Pacific
Journal of
Mathematics

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Volume 213 No. 2 February 2004

PACIFIC JOURNAL OF MATHEMATICS
Vol. 213, No. 2, 2004

THE INTEGRAL KERNEL IN THE KUZNETSOV SUM FORMULA FOR SU$(n+1,1)$ (II). THE CASE OF ONE DIMENSIONAL K-TYPES

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The Kuznetsov sum formula relates spectral data concerning automorphic forms to geometric data concerning the intersection of a discrete subgroup with the big cell in the Bruhat decomposition. An explicit formula for the integral kernel in the Kloosterman term of this formula is given for the groups isomorphic to SU$(n+1,1)$, $n \geq 2$ and arbitrary one dimensional K-types.

1. Introduction.

Let G be a connected semisimple Lie group of real rank one, and let Γ be a discrete subgroup of finite covolume of G. There is a Kuznetsov formula in this context (see [**MW**]) that relates spectral data concerning automorphic forms to geometric data concerning the intersection of a discrete subgroup with the big cell in the Bruhat decomposition. The τ-function is the kernel for the integral transformation relating test functions on the spectral side to those on the geometric side. In the classical case, this integral transformation can be described in terms of classical Bessel functions (see [**K**], [**GW**], [**MW**], Appendix), but in the general case the determination of the Kloosterman term in the Kuznetsov formula is much more complicated.

This function has been determined when $G = \mathrm{SO}(n+1,1)$ and $\mathrm{SU}(n+1,1)$ (see [**MW**] and [**Kb**] respectively) in the case of the trivial K-type. In the present paper we shall extend the methods in [**Kb**] to obtain a formula for τ (see (2.7)) in the case of the group $\mathrm{SU}(n+1,1)$ and an arbitrary one dimensional K-type. Similarly as in [**Kb**] the calculation requires solving complicated recurrence relations.

Let $G = NAK$ be an Iwasawa decomposition of G and let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ be the corresponding decomposition at the Lie algebra level. Let M be the centralizer of A in K and let χ be a nontrivial unitary character on N. As in [**Kb**], one necessary ingredient in our computations is the knowledge of generators for the M_χ-invariants in the universal Lie algebra $\mathcal{U}(\overline{\mathfrak{n}})$ for those groups (see [**MV**]). The main difference comes from the fact that in this case the recurrence formulas needed to compute the τ-function involve new and complicated terms that become zero only when the K-type is trivial.

However, the final formula for τ in the present case ends up being similar to that obtained in [**Kb**], but involving a real parameter that depends on the K-type.

A new feature of this case is the presence of poles in the right half plane of the form $\mu - 2$, $\mu - 4$, $\ldots$, where $\mu \in \mathbb{N}$ is the K-type parameter.

2. Preliminaries.

We consider the Lie subalgebra of $\mathfrak{gl}(n+2, \mathbf{C})$ given by

$$\mathfrak{g} = \{X \in \mathfrak{gl}(n+2, \mathbf{C}) \mid XJ + J\overline{X}^t = 0,\ \operatorname{tr} X = 0\}$$

where

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and I_n is the $n \times n$ identity matrix. Then $\mathfrak{g}$ is the Lie algebra $\simeq \mathcal{S}U(n+1,1)$. We denote by $G \simeq \mathrm{SU}(n+1,1)$ the connected Lie subgroup of $\mathrm{Gl}(n+2, \mathbf{C})$ with Lie algebra $\mathfrak{g}$. A Cartan involution of $\mathfrak{g}$ is given by $\theta(X) = -\overline{X}^t$.

This involution induces the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We take $\mathfrak{a}$ the maximal abelian subalgebra of $\mathfrak{p}$ given by $\mathfrak{a} = \mathbb{R}H$, where $H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0_n & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Let K and A be the connected Lie subgroups of G corresponding to $\mathfrak{k}$ and $\mathfrak{a}$, respectively. Let M be the centralizer of A in K, and let $\mathfrak{m}$ be the corresponding Lie algebra of M. If $\alpha \in \mathfrak{a}^*$ is such that $\alpha(H) = 1$, then let $\mathfrak{n}_\alpha$ and $\mathfrak{n}_{2\alpha}$ be the root spaces associated to α and 2α, respectively. Let $\rho(H_0) = \frac{1}{2}\mathrm{tr}\ \mathrm{ad}\,(H_0)\,|_{\mathfrak{n}}$, $H_0 \in \mathfrak{a}$. Then $\rho(H) = n+1$. We have $\mathfrak{n}_\alpha = \{X(x) \mid x \in \mathbb{C}^n\}$ and $\mathfrak{n}_{2\alpha} = \mathbb{R}Z(i)$, where

$$X(x) = \begin{pmatrix} 0 & x & 0 \\ 0 & 0_n & -\overline{x}^t \\ 0 & 0 & 0 \end{pmatrix}, \qquad Z(i) = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We also have

$$\mathfrak{m} = \left\{ M(A) = \begin{pmatrix} a & \ldots & 0 \\ . & A & . \\ 0 & \ldots & a \end{pmatrix} \mid A \in M_n(\mathbb{C}), A + \overline{A}^t = 0,\ 2a + \mathrm{tr}\ (A) = 0 \right\}.$$

If $\mathfrak{n} = \mathfrak{n}_\alpha \oplus \mathfrak{n}_{2\alpha}$, then $\mathfrak{g}$ has the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$. Let $G = NAK$ be the corresponding Iwasawa decomposition at the group level. If $\overline{\mathfrak{n}} = \theta\mathfrak{n}$, the $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \overline{\mathfrak{n}}$.

Let $B(X,Y) = \frac{1}{2}\mathrm{tr}\,(X.Y)$, $\langle X, Y\rangle = -B(X, \theta Y)$, X, $Y \in \mathfrak{g}$. Then B is $\mathfrak{g}$-invariant and $B(H,H) = 1$.

If $e_1, \ldots, e_n$ denotes the canonical basis in $\mathbb{R}^n$, we set $X_j = X(e_j)$ and $X'_j = X(ie_j)$, $Y_j = -\theta X_j$, $Y'_j = -\theta X'_j$, $Z = Z(i)$, $Z' = -\theta Z$. Then

$\{X_j,\ X'_k,\ \sqrt{2}Z \mid 1 \leq j,k \leq n\}$ is an orthonormal basis of $\mathfrak{n}$ with respect to $\langle\ ,\ \rangle$. Note that $[X_i, X'_j] = 2\ \delta_{ij} Z$.

If χ is a character of N, then there exists $X_\chi \in \mathfrak{n}_\alpha$ such that $d\chi(X) = i\langle X, X_\chi\rangle = -iB(X, \theta X_\chi)$, for $X \in \mathfrak{n}$. Set $M_\chi = \{m \in M \mid \mathrm{Ad}(u)X_\chi = X_\chi\}$. As M acts transitively on the unit sphere of $\mathfrak{n}_\alpha$ (cf. [**MV**, Introduction]) there is $u_0 \in M$ such that $\mathrm{Ad}(u_0)X_\chi = cX_1$, $c \in \mathbb{R}^+$. So $M_\chi = u_0 M_1 {u_0}^{-1}$, where $M_1 = \{u \in M \mid \mathrm{Ad}(u)X_1 = X_1\}$.

The compact subgroup K of G is given by the set of matrices of the form $k = k(\theta) = \left(\begin{smallmatrix} U & 0 \\ 0 & e^{i\theta}\end{smallmatrix}\right)$, with $U \in U(n)$, $\det(U) = e^{-i\theta}$. If $\mu \in \mathbb{Z}$ define $\phi_\mu(k) = \det(U)^\mu = e^{-i\mu\theta}$ and let

$$\xi_\mu = \phi_\mu|_M. \tag{2.1}$$

Now consider de *Verma module* $M(-\nu) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{C}_{-\nu-\rho}$, where $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $\mathbb{C}_{-\nu-\rho}$ denotes the $\mathfrak{p}$-module $\mathbb{C}$ with $\mathfrak{m}$ acting by $d\xi_\mu$, $\mathfrak{n}$ acting by 0 and $\mathfrak{a}$ acting by $-\nu-\rho$, $\nu \in {\mathfrak{a}_c}^*$. Let $M(-\nu)[\overline{\mathfrak{n}}]$ denote the $\overline{\mathfrak{n}}$-completion of $M(-\nu)$ (see [**GW**], §2). If $J = (j_1, j_2, \ldots, j_m) \in \mathbb{N}^m$, ($\mathbb{N} = \{0,\ 1,\ 2, \ldots\}$), $m = 2n+1$, and $Y(J) = {Y_1}^{j_1} \ldots {Y_n'}^{j_{2n}} {Z'}^{j_m}$, then by Poincaré-Birkhoff-Witt theorem, the set $\{Y(J) \mid J \in \mathbb{N}^m\}$ constitutes a basis of $\mathcal{U}(\overline{\mathfrak{n}})$. Hence every element in $M(-\nu)[\overline{\mathfrak{n}}]$ has an expansion of the type $\sum_J a_J Y(J) \otimes 1$, $a_J \in \mathbb{C}$. A χ-Whittaker vector is an element $v_\chi(-\nu)$ in $M(-\nu)[\overline{\mathfrak{n}}]$ that satisfies the equation

$$X.v = d\chi(X)v \qquad \forall X \in \mathfrak{n}. \tag{2.2}$$

Such a vector has an expression of the form

$$v_\chi(-\nu) = \sum_{I \in \mathbb{N}^m} a_I(\chi, -\nu) Y(I) \otimes 1 \tag{2.3}$$

where the coefficients $a_I(\chi, -\nu) \in \mathbb{C}$ are rational functions of ν. There is a unique such Whittaker vector with $a_0(\chi, -\nu) = 1$ (see [**BM**] §6, Lemma 11, for instance). Since $m \cdot v_\chi(-\nu) = v_\chi(-\nu)$ for $m \in M_\chi$, we have that for each I $Y(I)$ must be a polynomial in the M_χ-invariants of $\mathcal{U}(\overline{\mathfrak{n}})$.

Let χ be such that

$$d\chi(X_1) = \lambda, \quad d\chi(X_i) = d\chi({X_j}') = 0 \qquad i > 1,\ j \geq 1, \tag{2.4}$$

$\lambda \in i\mathbb{R}$. For this choice of χ, we shall use the notation $u(\lambda, -\nu)$ for the unique χ-Whittaker vector in $M(-\nu)[\overline{\mathfrak{n}}]$ such that $a_0(\chi, -\nu) = 1$. The coefficients of this χ-Whittaker vector will be denoted $a(\lambda, -\nu)$. Then $u(\lambda, -\nu)$ must be an element in $\mathcal{U}(\overline{\mathfrak{n}})^{M_1}$. We note that $M_1 \simeq \mathrm{SU}(n-1)$ is the subgroup of matrices in M of the form

$$u_1(b, B) = \left(\begin{smallmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & b \end{smallmatrix}\right),$$

$B \in \mathrm{U}(n-1)$. Hence $u(\lambda, -\nu) = \sum_j {Y_1}^j f_j$, where f_j is an M_1-invariant polynomial in $Y_2, \ldots, Y_n, {Y_1}', \ldots, {Y_n}'$ and Z'. Then, by [**MV**], Theorem B, $f_j \in \mathbb{C}[{Y_1}', Z', q_1]$, where $q_1 = \sum_{i=2}^n {Y_i}^2 + {Y_i'}^2$.

Let $E_{i,j}$ be the matrix in $\mathfrak{gl}(n+2, \mathbb{C})$ having the entry (i, j) equal to 1, and all the other entries zero. In order to have simpler formulas, it is convenient to change the basis of $\overline{\mathfrak{n}}_{\mathbb{C}}$ to

$$\{V_1, V_2, Y_2, \ldots, Y_n, {Y_2}', \ldots, {Y_n}', T\},$$

where $V_1 = E_{2,1}$, $V_2 = E_{n+2,2}$, $T = E_{n+2,1}$. Note that $Y_1 = V_1 - V_2$, ${Y_1}' = -iV_1 - iV_2$, and $Z' = -i\frac{1}{\sqrt{2}}T$. Let q be the element in $\mathcal{U}(\overline{\mathfrak{n}})^{M_1}$,

$$q = \sum_{i=1}^n {Y_i}^2 + {Y_i'}^2.$$

Now it is clear that we may write

$$u(\lambda, -\nu) = \sum_{j,k,l \geq 0} a_{j,k,l}(\lambda, -\nu){V_1}^j {V_2}^k T^l \otimes 1 \qquad n = 1 \tag{2.5}$$

$$u(\lambda, -\nu) = \sum_{j,k,l,m \geq 0} a_{j,k,l,m}(\lambda, -\nu){V_1}^j {V_2}^k T^l q^m \otimes 1 \qquad n > 1. \tag{2.6}$$

Now we use the $T(\nu)$-transform ([**GW**] and [**MW1**]) to compute the τ-function. We see in [**MW**] that this τ-function is the main ingredient in the Bessel transform in the Kloosterman term of the sum formula of Kuznetsov type and also it appears in the Fourier coefficients of the Poincaré series. To give the formula of the τ-function we consider two parabolic Γ-percuspidal subgroups P and P'. Then $P = NAM$ and $P' = N'A'M'$. Let χ and χ_1 be nontrivial unitary characters on N and N' respectively. If $W(A) = \{1, s\}$ is the Weyl group of (P, A) then we take s^* a representative of s in K. The τ-function is given by the following formula (see [**MW**] Proposition 1.2):

$$\tau(\chi_1, \chi, ua, \nu) = \sum_{I \in \mathbb{N}^m} a_I(\chi, \nu) d\chi_1(\mathrm{Ad}(uas^*)^{-1} Y(I)^T) \otimes 1 \tag{2.7}$$

$u \in M$, $a \in A$ and where the coefficients a_I are given by Formula (2.3). Here, $Y \mapsto Y^T$ is the automorphism of the universal enveloping algebra given by $X \mapsto -X$, for $X \in \mathfrak{g}$.

3. An explicit formula for the χ–Whittaker vector.

The aim of this section is to give a formula for a Whittaker vector $u(\lambda, \nu+\rho)$ in the $\overline{\mathfrak{n}}$-completion of the Verma module $M(\nu+\rho) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}\mathfrak{p}} \mathbb{C}_\nu$, where $\mathfrak{m}$ acts on $\mathbb{C}_\nu$ by $d\xi_\mu$ with ξ_μ as in (2.1), $\mathfrak{n}$ acts by 0 and $\mathfrak{a}$ acts by ν, $\nu \in \mathfrak{a}^*$. In what follows we shall write ν instead of $\nu(H)$.

Lemma 3.1. *Let* X_1, ${X_1}'$, V_1, V_2, T *and* H *be as above. Let* $U^+ = \frac{1}{2}[H + iM(2iE_{11})]$ *and* $U^- = \frac{1}{2}[-H + iM(2iE_{11})]$. *Then, the following commutation relations hold for* $j,\ k,\ l \geq 1$:

$$\begin{aligned}
[X_1, {V_1}^j] &= j{V_1}^{j-1}(U^+ - j + 1)\\
[X_1, {V_2}^k] &= k{V_2}^{k-1}(U^- + k - 1)\\
[X_1, T^l] &= -l(V_1 + V_2)T^{l-1}\\
[{X'}_1, {V_1}^j] &= j{V_1}^{j-1}(iU^+ - i(j-1))\\
[{X_1}', {V_2}^k] &= k{V_2}^{k-1}(-iU^- - i(k-1))\\
[{X_1}', T^l] &= il(V_1 - V_2)T^{l-1}.
\end{aligned}$$

Also we have: $[U^+, V_2] = V_2$, $[U^+, T] = -T$ *and* $[U^-, T] = T$.

Proof. The formulas are proved by the usual $\mathrm{Sl}(2,\mathbb{C})$-technique. They are based on the identities: $[X_1, V_1] = U^+$, $[X_1, V_2] = U^-$, $[{X_1}', V_1] = iU^+$, $[{X_1}', V_2] = -iU^-$ and on the fact that $[V_1, V_2] = -T$. □

Thus, it follows from Lemma 3.1 that:

$$\begin{aligned}
(3.1)\quad & X_1.{V_1}^j{V_2}^kT^l \otimes 1\\
&= (X_1.{V_1}^j){V_2}^kT^l \otimes 1 + {V_1}^j(X_1.{V_2}^k)T^l \otimes 1 + {V_1}^j{V_2}^k(X_1.T^l) \otimes 1\\
&= j{V_1}^{j-1}(U^+ - j + 1){V_2}^kT^l \otimes 1 + k{V_1}^j{V_2}^{k-1}(U^- + k - 1)T^l \otimes 1\\
&\quad - l{V_1}^j{V_2}^k(V_1 + V_2)T^{l-1} \otimes 1\\
&= j{V_1}^{j-1}{V_2}^kT^l(k - l - j + 1 + U^+) \otimes 1\\
&\quad + k{V_1}^j{V_2}^{k-1}T^l(l + k - 1 + U^-) \otimes 1\\
&\quad - l{V_1}^j{V_2}^kV_1T^{l-1} \otimes 1 - l{V_1}^j{V_2}^{k+1}T^{l-1} \otimes 1.
\end{aligned}$$

Using that $[V_1, {V_2}^k] = -k{V_2}^{k-1}T$ and the fact that U^+ and U^- act on $\mathbb{C}_\nu$ by $\frac{\nu+\mu}{2}$ and $\frac{-\nu+\mu}{2}$ respectively, the last expression is equal to

$$\begin{aligned}
(3.2)\qquad & j{V_1}^{j-1}{V_2}^kT^l\left(k - l - j + 1 + \frac{\nu+\mu}{2}\right) \otimes 1\\
&+ k{V_1}^j{V_2}^{k-1}T^l\left(k - 1 - \frac{\nu-\mu}{2}\right) \otimes 1\\
&- l{V_1}^{j+1}{V_2}^kT^{l-1} \otimes 1 - l{V_1}^j{V_2}^{k+1}T^{l-1} \otimes 1.
\end{aligned}$$

Now, Formula (2.2) implies that:

$$\sum_{j,k,l\geq 0} a_{j,k,l}(\lambda,\nu+\rho) X_1 . {V_1}^j {V_2}^k T^l \otimes 1 \tag{3.3}$$
$$= \lambda \sum_{j,k,l\geq 0} a_{j,k,l}(\lambda,\nu+\rho) {V_1}^j {V_2}^k T^l \otimes 1,$$

$$\sum_{j,k,l\geq 0} a_{j,k,l}(\lambda,\nu+\rho) {X_1}' . {V_1}^j {V_2}^k T^l \otimes 1 = 0. \tag{3.4}$$

From Formulas (3.2) and (3.3) we obtain the following recurrence relation for the coefficients $a_{j,k,l}(\lambda,\nu+\rho)$:

$$\begin{aligned} &\lambda\, a_{j,k,l}(\lambda,\nu+\rho) \\ &= (j+1)\left(k-l-j+\frac{\nu+\mu}{2}\right) a_{j+1,k,l}(\lambda,\nu+\rho) \\ &\quad + (k+1)\left(k-\frac{\nu-\mu}{2}\right) a_{j,k+1,l}(\lambda,\nu+\rho) \\ &\quad - (l+1) a_{j-1,k,l+1}(\lambda,\nu+\rho) - (l+1) a_{j,k-1,l+1}(\lambda,\nu+\rho). \end{aligned} \tag{3.5}$$

Also, we have

(3.6)

$$\begin{aligned} &{X_1}'.{V_1}^j {V_2}^k T^l \otimes 1 \\ &= ({X_1}'.{V_1}^j) {V_2}^k T^l \otimes 1 + {V_1}^j ({X_1}'.{V_2}^k) T^l \otimes 1 + {V_1}^j {V_2}^k ({X_1}'.T^l) \otimes 1 \\ &= i\, j {V_1}^{j-1} \big(U^+ - j + 1\big) {V_2}^k T^l \otimes 1 + {V_1}^j k {V_2}^{k-1} \big(U^- - i(k-1)\big) T^l \otimes 1 \\ &\quad + {V_1}^j {V_2}^k \left(il(V_1 - V_2)\right) T^{l-1} \otimes 1 \\ &= i\, j {V_1}^{j-1} {V_2}^k T^l \big(U^+ - j + 1 + k - l\big) \otimes 1 \\ &\quad + {V_1}^j k {V_2}^{k-1} T^l \big(U^- - i(k-1+l)\big) \otimes 1 \\ &\quad + il {V_1}^j {V_2}^k V_1 T^{l-1} \otimes 1 - il {V_1}^j {V_2}^{k+1} T^{l-1} \otimes 1 \\ &= i\, j {V_1}^{j-1} {V_2}^k T^l \left(-j+1+k-l+\frac{\nu+\mu}{2}\right) \otimes 1 \\ &\quad + i\, {V_1}^j k {V_2}^{k-1} T^l \left(-k+1+\frac{\nu-\mu}{2}\right) \otimes 1 \\ &\quad + il {V_1}^{j+1} {V_2}^k T^{l-1} \otimes 1 - il {V_1}^j {V_2}^{k+1} T^{l-1} \otimes 1. \end{aligned}$$

This computation together with Formula (3.4) gives a second recurrence relation for the coefficients $a_{j,k,l}(\lambda, \nu+\rho)$:

$$
\begin{aligned}
(3.7)\quad (j+1)\left(-j+k-l+\frac{\nu+\mu}{2}\right) a_{j+1,k,l}(\lambda,\nu+\rho) \\
+ (k+1)\left(-k+\frac{\nu-\mu}{2}\right) a_{j,k+1,l}(\lambda,\nu+\rho) \\
+ (l+1)a_{j-1,k,l+1}(\lambda,\nu+\rho) - (l+1)a_{j,k-1,l+1(\lambda,\nu+\rho)} = 0.
\end{aligned}
$$

Finally, from the identities (3.5) and (3.7) we get the following relations:

$$
\begin{aligned}
(3.8)\quad \lambda a_{j,k,l}(\lambda,\nu+\rho) &= (j+1)(\nu+\mu-2j+2k-2l)a_{j+1,k,l}(\lambda,\nu+\rho) \\
&\quad - 2(l+1)a_{j,k-1,l+1}(\lambda,\nu+\rho), \\
(3.9)\quad \lambda a_{j,k,l}(\lambda,\nu+\rho) &= (k+1)(-\nu+\mu+2k)a_{j,k+1,l}(\lambda,\nu+\rho) \\
&\quad - 2(l+1)a_{j-1,k,l+1}(\lambda,\nu+\rho).
\end{aligned}
$$

Proposition 3.2. *Let χ be as in (2.4). If $u(\lambda,\nu+\rho)$ denotes the canonical χ-Whittaker vector in $M(\nu+\rho)$ for $\mathrm{SU}(2,1)$, then the coefficients $a_{j,k,l}(\lambda,\nu+\rho)$ of $u(\lambda,\nu+\rho)$ are given by the formula:*

(3.10)

$$
a_{j,k,l}(\lambda,\nu+\rho) = \frac{\lambda^{j+k+2l}(-1)^{k+l}\prod_{i=k+l}^{j+k+l-1}(\nu+1-i)}{2^l j!k!l!\prod_{i=0}^{j+l-1}(\nu+\mu-2i)\prod_{i=0}^{k-1}(\nu-\mu-2i)\prod_{i=0}^{j+l-1}(\nu+1-i)}
$$

$$
a_{0,0,0}(\lambda,\nu+\rho) = 1.
$$

The proof of this proposition follows by induction on $n = j+k+2l$ and is very similar to the proof of Proposition 3.2 in [**Kb**].

Now we shall compute the coefficients of the χ-Whittaker vector for $\mathrm{SU}(n+1,1)$ in the case $n>1$. We set

$$
(3.11)\qquad M_{jk} = M(E_{jk}-E_{kj}), \qquad M^{jk} = M(iE_{jk}+iE_{kj}).
$$

Then $d\xi(M_{ij}) = 0$ and $d\xi(M^{ij}) = \delta_{ij}\,(-i\mu)$.

Lemma 3.3. *Let X_i, X_i', Y_i, $Y_i{}'$, V_1, V_2, T and q be as in Section 1. Then the following identities hold: For $i \geq 2$,*

$$\begin{aligned}
[X_i, V_1{}^j] &= jV_1{}^{j-1}M_i \quad \text{where } M_i = \frac{1}{2}(M(E_{i1} - E_{1i}) + iM(iE_{i1} + iE_{1i}))\\
[X_i, V_2{}^k] &= kV_2{}^{k-1}(M_i)^t\\
[X_i, T^l] &= -ilT^{l-1}(Y_i{}')\\
[X_i', V_1^j] &= ijV_1^{j-1}M_i\\
[X_i', V_2^k] &= -ikV_2{}^{k-1}(M_i)^t\\
[X_i', T^l] &= ilT^{l-1}(Y_i{}')\\
[M_i, V_2{}^k] &= (k/2)(-Y_i + iY_i{}')V_2{}^{k-1}.
\end{aligned}$$

For $i \geq 1$,

$$\begin{aligned}
(3.12)\quad [X_i, q^m] &= 2\sum_{s=0}^{m-1} 4^s U_i^{(s)} T^s q^{m-1-s} \sum_{j=s}^{m-1} \binom{j}{s} (n + H - 2(m-1-j))\\
&\quad + 2\sum_{s=0}^{m-1} 4^s \binom{m}{s+1} T^s q^{m-1-s} \Phi^{(s)}(X_i),
\end{aligned}$$

$$\begin{aligned}
(3.13)\quad [X_i', q^m] &= 2i\sum_{s=0}^{m-1} 4^s U_i^{(s-1)} T^s q^{m-1-s} \sum_{j=s}^{m-1} \binom{j}{s} (n + H - 2(m-1-j))\\
&\quad - 2\sum_{s=0}^{m-1} 4^s \binom{m}{s+1} T^s q^{m-1-s} \Phi^{(s)}(X_i')
\end{aligned}$$

where $U_k^{(s)} = Y_k$ if s is even and $-iY_k'$ if s is odd. $\Phi^{(s)}(X_i)$ and $\Phi^{(s)}(X_1')$ belong to $\mathcal{U}(\mathfrak{g})\mathfrak{m}$ and are given by the formulas

$$\Phi^{(s)}(X_i) = \begin{cases} \sum_{j=1}^{n} (Y_j M_{ij} + Y_j' M^{ij}), & \text{if } s \text{ is even},\\ i\sum_{j=1}^{n} (Y_j' M_{ij} - Y_j M^{ij}), & \text{if } s \text{ is odd}, \end{cases}$$

and

$$\Phi^{(s)}(X_i') = \begin{cases} \sum_{j=1}^{n} (Y_j M^{ij} + Y_j' M_{ij}), & \text{if } s \text{ is even},\\ i\sum_{j=1}^{n} (Y_j' M^{ij} - Y_j M_{ij}), & \text{if } s \text{ is odd}. \end{cases}$$

Proof. For the first formulas see [**Kb**], Lemma 3.1. To prove Formula (3.12) we proceed by induction on m. The proof of (3.13) is similar. We need the formulas:

$$\begin{aligned}
[[X_i, Y_j], Y_k] &= -\delta_{ij} Y_k + \delta_{jk} Y_i - \delta_{ik} Y_j \\
[[X_i, {Y_j}'], {Y_k}'] &= \delta_{ij} Y_k + \delta_{jk} Y_i + \delta_{ik} Y_j \\
[X_i, Y_j] &= \delta_{ij} H + M(E_{ij} - E_{ji}) \\
[X_i, {Y_j}'] &= M(i(E_{ij} + E_{ji})) \\
[X_i', Y_j] &= -M(i(E_{ij} + E_{ji})) \\
[X_i', Y_j'] &= \delta_{ij} H + M(E_{ij} - E_{ji}) \\
[[X_i', Y_j], Y_k] &= \delta_{ij} Y_k' + \delta_{jk} Y_i' + \delta_{ik} Y_j' \\
[[X_i', Y_j'], Y_j'] &= -\delta_{ij} Y_k' + \delta_{jk} Y_i' - \delta_{ik} Y_j'.
\end{aligned}$$

For $m = 1$ we have

(3.14)

$$\begin{aligned}
[X_i, q] &= \sum_{j=1}^{n} ([X_i, {Y_j}^2] + [X_i, {Y_j'}^2]) \\
&= \sum_{j=1}^{n} ([[X_i, Y_j], Y_j] + [[X_i, {Y_j}'], {Y_j}'] + 2Y_j [X_i, Y_j] + 2{Y_j}'[X_i, {Y_j}']) \\
&= \sum_{j=1}^{n} (-2\delta_{ij} Y_j + Y_i + 2\delta_{ij} Y_j + Y_i) + 2Y_i H + 2Y_j M_{ij} + 2Y_j' M^{ij} \\
&= 2Y_i(n + H) + 2\sum_{j=1}^{n} (Y_j M_{ij} + Y_j' M^{ij}) \\
&= 2Y_i(n + H) + 2\Phi^{(0)}(X_i).
\end{aligned}$$

If $m > 1$ we need the following identities:

(3.15)

$$\sum_{j=s}^{m-1} \binom{j}{s} \big(n + H - 2(m - 1 - j)\big)\, q = q \sum_{j=s}^{m-1} \binom{j}{s} \big(n + H - 2(m - j)\big),$$

$$[q^m, Y_i] = \sum_{s=1}^{m} 4^s U_i^{(s)} T^s q^{m-s}, \tag{3.16}$$

$$[\Phi^{(s)}, q] = 4T\Phi^{(s+1)}. \tag{3.17}$$

Formula (3.15) follows from the fact that $[H, q] = -2q$, and (3.16) was proved in [**Kb**], Lemma 3.3. To prove (3.17) we observe that if s is even then

$$[\Phi^{(s)}(X_i), q] = \sum_{j=1}^{n} \big([Y_j, q]M_{ij} + [Y_j', q]M^{ij}\big)$$
$$= 4iT \sum_{j=1}^{n} \big(Y_j' M_{ij} - Y_j M^{ij}\big) = 4T\Phi^{(s+1)}(X_i).$$

If s is odd then

$$[\Phi^{(s)}(X_i), q] = i \sum_{j=1}^{n} \big([Y_j', q]M_{ij} - [Y_j, q]M^{ij}\big)$$
$$= 4T \sum_{j=1}^{n} \big(Y_j' M_{ij} + Y_j M^{ij}\big) = 4T\Phi^{(s+1)}(X_i).$$

Thus (3.17) follows.

Using that $[X_i, q^{m+1}] = [X_i, q^m]q + q^m[X_i, q]$ we have

$$[X_i, q^{m+1}] = 2\sum_{s=0}^{m-1} 4^s U_i^{(s)} T^s q^{m-s} \sum_{j=s}^{m-1} \binom{j}{s} \big(n + H - 2(m-j)\big)$$
$$+ 2q^m Y_i(n+H) + 2\sum_{s=0}^{m-1} 4^s \binom{m}{s+1} T^s q^{m-s} \Phi^{(s)}(X_i)$$
$$+ 2\sum_{s=0}^{m-1} 4^{s+1} \binom{m}{s+1} T^{s+1} q^{m-(s+1)} \Phi^{(s+1)}(X_i) + 2q^m \Phi^{(0)}(X_i),$$

thus

(3.18)

$$[X_i, q^{m+1}]$$
$$= 2\sum_{s=0}^{m-1} 4^s U_i^{(s)} T^s q^{m-s} \sum_{j=s}^{m-1} \binom{j}{s} \big(n + H - 2(m-j)\big)$$
$$+ 2U_i^{(0)} q^m (n+H) + 2\sum_{s=1}^{m} 4^s \binom{m}{s} U_i^{(s)} T^s q^{m-s}(n+H)$$
$$+ 2\binom{m}{1} q^m \Phi^{(0)}(X_i) + 2\sum_{s=1}^{m-1} 4^s \left(\binom{m}{s+1} + \binom{m}{s}\right) T^s q^{m-s} \Phi^{(s)}(X_i)$$
$$+ 2\, 4^m \binom{m}{m} T^m \Phi^{(m)}(X_i) + 2q^m \Phi^{(0)}(X_i).$$

Now,

$$
\begin{aligned}
(3.19)\qquad & 2U_i^{(0)}q^m\sum_{j=0}^{m-1}\binom{j}{0}\big(n+H-2(m-j)\big)+2U_i^{(0)}q^m(n+H)\\
&=2U_i^{(0)}q^m\sum_{j=0}^{m}\binom{j}{0}\big(n+H-2(m-j)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.20)\qquad & 2\sum_{s=1}^{m}4^sU_i^{(s)}T^sq^{m-s}\sum_{j=s}^{m-1}\binom{j}{s}\big(n+H-2(m-j)\big)\\
&\quad+2\sum_{s=1}^{m-1}4^s\binom{m}{s}U_i^{(s)}T^sq^{m-s}(n+H)\\
&=2\sum_{s=1}^{m-1}4^sU_i^{(s)}T^sq^{m-s}\sum_{j=s}^{m}\binom{j}{s}\big(n+H-2(m-j)\big)\\
&\quad+2\,4^mU_i^{(m)}T^m(n+H)\\
&=2\sum_{s=1}^{m}4^sU_i^{(s)}T^sq^{m-s}\sum_{j=s}^{m}\binom{j}{s}\big(n+H-2(m-j)\big).
\end{aligned}
$$

Using the identity $\binom{m}{s+1}+\binom{m}{s}=\binom{m+1}{s+1}$ we have that the sum of the last four terms in Formula (3.18) equals

$$
2\sum_{s=0}^{m}4^s\binom{m+1}{s+1}T^sq^{m-s}\Phi^{(s)}(X_i).
$$

Thus we obtain

$$
\begin{aligned}
(3.21)\qquad [X_i,q^{m+1}]&=2\sum_{s=0}^{m}4^sU_i^{(s)}T^sq^{m-s}\sum_{j=s}^{m}\binom{j}{s}(n+H-2(m-j))\\
&\quad+2\sum_{s=0}^{m}4^s\binom{m+1}{s+1}T^sq^{m-s}\Phi^{(s)}(X_i),
\end{aligned}
$$

so the proof of Formula (3.12) is complete. Using similar computations and the identity

$$
[q^m,Y_i']=\sum_{s=1}^{m}4^s\binom{m}{s}U_i^{(s-1)}T^sq^{m-s},
$$

whose proof can also be found in [**Kb**], Lemma 3.3, one can obtain Formula (3.13). □

Now, it follows that for $i \geq 2$

$$X_i \cdot V_1^j V_2^k T^l q^m \otimes 1 \\ = jV_1^{j-1} M_i V_2^k T^l q^m \otimes 1 \\ + kV_1^j V_2^{k-1} M_i^t T^l q^m \otimes 1 - ilV_1^j V_2^k T^{l-1} Y_i' q^m \otimes 1 \\ + V_1^j V_2^k T^l \, X_i \cdot q^m \otimes 1. \tag{3.22}$$

As $\mathfrak{m}$ acts by $d\xi_\mu$ and from the definition of M_{kl} and M^{kl} in (3.11), we have that $M_i \otimes 1 = M_i^t \otimes 1 = 0$ for $i \geq 2$ and

$$\Phi^{(s)}(X_i) \otimes 1 = \begin{cases} Y_i' M^{ii} \otimes 1 = -i\mu Y_i', \text{ if } s \text{ is even} \\ -iY_i M^{ii} \otimes 1 = -\mu Y_i, \text{ if } s \text{ is odd.} \end{cases}$$

Using that $[\mathfrak{m}, T] = [\mathfrak{m}, q] = 0$ and setting $\sigma(m-1, s, \nu) = \sum_{j=s}^{m-1} \binom{j}{s}(n + \nu - 2(m-1-j))$ it follows that

(3.23)

$$X_i \cdot V_1^j V_2^k T^l q^m \otimes 1 \\ = \frac{jk}{2} V_1^{j-1}(-Y_i + iY_i')V_2^{k-1} T^l q^m \otimes 1 - ilV_1^j V_2^k T^{l-1} Y_i' q^m \otimes 1 \\ + 2Y_i \sum_{s=0, s \text{ even}}^{m-1} 4^s \sigma(m-1, s, \nu) V_1^j V_2^k T^{l+s} q^{m-1-s} \\ - 2iY_i' \sum_{s=0, s \text{ odd}}^{m-1} 4^s \sigma(m-1, s, \nu) V_1^j V_2^k T^{l+s} q^{m-1-s} \\ - 2i\mu Y_i' \sum_{s=0, s \text{ even}}^{m-1} 4^s \binom{m}{s+1} V_1^j V_2^k T^{l+s} q^{m-1-s} \\ - 2\mu Y_i \sum_{s=0, s \text{ odd}}^{m-1} 4^s \binom{m}{s+1} V_1^j V_2^k T^{l+s} q^{m-1-s} \\ = Y_i \Bigg(-\frac{jk}{2} V_1^{j-1} V_2^{k-1} T^l q^m \otimes 1 + 2 \sum_{s=0, s \text{ even}}^{m-1} 4^s \sigma(m-1, s, \nu) V_1^j V_2^k T^{l+s} q^{m-1-s} \\ -2\mu \sum_{s=0, s \text{ odd}}^{m-1} 4^s \binom{m}{s+1} V_1^j V_2^k T^{l+s} q^{m-1-s} \Bigg) \\ - iY_i' \Bigg(-\frac{jk}{2} V_1^{j-1} V_2^{k-1} T^l q^m \otimes 1 + lV_1^j V_2^k T^{l-1} q^m \otimes 1$$

$$+ 2 \sum_{s=0, s \text{ odd}}^{m-1} 4^s \sigma(m-1,s,\nu) V_1^j V_2^k T^{l+s} q^{m-1-s}$$

$$\cdot \sum_{s=0, s \text{ even}}^{m-1} 4^s \binom{m}{s+1} V_1^j V_2^k T^{l+s} q^{m-1-s} \Bigg) .$$

Using that $X_i \cdot u(\lambda, \nu) = 0$ for $i \geq 2$ and the linear independence of the monomials $Y_i V_1^j V_2^k T^l q^m$ and $Y_i' V_1^j V_2^k T^l q^m$ we obtain from the last identity two recurrence formulas for the coefficients $a_{j,k,l,m}$:

$$\begin{aligned}(3.24)\qquad &(j+1)(k+1)a_{j+1,k+1,l,m} \\ &= 2\sum_{s=0}^{l} 4^s \sigma(m+s,s,\nu) a_{j,k,l-s,m+1+s} \\ &\quad + (l+1)a_{j,k,l+1,m} + 2\mu \sum_{s=0}^{l} (-1)^s \binom{m+1+s}{s+1} a_{j,k,l-s,m+1+s}.\end{aligned}$$

$$\begin{aligned}(3.25)\qquad (l+1)a_{j,k,l+1,m} &= 2\sum_{s=0}^{l} (-1)^s 4^s \sigma(m+s,s,\nu) a_{j,k,l-s,m+1+s} \\ &\quad - 2\mu \sum_{s=0}^{l} 4^s \binom{m+s+1}{s+1} a_{j,k,l-s,m+1+s}.\end{aligned}$$

To compute the action of X_1 we need the identities

$$V_2^k Y_1 = k V_2^{k-1} T + V_1 V_2^k - V_2^{k+1}$$
$$V_2^k Y_1' = -ik V_2^{k-1} T + -i V_1 V_2^k - i V_2^{k+1}$$

and the commutation relations in Lemma 3.1. Thus

$$\begin{aligned}(3.26)\qquad &X_1 \cdot V_1^j V_2^k T^l q^m \otimes 1 \\ &= j\left(\frac{\nu+\mu}{2} + k - l - j - m + 1\right) V_1^{j-1} V_2^k T^l q^m \otimes 1 \\ &\quad + k\left(\frac{\mu-\nu}{2} + k + m - 1\right) V_1^j V_2^{k-1} T^l q^m \otimes 1 \\ &\quad - l V_1^{j+1} V_2^k T^{l-1} q^m \otimes 1 - l V_1^j V_2^{k+1} T^{l-1} q^m \otimes 1 \\ &\quad + 2 \sum_{s=0, s \text{ even}} 4^s \sigma(m-1,s,\nu) \Big(k V_1^j V_2^{k-1} T^{l+s+1} q^{m-1-s} \\ &\quad + V_1^{j+1} V_2^k T^{l+s} q^{m-1-s} - V_1^j V_2^{k+1} T^{l+s} q^{m-1-s} \Big)\end{aligned}$$

$$- 2 \sum_{s=0, s \text{ odd}} 4^s \sigma(m-1, s, \nu)\Big(k V_1^j V_2^{k-1} T^{l+s+1} q^{m-1-s}$$
$$+ V_1^{j+1} V_2^k T^{l+s} q^{m-1-s} + V_1^j V_2^{k+1} T^{l+s} q^{m-1-s}\Big)$$
$$- 2\mu \sum_{s=0, s \text{ even}} 4^s \binom{m}{s+1}\Big(k V_1^j V_2^{k-1} T^{l+s+1} q^{m-1-s}$$
$$+ V_1^{j+1} V_2^k T^{l+s} q^{m-1-s} + V_1^j V_2^{k+1} T^{l+s} q^{m-1-s}\Big)$$
$$- 2\mu \sum_{s=0, s \text{ odd}} 4^s \binom{m}{s+1}\Big(k V_1^j V_2^{k-1} T^{l+s+1} q^{m-1-s}$$
$$+ V_1^{j+1} V_2^k T^{l+s} q^{m-1-s} - V_1^j V_2^{k+1} T^{l+s} q^{m-1-s}\Big).$$

To compute the action of X_1' we shall use that

$$\Phi^{(s)}(X_1') = \begin{cases} Y_1 M^{11} \otimes 1 = -i\mu Y_1, & \text{if } s \text{ is even,} \\ iY_1' M^{11} \otimes 1 = \mu Y_1', & \text{if } s \text{ is odd.} \end{cases}$$

Therefore we get

$$X_1' \cdot V_1^j V_2^k T^l q^m \otimes 1 \tag{3.27}$$
$$= ij \left(\frac{\nu + \mu}{2} + k - l - j - m + 1 \right) V_1^{j-1} V_2^k T^l q^m \otimes 1$$
$$- ik \left(\frac{\mu - \nu}{2} + k + m - 1 \right) V_1^j V_2^{k-1} T^l q^m \otimes 1$$
$$+ il V_1^{j+1} V_2^k T^{l-1} q^m \otimes 1 - il V_1^j V_2^{k+1} T^{l-1} q^m \otimes 1$$
$$- 2i \sum_{s=0, s \text{ even}} 4^s \sigma(m-1, s, \nu)\Big(k V_1^j V_2^{k-1} T^{l+s+1} q^{m-1-s}$$
$$+ V_1^{j+1} V_2^k T^{l+s} q^{m-1-s} + V_1^j V_2^{k+1} T^{l+s} q^{m-1-s}\Big)$$
$$+ 2i \sum_{s=0, s \text{ odd}} 4^s \sigma(m-1, s, \nu)\Big(k V_1^j V_2^{k-1} T^{l+s+1} q^{m-1-s}$$
$$+ V_1^{j+1} V_2^k T^{l+s} q^{m-1-s} - V_1^j V_2^{k+1} T^{l+s} q^{m-1-s}\Big)$$
$$+ 2i\mu \sum_{s=0, s \text{ even}} 4^s \binom{m}{s+1}\Big(k V_1^j V_2^{k-1} T^{l+s+1} q^{m-1-s}$$
$$+ V_1^{j+1} V_2^k T^{l+s} q^{m-1-s} - V_1^j V_2^{k+1} T^{l+s} q^{m-1-s}\Big)$$
$$+ 2i\mu \sum_{s=0, s \text{ odd}} 4^s \binom{m}{s+1}\Big(k V_1^j V_2^{k-1} T^{l+s+1} q^{m-1-s}$$

$$+ V_1^{j+1}V_2^kT^{l+s}q^{m-1-s} + V_1^jV_2^{k+1}T^{l+s}q^{m-1-s}\Big).$$

Thus

(3.28)

$$\begin{aligned}&(X_1 - iX_1') \cdot V_1^jV_2^kT^lq^m \otimes 1\\ &= j\big(\nu + \mu + 2k - 2l - 2j - 2m + 2\big)V_1^{j-1}V_2^kT^lq^m \otimes 1\\ &\quad - 2lV_1^jV_2^{k+1}T^{l-1}q^m \otimes 1 - 4\sum_{s=0}^{m-1} 4^s\sigma(m-1, s, \nu)V_1^jV_2^{k+1}T^{l+s}q^{m-1-s}\\ &\quad - 4\mu\sum_{s=0}^{m-1}(-1)^s4^s\binom{m}{s+1}V_1^jV_2^{k+1}T^{l+s}q^{m-1-s} \otimes 1,\end{aligned}$$

and

$$\begin{aligned}(3.29)\quad &(X_1 + iX_1') \cdot V_1^jV_2^kT^lq^m \otimes 1 =\\ &k\big(\mu - \nu + 2k + 2m - 2\big)V_1^jV_2^{k-1}T^lq^m \otimes 1 - 2lV_1^{j+1}V_2^kT^{l-1}q^m \otimes 1\\ &\quad + 4k\sum_{s=0}^{m-1}(-1)^s4^s\sigma(m-1, s, \nu)V_1^jV_2^{k-1}T^{l+s+1}q^{m-1-s}\\ &\quad + 4\sum_{s=0}^{m-1}(-1)^s4^s\sigma(m-1, s, \nu)V_1^{j+1}V_2^kT^{l+s}q^{m-1-s}\\ &\quad - 4\mu k\sum_{s=0}^{m-1} 4^s\binom{m}{s+1}V_1^jV_2^{k-1}T^{l+s+1}q^{m-1-s} \otimes 1\\ &\quad - 4\mu\sum_{s=0}^{m-1} 4^s\binom{m}{s+1}V_1^{j+1}V_2^kT^{l+s}q^{m-1-s} \otimes 1.\end{aligned}$$

Proposition 3.4. *The coefficients $a_{j,k,l,m}(\nu+\rho, \lambda)$ for the χ-Whittaker $u(\lambda, \nu + \rho)$ in $M(\nu + \rho)$ satisfy the following recurrence relations:*

(3.30)

$$\begin{aligned}\lambda a_{j,k,l,m} &= (j + 1)(\nu + \mu + 2k - 2l - 2m)a_{j+1,k,l,m} - 2(l + 1)a_{j,k-1,l+1,m}\\ &\quad - 4\sum_{s=0}^{l} 4^s\left(\sigma(m + s, s, \nu) + \mu(-1)^s\binom{m + 1 + s}{s + 1}\right)a_{j,k-1,l-s,m+1+s},\end{aligned}$$

$$\lambda a_{j,k,l,m} = (k+1)(\mu-\nu+2k+2m)a_{j,k+1,l,m} - 2(l+1)a_{j-1,k,l+1,m} + 4(k+1)\sum_{s=0}^{l-1} 4^s \left((-1)^s\sigma(m+s,s,\nu) - \mu\binom{m+1+s}{s+1}\right) a_{j,k+1,l-s-1,m+1+s} + 4\sum_{s=0}^{l} 4^s \left((-1)^s\sigma(m+s,s,\nu) - \mu\binom{m+1+s}{s+1}\right) a_{j-1,k,l-s,m+1+s} \tag{3.31}$$

$$(j+1)(k+1)a_{j+1,k+1,l,m} = (l+1)a_{j,k,l+1,m} + 2\sum_{s=0}^{l} 4^s \left(\sigma(m+s,s,\nu) + \mu(-1)^s\binom{m+1+s}{s+1}\right) a_{j,k,l-s,m+1+s} \tag{3.32}$$

$$(l+1)a_{j,k,l+1,m} = 2\sum_{s=0}^{l} 4^s\Bigg((-1)^s\sigma(m+s,s,\nu) - \mu\binom{m+s+1}{s+1}\Bigg)a_{j,k,l-s,m+1+s}. \tag{3.33}$$

Proof. The equations follow from the definition of the Whittaker vector, Formulas (3.24), (3.25), (3.28) and (3.29). We note that if we set $\mu = 0$ we get the formulas given in [**Kb**], Proposition 3.5. □

Equation (3.33) implies that $a_{j,k,l,m} = b_{j,k,l,m}a_{j,k,0,l+m}$, where $b_{j,k,l,m}$ satisfies the formulas:

$$b_{j,k,0,m} = 1 \tag{3.34}$$

$$(l+1)b_{j,k,l,m} = 2\sum_{s=0}^{l} 4^s\Bigg((-1)^s\sigma(m+s,s,\nu) - \mu\binom{m+s+1}{s+1}\Bigg)b_{j,k,l-s,m+1+s}. \tag{3.35}$$

From (3.34) and (3.35) we conclude that $b_{j,k,l,m}$ doesn't depend on j and k. Therefore we shall use the notation $b_{l,m}$ instead of $b_{j,k,l,m}$.

Formulas (3.30) and (3.32) imply that

$$\lambda a_{j,k,l,m} = (j+1)(\nu+\mu-2l-2m)a_{j+1,k,l,m}, \tag{3.36}$$

hence

$$a_{j,k,l,m} = \frac{\lambda^j}{j! \prod_{i=l+m}^{j+l+m-1}(\nu+\mu-2i)} a_{0,k,l,m}. \tag{3.37}$$

From Formulas (3.31) and (3.33) we get

$$\lambda a_{j,k,l,m} = (k+1)(\mu-\nu+2k+2m+2l)a_{j,k,l,m}, \tag{3.38}$$

which implies

$$a_{j,k,l,m} = \frac{(-1)^k\lambda^k}{k!\prod_{i=l+m}^{k+l+m}(\nu-\mu-2i)}a_{j,0,l,m}. \tag{3.39}$$

Finally, (3.37) together with (3.39) implies

$$a_{j,k,l,m} = \frac{(-1)^k\lambda^{j+k}}{j!k!\prod_{i=l+m}^{j+l+m-1}(\nu+\mu-2i)\prod_{i=l+m}^{k+l+m}(\nu-\mu-2i)}a_{0,0,l,m}. \tag{3.40}$$

If we set $l=0$ in the Formulas (3.32) and (3.33) we get:

$$(j+1)(k+1)a_{j+1,k+1,0,m} = 4\sigma(m,0,\nu)a_{j,k,0,m+1}.$$

But $\sigma(m,0,\nu) = \sum_{j=0}^{m}(n+\nu-2(m-j)) = (m+1)(n+\nu-m)$ and this together with (3.36) and (3.38) imply

$$a_{j,k,0,m+1} = \frac{-\lambda^2}{4(m+1)(n+\nu-m)(\nu+\mu-2m)(\nu-\mu-2k-2m)}a_{j,k,0,m} \tag{3.41}$$

and then

$$a_{j,k,0,m} = \frac{(-1)^m\lambda^{2m}}{4^m m!\prod_{i=0}^{m-1}(n+\nu-i)\prod_{i=j}^{j+m-1}(\nu+\mu-2i)\prod_{i=k}^{k+m-1}(\nu-\mu-2i)}a_{j,k,0,0}. \tag{3.42}$$

Finally, we obtain the following explicit formula for the coefficient $a_{j,k,l,m}$:

$$\begin{aligned} &a_{j,k,l,m}(\nu,\lambda) \\ &= b_{l,m}(\nu)a_{j,k,0,l+m}(\nu,\lambda) \\ &= \frac{b_{l,m}(\nu)(-1)^{k+l+m}\lambda^{j+k+2l+2m}}{j!k!(l+m)!4^{l+m}\prod_{i=0}^{j+l+m-1}(\nu+\mu-2i)\prod_{i=0}^{k+l+m-1}(\nu-\mu-2i)\prod_{i=0}^{l+m-1}(n+\nu-i)}a_{0,0,0,0}. \end{aligned} \tag{3.43}$$

Now let $M(-\nu)$ be the Verma module $\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{p})}\mathbb{C}_{-\nu-\rho}$, where $\mathbb{C}_{-\nu-\rho}$ is the $\mathfrak{p}$-module with $\mathfrak{m}$ acting by $d\xi_\mu$, $\mathfrak{n}$ acting by 0 and a acting by $-\nu-\rho$.

Let $u(\lambda,-\nu)=\sum_I a_I(\lambda,\nu)Y(I)\otimes 1$ be the χ-Whittaker vector on the Verma module $M(-\nu)$, with χ as in Formula (2.4).

To obtain an explicit formula for the Whittaker vector $u(\lambda,-\nu)$ we change the parameter ν in Formulas (3.10) and (3.43) into $-\nu-\rho$, since H acts by $-\nu(H)-\rho(H)$. Recall that for $G\simeq \mathrm{SU}(n+1,1)$ we have $\rho(H)=n+1$. With this parametrization and setting $a_{0,0,0}=1$ in the case $n=1$ and $a_{0,0,0,0}=1$ in the case $n>1$ we get

(3.44)

$$a_{j,k,l}(\lambda,-\nu-\rho)=\frac{(-1)^{j+l}(\lambda/2)^{j+k+2l}\prod_{i=k+l+1}^{j+k+l}(\nu+i)}{j!k!l!\prod_{i=1}^{j+l}\left(\frac{\nu-\mu}{2}+i\right)\prod_{i=1}^{k}\left(\frac{\nu+\mu}{2}+i\right)\prod_{i=1}^{j+l}(\nu+i)}$$

(3.45)

$$\begin{aligned}&a_{j,k,l,m}(\lambda,-\nu-\rho)\\&=\frac{b_{l,m}(-\nu-\rho)(-1)^j\lambda^{j+k+2l+2m}}{j!k!(l+m)!4^{l+m}\prod_{i=1}^{j+l+m}\left(\frac{\nu-\mu+n}{2}+i\right)\prod_{i=1}^{k+l+m}\left(\frac{\nu+\mu+n}{2}+i\right)\prod_{i=1}^{l+m}(\nu+i)}.\end{aligned}$$

Furthermore, it will be convenient to multiply $u(\lambda,-\nu-\rho)$ by the normalizing factor $I(\nu)=[\Gamma(\frac{\nu-\mu+n}{2})]^{-1}[\Gamma(\frac{\nu+\mu+n}{2})]^{-1}[\Gamma(\nu+1)]^{-1}$ in order to obtain a holomorphic Whittaker vector $\widetilde{u}(\lambda,-\nu-\rho)$. We may now state the main result in this section:

Theorem 3.5. *Let G be locally isomorphic with* $\mathrm{SU}(n+1,1)$*. Then a holomorphic χ-Whittaker vector in $M(-\nu)$ is given by:*

(1) *If $n=1$ then*

(3.46)

$$\begin{aligned}&\widetilde{u}(\lambda,-\nu-\rho)\\&=\sum_{j,k,l\geq 0}\frac{(-1)^{j+l}(\lambda/2)^{j+k+2l}\prod_{i=k+l+1}^{j+k+l}(\nu+i)\,V_1^jV_2^kT^l}{j!k!l!\Gamma\left(\frac{\nu-\mu}{2}+j+l+1\right)\Gamma\left(\frac{\nu+\mu}{2}+k+1\right)\Gamma\left(\nu+j+l+1\right)},\end{aligned}$$

(2) *if* $n > 1$ *then*

$$\widetilde{u}(\lambda, -\nu-\rho) = \sum_{j,k,l,m\geq 0} \cdot \frac{(-1)^j b_{l,m}(-\nu-\rho)\lambda^{j+k+2l+2m}\, V_1^j V_2^k T^l q^m \left[\Gamma\left(\nu+l+m+1\right)\right]^{-1}}{j!k!(l+m)!4^{l+m}\Gamma\left(\frac{\nu-\mu+n}{2}+j+l+m+1\right)\Gamma\left(\frac{\nu+\mu+n}{2}+k+l+m+1\right)}. \tag{3.47}$$

Remark 1. We observe from the definition of $\sigma(m,s,\nu)$ that

$$\begin{aligned} \sigma(m+s,s,-\nu-\rho) \pm \mu\binom{m+1+s}{s+1} \\ = \binom{m+1+s}{s}(m+1)\left(\frac{-\nu-1-2m\pm\mu}{s+1}\right) + \frac{2m}{s+2}. \end{aligned} \tag{3.48}$$

This together with (3.35) imply that $b_{l,m}(-\nu-\rho)$ is a polinomial in ν of degree l.

4. An explicit formula for the τ-function.

Let G be a Lie group locally isomorphic to $\mathrm{SU}(n+1,1)$. We recall from §2 the definition of the τ-function:

$$\tau(\chi_1,\chi,ua,\nu) = \sum_{I\in\mathbb{N}} a_I(-\nu) d\chi_1\big(\mathrm{Ad}(uas)^{-1}Y(I)^T\big) \tag{4.1}$$

where $u \in M$, $a \in A$ and $u(\lambda,-\nu) = \sum_I a_I(\lambda,-\nu)Y(I)\otimes 1$ is the χ-Whittaker vector on the Verma module $M(-\nu)$. This function appears in the χ_1-Fourier coefficient $D^{\chi}_{\chi_1}$ of the Poincaré series studied in [**MW**] and in the integral kernel of the Kuznetsov type formula in [**MW1**]. The case $\chi = \chi_1$ has special interest because the poles of the meromorphic continuation of $D^{\chi}_{\chi}(P,P,\nu)$ to $\mathbb{C}$ lie exactly at spectral parameters, that is, the nonzero eigenvalues of the Casimir operator C on $L^2_d(\Gamma\backslash G/K)$ have the form $\nu_j(H)^2 - \rho(H)^2$, where ν_j is a pole of $\{D^{\chi}_{\chi}(P,P,\nu) \mid \chi \in (\Gamma_N\backslash N)\hat{} - 1\}$ in the closed right half plane. Our main goal in this section will be to give an explicit formula for $\tau(\chi,\chi,ua,\nu)$.

For this purpose we need to compute $\mathrm{Ad}(uas^*)^{-1}Y(I)$. If we take $s^* = \left(\begin{smallmatrix} 0 & 0 & i \\ 0 & I_n & 0 \\ i & 0 & 0\end{smallmatrix}\right)$, where I_n denotes the $n\times n$ identity matrix, we have that $\mathrm{Ad}(s^*)^{-1}Y_i = X_i{}'$, $\mathrm{Ad}(s^*)^{-1}Y_i{}' = -X_i$, and therefore $\mathrm{Ad}(s^*)^{-1}V_1 = 1/2(X_1{}' - iX_1)$ and $\mathrm{Ad}(s^*)^{-1}V_2 = 1/2(-X_1{}' - iX_1)$. Also, since

$$\mathrm{Ad}(u^{-1})V_1 = c_1V_1 + d_1V_2 + \sum_{j=2}^{n}(c_jY_j + d_jY'{}_j)$$

for some coefficients c_j and d_j and using the fact that $\langle V_1, V_1\rangle = \langle V_2, V_2\rangle = 1/2$, we may write

$$\mathrm{Ad}(u^{-1})V_1 = 2\langle V_1, \mathrm{Ad}(u)V_1\rangle V_1 + 2\langle V_1, \mathrm{Ad}(u)V_2\rangle V_2 + \sum_{j=2}^{n}(c_jY_j + d_jY'^j).$$

In the same way we obtain

$$\mathrm{Ad}(u^{-1})V_2 = 2\langle V_2, \mathrm{Ad}(u)V_1\rangle V_1 + 2\langle V_2, \mathrm{Ad}(u)V_2\rangle V_2 + \sum_{j=2}^{n}(\widetilde{c}_jY_j + \widetilde{d}_jY'_j).$$

As $d\chi(\mathrm{Ad}(s^*)^{-1}V_1) = d\chi(\mathrm{Ad}(s^*)^{-1}V_2) = \frac{-i\lambda}{2}$ we get the formula

(4.2)

$$\begin{aligned} &d\chi\big(\mathrm{Ad}(uas^*)^{-1}{V_1}^j{V_2}^k\big)\\ &= a^{(j+k)\alpha}\left(\frac{-i\lambda}{2}\right)^{j+k} 2^{j+k}\langle V_1, \mathrm{Ad}(u)(V_1+V_2)\rangle^j\langle V_2, \mathrm{Ad}(u)(V_1+V_2)\rangle^k\\ &= (-i\lambda a^{\alpha})^{j+k}\,\langle V_1, \mathrm{Ad}(u)(V_1+V_2)\rangle^j\langle V_2, \mathrm{Ad}(u)(V_1+V_2)\rangle^k. \end{aligned}$$

In the case of $\mathrm{SU}(2,1)$ the last formula can be simplified. If we take $u = u_\theta = \begin{pmatrix} e^{i\theta} & 0 & 0\\ 0 & e^{-2i\theta} & 0\\ 0 & 0 & e^{i\theta}\end{pmatrix}$, then $\mathrm{Ad}(u)V_1 = e^{3i\theta}V_1$, $\mathrm{Ad}(u)V_2 = e^{-3i\theta}V_2$ and so

$$\begin{aligned} (4.3)\quad &d\chi\big(\mathrm{Ad}(uas^*)^{-1}{V_1}^j{V_2}^k\big)\\ &= (-i\lambda a^{\alpha})^{j+k}e^{3i(j-k)\theta}\langle V_1, V_1\rangle^j\langle V_2, V_2\rangle^k = \left(\frac{-i\lambda a^{\alpha}}{2}\right)^{j+k} e^{3i(j-k)\theta}. \end{aligned}$$

Now, T is M-invariant and it belongs to $\mathfrak{g}_{-2\alpha}$. Then $d\chi(\mathrm{Ad}(s^*)^{-1}T^l) = 0$ if $l > 0$. Also, from the computations above we get:

$$d\chi(\mathrm{Ad}(uas^*)^{-1}q^m) = a^{2m\alpha}d\chi({X_1}^2)^m = \big(\lambda a^{\alpha}\big)^{2m}.$$

We shall use the following notation: Let be $\lambda \in i\mathbb{R}$, $a \in A$ and $u \in M$. We shall write

$$z = z(\lambda, a) = -i(\lambda/2)^2 a^{\alpha}, \tag{4.4}$$

$$\omega_i = \omega_i(u) = \langle V_i, \mathrm{Ad}(u)(V_1+V_2)\rangle,\ i = 1,\ 2. \tag{4.5}$$

We note that $z \in i\mathbb{R}^{>0}$, thus $\overline{z} = -z$. We also recall the definition of the generalized hypergeometric functions. We shall follow the notation of [**Sl**]. Let $(a) = (a_1, a_2, \ldots, a_p)$ and $(b) = (b_1, \ldots, b_q)$. The generalized hypergeometric function ${}_pF_q((a);(b);y)$ is defined as follows:

$${}_pF_q((a);(b);y) = \sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{n!(b_1)_n\ldots(b_q)_n}y^n$$

where $(c)_n = c(c+1)\dots(c+n-1) = \dfrac{\Gamma(c+n)}{\Gamma(c)}$ if $n \geq 1$ and $(c)_0 = 1$. We recall that if $p \leq q$ then ${}_pF_q((a);(b);y)$ converges for every finite value of y. For $p = 0$ and $q = 1$ then ${}_0F_1$ corresponds to the classical Bessel function I_ν, that is

$$ {}_0F_1(\ ;b;z) = \Gamma(b)(z)^{b-1} I_{b-1}\left(2z^{\frac{1}{2}}\right). $$

We now state the main result of this work.

Theorem 4.1. *Let G be isomorphic to* $\mathrm{SU}(n+1,1)$*, and let $\tau(\chi,\chi,ua,\nu)$ be as in* (4.1)*, $u \in M$, $a \in A$ and χ a character on N as in* (2.4)*. Let z, ω_1 and ω_2 be as in* (4.4) *and* (4.5)*. If $n = 1$ let $u = u_\theta$ as in* (4.3)*. Then*

(4.6)

$$ \begin{aligned} &\tau(\chi,\chi,u_\theta,\nu) \\ &= \sum_{j,k\geq 0} \frac{(-1)^j \prod_{s=1}^{j+k}(\nu+s)\, z^{j+k} e^{3i(j-k)\theta}}{j!k!\prod_{s=1}^{j}(\nu+s)\left(\frac{\nu-\mu}{2}+s\right)\prod_{s=1}^{k}(\nu+s)\left(\frac{\nu+\mu}{2}+s\right)} \\ &= \sum_{j\geq 0} \frac{\left(\overline{ze^{-3i\theta}}\right)^j}{j!\left(\frac{\nu-\mu}{2}+1\right)_j} {}_1F_2\left(\nu+j+1;\nu+1,\frac{\nu+\mu}{2}+1, ze^{-3i\theta}\right) \\ &= \sum_{m\geq 0}(\nu+1)_m \sum_{j=0}^{m} \frac{(\overline{ze^{-3i\theta}})^j (ze^{3i\theta})^{m-j}}{j!(m-j)!(\nu+1)_j\left(\frac{\nu-\mu}{2}+1\right)_j(\nu+1)_{m-j}\left(\frac{\nu+\mu}{2}+1\right)_{m-j}}. \end{aligned} $$

If $n > 1$ then

$\tau(\chi,\chi,ua,\nu)$

(4.7)

$$ = \sum_{j,k,m\geq 0} \frac{(-1)^k (\overline{z}/2)^{j+k+2m}\, \omega_1^j \omega_2^k}{j!k!m!\prod_{s=1}^{j+m}\left(\frac{\nu-\mu+n-1}{2}+s\right)\prod_{s=1}^{k+m}\left(\frac{\nu+\mu+n-1}{2}+s\right)\prod_{s=1}^{m}(\nu+s)} $$

(4.8)

$$ \begin{aligned} &= \sum_{m\geq 0} \frac{(z/2)^{2m}}{m!(\nu+1)_m\left(\frac{\nu+\mu+n+1}{2}\right)_m\left(\frac{\nu-\mu+n+1}{2}\right)_m} \\ &\quad \cdot {}_0F_1\left(\frac{\nu-\mu+1}{2}+m+1,\frac{\overline{z}\omega_1}{2}\right) {}_0F_1\left(\frac{\nu+\mu+1}{2}+m+1,\frac{z\omega_2}{2}\right) \end{aligned} $$

(4.9)

$$= \Gamma\left(\frac{\nu-\mu+n+1}{2}\right)\Gamma\left(\frac{\nu+\mu+n+1}{2}\right)\left(\frac{\overline{z}\omega_1}{2}\right)^{\frac{\nu-\mu+n+1}{2}}\left(\frac{z\omega_2}{2}\right)^{\frac{\nu+\mu+n+1}{2}}$$

$$\cdot\sum_{m\geq 0}\frac{(\overline{z}/2)^{2m}\left(\frac{\overline{z}\omega_1}{2}\right)^m\left(\frac{z\omega_2}{2}\right)^m}{m!(\nu+1)_m} I_{\frac{\nu-\mu+n-1}{2}+m}\left((2\overline{z}\omega_1)^{1/2}\right) I_{\frac{\nu+\mu+n-1}{2}+m}\left((2z\omega_2)^{1/2}\right).$$

The first expressions for the τ-functions are easily obtained from the definition of the τ-function and the formulas in (4.2) and (4.3). Reordering these series and using the definition of the hypergeometric functions one gets the other alternative formulas.

Remark 2. The formula for the τ-function in (4.6) suggests that if $\mu \in \mathbb{N}$ is fixed, then there are a finite number of possible poles of the τ function in $Re\ \nu > 0$. In particular, for $\mathrm{SU}(2,1)$ there should be simple poles at $\nu = \mu - 2t$, $t \in \mathbb{N}$, provided that $\mu - 2t > 0$. We shall prove that these poles really exist by computing the residue at the point and showing that it is not zero.

Let $\mu \in \mathbb{N}$, and let $\nu \in \mathfrak{a}^*$, $\nu > \rho$ such that $\nu = \mu - 2t$ for some $t \in \mathbb{N}$. Note that as $\lambda \in i\mathbb{R}$ we have that $-i(\lambda/2)^2 a^\alpha = ix$, for some $x \in \mathbb{R}^{>0}$. Thus

$$\begin{aligned}
&\operatorname{Res}_{\nu=\nu_t}\tau(\chi,\chi,ua,\nu)\\
&= \lim_{\nu\to\nu_t}(\nu-\nu_t)\tau(\chi,\chi,ua,\nu)\\
&= \lim_{\nu\to\nu_t}(\nu-\nu_t)\sum_{\substack{0\leq j<t\\ k\geq 0}}\frac{(-1)^j\prod_{s=1}^{k}(\nu+j+s)\ (ix)^{j+k}e^{3i(j-k)\theta}}{j!k!\prod_{s=1}^{j}\left(\frac{\nu-\mu}{2}+s\right)\prod_{s=1}^{k}(\nu+s)\left(\frac{\nu+\mu}{2}+s\right)}\\
&\quad+\lim_{\nu\to\nu_t}(\nu-\nu_t)\sum_{\substack{j\geq t\\ k\geq 0}}\frac{(-1)^j\prod_{s=1}^{k}(\nu+j+s)\ (ix)^{j+k}e^{3i(j-k)\theta}}{j!k!\prod_{s=1}^{j}\left(\frac{\nu-\mu}{2}+s\right)\prod_{s=1}^{k}(\nu+s)\left(\frac{\nu+\mu}{2}+s\right)}.
\end{aligned}$$

The first term in the last expression of the residue is zero because the infinite sum involved is holomorphic at $\nu = \nu_t$. Then

$$\operatorname{Res}_{\nu=\nu_t}\tau(\chi,\chi,ua,\nu)$$

$$= \lim_{\nu\to\nu_t}\sum_{\substack{j\geq t\\ k\geq 0}} \frac{2(-1)^j\prod_{s=1}^{k}(\nu+j+s)(ix)^{j+k}e^{3i(j-k)\theta}}{j!k!\prod_{s=1}^{t-1}\left(\frac{\nu-\mu}{2}+s\right)\prod_{s=t+1}^{j}\left(\frac{\nu-\mu}{2}+s\right)\prod_{s=1}^{k}(\nu+s)\left(\frac{\nu+\mu}{2}+s\right)}.$$

Putting $\theta = \pi/6$ and using that $(-1)^j = i^{-2j}$ we get

$$(-1)^j(ix)^{j+k}e^{3i(j-k)\theta} = i^{-j+k}x^{j+k}e^{i\pi/2(j-k)} = x^{j+k}(-i)^{j-k}i^{j-k} = x^{j+k},$$

which is a positive real number. Furthemore, the factor $\prod_{s=1}^{t-1}(\frac{\nu-\mu}{2}+s)$ does not depend on j and k. Then

(4.10) $$\operatorname{Res}_{\nu=\nu_t}\tau(\chi,\chi,u(\pi/6)a,\nu)$$

$$= \frac{2}{\prod_{s=1}^{t-1}(-t+s)}\sum_{\substack{j\geq t\\ k\geq 0}}\frac{\prod_{s=1}^{k}(\nu_t+j+s)x^{j+k}}{j!k!\prod_{s=t+1}^{j}(-t+s)\prod_{s=1}^{k}(\nu_t+s)\left(\frac{\nu_t+\mu}{2}+s\right)}$$

$$= (-1)^{t-1}(t-1)!\sum_{\substack{j\geq t\\ k\geq 0}}\frac{\prod_{s=1}^{k}(\mu-2t+j+s)\;x^{j+k}}{j!k!(j-t)!\prod_{s=1}^{k}(\mu-2t+s)(\mu+t+s)}.$$

Now, the series involved in the last expression is convergent and the terms are strictly greater than zero. This shows that the residue of the τ function at $\nu = \nu_t$ is not zero.

This proves the existence of simple poles of the τ function for $SU(2,1)$ in $\{\operatorname{Re}\nu > 0\}$ at the real points $\mu-2$, $\mu-4$, ... The same should be true for $SU(n+1,1)$, for $n > 1$ but we have not carried out the verification.

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Received October 10, 2002.

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