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Let (G, K) be a Hermitian symmetric pair and let $\mathfrak{g}$ and $\mathfrak{k}$ denote the corresponding complexified Lie algebras. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ be the usual decomposition of $\mathfrak{g}$ as a $\mathfrak{k}$-module. K acts on the symmetric algebra $S(\mathfrak{p}^-)$. We determine the K-structure of all K-stable ideals of the algebra. Our results resemble the Pieri rule for Young diagrams. The result implies a branching rule for a class of finite dimensional representations that appear in the work of Enright and Willenbring (preprint, 2001) and Enright and Hunziker (preprint, 2002) on Hilbert series for unitarizable highest weight modules.

1. Introduction.

The Pieri rule for the unitary group $U(m)$ gives the decomposition of the tensor product of an irreducible representation F and a symmetric power of $\mathbb{C}^m$. In the simplest case, if $(n_1, n_2, \ldots, n_m)$ with $n_1 \geq n_2 \geq \cdots \geq n_m \geq 0$, $n_i \in \mathbb{Z}$, is the highest weight of F then $F \otimes \mathbb{C}^m$ decomposes multiplicity free with highest weights $(n_1+1, n_2, \ldots, n_m)$ and $(n_1, n_2, \ldots, n_{i-1}, n_i+1, \ldots, n_m)$ for $2 \leq i \leq m$ with $n_{i-1} > n_i$. Here we consider a related question about the module structure of ideals in the context of Hermitian symmetric pairs.

Let (G, K) be an irreducible symmetric pair of Hermitian type. That is, if $\mathfrak{g} = \mathrm{Lie}(G) \otimes \mathbb{C}$ then it is a simple Lie algebra over $\mathbb{C}$ and if $\mathfrak{k} = \mathrm{Lie}(K) \otimes \mathbb{C}$ then $\mathfrak{k} = \mathbb{C}H \oplus [\mathfrak{k}, \mathfrak{k}]$ with $ad(H)$ having eigenvalues $0, 1, -1$ on $\mathfrak{g}$. We write

$$\mathfrak{p}^{\pm} = \{X \in \mathfrak{g} \mid [H, X] = \pm X\}.$$

Then $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$ is a direct sum of $\mathfrak{k}$-modules. Let B denote the Killing form of $\mathfrak{g}$. Then B induces a perfect pairing between $\mathfrak{p}^+$ and $\mathfrak{p}^-$. Thus we can look upon the symmetric algebra $S(\mathfrak{p}^-)$ as polynomials on $\mathfrak{p}^+$ and $S(\mathfrak{p}^+)$ as differential operators with constant coefficients on $\mathfrak{p}^+$.

The main result of this article describes the $\mathfrak{k}$-module structure of any K-stable ideal in the symmetric algebra $S(\mathfrak{p}^-)$. This result is a consequence of Theorem 2.1 whose form resembles the Pieri rule described above and indicates that K-invariant ideals are similar in K-structure to monomomial ideals. This question regarding K structure of ideals in $S(\mathfrak{p}^-)$ arose from the study of the the Hilbert series of unitarizable highest weight representations (with respect to the grading induced by H). By the Transfer Theorem of

[**EW**] and [**EH**], many unitarizable highest weight representations L have Hilbert series of the form:

$$h_L(t) \;=\; R\,\frac{f(t)}{(1-t)^d},$$

where d equals the Gelfand Kirillov dimension of L, $f(t)$ is the (polynomial) Hilbert series of a finite dimensional representation E_L of the reduced Hermitian symmetric pair (G_L, K_L) and R is the ratio of the dimensions of the zero grade in L and E_L. For the Wallach representations [**EW**] this ratio is $1/f(0)$. The connection with this article comes when we note that for a Wallach representation L of $SU(p,q)$ $Sp(n,\mathbb{R})$ or $SO(2,m)$, the G_L-representation E_L has a K_L-stable highest weight space (i.e., $f(0)=1$). Therefore E_L is isomorphic to the quotient of $S(\mathfrak{p}_L^-)$ by a K_L-invariant ideal. For these cases the results proved here are used in [**EH**] to offer explicit formulas for the Hilbert series of the Wallach representations.

In the literature there has been considerable interest in the question of the K-structure of ideals in $S(\mathfrak{p}^-)$, as pointed out to us by the referee. Roughly speaking half of the Hermitian symmetric cases have been treated in case by case studies: The case corresponding to $U(p,q)$ is treated in [**DEP**], the case corresponding to $SO^*(2n)$ in [**AD**] and the case corresponding to $Sp(n,\mathbb{R})$ in [**A**]. A related example not covered in the Hermitian symmetric setting is the action of $GL(n)$ on the skew symmetric $(n+1)\times(n+1)$ matrices which is treated in [**D**].

2. *K*-invariant ideals.

We continue with the Hermitian symmetric setting and notation of the introduction. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$ and a system of positive roots Φ^+ such that if $\alpha \in \Phi^+$ then $\alpha(H) \geq 0$. We set $\Phi_n^+ = \{\alpha \in \Phi \mid \alpha(H) = 1\}$. Let $\Phi_c^+ = \Phi^+ - \Phi_n^+$ then Φ_c^+ is a system of positive roots for $\mathfrak{k}$ on $\mathfrak{h}$. Let $\gamma_1 < \gamma_2 < \cdots < \gamma_r$ denote Harish-Chandra's strongly orthogonal roots. Schmid [**S**] has shown that as a K-module under the restriction of the adjoint representation $S(\mathfrak{p}^-)$ is multiplicity free and the irreducible constituents are exactly the K-modules with highest weights of the form

$$-(n_1\gamma_1 + \cdots + n_r\gamma_r), \quad n_1 \geq n_2 \geq \cdots \geq n_r \geq 0,\; n_i \in \mathbb{Z}.$$

We will write $S(\mathfrak{p}^-)[n_1,\ldots,n_r]$ for the corresponding isotypic component. We also note that if $d = \sum n_i$ then $S(\mathfrak{p}^-)[n_1,\ldots,n_r] \subset S^d(\mathfrak{p}^-)$ (the homogeneous elements of degree d). The main result is:

Theorem 2.1. *Let $n_1 \geq n_2 \geq \cdots \geq n_r \geq 0$, $n_i \in \mathbb{Z}$. Then*

$$S(\mathfrak{p}^-)[n_1,\ldots,n_r]\mathfrak{p}^- =$$
$$S(\mathfrak{p}^-)[n_1+1,n_2,\ldots,n_r] \oplus \bigoplus_{\substack{2\leq i\leq r\\ n_i<n_{i-1}}} S(\mathfrak{p}^-)[n_1,\ldots,n_{i-1},n_i+1,\ldots,n_r],$$

with multiplication in $S(\mathfrak{p}^-)$.

Before turning to the proof we first recall four well-known results on the strongly orthogonal roots (mostly due to C. Moore). Let $(.,.)$ denote the dual form to $B_{|\mathfrak{h}}$.

$$(1) \qquad (\gamma_i,\gamma_i) = (\gamma_j,\gamma_j) \quad \text{for all} \quad i,j.$$

$$(2) \qquad \text{If } \ \alpha \in \Phi^+ \ \text{ then } \ (\alpha,\alpha) \leq (\gamma_1,\gamma_1).$$

Let $\mathfrak{h}^-$ denote the linear span of the coroots, $\gamma_i^\vee$, of the γ_i. Then $\dim \mathfrak{h}^- = r$ and the $\gamma_i^\vee$ form an orthogonal basis of $\mathfrak{h}^-$.

(3) If $\alpha \in \Phi_n^+$ and $\alpha \neq \gamma_i$ for any i, then $\alpha_{|\mathfrak{h}^-}$ is either of the form $\frac{1}{2}(\gamma_i+\gamma_j)$ with $i<j$ or $\frac{1}{2}\gamma_i$.

(4) If $\alpha \in \Phi_c^+$ then $\alpha_{|\mathfrak{h}^-}$ is either of the form $-\frac{1}{2}(\gamma_i-\gamma_j)$ with $i<j$ or $-\frac{1}{2}\gamma_i$.

With this notation in place we can prove:

Lemma 2.2. *Let $n_1 \geq n_2 \geq \cdots \geq n_r \geq 0$, $n_i \in \mathbb{Z}$. Then*

$$S(\mathfrak{p}^-)[n_1,\ldots,n_r]\mathfrak{p}^- \subset$$
$$S(\mathfrak{p}^-)[n_1+1,\ldots,n_r] \oplus \bigoplus_{\substack{2\leq i\leq r\\ n_i<n_{i-1}}} S(\mathfrak{p}^-)[n_1,\ldots,n_{i-1},n_i+1,\ldots,n_r].$$

Proof. Let $\lambda = -\sum n_i\gamma_i$ then it is standard that the possible highest weights that can occur in $S(\mathfrak{p}^-)[n_1,\ldots,n_r]\otimes\mathfrak{p}^-$ are of the form $\lambda-\alpha$ with $\alpha\in\Phi_n^+$. Schmid's result implies that if this highest weight occurs in $S(\mathfrak{p}^-)[n_1,\ldots,n_r]\mathfrak{p}^-$ then

$$\lambda-\alpha = -\sum m_i\gamma_i \text{ with } m_1\geq m_2\geq\cdots\geq m_r\geq 0, m_i\in\mathbb{Z}.$$

Thus, in particular, we have

$$\lambda-\alpha_{|\mathfrak{h}^-} = -\sum m_i\gamma_i$$

with the conditions above satisfied. Using the forms of $\alpha_{|\mathfrak{h}^-}$ in (3) above we see that since the coefficients on the right-hand side of the equation are all integers the only possibility for α is one of the γ_i. Now the Schmid conditions on the highest weights imply the lemma.

This simple result reduces the proof to showing that the predicted components actually occur. For our proof of this assertion (and hence of the theorem) we recall several results from [**W**]. Let $\mathfrak{n}_c^+$ denote the sum of the root spaces for the elements of Φ_c^+. We choose a nonzero element u_i in $S(\mathfrak{p}^-)^{\mathfrak{n}_c^+} \cap \ S(\mathfrak{p}^-)[n_1, \ldots, n_r]$ with $n_j = 1$ for $j \leq i$ and $n_j = 0$ for $j > i$. Then one can easily see from Schmid's result that:

(5) $\quad S(\mathfrak{p}^-)^{\mathfrak{n}_c^+}$ is the polynomial ring on the algebraically independent elements $u_1, \ldots, u_r$.

We will now analyze these covariants in further detail. For each $1 \leq j \leq r$, let $\Phi_{0,j}^+$ denote the set of elements, α, of Φ^+ such that $\alpha_{|\mathfrak{h}^-}$ is of the form $\frac{1}{2}(\gamma_p \pm \gamma_q)$ with $q \leq p \leq j$. Then $\Phi_{0,j} = \Phi_{0,j}^+ \cup -\Phi_{0,j}^+$ is a subrootsystem of Φ. Let $\mathfrak{g}_{0,j}$ denote the subalgebra of $\mathfrak{g}$ generated by the root spaces for the roots in $\Phi_{0,j}$. If $\mathfrak{u}_j$ is the sum of all ideals of $\mathfrak{g}_{0,j}$ contained in $\mathfrak{g}_{0,j} \cap \mathfrak{k}$ then $(\mathfrak{g}_{0,j}/\mathfrak{u}_\mathfrak{j}, (\mathfrak{k} \cap \mathfrak{g}_{0,j})/\mathfrak{u}_\mathfrak{j})$ is also an irreducible symmetric pair of Hermitian type. Set $\mathfrak{p}_{0,j}^\pm = \mathfrak{g}_{0,j} \cap \mathfrak{p}^\pm$. One has (see [**W**])

$$u_j \in S(\mathfrak{p}_{0,j}^-). \tag{6}$$

We denote by $x \mapsto \overline{x}$ the conjugation of $\mathfrak{g}$ with respect to $\mathrm{Lie}(G)$. Then $\overline{\mathfrak{p}^-} = \mathfrak{p}^+$. If $x \in \mathfrak{p}^+$ then we denote by $\partial(x)$ the derivation of $S(\mathfrak{p}^-)$ defined by $\partial(x)y = B(x, y)$ for $y \in \mathfrak{p}^-$. We will also denote the extension of ∂ to $S(\mathfrak{p}^+)$ by ∂. In addition we will use the notation $u \mapsto u(0)$ for the augmentation map of $S(\mathfrak{p}^-)$ to $\mathbb{C}$ given as the extension to a homomorphism of $y \mapsto 0$ for $y \in \mathfrak{p}^-$. We define for $u, v \in S(\mathfrak{p}^-)$, $\langle u, v\rangle = (\partial(\overline{v})u)(0)$. The following observation is well-known and easily checked:

(7) $\quad$ The Hermitian form $\langle \cdot, \cdot \rangle$ is positive definite and K-invariant. Furthermore, if $u, v, w \in S(\mathfrak{p}^-)$ then $\langle uv, w\rangle = \langle v, \partial(\overline{u})w\rangle$.

We now begin the proof that the predicted representations in Theorem 2.1 actually occur. If $n_1 \geq \cdots \geq n_r \geq 0$ then we note that

$$u_1^{n_1-n_2}u_2^{n_2-n_3}\ldots u_{r-1}^{n_{r-1}-n_r}u_r^{n_r}$$

is a basis of the highest weight space of $S(\mathfrak{p}^-)[n_1, \ldots, n_r]$. Thus to prove the result we must show that

$$u_1^{n_1-n_2+1}u_2^{n_2-n_3}\ldots u_{r-1}^{n_{r-1}-n_r}u_r^{n_r} \in S(\mathfrak{p}^-)[n_1, \ldots, n_r]\mathfrak{p}^- \tag{8}$$

and if $r \geq j > 1$ and $n_{j-1} > n_j$ then

$$u_1^{n_1-n_2} u_2^{n_2-n_3} \dots u_{j-1}^{n_{j-1}-n_j-1} u_j^{n_j-n_{j+1}+1} \dots u_{r-1}^{n_{r-1}-n_r} u_r^{n_r} \in S(\mathfrak{p}^-)[n_1, \dots, n_r]\mathfrak{p}^-. \tag{9}$$

Equation (8) is obvious since $u_1 = X_{-\gamma_1}$. This proves the theorem if $r = 1$. We proceed by induction on r. If $r = 1$ then we have already observed that the result is true. Assume that the result is true for $r - 1 \geq 1$. If $n_r = 0$ and if $n_{j-1} > n_j$ with $j < r$ then the inductive hypothesis and (6) above implies that

$$u_1^{n_1-n_2} u_2^{n_2-n_3} \dots u_{j-1}^{n_{j-1}-n_j-1} u_j^{n_j-n_{j+1}+1} \dots u_{r-1}^{n_{r-1}} \in S(\mathfrak{p}_{0,r-1}^-)[n_1, \dots, n_{r-1}]\mathfrak{p}_{0,r-1}^-$$

and so (9) is true in this case. By (6), the highest weights although not the full $\mathfrak{k}$-modules are contained in $S(\mathfrak{p}_{0,r}^-)$ and thus we may reduce to the case $\mathfrak{g} = \mathfrak{g}_{0,r}$ which we now assume. This implies that the $\mathfrak{k}$-module $S(\mathfrak{p}^-)[n, n, \dots, n]$ is one dimensional. We have

$$S(\mathfrak{p}^-)[n_1, \dots, n_r] = S(\mathfrak{p}^-)[n_1 - n_r, \dots, n_{r-1} - n_r, 0]u_r^{n_r}.$$

So the result for $j < r$ and $n_r > 0$ follows from the case when $n_r = 0$. To complete the proof we look at the remaining case; when $n_{r-1} > n_r \geq 0$. As before we are reduced to proving (9) for $j = r$, $n_r = 0$ and $n_{r-1} > 0$ to complete the induction. Let $D = \partial(\overline{u_r})$. Then

$$D : S(\mathfrak{p}^-)[n_1, \dots, n_r] \to S(\mathfrak{p}^-)[n_1 - 1, \dots, n_r - 1].$$

Here if $n_r < 0$ then we write $S(\mathfrak{p}^-)[n_1, \dots, n_r] = \{0\}$. Suppose in this last case, that (9) is not true. Then we would have:

$$u_1^{n_1-n_2} u_2^{n_2-n_3} \dots u_{r-1}^{n_{r-1}-1} u_r \notin S(\mathfrak{p}^-)[n_1, \dots, n_{r-1}, 0]\mathfrak{p}^-. \tag{10}$$

Now (10) implies that $D\left(S(\mathfrak{p}^-)[n_1, \dots, n_{r-1}, 0]\mathfrak{p}^-\right) = 0$ and hence

$$\begin{aligned} 0 &= \langle D\, S(\mathfrak{p}^-)[n_1, \dots, n_{r-1}, 0]\mathfrak{p}^-, S(\mathfrak{p}^-)\rangle \\ &= \langle S(\mathfrak{p}^-)[n_1, \dots, n_{r-1}, 0], \partial(\mathfrak{p}^+)\ u_r\ S(\mathfrak{p}^-)\rangle \\ &= \langle S(\mathfrak{p}^-)[n_1, \dots, n_{r-1}, 0], (\partial(\mathfrak{p}^+)\ u_r)\ S(\mathfrak{p}^-)\rangle. \end{aligned}$$

Again since we have reduced to the case where $\mathfrak{g} = \mathfrak{g}_{0,r}$, u_r spans a one dimensional $\mathfrak{k}$-module and so

$$\partial(\mathfrak{p}^+)\ u_r = S(\mathfrak{p}^-)[1, 1, \dots, 1, 0].$$

Combining these last two identities gives

$$0 = \langle S(\mathfrak{p}^-)[n_1, \dots, n_{r-1}, 0],\ u_{r-1}\ S(\mathfrak{p}^-)\rangle.$$

By induction on rank we conclude that

$$0 = \langle S(\mathfrak{p}^-)[n_1, \dots, n_{r-1}, 0],\ S(\mathfrak{p}^-)[n_1, \dots, n_{r-1}, 0]\rangle,$$

for all $n_1 \geq n_2 \geq \cdots \geq n_{r-1} \geq 1$. This is a contradiction and so (10) does not hold. This completes the induction and the proof of the theorem. $\square$

As a consequence of this result we can describe the ideals generated by isotypic components in $S(\mathfrak{p}^-)$.

Corollary 2.3. *Let $n_1 \geq n_2 \geq \cdots \geq n_r$ then*

$$S(\mathfrak{p}^-)(S(\mathfrak{p}^-)[n_1, \ldots, n_r]) = \bigoplus_{\substack{m_i \geq n_i \\ m_1 \geq \cdots \geq m_r \geq 0}} S(\mathfrak{p}^-)[m_1, \ldots, m_r].$$

Proof. Let I denote the ideal on the left and let I^d denote the d-th level in the grading. Multiplication by elements of $\mathfrak{p}^-$ shifts the grade by one. Set $d_0 = \sum n_i$. Then d_0 is the minimal value for $I^d \neq 0$. The corollary holds when restricted to this level of the grade. Now suppose $d > d_0$ and the corollary holds for I^{d-1}:

$$I^{d-1} = \bigoplus_{\substack{m_i \geq n_i \\ m_1 \geq \cdots \geq m_r \geq 0 \\ \sum m_i = d-1}} S(\mathfrak{p}^-)[m_1, \ldots, m_r]. \tag{11}$$

Now multiply by $\mathfrak{p}^-$. Then $I^d = \mathfrak{p}^- \ I^{d-1}$ which by the theorem gives

$$I^{d} \subset \bigoplus_{\substack{m_i \geq n_i \\ m_1 \geq \cdots \geq m_r \geq 0 \\ \sum m_i = d}} S(\mathfrak{p}^-)[m_1, \ldots, m_r]. \tag{12}$$

For the opposite inclusion suppose the indices $m_1, \ldots, m_r$ satisfy the conditions in (12). Since $d > d_0$ choose i, $1 \leq i \leq r$, maximal with $m_i > n_i$. Then, by the induction hypothesis,

$$S(\mathfrak{p}^-)[m_1, \ldots, m_i - 1, m_{i+1}, \ldots, m_r] \subset I^{d-1}.$$

Multiplying by $\mathfrak{p}^-$ and applying Theorem 2.1 implies $S(\mathfrak{p}^-)[m_1, \ldots, m_r] \subset I^d$. This proves equality in (12) and completes the induction.

Let I be a general K-invariant ideal in $S(\mathfrak{p}^-)$. Then I is generated as an ideal by a finite set of K-isotypic subspaces, say $S(\mathfrak{p}^-[\underline{n}^j])$, where $\underline{n}^j = [n^j_1, \ldots, n^j_m]$, $1 \leq j \leq t$. If $\underline{n}, \underline{m} \in \mathbb{Z}^r$ then we write $\underline{n} \succ \underline{m}$ if $n_i \geq m_i$ for all i, $1 \leq i \leq n$. Let $L_j = \{\underline{n} \in \mathbb{Z}^r | \underline{n} \succ \underline{n}^j\}$ and $L_I = \bigcup\limits_j L_j$. With this notation in place we have:

Corollary 2.4. *The ideal I is multiplicity free with K-decomposition:*

$$I = \bigoplus_{\substack{\underline{m} \in L_I \\ m_1 \geq \cdots \geq m_r \geq 0}} S(\mathfrak{p}^-)[m_1, \ldots, m_r].$$

3. A branching formula.

Let ω be the unique fundamental weight orthogonal to the compact roots. For any Φ_c^+-dominant integral weight ξ, let F_ξ be the irreducible finite dimensional $\mathfrak{k}$-module with highest weight ξ. Let $N(\xi+\rho)$ denote the generalized Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{k}\oplus\mathfrak{p}^+)} F_\xi$.

Theorem 3.1. *For $m \in \mathbb{N}$ let $E_{m\omega}$ be the irreducible finite dimensional $\mathfrak{g}$-module with highest weight $m\omega$. Then as a $\mathfrak{k}$-module,*

$$E_{m\omega} \simeq \bigoplus_{m\geq n_1\geq\cdots\geq n_r\geq 0} F_{-n_1\gamma_1-n_2\gamma_2-\cdots-n_r\gamma_r} \otimes F_{m\omega}.$$

Proof. Let α denote the unique simple noncompact root. Then $\alpha = \gamma_1$. From the BGG resolution we obtain the exact sequence:

$$N(s_\alpha(m\omega+\rho)) \to N(m\omega+\rho) \to E_{m\omega} \to 0.$$

The $\mathfrak{k}$-module $F_{m\omega}$ is one dimensional, so the image of the left term has the form $I\otimes F_{m\omega}$ where I is the ideal in $S(\mathfrak{p}^-)$ generated by $S(\mathfrak{p}^-)[m+1, 0, \ldots, 0]$. The result now follows from Corollary 2.3. $\square$

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