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We prove that the lattice of subgroups of every finite simple group is a complemented lattice.

1. Introduction.

A group G is called a K-group (a *complemented* group) if its subgroup lattice is a complemented lattice, i.e., for a given $H \leq G$ there exists a $X \leq G$ such that $\langle H, X\rangle = G$ and $H \wedge X = 1$. The main purpose of this Note is to answer a long-standing open question in finite group theory, by proving that:

Every finite simple group is a K-group.

In this context, it was known that the alternating groups, the projective special linear groups and the Suzuki groups are K-groups ([**P**]).

Our proof relies on the FSGC-theorem and on structural properties of the maximal subgroups in finite simple groups. The rest of this paper is divided into four sections. In Section 2 we collect some criteria for a subgroup of a group G to have a complement and recall some useful known results. In Section 3 we deal with the classical groups, in 4 with the exceptional groups of Lie type and in Section 5 with the sporadic groups.

With reference to notation and terminology, we shall follow closely those in use in [**P**] and [**S**]. All groups are meant to be finite.

2. Preliminaries.

We begin with the following:

Proposition 2.1. *Given the group G, let T, X be subgroups of G such that $T \leq X < G$. If the interval $[X/T]$ is a complemented lattice and if X is contained in only one maximal subgroup M of G, then every $H \leq G$ with $H \not\leq M$ and $H \wedge T = 1$ has a complement in G.*

Proof. Let C be a complement of $\langle H, T\rangle \wedge X$ in $[X/T]$. Then $\langle H, C\rangle = \langle H, T, C\rangle \geq \langle \langle H, T\rangle \wedge X, C\rangle = X$. Since $H \not\leq M$, we conclude that $\langle H, C\rangle = G$. Moreover $H \wedge C = H \wedge X \wedge C \leq \langle H, T\rangle \wedge X \wedge C = T$, hence $H \wedge C \leq H \wedge T = 1$. □

The condition on M in Proposition 2.1 means that $[G/X]$ is a *monocoatomic* interval with *coatom* M.

Corollary 2.2. *Let X be a K-subgroup and $[G/X]$ a monocoatomic interval with coatom M. Then every $H \leq G$ not contained in M_G has a complement in G. In particular G/M_G is a K-group.*

Proof. There exists a $g \in G$ such that $H^g \not\leq M$. By Proposition 2.1 with $T = 1$, H^g has a complement. Hence also H has a complement C in G. Moreover, if $M_G < H$, then CM_G/M_G is a complement of H/M_G in G/M_G. □

Proposition 2.3. *Let G be a simple group and $[G/X]$ a monocoatomic interval with coatom M. If N is a central subgroup of M of prime order with $N \leq X$ and if X/N is a K-group, then G is a K-group.*

Proof. Let H be a proper subgroup of G. Since $M_G = 1$, without loss of generality we may assume $H \not\leq M$. If now $H \wedge N = 1$, by Proposition 2.1 H has a complement in G. Assume now $N \leq H$; there exists a $g \in G$ such that $N^g \wedge H = 1$. So if H has no complement in G, by Proposition 2.1 we must have $N^g \leq \mathcal{C}(H)$. It follows that if $\mathcal{F} = \{N^x \mid x \in G\}$ and $\mathcal{F}_1 = \{N^x \mid N^x \not\leq H\}$, then $\mathcal{N}(H) \geq \langle H, \mathcal{F}_1 \rangle \geq \langle \mathcal{F} \rangle = G$, a contradiction. □

We finally recall:

(2.1) *The direct product of a family of groups is a K-group if and only if each factor is a K-group*,

see Corollary 3.1.5 in [**S**].

(2.2) *If G contains an abelian subgroup A generated by minimal normal subgroups of G and a complement K to A that is a K-group, then G is a K-group*,

see Lemma 3.1.9 in [**S**].

(2.3) *The symmetric and alternating groups, the projective special linear groups $L_n(q)$ and the simple Suzuki groups ${}^2B_2(q)$ are K-groups*,

see [**P**].

For our purpose it will be convenient to know which non-simple groups of Lie type ([**C**], p. 175, p. 268) are complemented.

Proposition 2.4. *The following non-simple groups of Lie type are K-groups:*

$$L_2(2),\ L_2(3),\ Sp_4(2),\ G_2(2),\ {}^2G_2(3).$$

The following non-simple groups of Lie type are not K-groups:

$${}^2B_2(2),\ {}^2F_4(2),\ U_3(2).$$

Proof. In fact $L_2(2) \cong S_3$, $L_2(3) \cong A_4$, $Sp_4(2) \cong S_6$, and we are done by (2.3). In $G_2(2)$ there is a monocoatomic interval $[G_2(2)/H]$ with $H \cong L_3(2)$ and corefree coatom, by Theorem 2.5 in [**Co**]: Hence $G_2(2)$ is a K-group by

(2.3) and Corollary 2.2. The group ${}^2G_2(3)$ has a corefree maximal subgroup isomorphic to $Z_7 : Z_6$ ([**K3**]): Hence it is a K-group by (2.2). On the other hand, we have ${}^2B_2(2) \cong Z_5 : Z_4$ ([**A**]), $U_3(2) \cong 3^2 : Q_8$ ([**KL**], p. 43) and finally $|\,{}^2F_4(2) : {}^2F_4(2)'\,| = 2$, but all involutions of ${}^2F_4(2)$ are contained in ${}^2F_4(2)'$ ([**AS**], p. 75). □

To prove the main theorem, we take a counterexample L of minimal order and show that such a group L does not exist.

3. The simple classical groups.

We are going to assume in this section that $L = G_0(n,q)$, a (simple) classical group as in [**KL**].

a) $G_0(n,q)$ is not of type A_m, $n = m+1$, $m \geq 1$.
See (2.3).

b) $G_0(n,q)$ is not of type C_m, $n = 2m$, $m \geq 2$.

Proof. Let r be a prime divisor of m, so that $m = rt$, $t \geq 1$. By Theorem 1 and Theorem 2 in [**L**], the interval $[PSp(2m,q)/PSp(2t,q^r)]$ is monocoatomic. Moreover $PSp(2t,q^r)$ is simple, since $q^r \geq 4$, of order less than the order of L, hence a K-group. But then by Corollary 2.2, L is a K-group, a contradiction. □

c) $G_0(n,q)$ is not of type ${}^2A_m$, $n = m+1$, $m \geq 2$.

Proof. We consider first the cases $(n,q) = (3,3), (3,5)$. The groups $U_3(3)$ and $U_3(5)$ are K-groups: In fact one has $PSL_2(7) <\cdot\, U_3(3)$ and $A_7 <\cdot\, U_3(5)$ ([**K1**], §5). Assume now $(n,q) \neq (3,3), (3,5)$. With reference to the notation in [**BGL**], p. 388, let G be the simple adjoint algebraic group over $\overline{\mathbb{F}}_q$ with associated Dynkin diagram of type A_m, $\lambda = \sigma_q$ and $\mu = {}^2\sigma_q$: We have $G_\lambda = PGL_n(q)$, $G_\mu = PGU_n(q)$, $O^{p'}(G_\lambda) = L_n(q)$, $O^{p'}(G_\mu) = U_n(q) = G_0(n,q)$,

$$T := O^{p'}(G_\mu \cap G_\lambda) = \begin{cases} PSp_n(q) & \text{if } n \text{ is even} \\ \Omega_n(q) & \text{if } nq \text{ is odd} \\ Sp_{n-1}(q) & \text{if } n \text{ is odd and } q \text{ is even.} \end{cases}$$

From Theorem 2 in [**BGL**] it follows that $[U_n(q)/T]$ is monocoatomic. Moreover, T is a K-group, either because it is simple of order less than $|\,L\,|$, or because it is isomorphic to $Sp_4(2)$ (Proposition 2.4): Hence $G_0(n,q)$ is a K-group, a contradiction. □

d) $G_0(n,q)$ is not of type B_m, $n = 2m+1$, $m \geq 3$, q odd.

Proof. Assume $q = p^f$, with $f > 1$ and let r be a prime divisor of f. Then by Theorem 1 in [**BGL**], $[P\Omega_n(q)/P\Omega_n(q^{1/r})]$ is monoatomic, a contradiction. Therefore we must have $q = p$. Now, by §5 in [**K1**] and Proposition 4.2.15 in [**KL**], $G_0(n, q)$ contains a maximal subgroup M which is a split extension of an irreducible elementary abelian 2-group by A_n or S_n. Therefore M is a K-group by (2.2), and $G_0(n, q)$ is a K-group, a contradiction. □

e) $G_0(n, q)$ is not of type D_m, $n = 2m$, $m \geq 4$.

Proof. Let $V = \mathbb{F}_q^n$ be the natural (projective) module for $G_0(n, q)$, and let W be a nonsingular subspace of V of dimension 1. Since $\overline{\Omega} := G_0(n, q)$ is a counterexample of minimal order, the socle $\operatorname{soc} H_{\overline{\Omega}}$ of the stabilizer $H_{\overline{\Omega}}$ of W in $\overline{\Omega}$, which is isomorphic to $\Omega_{n-1}(q)$ if q is odd, and to $Sp_{n-2}(q)$ if q is even, must be contained, by Corollary 2.2, in an element $K_{\overline{\Omega}}$ of $\mathcal{C}(\overline{\Omega}) \cup \mathcal{S}$ different from $H_{\overline{\Omega}}$ (for the definition of the family $\mathcal{C}(\overline{\Omega}) \cup \mathcal{S}$ we refer to §1.1 and §3.1 in [**KL**]).

By order considerations, one can prove that only condition (i) of Theorem 4.2 in [**Li**] applies: This means that $K_{\overline{\Omega}}$ must be an element of $\mathcal{C}(\overline{\Omega})$. Since $H_{\overline{\Omega}} \in \mathcal{C}_1$, one is left to show that there does not exist an element $K_{\overline{\Omega}}$ in $\mathcal{C}_i$, for an $i \neq 1$, such that $\operatorname{soc} H_{\overline{\Omega}} < K_{\overline{\Omega}} < \overline{\Omega}$.

For q odd, the arguments used in the proof of Proposition 7.1.3 in [**KL**] show that such a $K_{\overline{\Omega}}$ does not exist, taking into account that in our situation $n_2 = n - 1 \geq 7$. To deal with the case when q is even, again one can proceed using arguments suggested in the proof of Lemma 7.1.4 in [**KL**]. □

f) $G_0(n, q)$ is not of type ${}^2D_m$, $n = 2m$, $m \geq 4$.

Proof. Following the notation in [**BGL**], let G be the simple adjoint algebraic group over $\overline{\mathbb{F}}_q$ with associated Dynkin diagram of type D_m, $\lambda = \sigma_q$ and $\mu = {}^2\sigma_q$. Then $O^{p'}(G_\lambda) = P\Omega_n^+(q)$, $O^{p'}(G_\mu) = P\Omega_n^-(q) = G_0(n, q)$,

$$T := O^{p'}(G_\mu \cap G_\lambda) = \begin{cases} \Omega_{n-1}(q) & \text{if } q \text{ is odd} \\ Sp_{n-2}(q) & \text{if } q \text{ is even.} \end{cases}$$

By Theorem 2 in [**BGL**], $[G_0(n, q)/T]$ is monocoatomic. Since $n \geq 8$, T is simple, hence $G_0(n, q)$ is a K-group, a contradiction. □

We have therefore completed the proof that L is not a classical group.

4. The simple exceptional groups of Lie type.

Now we are going to show that the minimal counterexample L cannot be an exceptional group of Lie type $G(q)$.

a) $G(q)$ is not of type G_2, ${}^2G_2$.

Proof. If r is a prime divisor of f, where $q = p^f$, write $q = q_0^r$. Then $G(q_0) <\!\cdot\, G(q)$ ([**Co**], Theorem 2.3, 2.4, [**K3**], Theorem A, C). Hence by Proposition 2.4, we have $L = G_2(p)$, for an odd prime p. But then $G_2(2)$ is maximal in $G_2(p)$ by [**K3**], and we are done by Proposition 2.4. □

b) $G(q)$ is not of type F_4.

Proof. $F_4(q)$ contains a quasisimple maximal subgroup M of type $B_4(q)$, with $|\,Z(M)\,| = (2, q-1)$ ([**LSS**], p. 322). But then, by Proposition 2.3, $F_4(q)$ is a K-group. □

c) $G(q)$ is not of type E_6, E_7, E_8.

Proof. We have $F_4(q) <\!\cdot\, E_6(q)$ ([**LS**], Table 1), which excludes E_6.

If L is of type E_7, there exist subgroups $H \leq M <\!\cdot\, G$ such that $|M : H| = |\,Z(H)\,| = (2, q-1)$ and $H/Z(H) \cong L_2(q) \times P\Omega^+_{12}(q)$ ([**LS**], Table 1). Hence $H/Z(H)$ is a K-group by (2.1). We claim that $[G/H]$ is monocoatomic. Clear if q is even. For q odd, suppose $H < M_1 <\!\cdot\, G$, with $M_1 \neq M$. Since $|\,M : H\,| = 2$, we have $|\,M_1\,| > |\,M\,| \geq q^{64}$. By the Theorem in [**LS**], M_1 either is a parabolic subgroup, or it appears in Table 1 in [**LS**]: However, both situations are excluded by rank or order considerations. So again by Proposition 2.3, G is a K-group, a contradiction.

Finally assume G is of type E_8. There exist subgroups $H \leq M <\!\cdot\, G$ such that $|\,M : H\,| = |\,Z(H)\,| = (2, q-1)$, with $H/Z(H) \cong P\Omega^+_{16}(q)$ ([**I**], p. 286, [**LS**], Table 1), hence a K-group. Using the Theorem in [**LS**] again one shows that $[G/H]$ is monoatomic, giving rise to a contradiction. □

d) $G(q)$ is not of type ${}^2B_2$.
See (2.3).

e) $G(q)$ is not of type ${}^2F_4$.

Proof. The group ${}^2F_4(2)$ is not simple, and we have seen that it is not a K-group (Proposition 2.4). Its derived subgroup (the Tits group) is simple and it is a K-group, since it has a maximal subgroup isomorphic to $L_2(25)$ ([**A**]). So now assume $L = {}^2F_4(2^{2m+1})$, with $m \geq 1$. By the Main Theorem in [**M**], there exist $H <\!\cdot\, M <\!\cdot\, L$ such that $|\,M : H\,| = 2$ and $H \cong Sp_4(2^{2m+1})$. Since the nonabelian composition factors of maximal subgroups of L not conjugate to M are of type $A_1(q)$, ${}^2B_2(q)$, $U_3(q)$ and ${}^2F_4(q^{1/r})$, r an odd prime, one concludes that $[G/H]$ is monocoatomic. □

f) $G(q)$ is not of type ${}^2E_6$.

Proof. In fact we have $F_4(q) <\!\cdot\, {}^2E_6(q)$ from Table 1 in [**LS**]. □

g) $G(q)$ is not of type ${}^3D_4$.

Proof. From the Theorem in [**K2**], we have $G_2(q) <\cdot\, {}^3D_4(q)$. Since $G_2(q)$ is a K-group, we get a contradiction. □

This concludes the proof that L is not a group of Lie type.

5. Sporadic simple groups.

We are left to deal with the sporadic groups: To this end, for each group we exhibit a maximal subgroup which is a K-group. From the tables in [**A**] we have:

$$L_2(11) <\cdot M_{11},\ \ L_2(11) <\cdot M_{12},\ \ A_7 <\cdot M_{22},\ \ M_{22} <\cdot M_{23},\ \ M_{23} <\cdot M_{24},$$
$$L_2(11) <\cdot J_1,\quad A_5 <\cdot J_2,\quad L_2(19) <\cdot J_3,\quad 43:14 <\cdot J_4,\quad M_{22} <\cdot HS,$$
$$A_7 <\cdot Suz,\ \ M_{22} <\cdot McL,\ \ A_8 <\cdot Ru,\ \ S_4 \times L_3(2) <\cdot He,\ \ 67:22 <\cdot Ly,$$
$$A_7 <\cdot O'N,\quad M_{23} <\cdot Co_2,\quad M_{23} <\cdot Co_3,\quad Co_3 <\cdot Co_1,\quad S_{10} <\cdot Fi_{22},$$
$$S_{12} <\cdot Fi_{23},\quad Fi_{23} <\cdot Fi'_{24},\quad A_{12} <\cdot HN,\quad S_5 <\cdot Th,\quad 31:15 <\cdot BM,$$
$$31:15 \times S_3 <\cdot M\quad .$$

We have thus completed the proof of the main theorem:

Theorem. *Every finite simple group is a K-group.*

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