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We provide a rigorous mathematical foundation to the study of strongly rational, holomorphic vertex operator algebras V of central charge $c = 8, 16$ and 24 initiated by Schellekens. If $c = 8$ or 16 we show that V is isomorphic to a lattice theory corresponding to a rank c even, self-dual lattice. If $c = 24$ we prove, among other things, that either V is isomorphic to a lattice theory corresponding to a Niemeier lattice or the Leech lattice, or else the Lie algebra on the weight one subspace V_1 is semisimple (possibly 0) of Lie rank less than 24.

1. Introduction.

One of the highlights of discrete mathematics is the classification of the positive-definite, even, unimodular lattices L of rank at most 24, due originally to Minkowski, Witt and Niemeier (cf. [**CS**] for more information and an extensive list of references). One knows (loc. cit.) that such a lattice has rank divisible by 8; that the E_8 root lattice is (up to isometry) the unique example of rank 8; that the two lattices of type $E_8 + E_8$ and Γ_{16} are the unique examples of rank 16; and that there are 24 inequivalent such lattices of rank 24. In each case, the lattice may be characterized by the nature of the semi-simple root system naturally carried by the set of minimal vectors (i.e., those of squared length 2).

The theory of vertex operator algebras is a newer subject which enjoys several parallels with lattice theory. Already in [**B**], Borcherds pointed out that one can naturally associate a vertex operator algebra V_L to any positive-definite, even lattice L (cf. [**FLM**] for a complete discussion). It is known that V_L is rational and that it is holomorphic precisely when L is self-dual ([**D**] and [**DLM1**]) (we defer the formal introduction of technical definitions concerning vertex operator algebras until Section 2). Since the central charge c of the vertex operator algebra V_L is precisely the rank of L, the classification of holomorphic vertex operator algebras of central charge at most 24 may be construed as a generalization of the corresponding problem for even, unimodular lattices. We will say that a holomorphic vertex operator algebra has *small central charge* in case it satisfies $c \leq 24$.

Schellekens was the first to consider the problem of classifying holomorphic vertex operator algebras V of small central charge [**Sch**]. Based on extensive computation, Schellekens wrote down a list of 71 integral q-expansions $q^{-1}+$ constant $+196884q+\cdots$ and, among other things, conjectured that the graded character of a V satisfying $c=24$ is necessarily equal to one of these 71 q-expansions. As is well-known, it is only the constant term that distinguishes the q-expansions from each other, the constant in question being the dimension of the Lie algebra naturally defined on the weight one subspace V_1 of V (see below for more details). Schellekens in fact wrote down a list of 71 Lie algebras (including levels) which are the candidates for V_1. (It turns out that if V has central charge strictly less that 24 then the graded character of V is uniquely determined, so at least as far as the dimension of the weight one subspace is concerned, the case $c=24$ carries the most interest.)

The purpose of the present paper is to put the Schellekens program on a firm mathematical foundation within the general context of rational conformal field theory, and to make a start towards the classification (up to isomorphism) of the holomorphic vertex operator algebras of small central charge. In addition to unitarity, there are other (unstated) assumptions in [**Sch**]. This circumstance means that we cannot assume the results of (loc. cit.) and need to find new approaches. In some ways we will go much further than [**Sch**] in that we will be able to give an adequate characterization of the holomorphic lattice theories among all holomorphic theories of small central charge. On the other hand, although we are able to establish numerical restrictions on the nature of the Lie algebra on V_1 which show that there are only a finite number of possibilities, we are at present unable to show, assuming $c=24$, that it is necessarily one of the 71 on Schellekens' list. The remaining obstacle is essentially that of establishing[1] that the levels of the associated Kac-Moody Lie algebras are positive integers (which is immediate if unitarity is assumed).

We now state some of our main results, and for this purpose we will take V to be a holomorphic vertex operator algebra of CFT type (see Section 2) which is C_2-cofinite. (In the language of [**DM**], V is strongly rational and holomorphic.) By results of Zhu [**Z**], this implies that c is a positive integer divisible by 8.

Theorem 1. *Suppose that $c=8$. Then V is isomorphic to the lattice theory V_{E_8} associated to the E_8 root lattice.*

Theorem 2. *Suppose that $c=16$. Then V is isomorphic to a lattice theory V_L where L is one of the two unimodular rank 16 lattices.*

[1] However, see comments added in proof at the end of the paper.

Thus for the cases $c = 8$ and 16, the classification of the holomorphic vertex operator algebras completely mirrors that of the corresponding lattices. These two theorems are commonly assumed in the physics literature, and are related to the uniqueness of the heterotic string ([**Sch**] and [**GSW**]).

Theorem 3. *Suppose that $c = 24$. Then the Lie algebra on V_1 is reductive, and exactly one of the following holds:*

(a) $V_1 = 0$.
(b) *V_1 is abelian of rank 24. In this case V is isomorphic to the lattice theory V_Λ where Λ is the Leech lattice.*
(c) *V_1 is a semi-simple Lie algebra of rank 24. In this case V is isomorphic to the lattice theory V_L where L is the even, unimodular rank 24 lattice whose root system is the same as the root system associated to V_1.*
(d) *V_1 is a semi-simple Lie algebra of rank less than 24.*

We actually establish more than is stated here. It is a well-known conjecture [**FLM**] that the Moonshine module is characterized among all $c = 24$ holomorphic theories by the condition $V_1 = 0$. Our methods are less effective when there is no Lie algebra, however we will show that in this case V_2 carries the structure of a simple, commutative algebra (of dimension 196,884). The commutativity and dimension formula are well-known; it is the simplicity that is novel here. (The inherent difficulty in dealing with V_2 is that it is not an associative algebra, indeed it is not even power associative, and there seem to be no useful identities which are satisfied.) In Case (d) we show that the simple components $\mathfrak{g}_i$ of V_1 have levels k_i and dual Coxeter numbers $h_i^\vee$ such that the identity

$$\frac{h_i^\vee}{k_i} = \frac{\dim V_1 - 24}{24} \tag{1.1}$$

holds for each $\mathfrak{g}_i$. This implies that there are only finitely many choices for the family of pairs $(\mathfrak{g}_i, k_i)$ determined by V_1. Note that the condition (1.1) was already identified by Schellekens [**Sch**]. We are in fact able to extract some further numerical restrictions on the levels k_i, but these fall well short of the expectation[1] that they are all positive integers, and we forgo any discussion of this beyond (1.1). We also establish that if V_1 is semisimple in Theorem 3, then the Virasoro element in V coincides with the usual Virasoro element associated with an affine Lie algebra defined by the Sugawara construction. We expect the result to be useful in further analysis of the situation.

The main inspiration for the proof of our results originates from our recent paper [**DM**]. There, we introduced methods based on the theory of modular forms used in tandem with techniques from vertex operator algebra theory. In particular we obtained a simple numerical characterization of the lattice vertex operator algebras among all rational vertex operator algebras. The present paper is in many ways a continuation and elaboration of [loc.cit.]. When the central charge is small, the theory of modular forms gives very precise numerical information about the vertex operator algebra which allows us to show, under certain circumstances, that the numerical characterizations of [**DM**] are applicable. This leads to Theorems 1-3. Modular-invariance also underlies the other results that we obtain.

In addition to the Schellekens program that we have already discussed, another potential application of Theorem 3 is to the FLM conjecture regarding the Moonshine module alluded to above. Namely, Theorem 3 shows that the Leech lattice theory V_Λ is the only $c = 24$ holomorphic theory for which the Lie algebra V_1 is both nonzero and not semi-simple. On the other hand, the Moonshine module is closely related to V_Λ, being built from it by a $\mathbb{Z}_2$-orbifold construction [**FLM**].

The paper is organized as follows: We gather together some preliminaries in Section 2, and prove Theorems 1-3 together with the supplementary result in Case (a) of Theorem 3 in Section 3. In Section 4 we identify the Virasoro element with the Sugawara construction.

2. Preliminary results.

For general background on the theory of vertex operator algebras, we refer the reader to [**FLM**] and [**FHL**]. As usual, for a state v in a vertex operator algebra V, we denote the corresponding vertex operator by

$$Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1},$$

while the vertex operator corresponding to the conformal (Virasoro) vector is

$$Y(\omega,z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}.$$

V is called *rational* if all admissible V-modules are completely reducible (cf. [**DLM1**] for the definition of admissible module). It was shown in [**DLM2**] that this implies that V has only finitely many inequivalent simple modules. A rational vertex operator algebra is called *holomorphic* if it has a unique simple module, namely the adjoint module V. V is said to be C_2-*cofinite* in case the subspace spanned by elements $u_{-2}v$ for $u, v \in V$ is of finite codimension. Finally, we say that V is of CFT-*type* in case the natural $\mathbb{Z}$-grading on V takes the form

$$V = V_0 \oplus V_1 \oplus \cdots \quad \text{with} \quad V_0 = \mathbb{C}\mathbf{1}. \tag{2.1}$$

Throughout the rest of this paper, we assume that V is a C_2-cofinite, holomorphic vertex operator algebra of CFT-type of small central charge $c \le 24$. In order to avoid the case of the trivial vertex operator algebra $V = \mathbb{C}$, we also assume that V has dimension greater than one.

Next we discuss some consequences of these assumptions for the structure of V. Many of them are well-known. First, the weight 1 subspace V_1 of V carries a natural structure of Lie algebra given by

$$[u, v] = u_0 v \tag{2.2}$$

for $u, v \in V_1$. Because the adjoint module is the unique simple V-module, then the contragredient module V' is necessarily isomorphic to V. This is equivalent to the existence of a nondegenerate, invariant bilinear pairing

$$\langle , \rangle : V \times V \to \mathbb{C}$$

which is necessarily symmetric. (For the theory of contragredient modules, cf. Section 3 of [**FHL**].) Because of (2.1), Li's theory of invariant bilinear forms [**L1**] shows that we have $L(1)V_1 = 0$ and that $\langle , \rangle$ is uniquely determined up to an overall scalar. It is convenient to fix the normalization so that

$$\langle \mathbf{1}, \mathbf{1} \rangle = -1. \tag{2.3}$$

In particular, the restriction of $\langle , \rangle$ to V_1 is given by

$$\langle u, v \rangle \mathbf{1} = u_1 v \tag{2.4}$$

for $u, v \in V_1$.

In the language of [**DM**], it follows from what we have said that V is *strongly rational,* so that the results of (loc. cit.) apply. They tell us that the following hold:

(I) V_1 is a reductive Lie algebra of Lie rank $l \leq c$.
(II) $l = c$ if, and only if, V is isomorphic to a lattice theory V_L for some positive-definite, even, unimodular lattice L.

We also note that by results of Zhu [**Z**] (cf. [**DLM3**]), c is necessarily a positive integer divisible by 8. So in fact $c = 8, 16$, or 24.

We refer the reader to [**Z**] and [**DM**] for an extended discussion of the role of modular-invariance in the theory of rational vertex operator algebras. We need to recall the vertex operator algebra $(V, Y[\ ], \mathbf{1}, \omega - c/24)$ from [**Z**]. The new vertex operator associated to a homogeneous element a is given by

$$Y[a, z] = \sum_{n \in \mathbb{Z}} a[n] z^{-n-1} = Y(a, e^z - 1) e^{z \mathrm{wt} a}$$

while the Virasoro element is $\widetilde{\omega} = \omega - c/24$. Thus

$$a[m] = \mathrm{Res}_z \left(Y(a, z)(\ln(1+z))^m (1+z)^{\mathrm{wt} a - 1} \right)$$

and

$$a[m] = \sum_{i=m}^{\infty} c(\mathrm{wt} a, i, m) a(i)$$

for some scalars $c(\mathrm{wt}a, i, m)$ such that $c(\mathrm{wt}a, m, m) = 1$. In particular,

$$a[0] = \sum_{i \geq 0} \binom{\mathrm{wt}a - 1}{i} a(i).$$

We also write

$$Y[\omega - c/24, z] = \sum_{n \in \mathbb{Z}} L[n] z^{-n-2}.$$

Then the $L[n]$ again generate a copy of the Virasoro algebra with the same central charge c. Now V is graded by the $L[0]$-eigenvalues, that is

$$V = V_{[0]} \oplus V_{[1]} \oplus \cdots$$

where $V_{[n]} = \{v \in V | L[0]v = nv\}$.

We will need the following facts: Let $v \in V$ satisfy $L[0]v = kv$. Then the graded trace

$$Z(v, \tau) = \mathrm{Tr}_V o(v) q^{L(0)-c/24} = q^{-c/24} \sum_{n \geq 0} \mathrm{Tr}_{V_n} o(v) q^n \tag{2.5}$$

is a modular form on $\mathrm{SL}(2, \mathbb{Z})$ (possibly with character), holomorphic in the complex upper half-plane H and of weight k. Here, we have set $o(v)$ to be the *zero mode* of v, defined via $o(v) = v_{\mathrm{wt}v-1}$ if v is homogeneous, and extended by linearity to the whole of V. Moreover, τ will denote an element in H and $q = e^{2\pi i \tau}$. In particular, if we take v to be the vacuum element then (2.5) is just the graded trace

$$\mathrm{ch}_q V = q^{-c/24} \sum_{n \geq 0} (\dim V_n) q^n \tag{2.6}$$

and is a modular function of weight zero on $\mathrm{SL}(2, \mathbb{Z})$. Because of the holomorphy of $\mathrm{ch}_q(V)$ in H and our assumption that $c \leq 24$, one knows [**L**] that (2.6) is uniquely determined up to an additive constant, and indeed is determined uniquely if $c \leq 16$. The upshot is this:

Lemma 2.1. *One of the following holds:*

(a) $c = 8$ *and* $\mathrm{ch}_q(V) = \Theta_{E_8}(q)/\eta(q)^8 = q^{-1/3}(1 + 248q + \cdots)$
(b) $c = 16$ *and* $\mathrm{ch}_q(V) = (\Theta_{E_8}(q))^2/\eta(q)^{16} = q^{-2/3}(1 + 496q + \cdots)$
(c) $c = 24$ *and* $\mathrm{ch}_q(V) = J(q) + \mathrm{const} = q^{-1} + \mathrm{const} + 196884q + \cdots$.

Here, we have introduced the theta function $\Theta_{E_8}(q)$ of the E_8 root lattice

$$\Theta_{E_8}(q) = \sum_{\alpha \in E_8} q^{(\alpha,\alpha)/2}$$

(where E_8 denotes the root lattice of type E_8 normalized so that the squared length of a root is 2) as well as the eta function

$$\eta(q) = q^{1/24} \prod_{n \geq 1} (1 - q^n);$$

$$J(q) = q^{-1} + 0 + 196884q + \cdots$$

is the absolute modular invariant normalized to have constant term zero (alias the graded character of the Moonshine Module [**FLM**]). We also need the 'unmodular' Eisenstein series of weight two, namely

$$E_2(q) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n$$

where $\sigma_1(n)$ is the sum of the divisors of n.

Lemma 2.2. *For states u, v in V_1 we have*

$$\mathrm{Tr}_V o(u)o(v)q^{L(0)-c/24} = \frac{\langle u, v\rangle}{24} \left(\frac{24}{c} D_q(ch_q V) + E_2(q) ch_q V \right)$$

where $D_q = q\frac{d}{dq}$.

Proof. First recall the important identity, Proposition 4.3.5 in [**Z**]. If we pick a pair of states $u, v \in V_1$ this result yields the following:

$$\mathrm{Tr}_V o(u)o(v)q^{L(0)-c/24} = Z(u[-1]v, \tau) + 1/12 E_2(\tau) Z(u[1]v, \tau). \tag{2.7}$$

The term $Z(u[-1]v, \tau)$ is a modular form of weight 2, and we shall be able to write it down explicitly. Indeed, the l.h.s. of (2.7) has leading term $\kappa(u, v)q^{1-c/24}$ where $\kappa(u, v)$ denotes the usual Killing form on V_1, whereas the leading term of the second summand on the r.h.s. of (2.7) is equal to $\frac{-1}{12}\langle u, v\rangle q^{-c/24}$. From this discussion we conclude that the leading term of $Z(u[-1]v, \tau)$ is equal to $\frac{1}{12}\langle u, v\rangle q^{-c/24}$. We claim that up to scalars, the unique form of weight 2 on $\mathrm{SL}(2, \mathbb{Z})$ which is holomorphic in H and has a pole of order $c/24$ at infinity is $D_q(ch_q(V))$. To see this, suppose that $f(\tau)$ and $g(\tau)$ are two such forms. Then for some scalar α, $f(\tau) - \alpha g(\tau)$ is a form of weight 2 on $\mathrm{SL}(2, \mathbb{Z})$ which is holomorphic both in H and at infinity. But it is well-known [**L**] that such a form is identically zero, so that $f(\tau) = \alpha g(\tau)$. So, up to scalars, there is at most one form $f(\tau)$ with the stated properties. On the other hand, as $ch_q V$ is a form of weight zero on $\mathrm{SL}(2, \mathbb{Z})$ which is holomorphic in H and has a pole of order $c/24$ at infinity, then $D_q(ch_q V)$ is a weight 2 form [**L**] with the same divisor. Thus $D_q(ch_q V)$ is weight 2 form on $\mathrm{SL}(2, \mathbb{Z})$ with the desired properties.

As a result, we see that the first term on the r.h.s. of (2.7) is equal to $\frac{2\langle u,v\rangle}{c} D_q(ch_q(V))$. The lemma follows from this, (2.7), and (2.4). □

Identifying coefficients of $q^{1-c/24}$ in the formula of Lemma 2.2 yields:

Corollary 2.3. $\kappa(u, v) = 2\langle u, v\rangle(\frac{\dim V_1}{c} - 1)$.

3. Proof of the theorems.

We consider the three possibilities $c = 8, 16, 24$ in turn. In the first two cases the idea is to show that (II) *always* applies, while in the third case we study *when* it applies.

Case 1. $c = 8$. In this case dim $V_1 = 248$ by Lemma 2.1 (a), so that $\kappa(u, v) = 60\langle u, v\rangle$ by Corollary 2.3. Since $\langle , \rangle$ is nondegenerate then so too is the Killing form. We conclude that in fact V_1 is semi-simple of dimension 248, and by (I) the Lie rank is no greater than 8. By the classification of semi-simple Lie algebras, we conclude that in fact V_1 is the Lie algebra of type E_8 and Lie rank 8. Now (II) and the fact that there is a unique positive-definite, even, unimodular lattice of rank 8, namely the E_8 root lattice, completes the proof of Theorem 1.

Case 2. $c = 16$. This is very similar to Case 1. Namely, we have dim $V_1 = 496$ by Lemma 2.1 (b), whence $\kappa(u, v) = 60\langle u, v\rangle$ once more. So again V_1 is semi-simple, and we proceed as in Case 1, now using the fact that there are just the two positive-definite, even, unimodular lattices. Theorem 2 follows.

Case 3. $c = 24$. Set $\dim V_1 = d$. From Corollary 2.3 we see in this case that

$$\kappa(u, v) = \langle u, v\rangle(d - 24)/12. \tag{3.1}$$

If $d = 24$ then the Killing form is identically zero. Then V_1 is solvable by Cartan's criterion and therefore abelian since V_1 is in any case reductive by (I). So if $d = 24$ then we have shown that V_1 has rank 24 and is therefore a lattice theory V_L for suitable L by (II). The fact that $(V_L)_1$ is abelian tells us that L has no roots i.e., no vectors of squared length two, and that L is therefore the Leech lattice (cf. [**CS**]). This deals with Case (b) of Theorem 3.

Let us assume from now on that $d \neq 0, 24$. Together with (3.1), this tells us that the Killing form on V_1 is again nondegenerate, so that V_1 is a semi-simple Lie algebra of Lie rank l no greater than 24 by (I). Moreover, if $l = 24$ then V is a lattice theory by (II) once more. This confirms Part (c) of Theorem 3. □

Next we consider the levels of the affine Lie algebras spanned by the vertex operators $Y(u, z)$, u in V_1. For states $u, v \in V_1$ and integers m, n we have

$$[u_m, v_n] = (u_0 v)_{m+n} + m u_1 v \delta_{m,-n}, \tag{3.2}$$

whereas the usual relations for a Kac-Moody Lie algebra of level k associated to a simple Lie algebra $\mathfrak{g}$ take the form

$$[a_m, b_n] = [a, b]_{m+n} + k(a, b) m \delta_{m,-n} \tag{3.3}$$

where (a, b) is the nondegenerate form on $\mathfrak{g}$ normalized so that $(\alpha, \alpha) = 2$ for a long root α.

Let V_1 be a direct sum

$$V_1 = \mathfrak{g}_{1,k_1} \oplus \mathfrak{g}_{2,k_2} \oplus \cdots \oplus \mathfrak{g}_{n,k_n} \tag{3.4}$$

of simple Lie algebras $\mathfrak{g}_i$ whose corresponding affine Lie algebra has level k_i. By comparing (3.2) and (3.3), using Corollary 2.3 we obtain for $u, v \in \mathfrak{g}_i$ that

$$\kappa_{\mathfrak{g}_i}(u, v) = (d - 24)k_i(u, v)/12 \tag{3.5}$$

where $\kappa_{\mathfrak{g}_i}$ denotes the restriction of the Killing form to $\mathfrak{g}_i$. Now $\kappa_{\mathfrak{g}_i}(h_\alpha, h_\alpha) = 4h_i^\vee$ for a long root α, where $h_i^\vee$ is the dual Coxeter number of the root system associated to $\mathfrak{g}_i$. Therefore, (3.5) tells us that for each simple component $\mathfrak{g}_i$ of V_1, of level k_i and dual Coxeter number $h_i^\vee$, the ratio

$$h_i^\vee / k_i = (d - 24)/24 \tag{3.6}$$

is independent of $\mathfrak{g}_i$.

Proposition 3.1. *Assume that Case* (a) *of Theorem* 3 *holds. Then* $B = V_2$ *carries the structure of a* (*non-associative*) *simple, commutative algebra with respect to the product* $a.b = a_1 b$.

Proof. We take for granted the well-known facts that B is indeed a non-associative, commutative algebra with respect to the indicated product, and that the pairing $\langle , \rangle : B \times B \to \mathbb{C}$ defined by

$$a_3 b = \langle a, b\rangle \mathbf{1} \tag{3.7}$$

endows B with a nondegenerate, invariant trace form. That is, $\langle , \rangle$ is symmetric and satisfies $\langle ab, c\rangle = \langle a, bc\rangle$. Moreover B has an identity element $1/2\omega$. Set $d = \dim B = 196884$.

Next we state two more results that we will need. Each can be established using modular-invariance arguments along the same lines as before. Alternatively, we may use results of Section 4 of [**M**]:

$$\mathrm{Tr}_B o(ab) = (d/3)\langle a, b\rangle. \tag{3.8}$$

If $e^2 = e$ is in B then

$$\mathrm{Tr}_B o(e)^2 = 4620\langle e, e\rangle + 20336\langle e, e\rangle^2. \tag{3.9}$$

Turning to the proof of the Proposition, we first show that B is semisimple. Indeed, (3.8) guarantees that the form $\mathrm{Tr}_B o(ab)$ is nondegenerate, and this is sufficient to establish that B is semi-simple. To see this, recall a well-known result of Dieudonne (cf. [**S**, Theorem 2.6]) that an arbitrary algebra B is semisimple if it has a nondegenerate trace form and contains no nonzero nilpotent ideals. We will show that indeed there are no nonzero nilpotent ideals in B. If not, we may choose a minimal nonzero nilpotent ideal M in B. Note that $M^2 = 0$. Let m be a nonzero element in M, and let b in B be arbitrary. Then mb lies in M, so that $(mb)B \subset M$ and $(mb)M = 0$. This shows that each element mb is nilpotent as a multiplication operator on B, and hence $\mathrm{Tr}_B mb = 0$ for all b. This contradicts the non-degeneracy of $\langle , \rangle$.

From the last paragraph we know that B can be written as an (orthogonal) direct sum of simple ideals

$$B = B_1 + B_2 + \cdots + B_t.$$

We must show that $t = 1$. Write $1/2\omega = e_1 + e_2 + \cdots + e_t$ where e_i is the identity element of B_i. In particular, e_i is a central idempotent of B. By (3.8) and (3.9) we obtain

$$(d/3)\langle e, e\rangle = 4620\langle e, e\rangle + 20336\langle e, e\rangle^2 \tag{3.10}$$

where e is any of the idempotents e_i. Notice that (3.10) yields that

$$\langle e, e\rangle = 3. \tag{3.11}$$

It is well-known that (3.11) implies that the components of the vertex operator $Y(2e, z)$ generate a Virasoro algebra of central charge 24, so that the total central charge must be $24t$. Hence, $t = 1$ as required. □

4. Some Virasoro elements.

Throughout this section we assume that V is a strongly rational, holomorphic vertex operator algebra of central charge 24 as before, and we assume in addition that V_1 is a (nonzero) semisimple Lie algebra of Lie rank l. Let V_1 have decomposition (3.4) into simple Lie algebras. Set $d_i = \dim \mathfrak{g}_{i,k_i}$. Let $(,)$ denote the normalized invariant bilinear form on $\mathfrak{g}_{i,k_i}$ with the property that $(\alpha, \alpha) = 2$ for a long root α in $\mathfrak{g}_{i,k_i}$. It is known (cf. [**DL**], [**FZ**], [**K**] and [**L2**]) that the element

$$\omega_i = \frac{1}{2(k_i + h_i^\vee)} \sum_{j=1}^{d_i} u_{-1}^j u_{-1}^j \mathbf{1} \tag{4.1}$$

is a Virasoro element of central charge $c_i = \frac{k_i \dim \mathfrak{g}_i}{k_i + h_i^\vee}$, where $u^1, u^2, \ldots, u^{d_i}$ is an orthonormal basis of $\mathfrak{g}_i$ with respect to $(,)$.

There are three Virasoro elements in V that are relevant to a further analysis of the situation. Namely, in addition to the original Virasoro element ω in V, we have

$$\omega_{aff} = \sum_{i=1}^{n} \omega_i \tag{4.2}$$

and

$$\omega_H = \frac{1}{2} \sum_{i=1}^{l} (h_{-1}^i)^2 \mathbf{1} \tag{4.3}$$

where $h^1, \dots, h^l$ is an orthonormal basis of a maximal abelian subalgebra H of V_1 with respect to the inner product $\langle , \rangle$. Note that as a consequence of Equation (3.6), ω_{aff} has central charge $\sum c_i = 24$. We omit further discussion of ω_H, but will prove:

Proposition 4.1. *$\omega_{aff} = \omega$.*

Proof. Consequences of modular-invariance again underlie the proof of the Proposition, notably the absence of cusp-forms of small weight on $\mathrm{SL}(2,\mathbb{Z})$. We introduce some notation for Virasoro operators: In addition to the usual operators $L(n)$ associated to ω , we use $L^{aff}(n)$ for the corresponding operators associated to ω_{aff}. We also set $\omega' = \omega - \omega^{aff}$. We will soon see that ω' is itself a Virasoro element, and define its component operators to be $L'(n)$. We eventually want to prove of course that $\omega' = 0$. We proceed in a series of steps.

Step 1. ω' is a highest weight vector of weight 2 for the Virasoro algebra generated by $L(n)$.

The definition of ω' shows that it lies in V_2, so it suffices to show that ω' is annihilated by the operators $L(1)$ and $L(2)$. First calculate that if u is an element of V_1 then $[L(1), u_{-1}] = u_0$. Then from the definitions it follows easily that $L(1)$ annihilates both ω_{aff} and ω_H, whence it also annihilates ω'. To establish that $L(2)$ annihilates ω' we must show that

$$L(2)\omega_{aff} = L(2)\omega_H = 12. \tag{4.4}$$

Now for u in V_1 we have $[L(2), u_{-1}] = u_1$. Then

$$L(2)u^j_{-1}u^j_{-1}\mathbf{1} = -\langle u^j, u^j \rangle = k_i,$$

and (4.4) follows.

Step 2. ω' is a Virasoro vector of central charge 0.

Use Step 1, in particular that $L(1)$ annihilates ω_{aff}, to see that

$$[L(m), L^{aff}(n)] = (m-n)L^{aff}(m+n) - 2(m^3 - m)\delta_{m,-n}.$$

The result follows easily from this.

Step 3. $Z_V(\omega', \tau) = 0$.

Since ω' is a highest weight vector of weight 2 for the Virasoro operators corresponding to ω then from our earlier discussion of (2.5) we see that $Z_V(\omega', \tau)$ is a modular form on $\mathrm{SL}(2, Z)$ of weight 2 and is holomorphic in the upper half-plane. Moreover there is no pole in the q-expansion of $Z_V(\omega', \tau)$, so $Z_V(\omega', \tau)$ is in fact a holomorphic modular form of weight 2 on $\mathrm{SL}(2, Z)$, hence must be zero.

Step 4. $\mathrm{Tr}_V L'(0)^2 q^{L(0)-1} = 0$.

By Proposition 4.3.5 in [**Z**] we have

$$\mathrm{Tr}_V L'(0)^2 q^{L(0)-1} = Z_V(L'[-2]\omega', \tau) - \sum_{k\geq 1} E_{2k}(\tau) Z_V(L'[2k-2]\omega', \tau)$$

where $Y[\omega', z] = \sum_{m\in\mathbb{Z}} L'[m]z^{-m-2}$ and the functions $E_{2k}(\tau)$ are Eisenstein series of weight $2k$, normalized as in [**DLM3**]. $E_{2k}(\tau)$ is a holomorphic modular form on $\mathrm{SL}(2,\mathbb{Z})$ if $k > 1$. Since $L'[2k-2]\omega' = 0$ if $k \geq 2$, $L'[0]\omega' = 2\omega'$, and $L[0]L'[-2]\omega' = 4L'[-2]\omega'$, we see that

$$\mathrm{Tr}_V L'(0)^2 q^{L(0)-1} = Z_V(L'[-2]\omega', \tau)$$

is a modular form of weight 4 for $\mathrm{SL}(2,\mathbb{Z})$. Since $L'(0)V_1 = 0$, it is in fact a cusp form, hence equal to zero.

Step 5. All the eigenvalues of $L^{aff}(0)$ on V are real.

In order to see this recall the decomposition (3.4) and set $\mathfrak{g} = V_1$. Consider the affine Lie algebra

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}$$

with bracket

$$[x\otimes t^p, y \otimes t^q] = [x, y] \otimes t^{p+q} + p\delta_{p+q,0}\langle x, y\rangle$$

for $x, y \in \mathfrak{g}$ and $p, q \in \mathbb{Z}$. Since each V_m is finite dimensional we see that V has a composition series as a module for $\hat{\mathfrak{g}}$ such that each factor is an irreducible highest weight $\hat{\mathfrak{g}}$-module. So it is enough to show that $L^{aff}(0)$ has only real eigenvalues on any irreducible highest weight $\hat{\mathfrak{g}}$-module. Note that such an irreducible highest weight $\hat{\mathfrak{g}}$-module is a tensor product of irreducible highest weight $\hat{\mathfrak{g}}_i$-modules $L(k_i, \Lambda_i)$ of level k_i $(i = 1, \ldots, n)$ for some dominant weight Λ_i in the weight lattice of $\mathfrak{g}_i$ as V is a completely reducible $\mathfrak{g}_i$-module. Here $L(k_i, \Lambda_i)$ is the unique irreducible quotient of the generalized Verma module $U(\hat{\mathfrak{g}}_i) \otimes_{U(\mathfrak{g}_i\otimes\mathbb{C}[t]+\mathbb{C})} L(\Lambda_i)$ where $L(\Lambda_i)$ is the highest weight module for $\mathfrak{g}_i$ with highest weight Λ_i and $x \otimes t^m$ acts as zero if $m > 0$ and $x \otimes t^0$ acts as x. So it is enough to show that $L_i(0)$ has only real eigenvalues on $L(k_i, \Lambda_i)$ where $Y(\omega_i, z) = \sum_{m\in\mathbb{Z}} L_i(m)z^{-m-2}$.

It is well-known that the eigenvalues of $L_i(0)$ on $L(k_i, \Lambda_i)$ are the numbers $\frac{(\Lambda_i+2\rho_i,\Lambda)}{2(k_i+h_i^\vee)} + m$ (cf. [**DL**] and [**K**]) for nonnegative integers m where ρ_i is the half-sum of the positive roots of $\mathfrak{g}_i$. Since k_i is rational, it is clear that $\frac{(\Lambda_i+2\rho_i,\Lambda)}{2(k_i+h_i^\vee)} + m$ is real, as required.

Step 6. $\omega' = 0$.

By Step 5, all eigenvalues of $L'(0)$ are real on V. This implies that $L'(0)^2$ has only nonnegative eigenvalues. By Step 4 we conclude that all the eigenvalues of $L'(0)$ are zero. Since $L'(0)\omega' = 2\omega'$, we must have $\omega' = 0$, as desired. □

Added in proof. The authors have recently been able to establish the conjectured integrality of the levels. Details will appear elsewhere.

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