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PARTIAL REGULARITY FOR WEAK SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH SUPERCRITICAL EXPONENTS

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Let Ω be an open subset in $\mathbf{R}^n$ $(n \geq 3)$. In this paper, we study the partial regularity for stationary positive weak solutions of the equation

$$\Delta u + h_1(x)u + h_2(x)u^\alpha = 0 \quad \text{in } \Omega. \tag{1.1}$$

We prove that if $\alpha > \frac{n+2}{n-2}$, and $u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega)$ is a stationary positive weak solution of (1.1), then the Hausdorff dimension of the singular set of u is less than $n - 2\frac{\alpha+1}{\alpha-1}$, which generalizes the main results in Pacard 1993 and Pacard 1994.

1. Introduction.

Let Ω be an open subset in $\mathbf{R}^n$ $(n \geq 3)$. In this paper, we prove a partial regularity result for positive weak solutions of the equation

$$\Delta u + h_1(x)u + h_2(x)u^\alpha = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $\alpha > \frac{n+2}{n-2}$, $h_i \in C^1(\Omega)$, $a_i \leq h_i(x) \leq b_i$, $0 < a_i < b_i$ and $|\nabla \log h_i(x)| \leq \beta$ $(i = 1, 2)$ for $x \in \overline{\Omega}$. As we know, there is not much known about the properties of the weak solutions of (1.1).

We say that u is a positive weak solution of (1.1) in Ω if $u(x) \geq 0$ for a.e. $x \in \Omega$ and for all $\phi \in C^\infty(\Omega)$ with compact support in Ω,

$$-\int_\Omega u\Delta\phi dx = \int_\Omega [h_1(x)u + h_2(x)u^\alpha]\phi(x)dx. \tag{1.2}$$

We say that a weak solution u is stationary, if it satisfies

$$\begin{aligned}\int_\Omega \Big[\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial \phi^j}{\partial x_i} - \frac{1}{2}|\nabla u|^2\frac{\partial \phi^i}{\partial x_i} + \frac{1}{2}u^2\frac{\partial h_1}{\partial x_i}\phi^i + \frac{1}{2}h_1 u^2\frac{\partial \phi^i}{\partial x_i} \\ + \frac{1}{\alpha+1}u^{\alpha+1}\frac{\partial h_2}{\partial x_i}\phi^i + \frac{1}{\alpha+1}h_2 u^{\alpha+1}\frac{\partial \phi^i}{\partial x_i}\Big] dx = 0\end{aligned} \tag{1.3}$$

for all regular vector field ϕ with compact support in Ω (summation over i and j is understood).

For weak solutions in $H^1(\Omega)\cap L^{\alpha+1}(\Omega)$ this identity is obtained by assuming that the functional $E(u)$ is stationary with respect to domain variations, that is,

$$\frac{d}{dt}E(u_t)|_{t=0} = 0$$

where

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2}\int_\Omega h_1 u^2 - \frac{1}{\alpha+1}\int_\Omega h_2 u^{\alpha+1} dx$$

and $u_t(x) = u(x + t\phi(x))$.

Let $u \in H^1(\Omega)\cap L^{\alpha+1}(\Omega)$ be a positive weak solution of (1.1). We denote by Σ the set of points $x \in \Omega$ such that u is not bounded in any neighborhood W of x in Ω. If u is bounded in a neighborhood of x then the classical regularity theory ensures that u is regular in the neighborhood of x. Therefore Σ is the singular set of u. Moreover, Σ is a closed subset of Ω.

If $\alpha < \frac{n}{n-2}$, a simple bootstrap argument shows that all positive weak solutions of (1.1) are regular. It is well-known that the singular set may not be empty if $\alpha \geq \frac{n}{n-2}$. Pacard [**Pa2**] constructed solutions with singular sets of Hausdorff dimension $d < n - \frac{2\alpha}{\alpha-1}$. Schoen and Yau proved in [**SY**] that the singular set of a positive weak solution of (1.1) is not always as simple as in the examples given in [**Pa2**].

In [**Pa1**] and [**Pa3**], Pacard showed that the Hausdorff dimension of the singular set of a stationary positive weak solution u of the equation $-\Delta u = u^\alpha$ in Ω is less than $n - 2\frac{\alpha+1}{\alpha-1}$.

In a recent paper [**GL**], we considered the compactness for positive solutions of Equation (1.1). Using the ideas in [**LT1**] and [**LT2**], we obtained the measure estimate of the blow up set of a sequence of positive smooth solutions $\{u_i\}$ of (1.1) with $\{\|u_i\|_{H^1(\Omega)} + \|u_i\|_{L^{\alpha+1}(\Omega)}\}$ bounded. We applied such result to a semilinear eigenvalue problem

$$-\Delta u = \lambda(u + u^\alpha) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.4}$$

when Ω is a smooth star-shaped domain and obtained that any branch of positive solutions $(\lambda(s), u(s))$ of (1.4) must converge to a (singular) positive solution u_0 of the equation

$$-\Delta u = \lambda_0(u + u^\alpha) \text{ in } \Omega \tag{1.5}$$

as $\lambda(s) + \|u(s)\|_{L^\infty(\Omega)} \to \infty$, $s \to \infty$, where $\lambda_0 = \lim_{s\to\infty}\lambda(s)$ and $0 < \lambda_0 < \infty$. The existence of such branches of positive solutions is obtained by Rabinowitz. It was proved in [**BDT**] and [**Da**] that some branches are simple curves.

In this paper, we shall prove a partial regularity theorem for a stationary positive weak solution of (1.1) with $\alpha > \frac{n+2}{n-2}$.

Theorem A. *Let $\alpha > \frac{n+2}{n-2}$. If $u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega)$ is a stationary positive weak solution of* (1.1), *then the Hausdorff dimension of the singular set of u is less than $n - 2\frac{\alpha+1}{\alpha-1}$.*

Our result covers the main results in [**Pa1**] and [**Pa3**]. The proof is quite different, we used the duality of a weighted Hardy space and a weighted BMO, which was used in [**CLL**] to get a partial regularity result for a weak heat flow.

When $h_1 = 0$ and h_2 is a constant, it is not hard to construct solutions of (1.1) which are singular (see [**Lin**] and [**Re**]). However, when h_2 is not a constant, the problem is much harder. A singular solution was given in this case by Johnson-Pan-Yi [**JPY**]. Let $\Omega = B_R$, here $B_R \subset \mathbf{R}^n$ ($n \geq 3$) is a ball with center at 0 and radius of $R > 0$. Consider the equation

$$\Delta u + K(|x|)u^\alpha = 0 \text{ in } B_R \tag{1.6}$$

with $K(|x|)$ satisfying the following conditions in [**JPY**]:

(K1) $K \in C^1[0, \infty)$, $K'(0) = 0$, $K(r) > 0$ for $r \geq 0$, and $\lim_{r\to\infty} K(r) = K(\infty) > 0$;

(K2) There is a $\delta > 0$ such that $\lim\limits_{r\to\infty} r^\delta(K(r) - K(\infty)) = 0$, $\lim\limits_{r\to\infty} r^{1+\delta}K'(r) = 0$;

(K3) $K'(r) \leq 0$ for $r > 0$.

It is proved in [**JPY**] (Theorem 1) that the equation

$$\Delta u + K(|x|)u^\alpha = 0 \text{ in } \mathbf{R}^n$$

has a singular solution $U_0(r)$ with $r = |x|$, which satisfies

$$\lim_{r\to 0} r^{\frac{2}{\alpha-1}} U_0(r) = \left[\frac{1}{K(0)} \cdot \frac{2}{\alpha-1}\left(n - 2 - \frac{2}{\alpha-2}\right)\right]^{\frac{1}{\alpha-1}},$$

$$\lim_{r\to 0} r^{\frac{2}{\alpha-1}+1} U_0'(r) = -\frac{2}{\alpha-1}\left[\frac{1}{K(0)} \cdot \frac{2}{\alpha-1}\left(n - 2 - \frac{2}{\alpha-1}\right)\right]^{\frac{1}{\alpha-1}}.$$

It is clear that $U_0(|x|)$ for $x \in B_R$ is a singular solution of Equation (1.6).

Throughout this paper, C will denote a universal constant depending only on α, β, n and a_i, b_i ($i = 1, 2$), unless it is explicitly stated.

2. $H^1_w(\mathbf{R}^n)$ and $M^\sharp_{1,\nu} g(x)$.

In this section we review definitions and properties of the space $H^1_w(\mathbf{R}^n)$ and the function $M^\sharp_{1,\nu} g(x)$. See Strömberg & Torchinsky [**ST**] for more details.

Let μ be the Lebesgue measure in $\mathbf{R}^n$ and $d\mu(x) = dx$. Let ν be a weighted measure with respect to the Lebesgue measure in $\mathbf{R}^n$ with weight $w(x)$. Then

$$H^1_w(\mathbf{R}^n) = \{f \in \mathcal{S}'(\mathbf{R}^n) : \ M_1(F_\phi) \in L^1_w(\mathbf{R}^n), \quad \|f\|_{H^1_w} = \|M_1(F_\phi)\|_{L^1_w}\},$$

where

$$F_\phi(x) = \frac{1}{t^n} \int_{\mathbf{R}^n} f(y)\phi\left(\frac{y-x}{t}\right) dy,$$

ϕ is any smooth function with support in the unit ball and $M_1(F_\phi(x)) = \sup_{t>0} F_\phi(x)$.

For $g \in L^1_{\text{loc}}(\mathbf{R}^n)$, define

$$M^\sharp_{1,\nu} g(x) = \sup_{t>0} \frac{1}{\nu(B(x,t))} \int_{B(x,t)} |g(y) - (g)_{x,t}| dy,$$

where

$$(g)_{x,t} \equiv \frac{1}{B(x,t)} \int_{B(x,t)} g dy,$$

and $B(x,t) \subset \mathbf{R}^n$ is the ball centered at x with radius t. It follows from Theorem 2 in Chapter IX in [**ST**] that for $f \in \hat{\mathcal{D}}_0$, $g \in L^1_{\text{loc}}(\mathbf{R}^n)$ and $\nu \in D_d$ for some $d > 0$ (see Doubling D_d condition in Chapter I in [**ST**]), there exists $C > 0$ independent of f and g such that

$$\int_{\mathbf{R}^n} f(x)g(x)dx \le C\left(\int_{\mathbf{R}^n} M_1(F_\phi(x)) M^\sharp_{1,\nu} g(x) w(x) dx\right). \tag{2.1}$$

Since $\hat{\mathcal{D}}_0$ is dense in $H^1_w(\mathbf{R}^n)$ (see Theorem 1 of Chapter VII in [**ST**]), we conclude that (2.1) holds for $f \in H^1_w(\mathbf{R}^n)$ and $g \in L^1_{\text{loc}}(\mathbf{R}^n)$.

In this paper, we define $w(x) = |x|^{-2/(\alpha-1)}$ and $d\nu(x) = |x|^{-2/(\alpha-1)}dx$. Then ν is a doubling weighted measure with respect to the Lebesgue measure of $\mathbf{R}^n$ with weight $|x|^{-2/(\alpha-1)}$ and $\nu \in D_{n-\frac{2}{(\alpha-1)}}$. Moreover,

$$\nu(B(x,t)) = \frac{(\alpha-1)\omega_n}{n(\alpha-1)-2} t^{n-\frac{2}{\alpha-1}},$$

where ω_n is the area of the $(n-1)$-dimensional unit sphere in $\mathbf{R}^n$.

3. A monotonicity inequality and blow up.

In this section, we first recall a monotonicity inequality for stationary positive weak solutions of (1.1) established in [**GL**], using this monotonicity inequality and a blow up argument, we then obtain a decay property of the scaled energy. Assume henceforth that $u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega)$ is a stationary positive solution of (1.1).

For any $x_0 \in \Omega$ and $r > 0$, define

$$
\begin{aligned}
E_u(x_0, r) \equiv & \frac{(\alpha-1)}{2(\alpha+1)} e^{Cr} r^{-\mu} \int_{B(x_0,r)} h_2 u^{\alpha+1} dx \\
& + \frac{1}{4}\left(e^{Cr} \frac{d}{dr}\left(r^{-\mu} \int_{\partial B(x_0,r)} u^2 ds \right)\right) \\
& + \frac{1}{4}\left(e^{Cr} r^{-\mu-1}(-1+Cr) \int_{\partial B(x_0,r)} u^2 ds \right) \\
& + C \int_0^r e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi,
\end{aligned}
$$

where $\mu = n - 2\frac{\alpha+1}{\alpha-1}$ and C depends only upon α, β, n and a_i, b_i $(i = 1, 2)$. It is proved in [**GL**] that $E_u(x_0, r)$ can be written to the equivalent forms:

$$
\begin{aligned}
E_u(x_0, r) \equiv & \frac{1}{2} e^{Cr} r^{-\mu} \int_{B(x_0,r)} |\nabla u|^2 dx - \frac{1}{2} e^{Cr} r^{-\mu} \int_{B(x_0,r)} h_1 u^2 dx \\
& - \frac{1}{(\alpha+1)} e^{Cr} r^{-\mu} \int_{B(x_0,r)} h_2 u^{\alpha+1} dx \\
& + \frac{1}{(\alpha-1)} e^{Cr} r^{-\mu-1} \int_{\partial B(x_0,r)} u^2 ds \\
& + \frac{C}{4} e^{Cr} r^{-\mu} \int_{\partial B(x_0,r)} u^2 ds + C \int_0^r e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi
\end{aligned}
$$

and

$$
\begin{aligned}
E_u(x_0, r) \equiv & \left(\frac{\alpha-1}{\alpha+3}\right) e^{Cr} r^{-\mu} \Bigg[\frac{1}{(\alpha+1)} \int_{B(x_0,r)} h_2 u^{\alpha+1} dx \\
& + \frac{1}{2} \int_{B(x_0,r)} |\nabla u|^2 dx - \frac{1}{2} \int_{B(x_0,r)} h_1 u^2 dx \Bigg] \\
& + \frac{1}{(\alpha+3)} \frac{d}{dr} \left(e^{Cr} r^{-\mu} \int_{\partial B(x_0,r)} u^2 ds \right) \\
& + \left(\frac{C}{4} - \frac{C}{(\alpha+3)} \right) e^{Cr} r^{-\mu} \int_{\partial B(x_0,r)} u^2 ds \\
& + C \int_0^r e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi.
\end{aligned}
$$

All the derivatives in the above expressions are to be understood in the sense of distributions. Lemma 3.1 and Lemma 3.2 below are proved in [**GL**].

Lemma 3.1. *If $u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega)$ is a stationary positive weak solution of (1.1), then $E_u(x_0, r)$, defined above, is an increasing function of r.*

Lemma 3.2. *$E_u(x_0, r)$ is a continuous function of $x_0 \in \Omega$ and $r > 0$.*

Now we show the following lemma:

Lemma 3.3. *There exist $0 < r_0 < 1$ independent of $x_0 \in \Omega$ and some constant $C > 0$ depending only upon α, n, such that the following inequality holds:*

$$r^{-\mu} \int_{B(x_0,r)} |\nabla u|^2 dx \le C E_u(x_0, 2r) \le C E_u(x_0, r_0) \quad \text{for } r < r_0/2. \tag{3.1}$$

Proof. We consider the last one of the three equivalent formulations of $E_u(x_0, r)$ given above. By Lemma 2.3 in [**GL**] we know that there exists $0 < r_0 < 1$ such that

$$E_u(x_0, r) \ge 0 \quad \text{for all } x_0 \in \Omega, \quad 0 < r < r_0, \tag{3.2}$$

and for $r < r_0$,

$$\begin{aligned} &\frac{1}{2}\left(\frac{\alpha-1}{\alpha+3}\right) e^{Cr} r^{-\mu} \int_{B(x_0,r)} h_1 u^2 dx \\ &\le C \int_0^r e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi + \frac{1}{2(\alpha+1)} \left(\frac{\alpha-1}{\alpha+3}\right) e^{Cr} r^{-\mu} \int_{B(x_0,r)} h_2 u^{\alpha+1} dx. \end{aligned} \tag{3.3}$$

We denote by $\phi(r) = r^{-\mu} \int_{B(x_0,r)} |\nabla u|^2 dx$. Since $E_u(x_0, r)$ is an increasing function of r, we integrate it from 0 to $r < r_0$ and obtain that for almost every $x_0 \in \Omega$, (note that $e^{Cr} > 1$)

$$\frac{\alpha-1}{2} \int_0^r \phi(\rho) d\rho + e^{Cr} r^{-\mu} \int_{\partial B(x_0,r)} u^2 ds \le (\alpha+3) E_u(x_0, r) r \quad \text{for } r < r_0.$$

(Here we have used $\lim_{r\to 0} r^{-\mu} \int_{\partial B(x_0,r)} u^2 ds = 0$ a.e. $x_0 \in \Omega$. This fact is proved in [**GL**].) Now we use Remark 2 in [**Pa1**] and we see that there exists some $\sigma \in [r/2, r]$ such that

$$\phi(\sigma) \le \frac{8}{r} \int_0^r \phi(\rho) d\rho \le C E_u(x_0, r),$$

for some constant $C > 0$ depending only upon α, β and n. In addition we have $\phi(r/2) \le 2^\mu \phi(\sigma)$, if $\sigma \in [r/2, r]$. This gives us the desired result for almost every x_0 and, by continuity, for every x_0.

Proposition 3.4. *Assume that there exist $x_0 \in \Omega$ and $0 < r_1 < r_0$ such that $E_u(x_0, r_1) \le \delta$. Then*

$$r^{-\mu} \int_{B(y,r)} |\nabla u|^2 dx \le C\delta, \tag{3.4}$$

for all $y \in B(x_0, r_1/8)$ and $0 < r < r_1/4$, where C only depends upon n, α, β.

Proof. Let $0 < r < r_1$. We know that for any $y \in B(x_0, r/2)$, $B(y, r/2) \subset B(x_0, r) \subset B(x_0, r_1)$. Thus,

$$\int_{B(y,r/2)} |\nabla u|^2 dx \leq \int_{B(x_0,r)} |\nabla u|^2 dx.$$

Thus, (note that $e^{Cr} > 1$)

$$\begin{aligned} E_u(x_0, r) \geq\ & 2^{-\mu} \left(\frac{\alpha - 1}{2(\alpha+1)(\alpha+3)} \right) \left(\frac{r}{2} \right)^{-\mu} \int_{B(x_0,r)} h_2 u^{\alpha+1} dx \\ & + \left(\frac{\alpha - 1}{\alpha + 3} \right) e^{Cr} r^{-\mu} \left(\frac{1}{2} \int_{B(y,r/2)} |\nabla u|^2 dx - \widetilde{C} r^n \right) \\ & + \frac{1}{(\alpha+3)} \frac{d}{dr} \left(e^{Cr} r^{-\mu} \int_{\partial B(x_0,r)} u^2 dx \right) \\ & + C e^{Cr} r^{-\mu} \left(\frac{1}{4} - \frac{1}{(\alpha+3)} \right) \int_{\partial B(x_0,r)} u^2 ds + C \int_0^r e^{C\xi} \xi^{\frac{\alpha+1}{\alpha-1}} d\xi. \end{aligned}$$

Define $\psi(r) = \left(\frac{r}{2}\right)^{-\mu} \int_{B(y,r/2)} |\nabla u|^2 dx$. By the argument similar to that in the proof of Lemma 2.3 in [**GL**], we have, for almost every $x_0 \in \Omega$,

$$2^{\mu-1}(\alpha - 1) \int_0^r \psi(\rho) d\rho \leq (\alpha + 3) E_u(x_0, r_1) r. \tag{3.5}$$

Using Remark 2 in [**Pa1**] again, we see that there exists some $\sigma \in [r/2, r]$ such that

$$\psi(\sigma) \leq \frac{8}{r} \int_0^r \psi(s) ds \leq C E_u(x_0, r_1), \tag{3.6}$$

for some constant $C > 0$ only depending upon α, β and n. It is clear that $\psi(r/2) \leq 2^{\mu} \psi(\sigma)$. Since

$$\psi(r/2) = \left(\frac{r}{4} \right)^{-\mu} \int_{B(y,r/4)} |\nabla u|^2 dx, \tag{3.7}$$

we have the desired result for almost every $y \in B(x_0, r_1/8)$. By continuity, we see that it holds for every $y \in B(x_0, r_1/8)$.

Define

$$F_u(x_0, r) = r^{-\mu} \int_{B(x_0,r)} |\nabla u|^2 dx + C \int_0^r e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi,$$

where $C \int_0^r e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi$ is the function in the formulations of $E_u(x_0, r)$. Then we have the following lemma:

Lemma 3.5. *We have that*

$$r^{-\mu} \int_{B(x_0,r)} h_2 u^{\alpha+1} dx \leq C F_u(x_0, r) \quad \textit{for all } x_0 \in \Omega \textit{ and } 0 < r < r_0$$

and

$$r^{-\mu}\int_{B(x_0,r)} h_1 u^2 dx \le C F_u(x_0,r) \quad \text{for all } x_0 \in \Omega \text{ and } 0<r<r_0,$$

where C depends only upon α, n, a_i and b_i $(i=1,2)$.

Proof. We only show the first inequality, the second can be obtained by a similar argument. Since $E_u(x_0,r) \ge 0$ for all $x_0 \in \Omega$ and $0<r<r_0$, it can be seen from the second of the three equivalent formulations given above that

$$r^{-\mu}\int_{B(x_0,r)} h_2 u^{\alpha+1} dx \le C\left(F_u(x_0,r) + r^{-\mu-1}\int_{\partial B(x_0,r)} u^2 ds\right),$$

for some constant C depending only upon α, n, a_i and b_i $(i=1,2)$. On the other hand, the trace embedding theorem gives

$$H^1(B(x_0,r)) \hookrightarrow W^{\frac{1}{2},2}(\partial B(x_0,r)) \hookrightarrow L^{\frac{2(n-1)}{n-2}}(\partial B(x_0,r)).$$

Therefore,

$$\|u\|_{L^{\frac{2(n-1)}{n-2}}(\partial B(x_0,r))} \le C\|u\|_{H^1(B(x_0,r))}.$$

By Hölder inequality,

$$r^{-1}\int_{\partial B(x_0,r)} u^2 ds \le C\left(\int_{\partial B(x_0,r)} u^{\frac{2(n-1)}{n-2}}\right)^{\frac{n-2}{n-1}} \le C\|u\|^2_{H^1(B(x_0,r))},$$

so we obtain

$$r^{-\mu-1}\int_{\partial B(x_0,r)} u^2 ds \le C F_u(x_0,r).$$

This implies that the first inequality in the lemma holds.

Theorem 3.6. *There exist constants $0<\epsilon_0$, $\tau<1$, $0<r_2<r_0/4$, such that*

$$E_u(x_0,r) \le \epsilon_0 \tag{3.8}$$

implies

$$F_u(x_0,\tau r) \le \frac{1}{2}F_u(x_0,r) \quad \text{for all } x_0 \in \Omega \text{ and } 0<r<r_2. \tag{3.9}$$

Proof. It follows from Proposition 3.4 that if $E_u(x_0,r) \le \epsilon_0$, then for $\eta < r/4$,

$$\eta^{-\mu}\int_{B(x_0,\eta)} |\nabla u|^2 dx \le C\epsilon_0.$$

This implies $\lim_{\eta\to 0}\eta^{-\mu}\int_{B(x_0,\eta)}|\nabla u|^2 dx = 0$. (Otherwise we can choose ϵ_0 smaller to deduce a contradiction.)

If the result were false, there would exist balls $B(x_k, r_k) \subset \Omega$ with $r_k \to 0$ as $k \to \infty$ such that

$$F_u(x_k, r_k) \equiv \lambda_k^2 \to 0, \tag{3.10}$$

whereas

$$F_u(x_k, \tau r_k) > \frac{1}{2}\lambda_k^2, \tag{3.11}$$

for $\tau > 0$ selected as below. We rescale our variables to the unit ball $B(0,1) \subset \mathbf{R}^n$ as follows: For $z \in B(0,1)$, we set

$$v_k(z) \equiv r_k^{2/(\alpha-1)} \left(\frac{u(x_k + r_k z) - a_k}{\lambda_k} \right), \tag{3.12}$$

where

$$a_k \equiv \frac{1}{|B(x_k, r_k)|} \int_{B(x_k, r_k)} u dy = (u)_{x_k, r_k},$$

($|B(x_k, r_k)| = \mathrm{Vol}(B(x_k, r_k))$ denotes the average of u over $B(x_k, r_k)$, $k = 1, 2, \ldots$.)

Using (3.10), (3.11) and (3.12) we have

$$\sup_k \int_{B(0,1)} |v_k|^2 dz < \infty, \quad \sup_k \int_{B(0,1)} |\nabla v_k|^2 dz < \infty,$$

but

$$\frac{1}{\tau^\mu} \int_{B(0,\tau)} |\nabla v_k|^2 dz > \frac{1}{2} - e^C \tau^{2\alpha/(\alpha-1)} \geq 1/4 \ \ (k = 1, 2, \ldots), \tag{3.13}$$

if we choose $\tau < \left(\frac{1}{4} e^{-C}\right)^{\frac{\alpha-1}{2\alpha}}$. In fact, we know that

$$C \int_0^{\tau r_k} e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi \leq \frac{C e^C (\alpha - 1)}{2\alpha} (\tau r_k)^{\frac{2\alpha}{\alpha-1}}$$

and since $C \int_0^{r_k} e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi < \lambda_k^2$, it holds that

$$\frac{C(\alpha-1)}{2\alpha} r_k^{\frac{2\alpha}{\alpha-1}} < \lambda_k^2.$$

Thus, it follows from (3.11) that

$$\lambda_k^2 \left(\tau^{-\mu} \int_{B(0,\tau)} |\nabla v_k|^2 dz \right) \geq \left(\frac{1}{2} - e^C \tau^{\frac{2\alpha}{\alpha-1}} \right) \lambda_k^2.$$

The sequence $\{v_k\}_{k=1}^\infty$ is thus bounded in $H^1(B(0,1))$, so there exists a subsequence (still denoted by $\{v_k\}$) such that

$$v_k \to v \quad \text{strongly in } \ L^2(B(0,1)) \tag{3.14}$$

$$\nabla v_k \rightharpoonup \nabla v \quad \text{weakly in } \ L^2(B(0,1)). \tag{3.15}$$

Choose any function $w \in C_0^\infty(B(0,1))$. Define

$$w_k(y) \equiv w\left(\frac{y-x_k}{r_k}\right), \quad (y \in B(x_k,r_k)),$$
$$h_1(y) \equiv \widetilde{h}_1\left(\frac{y-x_k}{r_k}\right),$$
$$h_2(y) \equiv \widetilde{h}_2\left(\frac{y-x_k}{r_k}\right).$$

Since u is a weak solution of (1.1), we have

$$\int_{B(x_k,r_k)} \nabla u\, \nabla w_k dy = \int_{B(x_k,r_k)} \Big(h_1(y)u + h_2(y)u^\alpha\Big) w_k(y)dy. \tag{3.16}$$

Thus,

$$\int_{B(0,1)} \nabla v_k\, \nabla w dz = \int_{B(0,1)} \Big[r_k^2\widetilde{h}_1(z)\Big(v_k(z)+\frac{a_k}{A_k}\Big) + \lambda_k^{\alpha-1}\widetilde{h}_2(z)\Big(v_k(z)+\frac{a_k}{A_k}\Big)^\alpha\Big] w(z)dz, \tag{3.17}$$

where $A_k = \lambda_k r_k^{-2/(\alpha-1)}$. Since

$$\begin{aligned} I_k^1 :&= r_k^2\left|\int_{B(0,1)} \widetilde{h}_1\left(v_k+\frac{a_k}{A_k}\right) wdz\right| \\ &\le r_k^2\left(\int_{B(0,1)} \widetilde{h}_1\left(v_k+\frac{a_k}{A_k}\right)^2 dz\right)^{1/2}\left(\int_{B(0,1)} \widetilde{h}_1 w^2 dz\right)^{1/2} \\ &\le r_k^2\left[\lambda_k^{-2}r_k^{-2}\left(r_k^{-\mu}\int_{B(x_k,r_k)} h_1u^2dx\right)\right]^{1/2} \|\widetilde{h}_1 w^2\|_{L^2(B(0,1))} \\ &\le Cr_k\|\widetilde{h}_1w^2\|_{L^2(B(0,1))} \to 0 \end{aligned}$$

(here we used Lemma 3.5) as $k\to\infty$ and

$$\begin{aligned} I_k^2 :&= \lambda_k^{\alpha-1}\left|\int_{B(0,1)} \widetilde{h}_2\left(v_k+\frac{a_k}{A_k}\right)^\alpha wdz\right| \\ &\le \lambda_k^{\alpha-1}\left(\int_{B(0,1)} \widetilde{h}_2\left(v_k+\frac{a_k}{A_k}\right)^{\alpha+1} dz\right)^{\alpha/(\alpha+1)} \\ &\quad \cdot \left(\int_{B(0,1)} \widetilde{h}_2|w|^{\alpha+1}dz\right)^{1/(\alpha+1)} \\ &\le \lambda_k^{\alpha-1}\left(\lambda_k^{-(\alpha+1)}r_k^{-\mu}\int_{B(x_k,r_k)} h_2u^{\alpha+1}dx\right)^{\frac{\alpha}{\alpha+1}} \|\widetilde{h}_2^{1/(\alpha+1)}w\|_{L^{\alpha+1}(B(0,1))} \end{aligned}$$

$$\leq C\lambda_k^{(\alpha-1)/(\alpha+1)}\|\widetilde{h}_2^{1/(\alpha+1)}w\|_{L^{\alpha+1}(B(0,1))} \to 0$$

(here we used Lemma 3.5) as $k \to \infty$.

Letting $k \to \infty$ in (3.17), we get

$$\int_{B(0,1)} \nabla v \, \nabla w dz = 0. \tag{3.18}$$

Hence v is harmonic function, and hence smooth, and we have the bound

$$\|\nabla v\|_{L^\infty(B(0,\frac{1}{2}))} \leq \frac{C}{|B(0,1)|}\int_{B(0,1)} v^2 dz < \infty, \tag{3.19}$$

where $|B(0,1)| =$Vol$\,(B(0,1))$. We will show in next section that

$$\nabla v_k \to \nabla v \quad \textit{strongly} \text{ in } \; L^2\left(B\left(0,\frac{1}{4}\right)\right) \tag{3.20}$$

then we have,

$$\frac{1}{\tau^\mu}\int_{B(0,\tau)} |\nabla v|^2 dz \leq C\tau^{n-\mu} < \frac{1}{4} \tag{3.21}$$

provided $0 < \tau < \min\left\{\left(\frac{1}{4C}\right)^{\frac{\alpha-1}{2(\alpha+1)}}, \left(\frac{e^{-C}}{4}\right)^{(\alpha+1)/(2\alpha)}, \frac{1}{4}\right\}$, which contradicts (3.13).

4. Compactness.

In this section we turn our attention to (3.20). We choose a smooth cut-off function $\zeta : \mathbf{R}^n \to \mathbf{R}$ satisfying

$$0 \leq \zeta \leq 1,$$
$$\zeta \equiv 1 \;\text{ on }\; B\left(0,\frac{1}{4}\right),$$
$$\zeta \equiv 0 \;\text{ on }\; \mathbf{R}^n\backslash B\left(0,\frac{5}{16}\right).$$

Lemma 4.1. *The sequence* $\{\zeta v_k\}_{k=1}^\infty$ *is bounded in* $M^\sharp_{1,\nu}(\mathbf{R}^n,\mathbf{R})$.

Proof. We first show that for $0 < r < r_0 < 1$,

$$E_u(x_0,r) \leq CF_u(x_0,r) \;\text{ for all } x_0 \in \Omega. \tag{4.1}$$

In fact, it follows from the second of the three formulations of $E_u(x_0, r)$ given above that

$$E_u(x_0,r) \le \frac{1}{2} e^{Cr} r^{-\mu} \int_{B(x_0,r)} |\nabla u|^2 dx + \frac{1}{(\alpha-1)} e^{Cr} r^{-\mu-1} \int_{\partial B(x_0,r)} u^2 ds + \frac{C}{4} e^{Cr} r^{-\mu} \int_{\partial B(x_0,r)} u^2 ds + C \int_0^r e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi. \tag{4.2}$$

By the trace embedding theorem and the argument similar to the one used in the proof of Lemma 3.5, we obtain

$$r^{-\mu-1} \int_{\partial B(x_0,r)} u^2 ds \le C F_u(x_0, r). \tag{4.3}$$

Note that $e^{Cr} < e^C$. Our claim can be obtained from (4.2) and (4.3).

Fix any point $z_0 \in B(0, \frac{3}{4})$ and any radius $0 < r < \frac{1}{8}$, set

$$y_k = x_k + r_k z_0 \in B\left(x_k, \frac{3}{4} r_k\right).$$

By the claim above and an argument similar to the one used in the proof of Lemma 3.3, we obtain that

$$\begin{aligned} &\frac{1}{(rr_k)^\mu} \int_{B(y_k, rr_k)} |\nabla u|^2 dy \\ &\le C E_u\left(y_k, \frac{1}{4} r_k\right) \\ &\le C \left(r_k^{-\mu} \int_{B(y_k, \frac{1}{4} r_k)} |\nabla u|^2 dy + C \int_0^{\frac{1}{4} r_k} e^{C\xi} \xi^{(\alpha+1)/(\alpha-1)} d\xi \right) \\ &\le C F_u(x_k, r_k) = C \lambda_k^2. \end{aligned}$$

Rescaling this estimate we obtain,

$$r^{-\mu} \int_{B(z_0,r)} |\nabla v_k|^2 dz \le C \tag{4.4}$$

for $k = 1, 2, \dots$, all $0 < r < \frac{1}{8}$ and $z_0 \in B(0, \frac{3}{4})$. This implies that

$$\frac{1}{r^{n-\frac{2}{\alpha-1}}} \int_{B(z_0,r)} |v_k - (v_k)_{z_0,r}| dz \le C < \infty \tag{4.5}$$

for k, r and z_0 as above. This implies $v_k \in \mathcal{L}^{1,n-\frac{2}{\alpha-1}}(B(0, \frac{3}{4}))$ and $\mathcal{L}^{1,n-\frac{2}{\alpha-1}}(B(0, 3/4))$ is a Campanato space (see [**Gi**]). Since $B(0, \frac{3}{4})$ is type

(A) ([**Gi**], Chapter III, Definition 1.3), Proposition 1.2 in Chapter III in [**Gi**] implies that

$$\sup_{z_0\in B(0,3/4),\,0<r<1/8}\frac{1}{r^{n-\frac{2}{\alpha-1}}}\int_{B(z_0,r)}|v_k|dz \le C\sup_{z_0\in B(0,3/4),\,0<r<1/8}\frac{1}{r^{n-\frac{2}{\alpha-1}}}\int_{B(z_0,r)}|v_k-(v_k)_{z_0,r}|dz\le C. \tag{4.6}$$

Since ζ is smooth, then

$$|(\zeta v_k)_{z_0,r}-\zeta(v_k)_{z_0,r}|\le\frac{Cr}{|B(z_0,r)|}\int_{B(z_0,r)}|v_k|dz\ \text{ on } B(z_0,r) \tag{4.7}$$

for any ball $B(z_0,r)$. Thus, if $z_0\in B(0,\frac{3}{4})$, $0<r<\frac{1}{8}$, we have,

$$\frac{1}{|B(z_0,r)|}\int_{B(z_0,r)}|\zeta v_k-(\zeta v_k)_{z_0,r}|dz \le\frac{1}{|B(z_0,r)|}\int_{B(z_0,r)}|v_k-(v_k)_{z_0,r}|dz+\frac{Cr}{|B(z_0,r)|}\int_{B(z_0,r)}|v_k|dz. \tag{4.8}$$

On the other hand, if $z_0\in\mathbf{R}^n\backslash B(0,\frac{3}{4})$, $0<r<\frac{1}{8}$, we have

$$\int_{B(z_0,r)}|\zeta v_k-(\zeta v_k)_{z_0,r}|dz=0. \tag{4.9}$$

It follows from (4.6), (4.8) and (4.9) that

$$\zeta v_k\in\mathcal{L}^{1,n-\frac{2}{\alpha-1}}(\mathbf{R}^n). \tag{4.10}$$

This also implies that $\{\zeta v_k\}_{k=1}^{\infty}$ is bounded in $M^{\sharp}_{1,\nu}(\mathbf{R}^n,\mathbf{R})$ for $k=1,2,\dots$.

Proposition 4.2. *The rescaled functions $\{\nabla v_k\}_{k=1}^{\infty}$ converge strongly in $L^2(B(0,\frac{1}{4}))$.*

Proof. Subtracting (3.18) from (3.17) we obtain

$$\int_{B(0,1)}(\nabla v_k-\nabla v)\,\nabla w dz =r_k^2\int_{B(0,1)}\widetilde{h}_1\left(v_k+\frac{a_k}{A_k}\right)w+\lambda_k^{\alpha-1}\int_{B(0,1)}\widetilde{h}_2\left(v_k+\frac{a_k}{A_k}\right)^{\alpha}wdz \tag{4.11}$$

for $w\in C_0^{\infty}(B(0,1))$. Hence it holds for $w\in H_0^1(B(0,1))\cap L^{\infty}(B(0,1))$. We now insert $w\equiv\zeta^2(v_k-v)$ into (4.11). The left-hand side of (4.11) is

$$L_k\equiv\int_{B(0,1)}\zeta^2|\nabla v_k-\nabla v|^2dz+2\int_{B(0,1)}\zeta(v_k-v)\,(\nabla v_k-\nabla v)\,\nabla\zeta dz \ge\int_{B(0,\frac{1}{4})}|\nabla v_k-\nabla v|^2dz+o(1)$$

as $k \to \infty$, in view of (3.14) and (3.15). The right-hand side of (4.11) reads

$$\begin{aligned} R_k &\equiv r_k^2 \int_{B(0,1)} \widetilde{h}_1 \left(v_k + \frac{a_k}{A_k} \right) \zeta^2 (v_k - v) dz \\ &\quad + \lambda_k^{\alpha-1} \int_{B(0,1)} \widetilde{h}_2 \left(v_k + \frac{a_k}{A_k} \right)^\alpha \zeta^2 (v_k - v) dz \\ &= R_k^1 + R_k^2. \end{aligned}$$

$$\begin{aligned} R_k^1 &\le r_k^2 \left(\int_{B(0,1)} \widetilde{h}_1 \left(v_k + \frac{a_k}{A_k} \right)^2 \right)^{1/2} \left(\int_{B(0,1)} \widetilde{h}_1 \zeta^4 (v_k - v)^2 dz \right)^{1/2} \\ &= C r_k^2 \left(\lambda_k^{-2} r_k^{\frac{4}{\alpha-1} - n} \int_{B(x_k, r_k)} h_1 u^2 dx \right)^{1/2} \\ &\le C r_k^2 (r_k^{-2})^{1/2} = C r_k \to 0 \end{aligned}$$

as $k \to \infty$.

Now we show that

$$\zeta \Big(v_k + \frac{a_k}{A_k} \Big)^\alpha \in H_w^1(\mathbf{R}^n) \tag{4.12}$$

for $k = 1, 2, \ldots$. We first consider

$$M_1 \left(\zeta \left(v_k + \frac{a_k}{A_k} \right)^\alpha \right)(z) := \sup_{t>0} \frac{1}{t^n} \int_{\mathbf{R}^n} (\zeta^{1/\alpha} f_k)^\alpha (y) \phi \left(\frac{y-z}{t} \right) dy$$

where $f_k(y) := v_k(y) + \frac{a_k}{A_k}$, ϕ is a Schwartz function with nonvanishing integral (see [**ST**]).

If $t \ge 1 + \frac{|z|}{4}$, we have

$$\begin{aligned} &\frac{1}{t^n} \int_{\mathbf{R}^n} \left(\zeta^{1/\alpha} f_k \right)^\alpha (y) \phi \left(\frac{y-z}{t} \right) dy \\ &\le \frac{1}{t^n} \int_{B(0,1)} \left(\zeta^{1/\alpha} f_k \right)^\alpha (y) \phi_t dy \\ &\le \frac{1}{t^n} \left(\int_{B(0,1)} \left(\zeta^{1/\alpha} f_k \right)^{\alpha+1} dy \right)^{\alpha/(\alpha+1)} \left(\int_{B(0,1)} \phi_t^{\alpha+1} dy \right)^{1/(\alpha+1)} \\ &\le \frac{C}{t^n} \left[\lambda_k^{-(\alpha+1)} r_k^{-\mu} \int_{B(x_k, r_k)} u^{\alpha+1} dx \right]^{\alpha/(\alpha+1)} \\ &\le \frac{C}{t^n} \lambda_k^{\frac{\alpha(1-\alpha)}{\alpha+1}} \\ &\le \frac{C}{(4+|z|)^n} \lambda_k^{\frac{\alpha(1-\alpha)}{\alpha+1}}. \end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbf{R}^n} M_1\left(\zeta\left(v_k(z)+\frac{a_k}{A_k}\right)^{\alpha}\right) w(z)dz \\
&= \int_{\mathbf{R}^n} M_1\left(\zeta\left(v_k(z)+\frac{a_k}{A_k}\right)^{\alpha}\right)|z|^{-2/(\alpha-1)}dz \\
&\le C\lambda_k^{\frac{\alpha(1-\alpha)}{\alpha+1}}\int_{\mathbf{R}^n}|z|^{-2/(\alpha-1)}(4+|z|)^{-n}dz \\
&\le C\lambda_k^{\frac{\alpha(1-\alpha)}{\alpha+1}}.
\end{aligned}$$

If $t<1+\frac{|z|}{4}$, we have, if $|y-z|<t$, then $|y-z|<1+\frac{|z|}{4}$ and $|y|>\frac{3}{4}|z|-1$. Therefore, if $|z|>8/3$, then $|y|>1$. Thus, for $0<\epsilon<1$ and $z\in B(0,3)$,

$$\begin{aligned}
&\frac{1}{t^n}\int_{\mathbf{R}^n}\left(\zeta^{1/\alpha}f_k\right)^{\alpha}\phi\left(\frac{y-z}{t}\right)dy \\
&\le \frac{1}{t^n}\left(\int_{B(z,t)}\left(\zeta^{1/\alpha}f_k\right)^{\alpha+1-\epsilon}dy\right)^{\alpha/(\alpha+1-\epsilon)} \\
&\qquad\cdot\left(\int_{B(z,t)}\phi_t^{(\alpha+1-\epsilon)/(1-\epsilon)}dy\right)^{(1-\epsilon)/(\alpha+1-\epsilon)} \\
&\le C\left(M\left(\left(\zeta^{1/\alpha}f_k\right)^{\alpha+1-\epsilon}\right)\right)^{\alpha/(\alpha+1-\epsilon)},
\end{aligned}$$

where $M(\cdot)$ is the Hardy-Littlewood maximal function. If $z\in\mathbf{R}^n\backslash B(0,3)$,

$$\frac{1}{t^n}\int_{\mathbf{R}^n}\left(\zeta^{1/\alpha}f_k\right)^{\alpha}\phi_t dy=0.$$

Therefore,

$$\begin{aligned}
&\int_{\mathbf{R}^n} M_1\left(\zeta(z)\left(v_k(z)+\frac{a_k}{M_k}\right)^{\alpha}\right)|z|^{-2/(\alpha-1)}dz \\
&\le C\int_{B(0,3)}\left[M\left(\left(\zeta^{1/\alpha}f_k\right)^{\alpha+1-\epsilon}\right)\right]^{\alpha/(\alpha+1-\epsilon)}|z|^{-2/(\alpha-1)}dz \\
&\le C\left(\int_{B(0,3)}\left[M\left(\left(\zeta^{1/\alpha}f_k\right)^{\alpha+1-\epsilon}\right)\right]^{(\alpha+1)/(\alpha+1-\epsilon)}dz\right)^{\alpha/(\alpha+1)} \\
&\qquad\cdot\left(\int_{B(0,3)}|z|^{-2\frac{\alpha+1}{\alpha-1}}dz\right)^{1/(\alpha+1)} \\
&\le C\left(\int_{B(0,3)}\left(\zeta^{1/\alpha}f_k\right)^{\alpha+1}dz\right)^{\alpha/(\alpha+1)}\left(\int_0^3 r^{\mu-1}dr\right)^{1/(\alpha+1)}
\end{aligned}$$

$$\leq C\lambda_k^{\frac{\alpha(1-\alpha)}{\alpha+1}},$$

where we used the facts that $\mu > 0$ if $\alpha > \frac{n+2}{n-2}$, and

$$\begin{aligned}
&\int_{B(0,3)} \left[M\Big(\Big(\zeta^{1/\alpha} f_k\Big)^{\alpha+1-\epsilon}\Big)\right]^{(\alpha+1)/(\alpha+1-\epsilon)} dz \\
&= \int_{\mathbf{R}^n} \left[M\Big(\Big(\zeta^{1/\alpha} f_k\Big)^{\alpha+1-\epsilon}\Big)\right]^{(\alpha+1)/(\alpha+1-\epsilon)} dz \\
&\leq \int_{\mathbf{R}^n} \Big(\zeta^{1/\alpha} f_k\Big)^{\alpha+1} dz,
\end{aligned}$$

because

$$M\Big(\Big(\zeta^{1/\alpha} f_k\Big)^{\alpha-1+\epsilon}\Big)(z) \equiv 0 \quad \text{for } z \in \mathbf{R}^n \backslash B(0,3).$$

It concludes that $\Big(\zeta^{1/\alpha} f_k\Big)^{\alpha} \in H^1_w(\mathbf{R}^n)$. Therefore, it follows from (2.1) that

$$\begin{aligned}
R_k^2 &\leq \lambda_k^{\alpha-1} \int_{\mathbf{R}^n} M_1\Big(\Big(\zeta^{1/\alpha} f_k\Big)^{\alpha}\Big) M^{\sharp}_{1,\nu}(\zeta(v_k - v))|z|^{-2/(\alpha-1)} dz \\
&\leq C\lambda_k^{\alpha-1} \int_{\mathbf{R}^n} M_1\Big(\Big(\zeta^{1/\alpha} f_k\Big)^{\alpha}\Big)|z|^{-2/(\alpha-1)} dz \\
&\leq C\lambda_k^{\alpha-1}\lambda_k^{\frac{\alpha(1-\alpha)}{\alpha+1}} \\
&= C\lambda_k^{\frac{\alpha-1}{\alpha+1}} \to 0
\end{aligned}$$

as $k \to \infty$.

5. Proof of Theorem A.

In this section we shall prove Theorem A. We recall the definition of function space $L^{p,q}(\Omega)$:

$$L^{p,q}(\Omega) = \left\{ v \in L^p(\Omega) : \sup_{x\in\Omega, r>0} r^{-q} \int_{B(x,r)\cap\Omega} u^p dx < +\infty \right\}.$$

This is called Morrey space (see [**Gi**]). Now we recall a theorem in [**Pa2**].

Theorem 5.1. *Let u be a positive weak solution of* (1.1), *assume that* $u \in L^{\alpha,\lambda+\theta}(\Omega)$ *for* $\lambda = n - \frac{2\alpha}{\alpha-1}$ *and some* $\theta > 0$ *then u is regular in* Ω.

Note that Pacard [**Pa2**] proved this theorem only for the case that $h_1 \equiv 0$ and $h_2 \equiv 1$ in Ω, but we can easily see from his proof that this theorem still holds in our case.

Set

$$V \equiv \{x \in \Omega : \ E_u(x,r) < \epsilon_0 \ \text{ for some } 0 < r < r_2\},$$

where ϵ_0 and r_2 are constants in Theorem 3.6. Furthermore, using Theorem 3.6, we can show that (cf. [**Gi**]), if $x \in V$, there exists $r* > 0$ sufficiently small such that

$$F_u(y, r) \leq C r^{\gamma} \tag{5.1}$$

for some $0 < \gamma < \frac{2\alpha}{\alpha - 1}$, $C > 0$, all y near x, and all sufficiently small radii $0 < r < r*$. It follows from Lemma 3.5 that

$$r^{-\mu} \int_{B(x_0, r)} u^{\alpha+1} dx \leq C F_u(x_0, r) \tag{5.2}$$

for all $x_0 \in \Omega$ and $0 < r < r_0$. Note that $\gamma < \frac{2\alpha}{\alpha - 1}$, by (5.1) and (5.2), we have

$$r^{-\mu} \left(\int_{B(y,r)} \left(|\nabla u|^2 + u^{\alpha+1} \right) dx \right) \leq C r^{\gamma} \tag{5.3}$$

for all y near x, and $0 < r < r*$. Now we show that

$$u \in L^{\alpha, \lambda + \theta_0}(B(x, r*/2)) \tag{5.4}$$

for some $\theta_0 > 0$. In fact, choosing $\theta_0 = \frac{\alpha\gamma}{\alpha + 1}$, we have, for $0 < r < r*$,

$$r^{-(n+\theta_0)} \int_{B(x,r)} u^{\alpha} dy \leq r^{-(n+\theta_0)} \left(\int_{B(x,r)} u^{\alpha+1} dy \right)^{\alpha/(\alpha+1)} r^{1/(\alpha+1)} \leq C.$$

This implies (5.4) and therefore, by Theorem 5.1, u is regular at x. Hence u is regular in V.

Define $\Sigma = \Omega \backslash V$. Then

$$\Sigma \equiv \cap_{r>0} \{x \in \Omega : \ E_u(x, r) \geq \epsilon_0\}.$$

It is proved in [**GL**] that

$$\Sigma \subset \cap_{r>0} \left\{ x \in \Omega : \int_{B(x,r)} \left(u^{\alpha+1} + |\nabla u|^2 \right) dy \geq C \epsilon_0 r^{\mu} \right\}. \tag{5.5}$$

Thus, standard covering arguments imply that the Hausdorff dimension of Σ is less than $n - 2\frac{\alpha+1}{\alpha-1}$. This completes the proof of Theorem A.

Remark. The conclusion of Theorem A still holds for the stationary positive weak solutions of the equation

$$\Delta u + h_1(x) u^{\kappa} + h_2(x) u^{\alpha} = 0 \ \text{ in } \Omega$$

where $0 < \kappa < \alpha$, $\alpha > \frac{n+2}{n-2}$. It should be very interesting to know whether our partial regularity theorem holds for the equation

$$\Delta u + h(x) f(u) = 0 \ \text{ in } \Omega$$

where $f(s)$ satisfies that $f(s) > 0$ for $s > 0$ and f has the growth rate $\alpha > \frac{n+2}{n-2}$. The main difficulty is how to establish the monotonicity inequality.

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