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Journal of
Mathematics*

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PROBLEM IN C^1 DOMAINS

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Some non-existence result is established for bounded solutions of a Neumann problem on the upper half space. Based on this non-existence result, precise asymptotic behavior is given for the principal eigenvalue of some linear eigenvalue problem in bounded C^1 domains, and this answers a question that appeared in Lacey et al, 1998.

1. Introduction.

For any $\gamma > 0$, set

$$(1) \quad \Lambda(\gamma) = \sup_{u \in H^1(\Omega) \setminus \{0\}} \frac{\gamma \int_{\partial\Omega} u^2 - \int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2},$$

where Ω is a bounded domain in \mathbb{R}^n with boundary $\partial\Omega$. It is straightforward to show that the supremum of (1) is attained by some positive function $u_\gamma \in H^1(\Omega)$, which is a weak solution of

$$(2) \quad \Delta u = \Lambda(\gamma)u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = \gamma u \quad \text{on } \partial\Omega,$$

where ν is the outward unit normal vector on $\partial\Omega$; ν exists a.e. for Lipschitz domains. The goal of this paper is to understand the asymptotic behavior of $\Lambda(\gamma)$ as $\gamma \rightarrow \infty$ when $\partial\Omega \in C^1$. Since $\Lambda(\gamma) \rightarrow \infty$ when $\gamma \rightarrow \infty$, (2) can be viewed as a singularly perturbed linear eigenvalue problem.

The asymptotic behavior of $\Lambda(\gamma)$ was first studied by Lacey, Ockendon and Sabina in [3], where they investigated some reaction-diffusion model in which distributed nonlinear absorption mechanisms compete with nonlinear boundary sources. In order to describe the long time behaviors of solutions to this reaction-diffusion model, it is important to understand the asymptotic behavior of $\Lambda(\gamma)$ as $\gamma \rightarrow \infty$ (see [3] and the references therein). Among other things, Lacey, Ockendon and Sabina showed in [3] that

$$(3) \quad \lim_{\gamma \rightarrow \infty} \frac{\Lambda(\gamma)}{\gamma^2} = 1$$

if $\partial\Omega$ is C^2 and is differentially equivalent to the unit sphere. On the other hand, when Ω is a planar domain and $\partial\Omega$ is piecewise C^1 , they proved that

$$\liminf_{\gamma \rightarrow \infty} \frac{\Lambda(\gamma)}{\gamma^2} \geq \operatorname{cosec}^2 \alpha \geq 1,$$

where α is the smallest interior semiangle on $\partial\Omega$. These considerations indicate that the asymptotic behavior of $\Lambda(\gamma)$ is strongly affected by the smoothness of the boundary. In this connection, we prove:

Theorem 1.1. (3) holds for any bounded C^1 domain.

Remark 1.1. Similar result can be established for the problem

$$\Delta u = \Lambda(\gamma)u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = \gamma b(x)u \quad \text{on } \partial\Omega.$$

More precisely, if $b(x) \in C(\partial\Omega)$ is positive somewhere, then

$$\lim_{r \rightarrow \infty} \frac{\Lambda(\gamma)}{\gamma^2} = \max_{\partial\Omega} (b_+)^2.$$

In the following we briefly sketch our approach: Since (2) is a singularly perturbed problem, it is natural to “blow up” u_γ , the solution of (2), near its maximum which must be attained on $\partial\Omega$ (via the Maximum Principle). That is, straightening out the boundary and rescaling u_γ suitably, by passing to the limit we are led to the following Neumann problem on the upper half space:

$$(4) \quad \Delta u = au \quad \text{in } \mathbb{R}_+^n, \quad \frac{\partial u}{\partial x_n} = -u \quad \text{on } \partial\mathbb{R}_+^n,$$

where a is the limit of $\Lambda(\gamma)/\gamma^2$ (subject to a subsequence) as $\gamma \rightarrow \infty$. By adequate choice of test function in (1), one can show that $a \geq 1$. On the other hand, using some similar ideas as the sliding method developed in [1], we are able to show:

Theorem 1.2. If $a > 1$, (4) has no bounded nontrivial solutions.

By some non-degeneracy result in Section 3, the solution of (4) obtained via the blowup process is indeed nontrivial. Hence, Theorem 1.2 ensures that $a = 1$, which in turn yields Theorem 1.1.

Remark 1.2. It turns out that Theorem 1.2 is sharp: For every $a \leq 1$, (4) has bounded nontrivial solutions of the form $w(x')e^{-x_n}$, where $x = (x', x_n)$ and w is a solution of

$$(5) \quad \Delta w = (a - 1)w \quad \text{in } \mathbb{R}^{n-1};$$

Theorem 1.2 also fails without the boundedness condition: (4) has positive (unbounded) solutions of the form $w(x')e^{-x_n}$, where w is a positive solution of (5) for $a > 1$. We refer to [2] for the classification of positive solutions to (5).

The plan of this paper is as follows: Theorem 1.2 is established in Section 2. In Section 3, we first derive the relation between $\lim_{\gamma \rightarrow \infty} \Lambda(\gamma)/\gamma^2$ and (4), and then use it to complete the proof of Theorem 1.1.

2. The proof of Theorem 1.2.

We prove Theorem 1.2 in this section. Our idea is to construct some super-solution of (4), by employing some similar ideas as the sliding method of Berestycki, Caffarelli and Nirenberg (see, e.g., [1]).

Throughout this section, we assume that $u(x)$ is a bounded solution to (4) and $a > 0$; without loss of generality, we may assume that $\sup_{\mathbb{R}_+^n} u > 0$.

Lemma 2.1. $\sup_{\mathbb{R}_+^n} u = \sup_{\partial\mathbb{R}_+^n} u$.

Proof. Let $\{x^j\}_{j=1}^\infty$ be a sequence of points with $u(x^j) \rightarrow \sup_{\mathbb{R}_+^n} u$. Denote x_n^j the last component of x^j . We first show that $x_n^j \rightarrow 0$ as $j \rightarrow \infty$. If not, then there is a $\delta > 0$, such that $x_n^j > \delta$ (after passing to some subsequence). We consider $u^j(x) = u(x + x^j)$ for $|x| \leq \delta$. Notice that u^j is bounded in $|x| \leq \delta$. By standard elliptic estimates, we know that after passing to some subsequence, $u^j \rightarrow u_0$ in $C^2(B_{\delta/2}(0))$, where u_0 satisfies

$$\Delta u_0 = a u_0 \quad \text{in } B_{\delta/2}(0),$$

and u_0 assumes its positive maximum at the origin. This is clearly impossible.

Again, standard elliptic estimates yield that $|\nabla u| \leq C$ for some positive constant C . Therefore, $\sup_{\mathbb{R}_+^n} u = \sup_{\partial\mathbb{R}_+^n} u$.

We normalize $u(x)$ so that $\sup_{\mathbb{R}_+^n} u = 1$. By Lemma 2.1,

$$(6) \quad \sup_{\partial\mathbb{R}_+^n} u = 1.$$

Define

$$\begin{aligned} \Omega_h &= \{(x', x_n) \mid x' \in R^{n-1}, 0 < x_n < h\}, \\ \Omega_{h,r} &= \{(x', x_n) \mid |x'| < r, 0 < x_n < h\}, \\ \Gamma_{1,h,r} &= \partial\Omega_{h,r} \cap \{x_n > 0\}, \\ \Gamma_{2,h,r} &= \partial\Omega_{h,r} \cap \{x_n = 0\}. \end{aligned}$$

We first state a lemma concerning the sub- and super-solution method. Consider

$$(7) \quad \begin{aligned} -\Delta u &= f(x, u) && \text{in } \Omega_{h,r}, \\ u &= g(x) && \text{on } \Gamma_{1,h,r}, \\ \frac{\partial u}{\partial \nu} &= h(x, u) && \text{on } \Gamma_{2,h,r}, \end{aligned}$$

where $f(x, u)$, $h(x, u)$ are Carathéodory functions, $g(x)$ is continuous.

A function $u \in H^1(\Omega_{h,r})$ is called a sub-solution to (7) if $u \leq g(x)$ on $\Gamma_{1,h,r}$ and

$$\int_{\Omega_{h,r}} [\nabla u \nabla \eta - f(x, u)\eta] - \int_{\Gamma_{2,h,r}} h(x, u)\eta \leq 0,$$

for all $\eta \in C^\infty(\Omega_{h,r})$, $\eta \geq 0$, and $\eta = 0$ on $\Gamma_{1,h,r}$. Similarly $u \in H^1(\Omega_{h,r})$ is called a super-solution to (7) if the above inequalities are reversed. We refer the proof of the following result to [5]:

Lemma 2.2. *Suppose that $\bar{u}, \underline{u} \in H^1(\Omega_{h,r})$ are both bounded, $\bar{u} \geq \underline{u}$, and they are super-solution, sub-solution of (7), respectively. Then there is a solution $u \in H^1(\Omega_{h,r})$ to (7) such that $\underline{u} \leq u \leq \bar{u}$ holds a.e. in $\Omega_{h,r}$.*

We now apply Lemma 2.2 to construct a super-solution of (4).

Lemma 2.3. *For any fixed $h, r > 0$, there is a unique solution to*

$$(8) \quad \begin{aligned} \Delta \psi &= a\psi && \text{in } \Omega_{h,r}, \\ \psi &= 1 && \text{on } \Gamma_{1,h,r}, \\ \frac{\partial \psi}{\partial \nu} &= 1 && \text{on } \Gamma_{2,h,r}. \end{aligned}$$

Proof. It is easy to check that $\underline{u} = 0$ is a sub-solution and $\bar{u} = 1 + h - x_n$ is a super-solution to (8). The existence follows from Lemma 2.2, whereas the uniqueness follows from the Maximum Principle.

Lemma 2.4. *Let $u(x)$ be a solution to (4) and $\psi_{h,r}$ be the unique solution to (8). Then $\psi_{h,r} \geq u$ in $\Omega_{h,r}$.*

Proof. Set $w = \psi_{h,r} - u$. Then w satisfies

$$\begin{aligned} \Delta w &= aw && \text{in } \Omega_{h,r}, \\ w &\geq 0 && \text{on } \Gamma_{1,h,r}, \\ \frac{\partial w}{\partial \nu} &= 1 - u && \text{on } \Gamma_{2,h,r}. \end{aligned}$$

If $w(x_0) = \min_{\bar{\Omega}_{h,r}} w(x) < 0$, then $x_0 \in \Gamma_{2,h,r} \setminus \{|x'| = r\}$. By the Hopf Boundary Lemma, we know that $\partial w / \partial \nu(x_0) < 0$, which contradicts

$$\partial w / \partial \nu(x_0) = 1 - u(x_0) \geq 0.$$

This proves Lemma 2.4.

Now consider

$$(9) \quad \begin{aligned} \Delta \psi &= a\psi && \text{in } \Omega_{h,r}, \\ \psi &= 1 && \text{on } \Gamma_{1,h,r}, \\ \frac{\partial \psi}{\partial \nu} &= 0 && \text{on } \Gamma_{2,h,r}. \end{aligned}$$

It is easy to see that $\underline{u} = 0$ and $\bar{u} = 1$ are sub- and super-solutions to (9), respectively. Thus there is a unique solution to (9). We denote it by $\psi_{1,h,r}$. Decompose $\psi_{h,r}$ as

$$\psi_{h,r} = \psi_{1,h,r} + \psi_{2,h,r},$$

where $\psi_{2,h,r}$ is the unique solution to

$$\begin{aligned} \Delta\psi &= a\psi && \text{in } \Omega_{h,r}, \\ \psi &= 0 && \text{on } \Gamma_{1,h,r}, \\ \frac{\partial\psi}{\partial\nu} &= 1 && \text{on } \Gamma_{2,h,r}. \end{aligned}$$

It again follows from the Maximum Principle that

$$0 < \psi_{1,h,r} < 1, \quad 0 < \psi_{2,h,r} < 1 + h, \quad \forall x \in \Omega_{h,r}.$$

Furthermore, we can show:

Lemma 2.5.

a) $\psi_{1,h,r}$ is non-increasing in r . For any fixed $h > 0$, as $r \rightarrow \infty$, $\psi_{1,h,r}$ converges monotonically to $\psi_{1,h}$, where $\psi_{1,h}(x)$ is a function of x_n alone and satisfies

$$(10) \quad \begin{aligned} \Delta\psi &= a\psi && \text{in } \Omega_h, \\ \psi &= 1 && \text{on } \{x_n = h\}, \\ \frac{\partial\psi}{\partial\nu} &= 0 && \text{on } \{x_n = 0\}. \end{aligned}$$

b) $\psi_{2,h,r}$ is non-decreasing in r . For any fixed $h > 0$, as $r \rightarrow \infty$, $\psi_{2,h,r}$ converges monotonically to $\psi_{2,h}$, where $\psi_{2,h}(x)$ is a function of x_n alone and satisfies

$$(11) \quad \begin{aligned} \Delta\psi &= a\psi && \text{in } \Omega_h, \\ \psi &= 0 && \text{on } \{x_n = h\}, \\ \frac{\partial\psi}{\partial\nu} &= 1 && \text{on } \{x_n = 0\}. \end{aligned}$$

Proof. We only give the proof of Part (a) since Part (b) can be established in the same spirit. For any $r' > r$, set $w = \psi_{1,h,r'} - \psi_{1,h,r}$ in $\Omega_{h,r}$. Then w satisfies

$$(12) \quad \begin{aligned} \Delta w &= aw && \text{in } \Omega_{h,r}, \\ w &\leq 0 && \text{on } \Gamma_{1,h,r}, \\ \frac{\partial w}{\partial\nu} &= 0 && \text{on } \Gamma_{2,h,r}. \end{aligned}$$

It follows from the Maximum Principle and the Hopf Boundary Lemma that $w \leq 0$ in $\bar{\Omega}_{h,r}$. This proves the monotonicity of $\psi_{1,h,r}$.

Therefore, for any fixed $h > 0$, as $r \rightarrow \infty$, $\psi_{1,h,r}$ monotonically converges to $\psi_{1,h}$ in \mathbb{R}_+^n and $\psi_{1,h}$ satisfies (10).

We still need to show that $\psi_{1,h}$ is a function of x_n only. For any $P \in \mathbb{R}^{n-1}$ and $r', r > 0$ with $r' - r > |P|$, consider the difference $w_1 = \psi_{1,h,r'}(x' + P, x_n) - \psi_{1,h,r}(x', x_n)$ in $\Omega_{h,r}$. It is easy to see that w_1 satisfies (12), thus $w_1 \leq 0$ in $\Omega_{h,r}$. That is,

$$\psi_{1,h,r'}(x' + P, x_n) \leq \psi_{1,h,r}(x', x_n), \quad \forall x \in \Omega_{h,r}, \quad \forall P \in \mathbb{R}^{n-1}.$$

Sending $r, r' \rightarrow \infty$, we have

$$\psi_{1,h}(x' + P, x_n) \leq \psi_{1,h}(x), \quad \forall x \in \Omega_h.$$

Hence $\psi_{1,h}(x) = \psi_{1,h}(x_n)$.

Proof of Theorem 1.2. Denote $\psi_h = \psi_{1,h} + \psi_{2,h}$. Then by Lemmas 2.4 and 2.5, $\psi_h(x) \geq u(x)$ in Ω_h and ψ_h satisfies

$$\begin{aligned} \psi'' - a\psi &= 0 & \text{in } (0, h), \\ \psi'(0) &= -1, & \psi(h) = 1. \end{aligned}$$

Direct calculation shows that

$$\psi_h(x_n) = c_1 e^{\sqrt{a}x_n} + c_2 e^{-\sqrt{a}x_n},$$

where

$$\begin{aligned} c_1 &= \frac{1 - \frac{1}{\sqrt{a}}e^{-\sqrt{a}h}}{e^{\sqrt{a}h} + e^{-\sqrt{a}h}}, \\ c_2 &= \frac{1 + \frac{1}{\sqrt{a}}e^{\sqrt{a}h}}{e^{\sqrt{a}h} + e^{-\sqrt{a}h}}. \end{aligned}$$

Sending $h \rightarrow \infty$, we have $\psi_h(x_n) \rightarrow \frac{1}{\sqrt{a}}e^{-\sqrt{a}x_n}$. Thus by Lemma 2.4,

$$u(x) \leq \frac{1}{\sqrt{a}}e^{-\sqrt{a}x_n}, \quad \forall x \in \mathbb{R}_+^n.$$

It follows from (6) that

$$1 = \sup_{\partial\mathbb{R}_+^n} u(x) \leq \frac{1}{\sqrt{a}},$$

i.e., $a \leq 1$. This proves Theorem 1.2.

3. Asymptotic behaviors of eigenvalues.

We prove Theorem 1.1 in this section. For every piecewise smooth domain Ω , it was proved in [3] that

$$\underline{\lim}_{\gamma \rightarrow \infty} \frac{\Lambda(\gamma)}{\gamma^2} \geq 1.$$

To prove Theorem 1.1, we need to show that when $\partial\Omega$ is C^1 ,

$$(13) \quad \overline{\lim}_{\gamma \rightarrow \infty} \frac{\Lambda(\gamma)}{\gamma^2} \leq 1.$$

Proof of Theorem 1.1. For any $\gamma > 1$, let u_γ be a positive solution of (2) and u_γ attains its maximum at x_γ . By the Maximum Principle, we know that $x_\gamma \in \partial\Omega$. After normalization we can assume that $\max_{\bar{\Omega}} u_\gamma = 1$ and $x_\gamma \rightarrow 0 \in \partial\Omega$. Further, we can assume that there is a C^1 function ϕ such that $\partial\Omega \cap B_2(0)$ can be represented by $x_n = \phi(x')$ for $|x'| \leq 2$ with $\phi(0) = 0$ and $\partial\phi/\partial x_i(0) = 0$ for $i = 1, \dots, n - 1$.

For any $\eta \in C_0^\infty(B_2(0))$, u_γ satisfies

$$(14) \quad \int_{\Omega} \nabla u_\gamma \cdot \nabla \eta + \Lambda(\gamma) \int_{\Omega} u_\gamma \eta - \gamma \int_{\partial\Omega} u_\gamma \eta = 0.$$

Now we flatten $\partial\Omega$ near the origin. Let $y = \Phi(x) : \Omega \cap B_2(0) \rightarrow \Omega_\Phi \equiv \Phi(\Omega \cap B_2(0))$, be such that

$$\begin{aligned} \Phi_i(x) &= x_i, & i = 1, 2, \dots, n - 1, \\ \Phi_n(x) &= x_n - \phi(x'). \end{aligned}$$

Denote the inverse of $y = \Phi(x)$ by $x = \Psi(y)$. Then (14) can be rewritten as

$$\begin{aligned} \sum_{k,l=1}^n \int_{\Omega_\Phi} \frac{\partial u_\gamma}{\partial y_k} \frac{\partial \eta}{\partial y_l} \frac{\partial \Phi_k}{\partial x_i}(\Psi(y)) \frac{\partial \Phi_l}{\partial x_i}(\Psi(y)) |D\Psi| dy \\ + \Lambda(\gamma) \int_{\Omega_\Phi} u_\gamma \eta |D\Psi| dy - \gamma \int_{\partial\Omega_\Phi} u_\gamma \eta \sqrt{1 + |\nabla \phi(y')|^2} dy' = 0, \end{aligned}$$

where $|D\Psi|$ is the derterminant of $D\Psi$. Notice that $|\nabla \phi| = o(1)$ as $x' \rightarrow 0$. Thus $D\Psi \rightarrow I$ as $|y| \rightarrow 0$, where I is the $n \times n$ identity matrix. We now consider two different cases.

Case 1.

$$\overline{\lim}_{\gamma \rightarrow \infty} \frac{\Lambda(\gamma)}{\gamma^2} = a < +\infty.$$

Without loss of generality, we may assume that

$$\frac{\gamma \int_{\partial\Omega} u_\gamma^2 - \int_{\Omega} |\nabla u_\gamma|^2}{\gamma^2 \int_{\Omega} u_\gamma^2} = \sup_{u \in H^1(\Omega) \setminus \{0\}} \frac{\gamma \int_{\partial\Omega} u^2 - \int_{\Omega} |\nabla u|^2}{\gamma^2 \int_{\Omega} u^2} \rightarrow a,$$

and $u_\gamma(x_\gamma) = \max_{\Omega} u_\gamma(x) = 1$, $x_\gamma \rightarrow 0$. We let $z = \gamma(y - y_\gamma)$, where $y_\gamma = (x'_\gamma, 0)$, and set $v_\gamma(z) = u_\gamma(y)$. Then for any $R > 0$ and η with

compact support in B_{2R} , as γ becomes sufficiently large, v_γ satisfies

$$\begin{aligned}
 (15) \quad & \sum_{k,l=1}^n \int_{B_{2R}^+} \frac{\partial v_\gamma}{\partial z_k} \frac{\partial \eta}{\partial z_l} \\
 & \cdot \frac{\partial \Phi_k}{\partial x_i} \left(\Psi \left(y_\gamma + \frac{z}{\gamma} \right) \right) \frac{\partial \Phi_l}{\partial x_i} \left(\Psi \left(y_\gamma + \frac{z}{\gamma} \right) \right) |D\Psi| \left(y_\gamma + \frac{z}{\gamma} \right) dz \\
 & + \frac{\Lambda(\gamma)}{\gamma^2} \int_{B_{2R}^+} v_\gamma \eta |D\Psi| \left(y_\gamma + \frac{z}{\gamma} \right) dz \\
 & - \int_{z_n=0} v_\gamma \eta \sqrt{1 + |\nabla \phi(y'_\gamma + z'/\gamma)|^2} dz' = 0.
 \end{aligned}$$

Since for $z \in B_{2R}^+$, $y_\gamma + z/\gamma \rightarrow 0$ as $\gamma \rightarrow \infty$, we know that for sufficiently large γ ,

$$\left(\sum_{i=1}^n \frac{\partial \Phi_k}{\partial x_i} \left(\Psi \left(y_\gamma + \frac{z}{\gamma} \right) \right) \frac{\partial \Phi_l}{\partial x_i} \left(\Psi \left(y_\gamma + \frac{z}{\gamma} \right) \right) \right)_{kl} > \frac{I}{2}.$$

Let η_R be a cutoff function satisfying $\eta_R = 1$ in B_R with compact support in B_{2R} and $|\nabla \eta_R| \leq C$. Choosing $\eta = u \cdot \eta_R$ in (15), we have,

$$\|\nabla v_\gamma\|_{L^2(B_R)} \leq C \|v_\gamma\|_{L^\infty} = C.$$

Therefore, after passing to a subsequence, $v_\gamma \rightarrow v_0$ weakly in $H_{loc}^1(\mathbb{R}_+^n)$ as $\gamma \rightarrow \infty$, where $v_0 \in H_{loc}^1(\mathbb{R}_+^n)$ satisfies

$$\begin{aligned}
 \Delta v_0 &= a v_0 && \text{in } \mathbb{R}_+^n, \\
 \frac{\partial v_0}{\partial \nu} &= v_0 && \text{on } \partial \mathbb{R}_+^n, \\
 0 &\leq v_0 \leq 1 && \text{in } \mathbb{R}_+^n.
 \end{aligned}$$

To show that v_0 is nontrivial, we claim that there is a constant $C > 0$ such that

$$(16) \quad 1 = \|v_\gamma\|_{L^\infty(B_1^+)} \leq C \left(\|v_\gamma\|_{L^2(\partial \mathbb{R}_+^n \cap \{|x'| < 2\})} + \|v_\gamma\|_{L^{\frac{2n}{n-1}}(B_2^+(0))} \right).$$

Since the embeddings from $W^{1,2}(B_2^+(0))$ to $L^{\frac{2n}{n-1}}(B_2^+(0))$ and $L^2(\partial \mathbb{R}_+^n \cap \{|x'| < 2\})$ are both compact, we know that

$$1 \leq C \left(\|v_0\|_{L^2(\partial \mathbb{R}_+^n \cap \{|x'| < 1\})} + \|v_0\|_{L^{\frac{2n}{n-1}}(B_1^+(0))} \right),$$

from which it follows that $v_0 \neq 0$. From Theorem 1.2 we see that $a \leq 1$. Hence it suffices to establish (16).

Inequality (16) can be obtained via Moser iteration. Though it seems to be a standard result (see, e.g., [4]), we include a proof here for the completeness.

Direct calculation shows that

$$\begin{aligned} & \int_{B_2^+} \nabla v_\gamma \cdot \nabla (v_\gamma^k \xi^2) \\ &= \frac{4k}{(k+1)^2} \int_{B_2^+} \left| \nabla \left(v_\gamma^{\frac{k+1}{2}} \xi \right) \right|^2 + \frac{k-1}{(k+1)^2} \int_{B_2^+} v_\gamma^{k+1} \Delta \xi^2 \\ & \quad - \frac{4k}{(k+1)^2} \int_{B_2^+} v_\gamma^{k+1} |\nabla \xi|^2 - \frac{k-1}{(k+1)^2} \int_{\{x_n=0, |x'| \leq 2\}} v_\gamma^{k+1} \nabla \xi^2 \cdot \nu. \end{aligned}$$

Choosing $\eta = v_\gamma^k \xi^2$ in (15) with $k > 1$ and ξ having compact support in B_2^+ , we have

(17)

$$\begin{aligned} & \int_{B_2^+} \left| \nabla \left(v_\gamma^{\frac{k+1}{2}} \xi \right) \right|^2 \\ & \leq \frac{k-1}{4k} \int_{B_2^+} v_\gamma^{k+1} \Delta \xi^2 + \int_{B_2^+} v_\gamma^{k+1} |\nabla \xi|^2 + \frac{k-1}{4k} \int_{\{x_n=0, |x'| \leq 2\}} v_\gamma^{k+1} \nabla \xi^2 \cdot \nu \\ & \quad + \frac{(k+1)^2 C}{4k} \int_{\{x_n=0, |x'| \leq 2\}} v_\gamma^{k+1} \xi + \frac{(k+1)^2 C}{4k} \int_{B_2^+} v_\gamma^{k+1} \xi. \end{aligned}$$

Let

$$r_i = 1 + \frac{1}{2^{i-1}}, \quad i = 1, 2, \dots,$$

and choose ξ_i satisfying

$$\begin{aligned} \xi_i &= 1, & |x| &\leq r_{i+1}; \\ \xi_i &= 0, & |x| &> r_i; \\ |\nabla \xi_i| &\leq 2 \cdot 2^i, & |\nabla^2 \xi_i| &\leq 4 \cdot 4^i. \end{aligned}$$

Replacing ξ by ξ_i in (17) and using the Sobolev inequality and the trace inequality, we have, for $n \geq 3$,

$$\begin{aligned} & \left(\int_{B_{r_i}^+} \left(v_\gamma^{\frac{k+1}{2}} \xi_i \right)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} + \int_{\partial B_{r_i}^+ \cap \{x_n=0\}} \left(v_\gamma^{\frac{k+1}{2}} \xi_i \right)^{\frac{2(n-1)}{n-2}} \\ & \leq C \cdot \left(4^i + \frac{(k+1)^2}{4k} \right) \int_{B_{r_i}^+} v_\gamma^{k+1} + C \cdot \left(4^i + \frac{(k+1)^2}{4k} \right) \int_{\partial B_{r_i}^+ \cap \{x_n=0\}} v_\gamma^{k+1}, \end{aligned}$$

where C is some universal constant. By

$$\int_{B_{r_i}^+} v_\gamma^{k+1} \leq C \left(\int_{B_{r_i}^+} v_\gamma^{(k+1) \cdot \frac{n}{n-1}} \right)^{\frac{n-1}{n}},$$

we arrive at

$$\begin{aligned}
 (18) \quad & \left(\int_{B_{r_i}^+} \left(v_\gamma \frac{k+1}{2} \xi_i \right)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} + \int_{\partial B_{r_i}^+ \cap \{x_n=0\}} \left(v_\gamma \frac{k+1}{2} \xi_i \right)^{\frac{2(n-1)}{n-2}} \\
 & \leq C \cdot \left(4^i + \frac{(k+1)^2}{4k} \right) \left(\int_{B_{r_i}^+} v_\gamma^{(k+1) \cdot \frac{n}{n-1}} \right)^{\frac{n-1}{n}} \\
 & \quad + C \cdot \left(4^i + \frac{(k+1)^2}{4k} \right) \int_{\partial B_{r_i}^+ \cap \{x_n=0\}} v_\gamma^{k+1}.
 \end{aligned}$$

Define $\beta = (n - 1)/(n - 2)$, $q_0 = 2$, $q_{i+1} = \beta q_i$ and $p_i = nq_i/(n - 1)$ for $i = 0, 1, \dots$; choosing $k = q_i - 1$ in (18), we have

$$\begin{aligned}
 & \|v_\gamma\|_{L^{p_{i+1}}(B_{r_{i+1}}^+)}^{q_i} + \|v_\gamma\|_{L^{q_{i+1}}(\Gamma_{i+1})}^{q_i} \\
 & \leq C \cdot \left(4^i + \frac{q_i^2}{q_i - 1} \right) \cdot \left(\|v_\gamma\|_{L^{p_i}(B_{r_i}^+)}^{q_i} + \|v_\gamma\|_{L^{q_i}(\Gamma_i)}^{q_i} \right),
 \end{aligned}$$

where $\Gamma_i = \partial B_{r_i}^+ \cap \{x_n = 0\}$ for $i = 0, 1, \dots$; since $\beta > 1$, $(a^\beta + b^\beta)^{1/\beta} \leq a + b$. It follows that

$$\begin{aligned}
 (19) \quad & \left(\|v_\gamma\|_{L^{p_{i+1}}(B_{r_{i+1}}^+)}^{q_{i+1}} + \|v_\gamma\|_{L^{q_{i+1}}(\Gamma_{i+1})}^{q_{i+1}} \right)^{1/q_{i+1}} \\
 & \leq \left(C \cdot \left(4^i + \frac{q_i^2}{q_i - 1} \right) \right)^{1/q_i} \left(\|v_\gamma\|_{L^{p_i}(B_{r_i}^+)}^{q_i} + \|v_\gamma\|_{L^{q_i}(\Gamma_i)}^{q_i} \right)^{1/q_i}.
 \end{aligned}$$

Since $q_i = 2\beta^i$, it is easy to see that

$$\left(C4^i + \frac{Cq_i^2}{q_i - 1} \right)^{1/p_i} \leq [C(4^i + 2\beta^i)]^{1/(2\beta^i)} \leq C^{1/(2\beta^i)}(4 + \beta)^{i/(2\beta^i)}.$$

Thus

$$\prod_{i=1}^\infty \left(4^i C + \frac{q_i^2 C}{q_i - 1} \right)^{1/q_i} \leq C < \infty.$$

It follows that

$$\|v_\gamma\|_{L^{p_{i+1}}(B_{r_{i+1}}^+)} \leq C \left(\|v_\gamma\|_{L^2(\partial \mathbb{R}_+^n \cap \{|x'| < 2\})} + \|v_\gamma\|_{L^{\frac{2n}{n-1}}(B_2^+(0))} \right).$$

Sending $i \rightarrow \infty$, we obtain (16). For $n = 2$, we can obtain (16) in the same spirit. We thereby complete the proof of Theorem 1.1 in Case 1.

Case 2.

$$\overline{\lim}_{\gamma \rightarrow \infty} \frac{\Lambda(\gamma)}{\gamma^2} = \infty.$$

We will also rule out this possibility. Let u_γ be the sequence of positive functions such that

$$\frac{\gamma \int_{\partial\Omega} u_\gamma^2 - \int_{\Omega} |\nabla u_\gamma|^2}{\gamma^2 \int_{\Omega} u_\gamma^2} = \sup_{u \in H^1(\Omega) \setminus \{0\}} \frac{\gamma \int_{\partial\Omega} u^2 - \int_{\Omega} |\nabla u|^2}{\gamma^2 \int_{\Omega} u^2} = a(\gamma) \rightarrow \infty$$

as $\gamma \rightarrow \infty$, and $u_\gamma(x_\gamma) = \max_{\bar{\Omega}} u(x) = 1$. Define $z = \sqrt{a(\gamma)}\gamma(y - y_\gamma)$ and $v_\gamma(z) = u_\gamma(y)$. Then for any $R > 0$ and η with compact support in B_{2R}^+ , as γ becomes sufficiently large, v_γ satisfies

$$\begin{aligned} & \sum_{k,l=1}^n \int_{B_{2R}^+} \frac{\partial v_\gamma}{\partial z_k} \frac{\partial \eta}{\partial z_l} \frac{\partial \Phi_k}{\partial x_i} \left(\Psi \left(y_\gamma + \frac{z}{\gamma} \right) \right) \\ & \quad \cdot \frac{\partial \Phi_l}{\partial x_i} \left(\Psi \left(y_\gamma + \frac{z}{\gamma} \right) \right) |D\Psi| \left(y_\gamma + \frac{z}{\gamma} \right) dz \\ & + \frac{\Lambda(\gamma)}{a(\gamma)\gamma^2} \int_{B_{2R}^+} v_\gamma \eta |D\Psi| \left(y_\gamma + \frac{z}{\gamma} \right) dz \\ & - \frac{1}{\sqrt{a(\gamma)}} \int_{z_n=0} v_\gamma \eta \sqrt{1 + |\nabla \phi(y'_\gamma + z'/\gamma)|^2} dz' = 0. \end{aligned}$$

Similarly as in Case 1, we can show that $v_\gamma \rightarrow v_0$ weakly in $H_{loc}^1(B_R^+)$, where $0 \leq v_0 \leq 1$, $v_0 \neq 0$, and $v_0 \in H_{loc}^1(\mathbb{R}_+^n)$ is a weak solution of

$$(20) \quad \begin{aligned} \Delta v_0 &= v_0 && \text{in } \mathbb{R}_+^n, \\ \frac{\partial v_0}{\partial \nu} &= 0 && \text{on } \partial\mathbb{R}_+^n. \end{aligned}$$

Using even reflection, from (20) we know that there is a positive bounded function satisfying

$$\Delta v_0 = v_0 \quad \text{in } \mathbb{R}^n.$$

On the other hand, it is well-known that there is no nontrivial positive solution to the above equation. This finishes the proof of Theorem 1.1.

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Received December 15, 2000.

DEPARTMENT OF MATHEMATICS
THE OHIO STATE UNIVERSITY
COLUMBUS, OH 43210
E-mail address: lou@math.ohio-state.edu

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF OKLAHOMA
NORMAN, OK 73019
E-mail address: mzhu@aftermath.math.ou.edu