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In this paper we study a class of generalized Fock spaces associated with the Dunkl operators. Next we introduce the commutator relations between the Dunkl operators and multiplication operators which lead to a generalized class of Weyl commutation relations for the Dunkl kernel.

1. Introduction.

Fock space (called also Segal-Bargmann space [5]) is the Hilbert space of entire functions of $\mathbb{C}^d$ with inner product given by

$$(f,g) := \frac{1}{\pi^d}\int_{\mathbb{C}^d} f(z)\overline{g(z)}e^{-|z|^2}dxdy, \quad z = x+iy,$$

where

$$|z|^2 = \sum_{i=1}^{d} x_i^2 + y_i^2, \;\; dxdy = \prod_{i=1}^{d} dx_i dy_i.$$

This space which associated with Fock's [12] realization of the creation and annihilation operators of Bose particles is studied by Bargmann [4]. Next, the ordinary Fock space $\mathcal{A}$ is the subject of many works ([5, 7] and [8]).

In 2001, M. Sifi and F. Soltani [21] introduced a Hilbert space $\mathcal{A}_\gamma$ of entire functions where the inner product is weighted by a generalized Gaussian distribution. On $\mathcal{A}_\gamma$ the Dunkl operator on the real line:

$$T_\gamma(f)(z) := \frac{d}{dz}f(z) + \frac{2\gamma}{z}\left[\frac{f(z)-f(-z)}{2}\right], \quad \gamma > 0,$$

and the multiplication by z are adjoints and satisfy the commutation rule

$$[T_\gamma, z] = I + 2\gamma B, \;\; \text{where} \;\; Bf(x) = f(-x).$$

This commutator rule leads to a generalized class of Weyl commutation relations for the Dunkl kernel in the one dimensional.

In this paper we consider the differential-difference operators T_j, $j = 1, \dots, d$, on $\mathbb{R}^d$ introduced by C.F. Dunkl in [9] and called Dunkl operators in the literature. These operators are very important in pure Mathematics and in Physics. They provide a useful tool in the study of special functions associated with root systems [10]. They are closely related to certain representations of degenerated affine Hecke algebras ([6] and [16]). Moreover the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum Mechanics, namely the Calogero-Sutherland-Moser models, which deal with systems of identical particles in the one dimensional space ([2, 3] and [14]).

The Dunkl kernel $E_k(x,y)$ is the unique solution of the initial problem

$$T_j^x u(x,y) = y_j u(x,y); \;\; u(0,y) = 1; \;\; j = 1,\dots,d,$$

see [10, 17] and [18]. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$. Furthermore, the Dunkl kernel $E_k(z,w)\,;\ z,w \in \mathbb{C}^d$ can be expanded in a power series in the form

$$E_k(z,w) = \sum_{\nu\in\mathbb{N}^d} \varphi_\nu(z)\varphi_\nu(w),$$

with some homogeneous orthonormal basis $\{\varphi_\nu\}_{\nu\in\mathbb{N}^d}$ of polynomials ([17] and [19]).

We introduce in this paper the generalized Fock space $\mathcal{A}_k$ associated with the Dunkl operators. This is a Hilbert space of functions f on $\mathbb{C}^d$ which can be written $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu \varphi_\nu(z)$ with

$$\|f\|_k^2 = (f,f)_k := \sum_{\nu\in\mathbb{N}^d} |a_\nu|^2 < \infty.$$

If $f, g \in \mathcal{A}_k$, having series expansions $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu\varphi_\nu(z)$ and $g(z) = \sum_{\nu\in\mathbb{N}^d} b_\nu\varphi_\nu(z)$. Then the inner product is given by the generalized spherical harmonics

$$(f,g)_k = \Big(f(T)\widetilde{g}\Big)(0),$$

where $f(T) = f(T_1,\dots,T_d)$ and $\widetilde{g}(z) = \sum_{\nu\in\mathbb{N}^d} \overline{b_\nu}\varphi_\nu(z)$.

The generalized Fock space $\mathcal{A}_k$, has also a reproducing kernel $\mathcal{K}$ given by $\mathcal{K}(z,w) = E_k(z,\overline{w})\,;\ z,w \in \mathbb{C}^d$. If $f \in \mathcal{A}_k$, then we have

$$f(w) = (f, E_k(\cdot,\overline{w}))_k, \quad w \in \mathbb{C}^d.$$

Thus the Dunkl kernel serves as the generalized Dirac delta function in $\mathcal{A}_k$.

The associated operators for the generalized Fock space $\mathcal{A}_k$ are T_j and the multiplication operator by z_j . They are adjoints in $\mathcal{A}_k$ and satisfy a commutation rule:

$$[T_i, z_j] = \delta_{i,j} I + \sum_{\alpha\in R_+} k(\alpha)\alpha_i\alpha_j B_\alpha; \quad i,j = 1,\dots,d,$$

where B_α a reflection operator, $k(\alpha)$ a multiplicity function and R_+ is a positive root system.

These commutators rule lead to a generalized class of Weyl commutation relations for the Dunkl kernel.

These relations are studied in the classical case ($k = 0$) in [13].

Throughout this paper we shall use the standard multi-index notations. This for multi-indices $\nu, s \in \mathbb{N}^d$, we write $|\nu| = \sum_{i=1}^{d} \nu_i$, $\nu! = \prod_{i=1}^{d} \nu_i!$, $\begin{pmatrix} \nu \\ s \end{pmatrix} = \prod_{i=1}^{d} \begin{pmatrix} \nu_i \\ s_i \end{pmatrix}$ as well as $z^\nu = \prod_{i=1}^{d} z_i^{\nu_i}$, $D^\nu = \prod_{i=1}^{d} D_i^{\nu_i}$, for $z = (z_1, \dots, z_d) \in \mathbb{C}^d$ and any family $D = (D_1, \dots, D_d)$ of commuting operators. Finally, we will need the partial ordering $\leq$ on $\mathbb{N}^d$, which is defined by $s \leq \nu \Leftrightarrow s_i \leq \nu_i$, $i = 1, \dots, d$.

2. Preliminaries.

In this section we collect some notations and results on Dunkl operators and Dunkl kernel that will be important later on. General references here are [**9**, **17**, **18**, **19**] and [**20**].

We consider $\mathbb{R}^d$ with the Euclidean scalar $\langle .,. \rangle$ and $|x| = \sqrt{\langle x, x \rangle}$. On $\mathbb{C}^d$, $|.|$ denotes also the standard Hermitian norm, while $\langle z, w \rangle = \sum_{j=1}^{d} z_j w_j$ and $\ell(z) = \langle z, z \rangle$.

For $\alpha \in \mathbb{R}^d \backslash \{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α,

$$\sigma_\alpha x := x - \frac{2\langle \alpha, x \rangle}{|\alpha|^2}\, \alpha.$$

A finite set $R \subset \mathbb{R}^d \backslash \{0\}$ is called a root system if $R \cap \mathbb{R}.\alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. We assume that it is normalized by $|\alpha|^2 = 2$ for all $\alpha \in R$. For a given root system R the reflections σ_α, $\alpha \in R$ generated a finite group $G \subset O(d)$, the reflection group associated with R. All reflections in G correspond to suitable pairs of roots. For a given $\beta \in H = \mathbb{R}^d \backslash \bigcup_{\alpha \in R} H_\alpha$, we fix the positive subsystem $R_+ = \{\alpha \in R \ / \ \langle \alpha, \beta \rangle > 0\}$, then for each $\alpha \in R$ either $\alpha \in R_+$ or $-\alpha \in R_+$. The connected components of H are called the Weyl chambers of G.

A function $k : R \to \mathbb{C}$ on a root system R is called a multiplicity function if it is invariant under the action of the associated reflection group G. If one regards k as a function on the corresponding reflections, this means that k is constant on the conjugacy classes of reflections in G. For abbreviation, we introduce the index

$$\gamma = \gamma(k) := \sum_{\alpha \in R_+} k(\alpha).$$

Moreover, let w_k denotes the weight function:

$$w_k(x) := \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}, \quad x \in \mathbb{R}^d,$$

which is G-invariant and homogeneous of degree 2γ.

For $d = 1$ and $G = \mathbb{Z}_2$, the multiplicity function k is a simple parameter denoted $\gamma > 0$ and

$$w_k(x) = |x|^{2\gamma}, \quad x \in \mathbb{R}.$$

The Dunkl operators $T_j\,;\ j = 1, \dots, d$, on $\mathbb{R}^d$ associated with the finite reflection group G and multiplicity function k are given for a function f of class C^1 on $\mathbb{R}^d$, by

$$T_j f(x) := \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}.$$

In the case $k = 0$, the T_j; $j = 1 \dots, d$, reduce to the corresponding partial derivatives. In this paper we will assume throughout that $k \geq 0$.

For $y \in \mathbb{R}^d$, the initial problem

$$\begin{cases} T_j^x u(x, y) = y_j u(x, y); \quad j = 1, \dots, d, \\ u(0, y) = 1, \end{cases}$$

admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted $E_k(x, y)$ and called the Dunkl kernel ([**17**, **18**, **19**] and [**20**]). This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

Examples.

1) If $k = 0$, then $E_k(z, w) = e^{\langle z,w\rangle}$ for $z, w \in \mathbb{C}^d$. (Recall that $\langle .,. \rangle$ was defined to be bilinear on $\mathbb{C}^d \times \mathbb{C}^d$.)
2) If $d = 1$ and $G = \mathbb{Z}_2$, the Dunkl kernel is given by

$$E_\gamma(z, w) = \Im_{\gamma-\frac{1}{2}}(zw) + \frac{zw}{2\gamma + 1} \Im_{\gamma+\frac{1}{2}}(zw),$$

where

$$\Im_{\gamma-\frac{1}{2}}(zw) = \Gamma\left(\gamma + \frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n + \gamma + \frac{1}{2})} \left(\frac{zw}{2}\right)^{2n},$$

is the modified Bessel function of order $\gamma - \frac{1}{2}$ [**21**].

Let $\mathcal{P} = \mathbb{C}[\mathbb{R}^d]$ denotes the $\mathbb{C}$- Algebra of polynomial functions on $\mathbb{R}^d$ and $\mathcal{P}_n,\ n \in \mathbb{N}$, the subspace of homogeneous polynomials of degree n. In the context of generalized spherical harmonics, C.F. Dunkl in [**9**] introduced on $\mathcal{P}$ the following bilinear form:

$$(p, q)_k := \Big(p(T)q\Big)(0)\,; \quad p,\, q \in \mathcal{P}. \tag{1}$$

Here $p(T)$ is the operator derived from $p(x)$ by replacing x_i by T_i. A useful collection of its properties can be found in [**9**] and [**17**]. We recall that $(.,.)_k$ is symmetric, positive-definite and $(p, q)_k = 0$, for $p \in \mathcal{P}_n,\ q \in \mathcal{P}_m$ with $n \neq m$. Moreover, for all $j = 1, \dots, d$ and $p, q \in \mathcal{P}$,

$$(x_j p, q)_k = (p, T_j q)_k.$$

Let $\{\varphi_\nu\}_{\nu\in\mathbb{N}^d}$ be an orthonormal basis of $\mathcal{P}$ with respect to the scalar product $(.,.)_k$ such that $\varphi_\nu \in \mathcal{P}_{|\nu|}$ and the coefficients of the φ_ν are real. As $\mathcal{P} = \oplus_{n\in\mathbb{N}}\mathcal{P}_n$ and $\mathcal{P}_n \perp \mathcal{P}_n$ for $n \neq m$, the φ_ν with $|\nu| = n$ can for example be constructed by Gram-Schmidt orthogonalization within $\mathcal{P}_n$ from an arbitrary ordered real-coefficients basis of $\mathcal{P}_n$. If $k = 0$ the canonical choice of the homogeneous orthonormal basis φ_ν is just $\varphi_\nu(x) = \dfrac{x^\nu}{\sqrt{\nu!}}$.
As in the classical case, M. Rösler obtained in [17, p. 524] the following connection of the basis φ_ν with the Dunkl kernel:

$$E_k(z,w) = \sum_{\nu\in\mathbb{N}^d} \varphi_\nu(z)\varphi_\nu(w)\,; \quad z,w\in\mathbb{C}^d, \tag{2}$$

where the convergence is normal on $\mathbb{C}^d\times\mathbb{C}^d$.

Example. If $d = 1$ and $G = \mathbb{Z}_2$ every homogeneous orthonormal basis is of the form

$$\varphi_n(z) = \frac{z^n}{\sqrt{b_n(\gamma)}},\;\; b_n(\gamma) = \frac{2^n([n/2])!}{\Gamma(\gamma+\frac{1}{2})}\Gamma\left(\left[\frac{n+1}{2}\right]+\gamma+\frac{1}{2}\right). \tag{3}$$

Here $[n/2]$ is the integer part of $n/2$.

From (2), the Dunkl kernel E_k possesses the following properties ([17, 19] and [20]): For all $z,w\in\mathbb{C}^d$ and $\lambda\in\mathbb{C}$,

$$E_k(z,w) = E_k(w,z),\;\; E_k(\lambda z,w) = E_k(z,\lambda w), \tag{4}$$

$$\overline{E_k(z,w)} = E_k(\overline{z},\overline{w}),\;\; E_k(z,\overline{z}) = \sum_{\nu\in\mathbb{N}^d}|\varphi_\nu(z)|^2, \tag{5}$$

$$|E_k(z,w)| \leq e^{|z||w|}. \tag{6}$$

In [18], M. Rösler establish the Bochner-type representation of the Dunkl kernel

$$E_k(x,z) = \int_{\mathbb{R}^d} e^{\langle\xi,z\rangle}d\mu_x(\xi); \quad x\in\mathbb{R}^d,\; z\in\mathbb{C}^d, \tag{7}$$

where μ_x is a probability measure on $\mathbb{R}^d$ with support in $\{\xi\in\mathbb{R}^d/|\xi|\leq|x|\}$.

The Dunkl kernel E_k is analytic on $\mathbb{C}^d\times\mathbb{C}^d$. Therefore, there exist unique analytic functions m_ν, $\nu\in\mathbb{N}^d$, on $\mathbb{C}^d$ with

$$E_k(z,w) = \sum_{\nu\in\mathbb{N}^d}\frac{m_\nu(z)}{\nu!}w^\nu; \quad z,w\in\mathbb{C}^d. \tag{8}$$

The restriction of m_ν to $\mathbb{R}^d$ are called the ν-th moment functions ([18, 19] and [20]). It is given explicitly by

$$m_\nu(x) = \int_{\mathbb{R}^d}\xi^\nu d\mu_x(\xi), \quad x\in\mathbb{R}^d,$$

where μ_x is the measure given by (7).

The functions m_ν are homogeneous polynomials of degree $|\nu|$. Among the applications of these moments, we mention the construction of martingales from Dunkl-type Markov processes [**19**].

3. Fock spaces for the Dunkl kernel.

In this section we define and study the generalized Fock space for the Dunkl kernel in d-dimensions.

Definition 1. The generalized Fock space $\mathcal{A}_k$ associated with the Dunkl operators is the space of holomorphic functions f on $\mathbb{C}^d$ which can be written $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu \varphi_\nu(z)$ with

$$\|f\|_k^2 := \sum_{\nu\in\mathbb{N}^d} |a_\nu|^2 < \infty.$$

Hence the inner product in $\mathcal{A}_k$ is given for $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu \varphi_\nu(z) \in \mathcal{A}_k$ and $g(z) = \sum_{\nu\in\mathbb{N}^d} b_\nu \varphi_\nu(z) \in \mathcal{A}_k$, by

$$(f,g)_k := \sum_{\nu\in\mathbb{N}^d} a_\nu \overline{b_\nu}. \tag{9}$$

Remark. If $k = 0$, $\mathcal{A}_0$ is the ordinary Fock space $\mathcal{A}$ [**4**].

Proposition 1.

i) *If $f, g \in \mathcal{A}_k$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu \varphi_\nu(z)$ and $g(z) = \sum_{\nu\in\mathbb{N}^d} b_\nu \varphi_\nu(z)$, we have*

$$(f,g)_k = \Big(f(T)\widetilde{g}\Big)(0),$$

where $\widetilde{g}(z) = \sum_{\nu\in\mathbb{N}^d} \overline{b_\nu} \varphi_\nu(z)$.

ii) *If $f \in \mathcal{A}_k$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu \varphi_\nu(z)$, we have*

$$|f(z)| \le e^{|z|^2/2} \|f\|_k.$$

Proof. i) From [**17**, p. 529], we have

$$\Big(\varphi_\nu(T)\varphi_s\Big)(0) = \delta_{\nu,s},$$

where $\delta_{\nu,s}$ is the Kronecker symbol.

Thus

$$(f,g)_k = \sum_{\nu,s\in\mathbb{N}^d} a_\nu \overline{b_s} \Big(\varphi_\nu(T)\varphi_s\Big)(0).$$

Using the continuously of the inner product, we obtain the result.

ii) Using Cauchy-Schwarz's inequality, then

$$|f(z)|^2 \le \left[\sum_{\nu\in\mathbb{N}^d} |a_\nu|^2\right] \left[\sum_{\nu\in\mathbb{N}^d} |\varphi_\nu(z)|^2\right] = \|f\|_k^2 E_k(z,\overline{z}).$$

Thus

$$|f(z)| \leq [E_k(z,\overline{z})]^{1/2}\|f\|_k.$$

The result follows from the inequality (6). □

From Proposition 1 ii), the map $f \to f(z)$, $z \in \mathbb{C}^d$, is a continuous linear functional on $\mathcal{A}_k$. Thus from Riesz theorem [**1**], $\mathcal{A}_k$ has a reproducing kernel.

Proposition 2. *The function $\mathcal{K}$ given for $w, z \in \mathbb{C}^d$, by*

$$\mathcal{K}(z,w) = E_k(z,\overline{w}),$$

is a reproducing kernel for the generalized Fock spaces $\mathcal{A}_k$, that is:

i) *For every $w \in \mathbb{C}^d$, the function $z \to \mathcal{K}(z,w)$ belongs to $\mathcal{A}_k$.*

ii) *The reproducing property: For every $w \in \mathbb{C}^d$ and $f \in \mathcal{A}_k$, we have*

$$(f, \mathcal{K}(\cdot,w))_k = f(w).$$

Proof. i) Using (5) and (6), we deduce for $w \in \mathbb{C}^d$,

$$\|E_k(\cdot,\overline{w})\|_k^2 = \sum_{\nu\in\mathbb{N}^d} |\varphi_\nu(\overline{w})|^2 = E_k(w,\overline{w}) \leq e^{|w|^2},$$

which proves i).

ii) If $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu \varphi_\nu(z) \in \mathcal{A}_k$, it follows from (9) that

$$(f, E_k(\cdot,\overline{w}))_k = \sum_{\nu\in\mathbb{N}^d} a_\nu \overline{\varphi_\nu(\overline{w})} = f(w).$$

□

Corollary 1.

i) *The set $\{E_k(\cdot,\overline{w}),\ w \in \mathbb{C}^d\}$ is complete in $\mathcal{A}_k$.*

ii) *For all $z, w \in \mathbb{C}^d$, we have*

$$E_k(\overline{z},w) = (E_k(\cdot,\overline{z}), E_k(\cdot,\overline{w}))_k.$$

iii) *Let $m \in \mathbb{N}\backslash\{0\}$ and $z_1, z_2, \ldots, z_m \in \mathbb{C}^d$, with $z_i \neq z_j$, then*

$$\det\Big[E_k(\overline{z}_i, z_j)\Big]_{i,j=1}^m > 0.$$

Notation. We denote by $L^2(\mu_k)$ the Hilbert space of measurable functions on $\mathbb{R}^d$, for which

$$\|f\|_{2,k} := \left[\int_{\mathbb{R}^d} |f(x)|^2 d\mu_k(x)\right]^{1/2} < \infty.$$

Here μ_k is the measure defined on $\mathbb{R}^d$, by

$$d\mu_k(x) := c_k w_k(x)dx, \text{ with } c_k = \left(\int_{\mathbb{R}^d} e^{-|x|^2} d\mu_k(x)\right)^{-1},$$

is the Mehta-type constant.

In the next part of this section we establish the unitary equivalence of $L^2(\mu_k)$ and $\mathcal{A}_k$. First we recall some properties of the generalized Hermite functions ([17] and [19]):

Definition 2. The generalized Hermite polynomials $\{H_\nu\}_{\nu\in\mathbb{N}^d}$ associated with the basis $\{\varphi_\nu\}_{\nu\in\mathbb{N}^d}$ on $\mathbb{C}^d$, are given by

$$H_\nu(z) := 2^{|\nu|}e^{-\triangle_k/4}\varphi_\nu(z) = 2^{|\nu|}\sum_{n=0}^{[|\nu|/2]}\frac{(-1)^n}{2^{2n}n!}\triangle_k^n\varphi_\nu(z),$$

where $\triangle_k = \sum_{i=1}^d T_i^2$ is the Dunkl Laplacian [17].

Moreover, we define the generalized Hermite functions on $\mathbb{C}^d$, by

$$h_\nu(z) := 2^{-|\nu|/2}e^{-\ell(z)/2}H_\nu(z).$$

Examples.

1) If $k = 0$, we obtain

$$H_\nu(x) = \frac{2^{|\nu|}}{\sqrt{\nu!}}\prod_{i=1}^d e^{-\frac{1}{4}\frac{\partial^2}{\partial x_i{}^2}}(x_i^{\nu_i}) = \frac{1}{\sqrt{\nu!}}\prod_{i=1}^d \widehat{H}_{\nu_i}(x_i), \quad x\in\mathbb{R}^d,$$

where

$$\widehat{H}_{x_i} = (-1)^{\nu_i}e^{x_i^2}\frac{\partial^{\nu_i}}{\partial x_i^{\nu_i}}(e^{-x_i^2}).$$

2) If $d = 1$ and $G = \mathbb{Z}_2$, we obtain

$$H_n(z) = \sum_{i=0}^{[n/2]}\frac{(-1)^i}{i!\,b_{n-2i}(\gamma)}(2x)^{n-2i}, \quad x\in\mathbb{R},$$

where $b_n(\gamma)$ are the constants given by (3).

The following lemma is shown in [17, p. 525-529]:

Lemma 1.

i) *The set $\{h_\nu\}_{\nu\in\mathbb{N}^d}$ is an orthonormal basis of $L^2(\mu_k)$.*

ii) *For all $z, w \in \mathbb{C}^d$, there is a generating function for the generalized Hermite polynomials,*

$$e^{-\ell(w)}E_k(2z, w) = \sum_{\nu\in\mathbb{N}^d} h_\nu(z)\varphi_\nu(w).$$

Notation. We denote by U_k the kernel given for $z, w \in \mathbb{C}^d$, by

$$U_k(z, w) := e^{-(\ell(z)+\ell(w))/2}E_k(\sqrt{2}z, w). \tag{10}$$

Lemma 2. *For $w, z \in \mathbb{C}^d$, we have*

$$U_k(z, w) = \sum_{\nu\in\mathbb{N}^d} h_\nu(z)\varphi_\nu(w).$$

Proof. From Definition 2, we have

$$\sum_{\nu\in\mathbb{N}^d} h_\nu(z)\varphi_\nu(w) = e^{-\ell(z)/2}\sum_{\nu\in\mathbb{N}^d} 2^{-|\nu|/2}H_\nu(z)\varphi_\nu(w).$$

As φ_ν is homogeneous of degree $|\nu|$, then

$$\varphi_\nu\left(\frac{w}{\sqrt{2}}\right) = 2^{-|\nu|/2}\varphi_\nu(w).$$

Thus

$$\sum_{\nu\in\mathbb{N}^d} h_\nu(z)\varphi_\nu(w) = e^{-\ell(z)/2}\sum_{\nu\in\mathbb{N}^d} H_\nu(z)\varphi_\nu\left(\frac{w}{\sqrt{2}}\right).$$

Applying Lemma 1 ii) and (4), we obtain

$$\sum_{\nu\in\mathbb{N}^d} h_\nu(z)\varphi_\nu(w) = e^{-(\ell(z)+\ell(w))/2}E_k\Big(2z, \frac{w}{\sqrt{2}}\Big) = U_k(z,w).$$

□

Lemma 3.

i) *For all $z, w \in \mathbb{C}^d$, we have*

$$E_k(z,w) = \int_{\mathbb{R}^d} U_k(z,x)U_k(w,x)d\mu_k(x).$$

ii) *For all $z \in \mathbb{C}^d$, the function $x \to U_k(z,x)$ belongs to $L^2(\mu_k)$, and we have*

$$\|U_k(z,\cdot)\|^2_{2,k} = E_k(z,\overline{z}).$$

iii) *For all $x \in \mathbb{R}^d$, the function $z \to U_k(z,x)$ belongs to $\mathcal{A}_k$, and we have*

$$\|U_k(\cdot,x)\|^2_k = e^{-3|x|^2}E_k(2x,x).$$

Proof. i) We put

$$I = \int_{\mathbb{R}^d} U_k(z,x)U_k(w,x)d\mu_k(x).$$

From (10), we have

$$I = e^{-(\ell(z)+\ell(w))/2}\int_{\mathbb{R}^d} e^{-|x|^2}E_k(\sqrt{2}z,x)E_k(\sqrt{2}w,x)d\mu_k(x).$$

So from [**17**, p. 523] and (4), we get

$$\int_{\mathbb{R}^d} e^{-|x|^2}E_k(\sqrt{2}z,x)E_k(\sqrt{2}w,x)d\mu_k(x) = e^{(\ell(z)+\ell(w))/2}E_k(z,w),$$

which proves i).

ii) This assertion follows from i) and (5).

iii) For $z \in \mathbb{C}^d$, we put

$$\phi(z) := e^{-(\ell(z)+\ell(\overline{z}))/2}.$$

Let $x \in \mathbb{R}^d$, then from Proposition 2 ii), (10) and (4), we have

$$\|U_k(\cdot,x)\|_k^2 = e^{-|x|^2}(\phi(\cdot)E_k(\cdot,\sqrt{2}x), E_k(\cdot,\sqrt{2}x))_k = e^{-3|x|^2}E_k(2x,x).$$

□

Definition 3. The chaotic transform $\mathcal{C}_k$ (also called S-transform in the stochastic calculus [**15**]) is the transformation defined on $L^2(\mu_k)$, by

$$\mathcal{C}_k(f)(z) := \int_{\mathbb{R}^d} U_k(z,x)f(x)d\mu_k(x), \quad z \in \mathbb{C}^d.$$

Remark. The basis elements of $L^2(\mu_k)$ and $\mathcal{A}_k$ are called chaos. In the following theorem we shall prove that the transformation $\mathcal{C}_k$ maps the chaos of $L^2(\mu_k)$ to these of $\mathcal{A}_k$.

Theorem 1. *The chaotic transform $\mathcal{C}_k$ is a unitary mapping of $L^2(\mu_k)$ on $\mathcal{A}_k$. Moreover, the basis elements are related by*

$$\mathcal{C}_k(h_\nu) = \varphi_\nu.$$

Proof. It follows directly from Lemma 1 i) and Lemma 2, that for $\nu \in \mathbb{N}^d$,

$$\mathcal{C}(h_\nu)(z) = \int_{\mathbb{R}^d} U_k(z,x)h_\nu(x)d\mu_k(x) = \varphi_\nu(z), \quad z \in \mathbb{C}^d.$$

Consequently $\mathcal{C}_k$ maps the subspace generated by the family $\{h_\nu\}_{\nu\in\mathbb{N}^d}$ into the polynomials in $\mathcal{A}_k$. Thus $\mathcal{C}_k$ maps a dense set in $L^2(\mu_k)$ into a dense set in $\mathcal{A}_k$. Further, if $f \in L^2(\mu_k)$, then $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu h_\nu(x)$. For $\nu \in \mathbb{N}^d$, let $f_N(x) = \sum_{j=0}^N \sum_{|\nu|=j} a_\nu h_\nu(x)$, $x \in \mathbb{R}$. Then

$$\mathcal{C}_k(f_N)(z) = \sum_{j=0}^N \sum_{|\nu|=j} a_\nu \varphi_\nu(z); \ \lim_{N\to\infty} \|f - f_N\|_{2,k} = 0.$$

On the other hand, from Hölder's inequality and Lemma 3 ii), we have

$$|\mathcal{C}_k(f - f_N)(z)| \ \le [E_k(z,\overline{z})]^{1/2}\|f - f_N\|_{2,k}.$$

Thus we obtain

$$\mathcal{C}_k(f)(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu \varphi_\nu(z).$$

Hence

$$\|\mathcal{C}_k(f)\|_k^2 = \sum_{\nu\in\mathbb{N}^d} |a_\nu|^2 = \|f\|_{2,k}^2.$$

It follows that $\mathcal{C}_k$ is a unitary transformation from $L^2(\mu_k)$ into $\mathcal{A}_k$.

Clearly, if $g(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu \varphi_\nu(z) \in \mathcal{A}_k$, we have

$$\mathcal{C}_k^{-1}(g)(x) = \sum_{\nu\in\mathbb{N}^d} a_\nu h_\nu(x), \quad x \in \mathbb{R}^d. \tag{11}$$

Which completes the proof. □

Proposition 3. *If $g \in \mathcal{A}_k$, we have*

$$\mathcal{C}_k^{-1}(g)(x) = (g, U_k(\cdot, x))_k, \quad x \in \mathbb{R}^d.$$

Proof. Let $g \in \mathcal{A}_k$. We put for $x \in \mathbb{R}^d$,

$$\Psi_k(g)(x) = (g, U_k(\cdot, x))_k.$$

Using Lemma 2, Lemma 3 iii) and the same method as in the proof of Theorem 1 we obtain

$$\Psi_k(g)(x) = \sum_{\nu \in \mathbb{N}^d} a_\nu h_\nu(x) = \mathcal{C}_k^{-1}(g)(x), \quad x \in \mathbb{R}^d.$$

□

4. Commutators and Weyl relations for the Dunkl kernel.

We define the multiplication operators $Q_i;\ i = 1, \dots, d$ on $\mathcal{A}_k$ by

$$Q_i f(z) := z_i f(z), \quad z \in \mathbb{C}^d.$$

We denote also by $T_i;\ i = 1, \dots, d$ the operators defined on $\mathcal{A}_k$.

Let

$$\mathcal{D}(Q_i) = \{f \in \mathcal{A}_k \ / \ Q_i(f) \in \mathcal{A}_k\},$$
$$\mathcal{D}(T_i) = \{f \in \mathcal{A}_k \ / \ T_i f \in \mathcal{A}_k\}$$

denote the domains of Q_i and T_i respectively.

We denote by $[.,.]$ the commutator product ($[A, B] = AB - BA$). As in [**11**], we have the following relations:

Lemma 4. *The operators Q_i and T_i, $i = 1, \dots, d$ satisfy on $\mathcal{A}_k$ the commutation relations:*

$$[T_i, T_j] = [Q_i, Q_j] = 0; \quad i, j = 1, \dots, d, \tag{12}$$

$$[T_i, Q_j] = \delta_{i,j} I + \sum_{\alpha \in R_+} k(\alpha) \alpha_i \alpha_j B_\alpha; \quad i, j = 1, \dots, d, \tag{13}$$

where I the identity operator and B_α is the reflection operator ($B_\alpha^2 = I$) given by

$$B_\alpha f(z) = f(\sigma_\alpha z). \tag{14}$$

Proof. Using the fact that $\sigma_\alpha^2 = id$ and $\langle \alpha, \sigma_\alpha z \rangle = -\langle \alpha, z \rangle$, we obtain

$$T_i T_j f(z) = T_i \left(\frac{\partial}{\partial z_j} f \right)(z) + \sum_{\alpha \in R_+} k(\alpha) \alpha_j \frac{\partial}{\partial z_i} \left(\frac{f(z) - f(\sigma_\alpha z)}{\langle \alpha, z \rangle} \right).$$

Since

$$\frac{\partial}{\partial z_i}(f(\sigma_\alpha z)) = \frac{\partial}{\partial z_i} f(\sigma_\alpha z) - \sum_{\ell=1}^{d} \alpha_i \alpha_\ell \frac{\partial}{\partial z_\ell} f(\sigma_\alpha z),$$

we have

$$T_iT_jf(z) = -\frac{\partial^2}{\partial z_j \partial z_i} f(z) + T_i\left(\frac{\partial}{\partial z_j} f\right)(z) + T_j\left(\frac{\partial}{\partial z_i} f\right)(z)$$
$$- \sum_{\alpha\in R_+} k(\alpha)\alpha_i\alpha_j \left[\frac{f(z)-f(\sigma_\alpha z)}{(\langle\alpha, z\rangle)^2} - \sum_{\ell=1}^{d} \alpha_\ell \frac{\frac{\partial}{\partial z_\ell} f(\sigma_\alpha z)}{\langle\alpha, z\rangle}\right].$$

Thus

$$[T_i, T_j]f(z) = 0.$$

The other relations are evident. □

Proposition 4. *Let*

$$f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu \varphi_\nu(z) \in \mathcal{D}(Q_i) \qquad \textit{and}$$
$$g(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu \varphi_\nu(z) \in \mathcal{D}(T_i), \qquad \textit{then}$$
$$(Q_i f, g)_k = (f, T_i g)_k.$$

Proof. Applying Proposition 1 i), we get

$$(Q_i f, g)_k = (Q_i f(T)\widetilde{g})(0) = \sum_{\nu,s\in\mathbb{N}^d} a_\nu \overline{b_s} T_i \varphi_\nu(T)\varphi_s(0).$$

Then from (12) we obtain

$$(Q_i f, g)_k = \sum_{\nu,s\in\mathbb{N}^d} a_\nu \overline{b_s} \varphi_\nu(T) T_i \varphi_s(0) = (f, T_i g)_k.$$

□

Lemma 5. *If $f \in \mathcal{A}_k$, then $B_\alpha f \in \mathcal{A}_k$, and we have*

$$\|Q_i f\|_k^2 = \|T_i f\|_k^2 + \|f\|_k^2 + \sum_{\alpha\in R_+} k(\alpha)\alpha_i^2 (f, B_\alpha f)_k,$$

where B_α is the operator given by (14).

Proof. Let $f \in \mathcal{A}_k$. Applying the chaotic transform, in view of Theorem 1, it suffices to show that $\mathcal{C}_k^{-1}(B_\alpha f) \in L^2(\mu_k)$. From (11), we have

$$\mathcal{C}_k^{-1}(B_\alpha f)(x) = \mathcal{C}_k^{-1}(f)(\sigma_\alpha x), \quad x \in \mathbb{R}^d.$$

Putting $u = \sigma_\alpha x$, we get

$$d\mu_k(x) = |J_\alpha| d\mu_k(u) \quad \text{where} \quad J_\alpha = \det\left[\delta_{i,j} - \alpha_i\alpha_j\right]_{i,j=1}^{d}.$$

Since $J_\alpha = -1$, we obtain

$$\|\mathcal{C}_k^{-1}(B_\alpha f)\|_{2,k}^2 = \int_{\mathbb{R}^d} |\mathcal{C}_k^{-1}(f)(u)|^2 d\mu_k(u).$$

Which proves that $B_\alpha f \in \mathcal{A}_k$.

On the other hand, from Proposition 4, we deduce

$$\|Q_i f\|_k^2 = (f, T_i Q_i f)_k.$$

But from (13), we have

$$T_i Q_i f = Q_i T_i f + f + \sum_{\alpha\in R_+} k(\alpha)\alpha_i^2 B_\alpha f.$$

Thus

$$\|Q_i f\|_k^2 = (f, Q_i T_i f)_k + \|f\|_k^2 + \sum_{\alpha\in R_+} k(\alpha)\alpha_i^2 (f, B_\alpha f)_k.$$

Using another time Proposition 4, we obtain the result. □

Proposition 5. *The operators Q_i and T_i are closed densely defined operators on $\mathcal{A}_k$, and we have*

$$\mathcal{D}(Q_i) = \mathcal{D}(T_i); \;\; Q_i^* = T_i; \;\; T_i^* = Q_i,$$

where Q_i^* and T_i^* are the adjoints operators of Q_i and T_i, respectively.

Proof. These results follow from [**4**, Theorem 1.2], Lemma 5 and Proposition 4 by using the same method as [**21**, Proposition 6]. □

Lemma 6. *For $\nu \in \mathbb{N}^d\backslash\{0\}$, we have the following relations:*

i)

$$[T^\nu, Q_j] = \nu_j T_1^{\nu_1}\dots T_{i-1}^{\nu_{i-1}} T_i^{\nu_i-1} T_{i+1}^{\nu_{i+1}}\dots T_d^{\nu_d} + B_\alpha \sum_{i=1}^{d}\sum_{\ell=0}^{\nu_i-1}\sum_{\alpha\in R_+} k(\alpha)\alpha_i\alpha_j H_1^{\nu_1}\dots H_{i-1}^{\nu_{i-1}} H_i^{\ell} T_i^{\nu_i-\ell-1} T_{i+1}^{\nu_{i+1}}\dots T_d^{\nu_d},$$

where $H_i\,;\; i=1,\dots,d$, are given by the differential-difference operators

$$H_i = -T_i + 2\frac{\partial}{\partial x_i} - \sum_{\ell=1}^{d} \alpha_i\alpha_\ell \frac{\partial}{\partial x_\ell}.$$

ii)

$$[T_j, Q^\nu] = \nu_j Q_1^{\nu_1}\dots Q_{i-1}^{\nu_{i-1}} Q_i^{\nu_i-1} Q_{i+1}^{\nu_{i+1}}\dots Q_d^{\nu_d} + B_\alpha \sum_{i=1}^{d}\sum_{\ell=0}^{\nu_i-1}\sum_{\alpha\in R_+} k(\alpha)\alpha_i\alpha_j Z_1^{\nu_1}\dots Z_{i-1}^{\nu_{i-1}} Z_i^{\ell} Q_i^{\nu_i-\ell-1} Q_{i+1}^{\nu_{i+1}}\dots Q_d^{\nu_d},$$

where Z_i; $i=1,\dots,d$, are the multiplication operators given by

$$Z_i = Q_i - \sum_{\ell=1}^{d} \alpha_i\alpha_\ell Q_\ell.$$

Proof. From (13), we have

$$[T_i^{\nu_i}, Q_j] = \sum_{\ell=0}^{\nu_i-1} T_i^{\ell}[T_i, Q_j]T_i^{\nu_i-\ell-1}$$
$$= \nu_i\delta_{i,j}T_i^{\nu_i-1} + \sum_{\ell=0}^{\nu_i-1} k(\alpha)\alpha_i\alpha_j T_i^{\ell} B_\alpha T_i^{\nu_i-\ell-1}.$$

From this equality, we get

$$[T^\nu, Q_j] = \sum_{i=1}^{d} T_1^{\nu_1}\dots T_{i-1}^{\nu_{i-1}}[T_i^{\nu_i}, Q_j]T_{i+1}^{\nu_{i+1}}\dots T_d^{\nu_d}$$
$$= \nu_j T_1^{\nu_1}\dots T_{i-1}^{\nu_{i-1}}T_i^{\nu_i-1}T_{i+1}^{\nu_{i+1}}\dots T_d^{\nu_d}$$
$$+ \sum_{i=1}^{d}\sum_{\ell=0}^{\nu_i-1}\sum_{\alpha\in R_+} k(\alpha)\alpha_i\alpha_j T_1^{\nu_1}\dots T_{i-1}^{\nu_{i-1}}T_i^{\ell}B_\alpha T_i^{\nu_i-\ell-1}T_{i+1}^{\nu_{i+1}}\dots T_d^{\nu_d}.$$

But

$$T_i^{\nu_i}B_\alpha = B_\alpha H_i^{\nu_i},$$

where

$$H_i = -T_i + 2\frac{\partial}{\partial x_i} - \sum_{\ell=1}^{d}\alpha_i\alpha_\ell\frac{\partial}{\partial x_\ell}.$$

Thus we obtain Assertion i). And similarly we get ii). □

Notation. For $x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$, we denote by

$$I_k(z,x) := \frac{E_k(z,x) - E_k(z,\sigma_\alpha x)}{\langle\alpha, x\rangle}.$$

From [**17**, p. 533], we can write the function $I_k(z,x)$ in the form

$$I_k(z,x) = \langle\nabla_x E_k(z,x), \alpha\rangle + \frac{1}{2}\langle\alpha, x\rangle\alpha^t D_x^2 E_k(z,\xi)\alpha, \tag{15}$$

with some ξ on the line segment between x and $\sigma_\alpha x$.

(Here ∇ and $D^2 f(x)$ denote the usual gradient and Hessian of f in x.)

Lemma 7. *For $a, b \in \mathbb{C}^d$, we have the following commutation relations:*

i) $[E_k(a,T), Q_j] = a_j E_k(a,T) - R_{k,j}(a,T)$, *where*

$$R_{k,j}(a,T) =$$
$$\sum_{\alpha\in R_+} k(\alpha)\alpha_j I_k(a,T)$$
$$- B_\alpha \sum_{\substack{\nu\in\mathbb{N}^d\\ \alpha\in R_+}}\sum_{i=1}^{d}\sum_{\ell=0}^{\nu_i-1} k(\alpha)\alpha_i\alpha_j\frac{m_\nu(a)}{\nu!}H_1^{\nu_1}\dots H_{i-1}^{\nu_{i-1}}H_i^{\ell}T_i^{\nu_i-\ell-1}T_{i+1}^{\nu_{i+1}}\dots T_d^{\nu_d}.$$

ii) $[T_j, E_k(b,Q)] = b_j E_k(b,Q) - S_{k,j}(b,Q)$, *where*

$$S_{k,j}(b,Q) = \sum_{\alpha\in R_+} k(\alpha)\alpha_j I_k(b,Q) - B_\alpha \sum_{\substack{\nu\in\mathbb{N}^d\\ \alpha\in R_+}} \sum_{i=1}^{d}\sum_{\ell=0}^{\nu_i-1} k(\alpha)\alpha_i\alpha_j \frac{m_\nu(b)}{\nu!} Z_1^{\nu_1}\dots Z_{i-1}^{\nu_{i-1}} Z_i^{\ell} Q_i^{\nu_i-\ell-1} Q_{i+1}^{\nu_{i+1}}\dots Q_d^{\nu_d}.$$

Proof. Using (8) and Lemma 6 i), we obtain

$$\begin{aligned}
&[E_k(a,T), Q_j]\\
&= \sum_{\nu\in\mathbb{N}^d} \frac{m_\nu(a)}{\nu!}[T^\nu, Q_j]\\
&= \sum_{\nu\in\mathbb{N}^d} \frac{m_\nu(a)}{\nu!}\nu_j T_1^{\nu_1}\dots T_{i-1}^{\nu_{i-1}} T_i^{\nu_i-1} T_{i+1}^{\nu_{i+1}}\dots T_d^{\nu_d}\\
&\quad + B_\alpha \sum_{\nu\in\mathbb{N}^d}\sum_{i=1}^{d}\sum_{\ell=0}^{\nu_i-1}\sum_{\alpha\in R_+} k(\alpha)\alpha_i\alpha_j \frac{m_\nu(a)}{\nu!} H_1^{\nu_1}\dots H_{i-1}^{\nu_{i-1}} H_i^{\ell} T_i^{\nu_i-\ell-1} T_{i+1}^{\nu_{i+1}}\dots T_d^{\nu_d}.
\end{aligned}$$

Applying the relation

$$\frac{\partial}{\partial w_j} E_k(z,w) = z_j E_k(z,w) - \sum_{\alpha\in R_+} k(\alpha)\alpha_j I_k(z,w); \quad z,w\in\mathbb{C}^d,$$

we obtain

$$[E_k(a,T), Q_j] = a_j E_k(a,T) - R_{k,j}(a,T).$$

This proves i). Similarly, we can prove ii). □

Remark. If $d=1$ and $G=\mathbb{Z}_2$ [**21**], we have

$$R_\gamma(a,T_\gamma) = \frac{2\gamma}{2\gamma+1} a(I-B)\Im_{\gamma+\frac{1}{2}}(aT_\gamma),$$

$$S_\gamma(b,Q) = \frac{2\gamma}{2\gamma+1} b(I-B)\Im_{\gamma+\frac{1}{2}}(bQ),$$

where $Bf(x) = f(-x)$.

Since $E_k(a,0)=1$, the Dunkl kernel $E_k(a,z)$; $a,z\in\mathbb{C}^d$, is a unit in the integral domain formal power series over $\mathbb{C}^d$. We define

$$E_k^{-1}(a,z) := \sum_{\nu\in\mathbb{N}^d} \frac{t_\nu(a)}{\nu!} z^\nu.$$

Writing

$$E_k(a,z)E_k^{-1}(a,z) = E_k^{-1}(a,z)E_k(a,z) = 1,$$

we obtain

$$t_0(a) = 1, \quad \sum_{\nu\in\mathbb{N}^d} \left\{ \sum_{s\le\nu} \binom{\nu}{s} m_{\nu-s}(a) t_s(a) \right\} \frac{z^\nu}{\nu!} = 1.$$

Thus $\{t_\nu(a)\}_{\nu\in\mathbb{N}^d}$ is a sequence of moment functions in a determined by

$$t_0(a) = 1, \;\; t_\nu(a) = -\sum_{s\le\nu-1} \binom{\nu}{s} m_{\nu-s}(a) t_s(a).$$

The function $E_k^{-1}(a, z)$ occurs in the generalized Weyl commutation relations for the Dunkl kernel.

Theorem 2. *Let $a, b \in \mathbb{C}^d$, then:*

i) $E_k(b, Q)E_k(a, T) = E_k(a, T)E_k(b, P_a)$, $\;P_a = (P_{a,1}, \dots, P_{a,d})$, *where*

$$P_{a,j} = Q_j - a_j I + E_k^{-1}(a, T)R_{k,j}(a, T).$$

ii) $E_k(a, T)E_k(b, Q) = E_k(b, Q)E_k(a, L_b)$, $\;L_b = (L_{b,1}, \dots, L_{b,d})$, *where*

$$L_{b,j} = T_j + b_j I - E_k^{-1}(b, Q)S_{k,j}(b, Q).$$

iii) $E_k(a, Q)E_k(b, Q) = E_k(a\#b, Q)$, $\;E_k(a, T)E_k(b, T) = E_k(a\#b, T)$, *where $a\#b$ is the convolution of a and b given by*

$$m_\nu(a\#b) = \sum_{s\le\nu} \binom{\nu}{s} m_s(a) m_{\nu-s}(b).$$

Proof. We shall prove i), ii) follows in the same way. For $j = 1, 2, \dots, d$, we have

$$E_k^{-1}(a, T)Q_j E_k(a, T) = E_k^{-1}(a, T)\Big\{ E_k(a, T)Q_j - [E_k(a, T), Q_j] \Big\}.$$

Using Lemma 7 i), we obtain

$$E_k^{-1}(a, T)Q_j E_k(a, T) = Q_j - a_j I + E_k^{-1}(a, T)R_{k,j}(a, T).$$

Thus implies that for $\nu \in \mathbb{N}^d$:

$$E_k^{-1}(a, T)Q^\nu E_k(a, T) = P_a^\nu, \;\; P_a = (P_{a,1}, \dots, P_{a,d}),$$

where

$$P_{a,j} = Q_j - a_j I + E_k^{-1}(a, T)R_{k,j}(a, T).$$

Multiplying by $\dfrac{m_\nu(b)}{\nu!}$ and summing, we get

$$E_k^{-1}(a, T)E_k(b, Q)E_k(a, T) = E_k(b, P_a).$$

Then i) follows upon multiplication by $E_k(a, T)$.

iii) It suffices to prove the first relation.

Using (8) and (12), we can write

$$\begin{aligned}E_k(a,Q)E_k(b,Q) &= \sum_{\nu,s\in\mathbb{N}^d} \frac{m_\nu(a)m_s(b)}{\nu!\,s!}Q^{\nu+s}\\ &= \sum_{\nu\in\mathbb{N}^d} \left\{ \sum_{s\leq\nu} \begin{pmatrix} \nu \\ s \end{pmatrix} m_s(a)m_{\nu-s}(b) \right\} \frac{Q^\nu}{\nu!}\\ &= \sum_{\nu\in\mathbb{N}^d} \frac{m_\nu(a\#b)}{\nu!}Q^\nu.\end{aligned}$$

Thus we obtain

$$E_k(a,Q)E_k(b,Q) = E_k(a\#b,Q).$$

□

Remarks. 1) In the classical case ($k=0$) [**13**, p. 223], the Weyl commutation relations are given by

$$\begin{aligned}e^{\langle a,P\rangle}e^{\langle b,Q\rangle} &= e^{\langle a,b\rangle}e^{\langle b,Q\rangle}e^{\langle a,P\rangle},\\ e^{\langle a,P\rangle}e^{\langle b,P\rangle} &= e^{\langle a+b,P\rangle},\\ e^{\langle a,Q\rangle}e^{\langle b,Q\rangle} &= e^{\langle a+b,Q\rangle},\end{aligned}$$

where $P=(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_d})$ and $Q=(Q_1,\ldots,Q_d)$.

2) If $d=1$ and $G=\mathbb{Z}_2$ [**21**], the Weyl commutation relations are given by

$$\begin{aligned}E_\gamma(bQ)E_\gamma(aT_\gamma) &= E_\gamma(aT_\gamma)E_\gamma(bP_a);\\ E_\gamma(aT_\gamma)E_\gamma(bQ) &= E_\gamma(bQ)E_\gamma(aL_b),\end{aligned}$$

where

$$P_a = Q - aI + \frac{2\gamma}{2\gamma+1}aE_\gamma^{-1}(aT_\gamma)(I-B)\Im_{\alpha+1}(aT_\gamma),$$

and

$$L_b = T_\gamma + bI - \frac{2\gamma}{2\gamma+1}bE_\gamma^{-1}(bQ)(I-B)\Im_{\alpha+1}(bQ).$$

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