i,j) \in H_p} \frac{|\overline{G}|}{2} \left(1 - \frac{1}{pq_{ij}}\right) + \sum_{(i,j) \in H_1} \frac{|\overline{G}|}{2} \left(1 - \frac{1}{q_{ij}}\right),\end{aligned}$$

therefore $h = 0$, $k = 1$, $s(\Lambda) = (0, +, [pp_1, \ldots, pp_r], \{(pq_1, \ldots, pq_s)\})$, where $p_i, q_j \in \{1, p\}$. By setting K_1, K_p, H_1 and H_p in Equation (2.6) we obtain that p_i, q_j satisfy the equation

$$(2.7)\qquad |\overline{G}| - 2 = \sum_1^r |\overline{G}| \left(1 - \frac{1}{p_i}\right) + \sum_1^s \frac{|\overline{G}|}{2} \left(1 - \frac{1}{q_j}\right).$$

To find $s(\Lambda)$ it is enough to find the nontrivial solutions of (2.7). We divide the study of (2.7) in eight cases according to the factor group $\overline{G}$ in the epimorphism $\theta : \Lambda \to \overline{G}$ with $\mathrm{Ker}\,(\theta) = \Delta^+$:

(I) $\overline{G} = C_q \times C_2$, where $C_2 = \langle \overline{\sigma} \rangle$. The solution of Equation (2.7) is $p_1 = q$. Applying Riemann-Hurwitz formula to the covering $X \to X/G$ we obtain the signature $(0, +, [\overbrace{p, \ldots, p}^{r}, qp^{\epsilon}], \{(\overbrace{p, \ldots, p}^{s})\})$, where $\epsilon = 0$ or 1 and $2r + s = \frac{2g+2(1-\epsilon)(p-1)}{q(p-1)}$.

(II) $\overline{G} = D_q$. The solution of (2.7) is $q_{j_1} = q_{j_2} = q$. Therefore $s(\Lambda) = (0, +, [\overbrace{p, \ldots, p}^{r}], \{(qp^{\epsilon_1}, \overbrace{p, \ldots, p}^{s_1}, qp^{\epsilon_2}, \overbrace{p, \ldots, p}^{s_2})\})$, where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 = \frac{2g+(2-\epsilon_1-\epsilon_2)(p-1)}{q(p-1)}$.

(III) $\overline{G} = D_q \rtimes C_2$. The solution of (2.7) is $p_1 = 2$, and $q_1 = q$. Thus $s(\Lambda)$ becomes $(0, +, [\overbrace{p, \ldots, p}^{r}, 2p^{\epsilon_1}], \{(qp^{\epsilon_2}, \overbrace{p, \ldots, p}^{s})\})$, where $\epsilon_i = 0$ or 1 and $2r + s = \frac{g+(1-q\epsilon_1-\epsilon_2)(p-1)}{q(p-1)}$.

(IV) $\overline{G} = D_q \times C_2$. The solution in this case is $q_{j_1} = q_{j_2} = 2$ and $q_{j_3} = q$. This yields $s(\Lambda) = (0, +, [\overbrace{p, \ldots, p}^{r}], \{(2p^{\epsilon_1}, \overbrace{p, \ldots, p}^{s_1}, 2p^{\epsilon_2}, \overbrace{p, \ldots, p}^{s_2}, qp^{\epsilon_3}, \overbrace{p, \ldots, p}^{s_3})\})$, where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 + s_3 = \frac{2g+(2-q\epsilon_1-q\epsilon_2-2\epsilon_3)(p-1)}{2q(p-1)}$.

(V) $\overline{G} = S_4$. The solution of (2.7) is $q_{j_1} = 2$, $q_{j_2} = q_{j_3} = 3$. Then $s(\Lambda) = (0, +, [\overbrace{p, \dots, p}^{r}], \{(2p^{\epsilon_1}, \overbrace{p, \dots, p}^{s_1}, 3p^{\epsilon_2}, \overbrace{p, \dots, p}^{s_2}, 3p^{\epsilon_3}, \overbrace{p, \dots, p}^{s_3})\})$, where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 + s_3 = \frac{g+(1-3\epsilon_1-2\epsilon_2-2\epsilon_3)(p-1)}{6(p-1)}$.

(VI) $\overline{G} = A_4 \rtimes C_2$. The solution of (2.7) is $p_1 = 3$, and $q_1 = 2$. Thus $s(\Lambda)$ becomes $(0, +, [\overbrace{p, \dots, p}^{r}, 3p^{\epsilon_1}], \{(2p^{\epsilon_2}, \overbrace{p, \dots, p}^{s})\})$, where $\epsilon_i = 0$ or 1 and $2r + s = \frac{g+(1-4\epsilon_1-3\epsilon_2)(p-1)}{6(p-1)}$.

(VII) $\overline{G} = S_4 \times C_2$. The solution in this case is $q_{j_1} = 2$, $q_{j_2} = 3$ and $q_{j_3} = 4$. This yields $s(\Lambda) = (0,\ +,\ [\overbrace{p,\ \dots,\ p}^{r}],\ \{(2p^{\epsilon_1},\ \overbrace{p,\ \dots,\ p}^{s_1},\ 3p^{\epsilon_2},\ \overbrace{p,\ \dots,\ p}^{s_2},\ 4p^{\epsilon_3},\ \overbrace{p, \dots, p}^{s_3})\})$, where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 + s_3 = \frac{g+(1-6\epsilon_1-4\epsilon_2-3\epsilon_3)(p-1)}{12(p-1)}$.

(VIII) $\overline{G} = A_5 \times C_2$. The solution now is $q_{j_1} = 2$, $q_{j_2} = 3$ and $q_{j_3} = 5$. This yields $s(\Lambda) = (0, +, [\overbrace{p, \dots, p}^{r}], \{(2p^{\epsilon_1}, \overbrace{p, \dots, p}^{s_1}, 3p^{\epsilon_2}, \overbrace{p, \dots, p}^{s_2}, 5p^{\epsilon_3}, \overbrace{p, \dots, p}^{s_3})\})$, where $\epsilon_i = 0$ or 1 and $2r + s_1 + s_2 + s_3 = \frac{g+(1-15\epsilon_1-10\epsilon_2-6\epsilon_3)(p-1)}{30(p-1)}$. This finishes the proof.

3. Species of symmetries of real cyclic p-gonal Riemann surfaces.

Let X be a real cyclic p-gonal Riemann surface X with real equation. In the next theorem we study the topological types of the possible real forms of X.

Theorem 3. *Let X be a real cyclic p-gonal Riemann surface with p-gonal automorphism φ admitting a symmetry σ with fixed points and such that $\langle \sigma, \varphi \rangle = D_p$, p prime. If τ is another symmetry of X, then possible species of τ are* (*and all cases occur*):

(1) *$s(\Lambda)$ as in* (I).
  a) *$q \equiv 1 \mod (2)$. $\Sigma_\sigma = \Sigma_\tau$. If $r + \epsilon > 0$, then $\Sigma_\sigma = -1$. If $r + \epsilon = 0$, then $\Sigma_\sigma \in \{-1, +1\}$.*
  b) *$q \equiv 0 \mod (2)$. $\Sigma_\tau = \Sigma_\sigma$ as in case* (1a) *or $\Sigma_\tau = 0$.*

(2) *$s(\Lambda)$ as in* (II).
  a) *$q \equiv 1 \mod (2)$. $\Sigma_\sigma = \Sigma_\tau$ and $\Sigma_\sigma = -1$.*
  b) *$q \equiv 0 \mod (2)$, $q \neq 2$. $\Sigma_\sigma = -1$ and $\Sigma_\tau = -1$ or $\Sigma_\tau = +p, +1$.*

(3) *$s(\Lambda)$ as in* (III). *$\Sigma_\tau = 0$ or $\Sigma_\sigma = \Sigma_\tau$, besides $\Sigma_\sigma = -1$.*

(4) *$s(\Lambda)$ as in* (IV).
  a) *$q \equiv 1 \mod (2)$. $\{\Sigma_\sigma, \Sigma_\tau\} \subset \{\Sigma_1, \Sigma_2\}$, where $\Sigma_1 \in \{-1, +1, +p\}$. $\Sigma_2 \in \{-1, +1, +p\}$. In both cases $\Sigma_\sigma \neq +p$ and $\Sigma_\sigma \neq +1$ if σ is of the first type.*

b) $q \equiv 0 \mod (2)$, $q \neq 2$. $\{\Sigma_\sigma, \Sigma_\tau\} \subset \{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}$, *where* $\Sigma_4 = 0$, *and* $\Sigma_2 \in \{-1, +1, +p\}$ *and* $\Sigma_1, \Sigma_3 \in \{-1, +1, +p\}$. *In all cases* $\Sigma_\sigma \neq +p$ *and* $\Sigma_\sigma = +1$ *if* σ *is of the second type.*
$q = 2$. $\{\Sigma_\sigma, \Sigma_\tau\} \subset \{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}$, *where* $\Sigma_4 = 0$, *and* $\Sigma_i \in \{-1, +1, +p\}$ *for* $1 \leq i \leq 3$. *In all cases* $\Sigma_\sigma \neq +p$.

(5) $s(\Lambda)$ *as in* (V). $\Sigma_\tau = 0$ *or* $\Sigma_\sigma = \Sigma_\tau$, *with* $\Sigma_\sigma = -1$.
(6) $s(\Lambda)$ *as in* (VI). $\Sigma_\tau = 0$ *or* $\Sigma_\sigma = \Sigma_\tau$, *with* $\Sigma_\sigma = -1$.
(7) $s(\Lambda)$ *as in* (VII). $\{\Sigma_\sigma, \Sigma_\tau\} \subset \{\Sigma_1, \Sigma_2, 0\}$. $\Sigma_1 \in \{-1, +1, +p\}$ *and* $\Sigma_2 \in \{-1, +1, +p\}$, *but* $\Sigma_\sigma = -1$.
(8) $s(\Lambda)$ *as in* (VIII). $\Sigma_\tau = 0$ *or* $\Sigma_\sigma = \Sigma_\tau$, *with* $\Sigma_\sigma = -1$.

Proof. Consider the following chain of epimorphisms: $\theta : \Lambda \overset{\overline{\theta}}{\to} G \overset{\phi}{\to} \overline{G}$. The symmetries of X are symmetries τ in G which are lifts of symmetries $\overline{\tau}$ in $\overline{G}$. The species of τ is given by the conjugacy classes of reflections in $\overline{\Lambda} = \overline{\theta}^{-1}\langle\varphi, \tau\rangle = \overline{\theta}^{-1}\phi^{-1}\langle\overline{\tau}\rangle = \theta^{-1}\langle\overline{\tau}\rangle$ and the orientability of $\mathcal{H}/\overline{\Lambda}$. Notice that $\langle\varphi, \tau\rangle$ is either a cyclic group C_{2p} or a dihedral group D_p of order $2p$.

As in Theorem 2 we divide the proof in eight cases corresponding to the different types of groups $\overline{G}$ of conformal and anticonformal automorphisms of the Riemann sphere. The signature of Λ in each case is given by the corresponding case in Theorem 2.

(1a) $\overline{G} = C_q \times C_2$, $q \equiv 1 \mod (2)$. In this case $\overline{G}$ contains just one conjugacy class of symmetries and so does G: The one represented by σ. Moreover $D_p = \langle\varphi, \sigma\rangle$ is a normal subgroup of index q in G. By [5] the signature of $\overline{\theta}^{-1}(\langle\varphi, \sigma\rangle)$ is $(0, +, [\overbrace{p, \ldots, p}^{rq+\epsilon}], \{(\overbrace{p, \ldots, p}^{qs})\})$. By [14] (see also [4]) $\Sigma_\sigma = \pm 1$ as $D_p = \langle\varphi, \sigma\rangle$. The sign $+$ can only occur if $\overline{\theta}^{-1}(\langle\varphi, \sigma\rangle)$ has no proper periods, i.e., $r + \epsilon = 0$. If $s = 0$, then $r + \epsilon > 0$, the possible species is -1.

(1b) $\overline{G} = C_q \times C_2 = \langle\hat{\rho}, \overline{\sigma} \,|\, \hat{\rho}^q, \overline{\sigma}^2, \hat{\rho}^{-1}\overline{\sigma}\hat{\rho}\overline{\sigma}\rangle$, with $q \equiv 0 \mod (2)$. In this case $\overline{G}$ contains two conjugacy classes of symmetries, with representatives namely $\overline{\sigma}$ and $\hat{\rho}^{q/2}\overline{\sigma} = \overline{\tau}$, and so does G. To find the species of the symmetries we have to consider the normal subgroups $\overline{\theta}^{-1}(\langle\varphi, \sigma\rangle)$ and $\overline{\theta}^{-1}(\langle\varphi, \tau\rangle)$ of Λ with factor group C_q . By [5] they have signatures $(0, +, [\overbrace{p, \ldots, p}^{rq+\epsilon}], \{(\overbrace{p, \ldots, p}^{qs})\})$ and $(0, +, [\overbrace{p, \ldots, p}^{rq+\epsilon+sq/2}], \{-\})$ respectively. So species Σ_σ is as in (1a) and $\Sigma_\tau = 0$.

(2a) $\overline{G} = D_q = \langle\hat{\rho}, \overline{\sigma} \,|\, \hat{\rho}^q, \overline{\sigma}^2, (\hat{\rho}\overline{\sigma})2\rangle$, with $q \equiv 1 \mod (2)$. The group $\overline{G}$ contains one conjugacy class of symmetries and so does G. By the epimorphism $\theta : \Lambda \to D_q$ the images of reflections in Λ leave one fixed coset in D_q, so we get that $\overline{\Lambda}_\sigma$ has signature $(0,+,[\overbrace{p,\ldots,p}^{rq+s_1\frac{q-1}{2}+s_2\frac{q-1}{2}}],\{(\overbrace{p,\ldots,p}^{s_1+s_2+\epsilon_1+\epsilon_2})\})$. Now, $s_1+s_2+r>0$ since Λ is a NEC group, then $\Sigma_\sigma=-1$ by [4] and [14].

(2b) $\overline{G}=D_q=\langle\hat\rho,\overline{\sigma}\,|\,\hat\rho^q,\overline{\sigma}^2,(\hat\rho\overline{\sigma})^2\rangle$, with $q\equiv 0 \mod (2)$. The group $\overline{G}$ (and the group G) contains two conjugacy classes of symmetries, with representatives namely $\overline{\sigma}$ and $\hat\rho\overline{\sigma}=\overline{\tau}$. To find Σ_σ and Σ_τ we have to study the images of reflections by an epimorphism $\theta:\Lambda\to D_q$. Each of these images leaves either 2 $\overline{\sigma}$-cosets fixed and none from $\overline{\tau}$ or the other way round. Thus the signatures of $\overline{\Lambda}_\sigma$ and $\overline{\Lambda}_\tau$ are $(0,+,[\overbrace{p,\ldots,p}^{rq+s_1\frac{q-2}{2}+s_2\frac{q}{2}}],\{(\overbrace{p,\ldots,p}^{\epsilon_1+2s_1+\epsilon_2})\})$ and $(0,+,[\overbrace{p,\ldots,p}^{rq+s_1\frac{q}{2}+s_2\frac{q-2}{2}}],\{(\overbrace{p,\ldots,p}^{\epsilon_1+2s_2+\epsilon_2})\})$. Now σ has 1 oval and does not separate because $\overline{\theta}^{-1}(\langle\varphi,\sigma\rangle)$ contains proper periods since $s_1+s_2+r>0$ and $q>2$. If $\epsilon_1+s_2+\epsilon_2>0$, then $\langle\varphi,\tau\rangle=D_p$ and as before $\Sigma_\tau=-1$. If $\epsilon_1+s_2+\epsilon_2=0$ the signature of $\overline{\theta}^{-1}(\langle\varphi,\tau\rangle)$ becomes $(0,+,[\overbrace{p,\ldots,p}^{(2r+s_1)\frac{q}{2}}],\{(-)\})$. Thus $\Sigma_\tau=-1$, if $\langle\varphi,\tau\rangle=D_p$, and $\Sigma_\tau=+p,+1$ if $\langle\varphi,\tau\rangle=C_{2p}$.

(3) $\overline{G}=D_q\rtimes C_2=\langle\overline{\rho},\overline{\sigma_1},\overline{\sigma_2}\,|\,\overline{\rho}^2,\overline{\sigma_1}^2,\overline{\sigma_2}^2,(\overline{\sigma_1\sigma_2})^q,\overline{\rho\sigma_1\rho\sigma_2}\rangle$. The group $\overline{G}$ (and G) contains two conjugacy classes of symmetries, with representatives namely $\overline{\sigma}=\overline{\sigma}_1$ and $\overline{\rho\sigma}=\overline{\tau}$. The images of reflections in Λ are all mapped to conjugate reflections in $\overline{G}$. They are conjugate to $\overline{\sigma}$ as we know that σ has fixed points. Thus $\Sigma_\tau=0$. On the other hand $\overline{\Lambda}_\sigma$ has always proper periods. Therefore $\Sigma_\sigma=-1$.

(4) $\overline{G}=D_q\times C_2=\langle\overline{\sigma_1},\overline{\sigma_2},\overline{\sigma_3}\,|\,\overline{\sigma_i}^2,(\overline{\sigma_1\sigma_2})2,(\overline{\sigma_2\sigma_3})2,(\overline{\sigma_3\sigma_1})^q\rangle$, with $\overline{\sigma}_2$ central in $\overline{G}$. First of all the group $\langle\varphi,\sigma_2\rangle$ is a normal subgroup of G with factor group $D_q=\langle\overline{\sigma}_1,\overline{\sigma}_3\rangle$.

(4a) $q\equiv 1 \mod (2)$. In this case G has two conjugacy classes of reflections with representatives with images $\overline{\sigma}_1$ and $\overline{\sigma}_2$. Then there are two possible species for a symmetry of X: $\Sigma_{\sigma_1},\Sigma_{\sigma_2}$. The possible signatures for $\overline{\theta}^{-1}(\langle\varphi,\sigma_i\rangle)$ are given by the epimorphism $\Lambda\xrightarrow{\hat\theta}D_q$. By this epimorphism the images of c_0 and c_{s_1+i}, for $i\geq 2$, are conjugated to $\overline{\sigma_1}$ and the image of $c_1,\ldots,c_{s_1+1}$ is the identity (representing the central symmetry). Therefore c_0,c_1,c_{s+1} and c_{s+2} fixes q $\langle\overline{\sigma_2}\rangle$-cosets and one $\langle\overline{\sigma_1}\rangle$-coset each, each $c_1,\ldots,c_{s_1+1}$ fixes $2q$ $\langle\overline{\sigma_2}\rangle$-cosets (and none $\langle\overline{\sigma_1}\rangle$ -coset), and finally each c_{s_1+i}, $i\geq 2$ fixes two $\langle\overline{\sigma_1}\rangle$-cosets (and none $\langle\overline{\sigma_1}\rangle$-coset) in $\overline{G}$. Thus $\overline{\Lambda_1}$ and

$\overline{\Lambda_2}$ have signatures

$$(0,+,[\overbrace{p,\ldots,p}^{2rq+qs_1+(q-1)(s_2+s_3)+\frac{q-1}{2}(\epsilon_1+\epsilon_2)}],\{(\overbrace{p,\ldots,p}^{\epsilon_2+2s_2+2\epsilon_3+\epsilon_1+2s_3})\}) \quad \text{and}$$
$$(0,+,[\overbrace{p,\ldots,p}^{2rq+qs_2+\epsilon_3+qs_3}],\{(\overbrace{p,\ldots,p}^{q\epsilon_1+2qs_1+q\epsilon_2})\}),$$ see [10].

Altogether we have that Σ_1 is -1 if $s_2+s_3+\epsilon_1+\epsilon_2+\epsilon_3>0$, and Σ_1 is $+p,+1$ if $s_2+s_3+\epsilon_1+\epsilon_2+\epsilon_3=0$ and $\langle\varphi,\sigma_1\rangle$ is C_{2p}. On the other hand Σ_2 is -1 if $s_1+\epsilon_1+\epsilon_2>0$ and and $r+s_2+s_3+\epsilon_3>0$, Σ_2 is $+1$ if $s_1+\epsilon_1+\epsilon_2>0$ and $r=s_2=s_3=\epsilon_3=0$, and finally Σ_2 is $+p,+1$ if $s_1+\epsilon_1+\epsilon_2=0$ and $\langle\varphi,\sigma_2\rangle=C_{2p}$. In both cases $\Sigma_\sigma\neq+p$ since $\langle\varphi,\sigma\rangle=D_p$ and if σ is conjugate to σ_1 then again $\Sigma_\sigma\neq+1$. No further restrictions exist.

(4b) $q\equiv 0 \mod (2)$. In this case G has four conjugacy classes of reflections with representatives with homomorphic images $\overline{\sigma}_1$, $\overline{\sigma}_2$, $\overline{\sigma}_3$ and $(\overline{\sigma}_3\overline{\sigma})^{q/2}\overline{\sigma}_2=\overline{\sigma}_4$. Then there are four possible species for a symmetry of X: Σ_i, $1\leq i\leq 4$. The species are also given by the epimorphism $\Lambda\xrightarrow{\hat{\theta}}D_q$. By this epimorphism the images of c_0 and $c_{s_1+s_2+i}$, for $i\geq 3$, are conjugated to $\overline{\sigma_1}$, the image of $c_1,\ldots,c_{s_1+1}$ is the identity (representing the central symmetry), and the images of c_{s_1+i} , for $2\leq i\leq s_2+2$, are conjugate to $\overline{\sigma_3}$. First of all $\Sigma_4=0$ since no images of reflections by $\hat{\theta}$ are conjugate to $\overline{\sigma}_4$.

If $q=2$, then all the 3 symmetries are central and, as in (4a) the possible species for them are -1, $+1$ and $+p$.

If $q\neq 2$ then with the same procedure as in (4a) we get the following signatures for $\overline{\Lambda_1}$, $\overline{\Lambda_2}$ and $\overline{\Lambda_3}$:

$$(0,+,[\overbrace{p,\ldots,p}^{2rq+q(s_1+s_2)+\frac{q-2}{2}s_3+\frac{q-2}{2}\epsilon_1+\frac{q}{2}\epsilon_2}],\{(\overbrace{p,\ldots,p}^{2\epsilon_3+2\epsilon_1+4s_3})\}),$$
$$(0,+,[\overbrace{p,\ldots,p}^{2rq+qs_2+\epsilon_3+qs_3}],\{(\overbrace{p,\ldots,p}^{q\epsilon_1+2qs_1+q\epsilon_2})\}),$$
$$(0,+,[\overbrace{p,\ldots,p}^{2rq+q(s_1+s_3)+\frac{q-2}{2}s_2+\frac{q-2}{2}\epsilon_2+\frac{q}{2}\epsilon_1}],\{(\overbrace{p,\ldots,p}^{2\epsilon_3+2\epsilon_2+4s_2})\}).$$

Both $\overline{\Lambda}_1$ and $\overline{\Lambda}_3$ must have proper periods because otherwise all parameters in the signature of Λ except ϵ_3 are 0 and then Λ is a spherical group. Therefore Σ_1 is -1 if $s_3+\epsilon_1+\epsilon_3>0$, Σ_1 is $+p,+1$ if $s_3+\epsilon_1+\epsilon_3=0$ and $\langle\varphi,\sigma_1\rangle=C_{2p}$. Σ_2 is -1 if $s_1+\epsilon_1+\epsilon_2>0$ and $r+s_2+s_3+\epsilon_3>0$, Σ_2 is $+1$ if $s_1+\epsilon_1+\epsilon_2>0$ and $r=s_2=s_3=\epsilon_3=0$, and finally Σ_2 is $+p,+1$ if $s_1+\epsilon_1+\epsilon_2=0$ and $\langle\varphi,\sigma_2\rangle=C_{2p}$. Finally Σ_3 is -1 if $s_2+\epsilon_2+\epsilon_3>0$, Σ_3 is $+p,+1$ if $s_2+\epsilon_2+\epsilon_3=0$ and $\langle\varphi,\sigma_3\rangle=C_{2p}$. In all cases $\Sigma_\sigma\neq+p$ since $\langle\varphi,\sigma\rangle=D_p$. Again $\Sigma_\sigma\neq+1$ if σ is conjugate to σ_1 or σ_3 . No further restrictions exist.

(5) and (8) $\overline{G} = \langle \overline{\sigma_1}, \overline{\sigma_2}, \overline{\sigma_3} \,|\, \overline{\sigma_i}^2, (\overline{\sigma_1\sigma_2})^2, (\overline{\sigma_2\sigma_3})^3, (\overline{\sigma_3\sigma_1})^q \rangle$, where $q = 3$ in (5) and $q = 5$ in (8). $\overline{G}$, and thus G, contains two conjugacy classes of symmetries, with representatives namely $\overline{\sigma} = \overline{\sigma}_1$ and $\overline{\tau}$, with $\overline{\tau}$ conjugated to the antipodal map. Then $\Sigma_\sigma = \Sigma_{\sigma_1}$ and $\Sigma_\tau = 0$. As in case (2a), by [**10**] and [**14**], given the epimorphism $\overline{\theta}$, all the generating reflections of Λ induce reflections in $\overline{\theta}^{-1}(\langle \sigma_1 \rangle)$. So $\Sigma_\sigma = -1$ as they induce also proper periods.

(6) $\overline{G} = A_4 \rtimes C_2 = \langle \overline{\rho}, \overline{\sigma_1}, \overline{\sigma_2} \,|\, \overline{\rho}^3, \overline{\sigma_1}^2, \overline{\sigma_2}^2, (\overline{\sigma_1\sigma_2})^2, \overline{\rho\sigma_1\rho\sigma_2} \rangle$, where C_2 is generated by the antipodal map. With the same arguments as in (3) we obtain that G has two types of symmetries with representatives σ and τ where $\Sigma_\tau = 0$ and $\Sigma_\sigma = -1$.

(7) $\overline{G} = \langle \overline{\sigma_1}, \overline{\sigma_2}, \overline{\sigma_3} \,|\, \overline{\sigma_i}^2, (\overline{\sigma_1\sigma_2})^2, (\overline{\sigma_2\sigma_3})^3, (\overline{\sigma_3\sigma_1})4 \rangle = S_4 \times C_2$. This case is as case (4b) where the central symmetry is conjugated to the antipodal map and $\overline{\sigma_1}$ is conjugated to $\overline{\sigma_2}$. There are 3 conjugacy classes of symmetries with species 0, $\Sigma_1 = \Sigma_{\sigma_3}$ and $\Sigma_2 = \Sigma_{\sigma_2}$. Now Σ_1 is -1 if $s_3 + \epsilon_1 + \epsilon_3 > 0$, Σ_1 is $+p, +1$ if $s_3 + \epsilon_1 + \epsilon_3 = 0$ and $\langle \varphi, \sigma_3 \rangle$ is C_{2p}. On the other hand Σ_2 is -1 if $s_1 + s_2 + \epsilon_1 + \epsilon_2 + \epsilon_3 > 0$, Σ_2 is $+p$ if $s_1 + s_2 + \epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ and $\langle \varphi, \sigma_2 \rangle = C_{2p}$. $\Sigma_\sigma \neq +p, +1, 0$ since $\langle \varphi, \sigma \rangle = D_p$ and σ has fixed points.

To finish we show the existence of surfaces with the desired symmetries by listing appropriate groups G and epimorphisms θ. The p-gonal surfaces with the desired symmetries will be uniformised by the groups $\mathrm{Ker}\,(\theta)$. We distinguish the same eight cases as in Theorem 2.

(1) Let $G = \langle \varphi, \rho, \sigma \,|\, \varphi^t, \rho^q, \sigma^2, (\varphi\sigma)^2, (\varphi\rho)^{pq}, \rho^{-1}\sigma\rho\sigma \rangle$ and let $\theta : \Lambda \to G$ be defined by $\theta(x_i) = \varphi^{v_i}$, $1 \leq i \leq r$, $\theta(x_{r+1}) = \rho\varphi^{\epsilon + v_{r+1}}$, $\theta(c_{2j-1}) = \sigma$, $\theta(c_{2j}) = \varphi\sigma$, $\theta(e) = \rho^{-1}\varphi^l$, where $j_1 + \cdots + j_r + \epsilon + v_{r+1} + l \equiv 0 \mod p$.

(2) Let $G = \langle \varphi, \tau, \sigma \,|\, \varphi^p, \sigma^2, \tau^2, (\varphi\sigma)^2, (\sigma\tau)^q, (\varphi\tau)^2 \rangle$. Let $\theta : \Lambda \to G$ be defined by $\theta(x_i) = \varphi^{v_i}$, $1 \leq i \leq r$, $\theta(c_0) = \tau$, $\theta(c_j) = \sigma\varphi^{u_j}$ for $1 \leq j \leq s_1 + 1$, where $u_1 = \epsilon_1$ and $u_j = 1 - u_{j-1}$, $\theta(c_j) = \tau\varphi^{u_j}$, with $s_1 + 2 \leq j \leq s_1 + s_2 + 2$ where $u_{s_1+2} = \epsilon_2 + 1 - u_{s_1+1}$ and $u_j = 1 - u_{j-1}$, $\theta(e) = \varphi^l$, where $v_1 + \cdots + v_r + l \equiv 0 \mod p$.

To obtain the species $\Sigma_\tau = +p, +1$ we consider groups G with presentation $G = \langle \varphi, \tau, \sigma \,|\, \varphi^p, \sigma^2, \tau^2, (\varphi\sigma)^2, (\sigma\tau)^q, \varphi^{-1}\tau\varphi\tau \rangle$.

(3) Let $G = \langle \varphi, \rho, \tau, \sigma \,|\, \varphi^p, \rho^2, \tau^2, \sigma^2, \rho\sigma\rho\tau, (\varphi\rho)^2(\varphi\sigma)^2, (\sigma\tau)^q, (\varphi\tau)^2 \rangle$. Let $\theta : \Lambda \to G$ be defined by $\theta(x_i) = \varphi^{v_i}$, $1 \leq i \leq r$, $\theta(x_{r+1}) = \rho\varphi^{\epsilon_1 + v_{r+1}}$, $\theta(c_0) = \tau$, $\theta(c_j) = \sigma\varphi^{u_j}$ for $1 \leq j \leq s_1 + 1$, where $u_1 = \epsilon_1$ and $u_j = 1 - u_{j-1}$, $\theta(e) = \rho\varphi^l$, where $v_1 + \cdots + v_{r+1} + \epsilon_1 + l \equiv 0 \mod p$.

(4) $G = \langle \varphi, \sigma_1, \sigma_2, \sigma_3 \,|\, \sigma_i^2, (\sigma_1\sigma_2)^2, (\sigma_2\sigma_3)^2, (\sigma_3\sigma_1)^q, \varphi^p, (\varphi\sigma_i)^2 \rangle$. Let $\theta : \Lambda \to G$ be defined by $\theta(x_i) = \varphi^{v_i}$, $1 \leq i \leq r$, $\theta(c_0) = \sigma_1$, $\theta(c_j) = \sigma_2\varphi^{u_j}$ for $1 \leq j \leq s_1 + 1$, where $u_1 = \epsilon_1$ and $u_j = 1 - u_{j-1}$, $\theta(c_j) = \sigma_3\varphi^{u_j}$, with $s_1 + 2 \leq j \leq s_1 + s_2 + 2$ where $u_{s_1+2} = \epsilon_2 + 1 - u_{s_1+1}$ and $u_j = 1 - u_{j-1}$, $\theta(c_j) = \sigma_1\varphi^{u_j}$, with $s_1 + s_2 + 3 \leq j \leq s_1 + s_2 + s_3 + 3$ where $u_{s_1+s_2+3} =$

$\epsilon_3 + 1 - u_{s_1+s_2+2}$ and $u_j = 1 - u_{j-1}$, $\theta(e) = \varphi^l$, where $v_1 + \cdots + v_r + l \equiv 0 \mod p$.

To obtain the species $+p, +1$ one or two of the relations $(\varphi\sigma_i)^2$ in the presentation of G must be substituted by relations $\varphi^{-1}\sigma_i\varphi\sigma_i$.

(5) and (8) $G = \langle \varphi, \sigma_1, \sigma_2, \sigma_3 \mid \sigma_i^2, (\sigma_1\sigma_2)^2, (\sigma_2\sigma_3)^3, (\sigma_3\sigma_1)^q, \varphi^p, (\varphi\sigma_i)^2\rangle$, where $q = 3$ in (5) and $q = 5$ in (8) and let $\theta : \Lambda \to G$ be defined as in (4).

(6) $G = \langle \varphi, \rho, \sigma_1, \sigma_2 \mid \varphi^p, \rho^3, \sigma_1^2, \sigma_2^2, (\sigma_1\sigma_2)^2, \rho^2\sigma_1\rho\sigma_2, (\varphi\sigma_1)^2, (\varphi\sigma_2)^2, (\varphi\rho)^{3p}\rangle$. Let $\theta : \Lambda \to G$ be defined as $\theta(x_i) = \varphi^{v_i}$, $1 \le i \le r$, $\theta(x_{r+1}) = \rho\varphi^{\epsilon_1+v_{r+1}}$, $\theta(c_0) = \sigma_1$, $\theta(c_j) = \sigma_2\varphi^{u_j}$ for $1 \le j \le s+1$, where $u_1 = \epsilon_2$ and $u_j = 1 - u_{j-1}$, $\theta(e) = \rho^2\varphi^l$, where $v_1 + \cdots + v_{r+1} + \epsilon_1 + l \equiv 0 \mod p$.

(7) $G = \langle \varphi\sigma_1, \sigma_2, \sigma_3 \,|\, \sigma_i^2, (\sigma_1\sigma_2)^2, (\sigma_2\sigma_3)^3, (\sigma_3\sigma_1)^4, \varphi^p, (\varphi\sigma_i)^2\rangle$ and let $\theta : \Lambda \to G$ be defined as in (4). To obtain the species $+p, +1$ either the relations $(\varphi\sigma_1)^2$ and $(\varphi\sigma_1)^2$ or the relation $(\varphi\sigma_3)^2$ in the presentation of G must be changed to the corresponding commuting relation.

The kernels of the above epimorphisms will uniformise surfaces with a symmetry with species -1 for general groups Λ. The same epimorphisms yield the species $+1$ in cases 1 and 4 under the restrictions on Λ given in the first part of the theorem. Again, the same epimorphisms yield the species $+p, +1$ under the corresponding restrictions on Λ given in the first part of the theorem.

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