

*Pacific
Journal of
Mathematics*

DEFORMATIONS OF ASYMPTOTICALLY CONICAL
SPECIAL LAGRANGIAN SUBMANIFOLDS

TOMMASO PACINI

Volume 215 No. 1

May 2004

DEFORMATIONS OF ASYMPTOTICALLY CONICAL SPECIAL LAGRANGIAN SUBMANIFOLDS

TOMMASO PACINI

McLean, 1998, studied the deformations of compact special Lagrangian submanifolds, showing in particular that they come in smooth moduli spaces whose dimension depends only on the topology of the submanifold. In this article we study the analogous problem for noncompact, “asymptotically conical” SL submanifolds, with respect to various “boundary conditions at infinity”; these issues are related to the problem of compactifying McLean’s moduli spaces.

1. Introduction

In the context of Riemannian geometry, much research is directed to the study of minimal submanifolds. One of the questions that arises naturally in this field is the following:

Question: Can a given minimal submanifold be “deformed” to get new examples? What parameters might be involved in these deformations?

The naive approach is to parametrize these deformations as the zero-set of a “mean curvature operator”, then study them using the implicit function theorem. However, this entails a good understanding of the Jacobi operator of the initial submanifold Σ , which in general is not possible.

The work of Oh and, more recently, of McLean ([Oh] and [ML]) shows that, in the “right” geometric context, the problem simplifies and sometimes becomes tractable. We are thus lead to the study of minimal Lagrangian submanifolds in Kaehler–Einstein (KE) ambient spaces, and of special Lagrangian (SL) submanifolds in Calabi–Yau (CY) manifolds: See Section 2 for definitions and examples.

In [ML], McLean shows that every “infinitesimal SL deformation” of a (smooth) compact SL submanifold is “integrable”; i.e., it generates actual SL deformations. A corollary of this is, in the compact case, that the set

$$\mathcal{M}(\Sigma) := \{\text{SL immersions } \Sigma \hookrightarrow N\}$$

has a natural smooth structure. The dimension of this “moduli space” turns out to be related only to the topology of Σ .

For at least two reasons, it is interesting to understand if similar results hold also for noncompact SL submanifolds.

The first (perhaps simplistic) reason is that it is well-known that our basic “model” CY manifold, \mathbb{C}^n with its standard structures, does not admit compact minimal submanifolds. In particular, it cannot contain compact SL submanifolds.

The second reason is more interesting. Moduli spaces of smooth, compact, SL submanifolds are not, in general, compact; it is however expected that they may be compactified by adding “boundaries” consisting of singular, compact, SLs. Linear algebra, based at any point p of a CY manifold N , shows that $T_p N$, with all its structures, is isomorphic to \mathbb{C}^n . The simplest type of singularity of a SL submanifold is given by isolated points p whose “tangent space” in $T_p N$ is isomorphic to a SL cone in \mathbb{C}^n . The submanifold itself should locally be the limit of smooth “asymptotically conical” (AC) SL submanifolds. Understanding the class of AC SL submanifolds should thus be a fundamental element both in compactification and in desingularization procedures (such as gluing).

McLean’s result relies on the standard Hodge theory for compact manifolds. In the AC case, the basic tool is provided by the Fredholm theory of the Laplace operator, due to Nirenberg, Walker, Cantor, McOwen, Lockhart, Melrose, Bartnik et al.; see in particular [NW], [MO], [B], [M1] and [M2].

The most complete set of results in this direction has been developed by R. Melrose. In Section 3 we thus introduce asymptotically conical and asymptotically cylindrical manifolds using the language of sc - and b -geometry, following [M1] and [M2].

In Section 4, still following [M1], we present various results regarding intrinsic analysis on these manifolds (referring elsewhere for detailed proofs whenever the statements or techniques seem sufficiently well-known). In Section 4.2, we then focus on certain harmonic 1-forms, corresponding to the “infinitesimal AC SL deformations”: We prove a representation theorem and calculate the dimensions of these spaces, with respect to the two most interesting “rates of decay at infinity”.

In Section 5 we discuss the “AC” condition for submanifolds, showing how to preserve it under normal deformations. This requires a study of the exponential map and Jacobi field estimates, which may be of independent interest in b -geometry.

Section 6 presents the main result: As in the compact case, every infinitesimal AC SL deformation is integrable; i.e., each such infinitesimal deformation generates a 1-parameter family of SL submanifolds which, by the results of Section 5, are also AC. The results from Section 4.2 then show that the dimension of the space of deformations depends both on the topology and on the analytic properties of (Σ, g) .

We also discuss the role played by the curvature of the ambient space in the construction of spaces of deformations having different speeds of decay (e.g., L^2 decay) at infinity.

Note: Throughout the article, we will assume $n > 2$.

2. Minimal Lagrangian and special Lagrangian submanifolds

Recall the following standard definitions:

Definition 1. An immersion $\phi : \Sigma \hookrightarrow N$ of a manifold Σ into a Riemannian manifold (N, g) is “minimal” if the corresponding mean curvature vector field vanishes: $H_\phi \equiv 0$.

An immersion $\phi : \Sigma \hookrightarrow N$ into a symplectic manifold (N, ω) is “isotropic” if $\phi^*\omega \equiv 0$; if $\dim \Sigma = n$ and $\dim N = 2n$, isotropic submanifolds are called “Lagrangian”.

Usually the normal bundle $(T\Sigma)^\perp$ of a submanifold $\Sigma \subseteq (N, g)$ is not an intrinsic object: It depends on the immersion and on the ambient space.

However, assume that the ambient manifold is a Kaehler manifold: (N^{2n}, J, g, ω) , where $\omega(X, Y) := g(JX, Y)$ is the Kaehler 2-form. We will use the following notation:

- $K_N := \Lambda^{n,0}(N)$ is the canonical bundle of (N, J) .
- ∇ and Ric are the Levi–Civita connection and the symmetric Ricci 2-tensor associated to (N, g) .
- $\rho(X, Y) := \text{Ric}(JX, Y)$ is the “Ricci 2-form”.

Definition 2. A Kaehler manifold (N, J, g, ω) is “Kaehler–Einstein” if $\text{Ric} = c \cdot g$, for some $c \in \mathbb{R}$; equivalently, if $\rho = c \cdot \omega$.

If Σ is Lagrangian and N is Kaehler, the (restriction of the) symplectic form gives an isomorphism

$$(T\Sigma)^\perp \simeq \Lambda^1(\Sigma), \quad V \simeq \nu := \omega(V, \cdot)|_\Sigma.$$

This shows that $(T\Sigma)^\perp$ is actually independent of the ambient manifold N .

We now want to introduce “Calabi–Yau” manifolds and “special Lagrangian” submanifolds.

Given a smooth section Ω of K_N , notice that $d\Omega = (\partial + \bar{\partial})\Omega = \bar{\partial}\Omega$, since $\partial\Omega \in \Lambda^{n+1,0}(N) \equiv 0$. Thus Ω is closed iff Ω is holomorphic.

Furthermore, $\nabla\Omega \equiv 0 \Rightarrow d\Omega \equiv 0$.

Definition 3. A (differentiable) manifold N^{2n} is of “Calabi–Yau (CY) type” if it admits a Riemannian metric g with holonomy $\text{Hol}(N, g) \leq SU(n; \mathbb{C})$; equivalently, if it admits a Kaehler structure (J, g, ω) and a (nonzero) section Ω of K_N such that $\nabla\Omega \equiv 0$.

A choice of (J, g, ω, Ω) defines a “Calabi–Yau structure”.

In particular, CY manifolds are KE manifolds with $c = 0$, i.e., $\text{Ric} \equiv 0$.

Since Ω is parallel, it depends only on some $\Omega[p]$ and can thus be normalized in such a way that $\|\Omega\| = 1$. We will always assume this. Such an Ω is then unique up to a multiplicative factor $\theta \in S^1$.

Given a CY manifold (N^{2n}, J, g, Ω) , we will let α, β denote the real and imaginary parts of Ω : $\Omega = \alpha + i\beta$. Notice that $d\alpha = d\beta = 0$.

Definition 4. Let (N, Ω) be a CY manifold. An oriented, immersed submanifold $\phi : \Sigma^n \hookrightarrow N^{2n}$ is special Lagrangian (SL) iff $\Omega|_\Sigma = \text{vol}_\Sigma$.

It is simple to prove (cfr. [HL]) that

$$\Omega|_\Sigma = \pm \text{vol}_\Sigma \Leftrightarrow \phi^*\omega = 0, \phi^*\beta = 0.$$

In other words, up to a change of orientation, Σ is SL iff $\phi^*\omega = 0 = \phi^*\beta$. SL submanifolds are thus Lagrangian.

Furthermore, CY manifolds and SL submanifolds are one of the basic examples of “calibrated geometry” (see [HL]): In particular, it can easily be shown that special Lagrangians are minimal.

The following well-known proposition states that, in CY manifolds, minimal Lagrangian and SL submanifolds are closely related:

Proposition 1. *Let (N, J, g, Ω) be a CY manifold and $\Sigma \subseteq N$ be an oriented Lagrangian submanifold. Then Σ is minimal iff it is SL with respect to the CY structure $\theta \cdot \Omega$, for some $\theta \in S^1$.*

Finding examples of SL submanifolds is not easy; one reason for this is that most examples of CY ambient spaces are provided by an abstract existence theorem due to Yau [Y]. However, a few CY manifolds are known explicitly, allowing for explicit constructions of SL submanifolds.

The basic example is $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ with its standard structures $J_{\text{std}}, g_{\text{std}}, \omega_{\text{std}}$ and $\Omega_{\text{std}} := dz^1 \wedge \dots \wedge dz^n$. Many examples of SL submanifolds in \mathbb{C}^n are now known: See, e.g., [HL], [J2], [H] and [G]. In particular, [H] shows how minimal Lagrangian submanifolds in $\mathbb{C}\mathbb{P}^{n-1}$ give rise to families of SL submanifolds in \mathbb{C}^n ; an appropriate choice of parametrization then shows that they are “asymptotically conical” in the sense of Section 3.

A second example of an explicit CY is the following.

Given any compact KE manifold $(N^{2n}, J_N, g_N, \omega_N)$ such that $\rho_N = c \cdot \omega_N$, $c > 0$, Calabi [C] proved the existence of a complete Kaehler Ricci-flat metric on the canonical bundle K_N of N . K_N also has a natural global holomorphic volume form. With respect to these structures, K_N is an “asymptotically conical” CY manifold. Again, it is possible to show (see [G] and [P]) that minimal Lagrangian submanifolds $\Sigma^n \subseteq N^{2n}$ generate “asymptotically conical” SL submanifolds $\hat{\Sigma}^{n+1} \subseteq K_N^{2n+2}$: Each $\hat{\Sigma}$ is a \mathbb{R} -line bundle over Σ .

3. Asymptotically conical and cylindrical manifolds

In this section we want to introduce the basic objects of Melrose’s “*b*-geometry”. See [M1] for further details.

Let X^n be an oriented compact n -dimensional manifold with boundary ∂X and $\Sigma := X \setminus \partial X$ be its interior. We will use the following notation:

- $2X := X \cup_{\partial X} (-X)$ (compact oriented manifold without boundary).
- $C^\infty(X) :=$ space of functions on X having a smooth extension on a neighborhood of $X \subseteq 2X$.
- $TX := T(2X)|_X, T^*X := T^*(2X)|_X$.
- If E is a bundle on X , $\Lambda^0(E)$ will denote the space of smooth sections and $\Lambda_c^0(E)$ those with compact support in Σ .

Definition 5. $x \in C^\infty(X)$ is a boundary-defining function if $x \geq 0, \partial X = \{x = 0\}, dx|_{\partial X} \neq 0$.

Notice that if x' is a second boundary-defining function, then there exists $a \in C^\infty(X)$, nowhere zero, such that $x' = ax$. Fixing a boundary-defining function x gives a local trivialization of X near ∂X : $X \simeq [0, 1] \times \partial X$, where the function x corresponds to the coordinate x on $[0, 1]$. Now set:

$$\begin{aligned} \nu_b &:= \{Z \in \Lambda^0(TX) : Z|_{\partial X} \in T(\partial X)\} \\ \nu_{sc} &:= x \cdot \nu_b = \{Z \in \Lambda^0(TX) : Z = xZ', Z' \in \nu_b\}. \end{aligned}$$

Clearly, ν_{sc} is independent of the choice of boundary-defining function x .

Taylor expansion shows that, locally near ∂X , $\nu_b = \langle x\partial_x, \partial_{y^i} \rangle_{C^\infty(X)}$ and $\nu_{sc} = \langle x^2\partial_x, x\partial_{y^i} \rangle_{C^\infty(X)}$, where y^i denote local coordinates on ∂X .

Thus we have the following chain of spaces of vector fields:

$$\Lambda_c^0(T\Sigma) \subseteq \nu_{sc} \subseteq \nu_b \subseteq \Lambda^0(TX) \subseteq \Lambda^0(T\Sigma).$$

It turns out that one can define vector bundles bTX and ${}^{sc}TX$ over X , of constant rank n , such that ${}^bTX|_\Sigma \simeq T\Sigma \simeq {}^{sc}TX|_\Sigma$ and possessing differentiable structures such that $\nu_b = \Lambda^0({}^bTX), \nu_{sc} = \Lambda^0({}^{sc}TX)$.

Let ${}^bT^*X, {}^{sc}T^*X$ be the corresponding dual bundles. Then, locally near ∂X , we may define:

$$\begin{aligned} \left\{ \frac{dx}{x}, dy^i \right\} &:= \text{the local basis of } {}^bT^*X \text{ dual to } \{x\partial_x, \partial_{y^i}\} \\ \left\{ \frac{dx}{x^2}, \frac{dy^i}{x} \right\} &:= \text{the local basis of } {}^{sc}T^*X \text{ dual to } \{x^2\partial_x, x\partial_{y^i}\}. \end{aligned}$$

With these definitions, locally $\nu_b^* := \Lambda^0({}^bT^*X) = \langle \frac{dx}{x}, dy^i \rangle_{C^\infty(X)}$ and $\nu_{sc}^* := \Lambda^0({}^{sc}T^*X) = \langle \frac{dx}{x^2}, \frac{dy^i}{x} \rangle_{C^\infty(X)}$. Thus:

$$\Lambda_c^0(T^*\Sigma) \subseteq \Lambda^0(T^*X) \subseteq \nu_b^* = x \cdot \nu_{sc}^* \subseteq \nu_{sc}^* \subseteq \Lambda^0(T^*\Sigma).$$

The alternating products of ${}^bT^*X$, ${}^{sc}T^*X$ lead to the spaces of b - and sc -differential forms, whose spaces of sections satisfy:

$$\Lambda_c^i(\Sigma) \subseteq \Lambda^i(X) \subseteq {}^b\Lambda^i(X) = x^i \cdot {}^{sc}\Lambda^i(X) \subseteq {}^{sc}\Lambda^i(X) \subseteq \Lambda^i(\Sigma).$$

In this case, we will use the same notation for both the bundles of i -forms and their sections. Finally, we set:

- $\text{Diff}_b^m(X) :=$ space of linear operators $P : C^\infty(X) \longrightarrow C^\infty(X)$ such that, locally,

$$P = \sum_{i+|I| \leq m} p_{i,I}(x, y)(x D_x)^i D_y^I$$

where $I = (i_1, \dots, i_{n-1})$ is a multi-index, $p_{i,I} \in C^\infty(X)$, $D_x = -\sqrt{-1} \partial_x$, $D_y^i = -\sqrt{-1} \partial_{y^i}$, $D_y^I = (D_{y^1})^{i_1} \dots (D_{y^{n-1}})^{i_{n-1}}$ and $|I| = i_1 + \dots + i_{n-1}$.

More generally, for any vector bundles E, F over X , we define:

- $\text{Diff}_b^m(E; F) :=$ space of linear operators $P : \Lambda^0(E) \longrightarrow \Lambda^0(F)$ such that, locally, all components of P have the above form.
- $\text{Diff}_b^m(E) := \text{Diff}_b^m(E; E)$.

It may be useful to emphasize here that we are working in the category of manifolds with boundary. Thus, when checking that $P \in \text{Diff}_b^m(E; F)$ near the boundary, one must choose trivializations which hold up to the boundary (clearly, the classes Diff_b^m , Diff_{sc}^m and Diff^m coincide on Σ).

Analogously, we may define:

- $\text{Diff}_{sc}^m(X) :=$ space of linear operators $P : C^\infty(X) \longrightarrow C^\infty(X)$ such that, locally and with $p_{i,I} \in C^\infty(X)$,

$$P = \sum_{i+|I| \leq m} p_{i,I}(x, y)(x^2 D_x)^i (x D_y)^I.$$

- As above, we may also define $\text{Diff}_{sc}^m(E; F)$, $\text{Diff}_{sc}^m(E)$.

Notice that, when one commutes operators, lower-order terms appear; eg, $x(x\partial_x) = (x\partial_x - I)x$. However, it is easy to prove the following:

Lemma 1. $x^\delta \cdot \text{Diff}_b^m = \text{Diff}_b^m \cdot x^\delta$; $x^\delta \cdot \text{Diff}_{sc}^m = \text{Diff}_{sc}^m \cdot x^\delta$.

Also: $x^m \cdot \text{Diff}_b^m \subseteq \text{Diff}_{sc}^m$ (the opposite inclusion is, in general, false).

Consider, for example, the exterior derivative operator d acting on i -forms. Clearly, $d \in \text{Diff}^1(\Lambda^i \Sigma; \Lambda^{i+1} \Sigma)$ and $d \in \text{Diff}^1(\Lambda^i(X); \Lambda^{i+1}(X))$.

It is easy to show that also the following are true:

- $d \in \text{Diff}_b^1({}^b\Lambda^i(X); {}^b\Lambda^{i+1}(X))$,
- $d \in \text{Diff}_{sc}^1({}^{sc}\Lambda^i(X); {}^{sc}\Lambda^{i+1}(X))$,
- $d \in x \cdot \text{Diff}_b^1({}^{sc}\Lambda^i(X); {}^{sc}\Lambda^{i+1}(X))$.

In particular, this shows that there exist $\tilde{d}, \bar{d} \in \text{Diff}_b^1({}^{sc}\Lambda^i X; {}^{sc}\Lambda^{i+1} X)$ such that $d = x \cdot \tilde{d} = \bar{d} \cdot x$.

We now move on to define some particularly interesting categories of metrics on Σ .

Definition 6. An “exact b -metric” on X is any metric g_b on Σ such that, for some boundary-defining function x , $g_b = \frac{dx^2}{x^2} + h$, where h denotes a symmetric 2-tensor on TX such that $h|_{\partial X}$ is positive-definite on $T(\partial X)$.

The pair (X, g_b) is an “asymptotically cylindrical manifold with link $(\partial X, h|_{\partial X})$ ”.

A “scattering metric” on X is any metric g_{sc} on Σ such that, for some boundary-defining function x , $g_{sc} = \frac{dx^2}{x^4} + \frac{h}{x^2}$ where h has the same properties as above.

The pair (X, g_{sc}) is a “scattering manifold with link $(\partial X, h|_{\partial X})$ ”.

More explicitly, an exact b -metric has the form

$$g = \frac{dx^2}{x^2} + a_{00}dx^2 + a_{0j}dx dy^j + a_{j0}dy^j dx + a_{ij}dy^i dy^j$$

where $a_{00}, a_{0j} = a_{j0}, a_{ij} = a_{ji} \in C^\infty(X)$ and a_{ij} has the following property (with respect to its Taylor expansion in $x = 0$):

$$a_{ij}(x, y) = g_{ij}(y) + x \tilde{a}_{ij}(x, y)$$

where $g_{ij} dy^i dy^j$ is a metric on ∂X . Notice that, if $g = \frac{dx^2}{x^2}(1 + xa) + h$ for some $a \in C^\infty(X)$, the change of variables $\xi := x + \frac{\gamma}{2}x^2$, where $\gamma := a(0, y)$, gives $g = \frac{d\xi^2}{\xi^2} + h'$; i.e., such a g is an exact b -metric.

The analogous fact does not hold for sc -metrics. Scattering metrics may thus be generalized as follows:

Definition 7. An “asymptotically conical metric” on X is any metric g_{ac} on Σ such that, for some boundary-defining function x and some $a \in C^\infty(X)$, $g_{ac} = \frac{dx^2}{x^4}(1 + xa) + \frac{h}{x^2}$, where h denotes a symmetric 2-tensor on TX such that $h|_{\partial X}$ is positive-definite on $T(\partial X)$.

The pair (X, g_{ac}) is an “asymptotically conical manifold with link $(\partial X, h|_{\partial X})$ ”.

Clearly, the class of ac -metrics contains the class of sc -metrics as a subset; notice also that, if g is an ac -metric with respect to x , x^2g is an exact b -metric. Fixing one of these three types of metrics implies restricting the choice of boundary-defining functions to those that allow the metric to be brought to the standard form seen in the definition.

Notice that $\nu_b = \{Z \in \Lambda^0(TX) : \|Z\|_{g_b} \text{ is uniformly bounded on } \Sigma\}$.

Analogously, $\nu_{sc} = \{Z \in \Lambda^0(TX) : \|Z\|_{g_{ac}} \text{ is uniformly bounded on } \Sigma\}$.

Lemma 2. *Let (X, g) be an asymptotically conical manifold, with curvature tensor R and Levi-Civita connection ∇ . Then:*

1. *With respect to the basis $\{\partial_x, \partial_{y^i}\}$:*

- $g \in \left(\begin{array}{cc} x^{-4} + C^\infty x^{-3} & C^\infty x^{-2} \\ C^\infty x^{-2} & h_{ij}|_{\partial X} x^{-2} + C^\infty x^{-1} \end{array} \right)$ where $h_{ij} = h(\partial_{y^i}, \partial_{y^j})$
and the notation “ $f \in C^\infty x^\alpha$ ” means: $f = \phi \cdot x^\alpha$, for some $\phi \in C^\infty(X)$.

- $g^{-1} \in \left(\begin{array}{cc} x^4 + C^\infty x^5 & C^\infty x^4 \\ C^\infty x^4 & h_{|\partial X}^{ij} x^2 + C^\infty x^3 \end{array} \right)$ where h^{ij} denotes the inverse matrix of h_{ij} .
- The Christoffel symbols have the following property: $\Gamma_{ij}^k \in C^\infty$; $\Gamma_{ij}^x \in C^\infty x$; $\Gamma_{ix}^k \in C^\infty x^{-1}$; $\Gamma_{ix}^x \in C^\infty x$; $\Gamma_{xx}^k \in C^\infty x^{-1}$; $\Gamma_{xx}^x \in C^\infty x^{-1}$.
- Let $\tilde{\Gamma}_{\alpha\beta}^\gamma$ denote the Christoffel symbols with respect to the dual basis $\{dx, dy^i\}$; i.e., $\nabla_{\partial_x} dx = \tilde{\Gamma}_{xx}^x dx + \tilde{\Gamma}_{xx}^i dy^i$, etc. Then

$$\tilde{\Gamma}_{ij}^k = (\nabla_{\partial_{y^i}} dy^j)(\partial_{y^k}) = \partial_{y^i}[dy^j(\partial_{y^k})] - dy^j(\nabla_{\partial_{y^i}} \partial_{y^k}) = -\Gamma_{ik}^j.$$

Likewise, $\tilde{\Gamma}_{ij}^x = -\Gamma_{ix}^j$, $\tilde{\Gamma}_{xj}^i = -\Gamma_{xk}^j$, etc.

2. $\|R\| \in C^\infty x^2$.
3. Let $X \in \nu_b$. Then $\nabla_X \in \text{Diff}_b^1({}^{sc}TX)$ and $\nabla_X \in \text{Diff}_b^1({}^{sc}T^*X)$.
Let $X \in \nu_{sc}$. Then $\nabla_X \in x \cdot \text{Diff}_b^1({}^{sc}TX)$ and $\nabla_X \in x \cdot \text{Diff}_b^1({}^{sc}T^*X)$.

Before moving onto the next section, it is probably worthwhile making a few final remarks.

Recall the following:

Definition 8. Let Σ^n be a connected, oriented manifold and M^{n-1} be a (possibly not connected) compact oriented manifold. Σ is a “manifold with ends (with link M)” iff Σ admits a decomposition $\Sigma = \Sigma_0 \cup_M \Sigma_\infty$, where:

- Σ_0 is a compact manifold with boundary $\partial(\Sigma_0) = M$
- Σ_∞ is diffeomorphic to $M \times [a, b)$

and we are identifying $\partial(\Sigma_0)$ with $M \times \{a\}$.

Remarks.

- 1) Let Σ be a manifold with ends. Then $X := \Sigma_0 \cup_M M \times [a, b)$ is a compact manifold with boundary and $\Sigma = X \setminus \partial X$. Conversely, given any compact manifold X with boundary, the local trivialization of X near ∂X given by a boundary-defining function shows that $\Sigma := X \setminus \partial X$ is a manifold with ends, with link $M := \partial X$.
- 2) Notice that any two intervals $[a, b), [a', b')$ are diffeomorphic. The ends of Σ can thus be parametrized in countless ways.

However, not all these diffeomorphisms extend smoothly up to the boundary. Different parametrizations of Σ_∞ may thus lead to different differentiable structures on the compactification X of Σ . When starting out with a manifold with ends, it is important to specify which compactification is being used.

- 3) Let (X, g_{ac}) be asymptotically conical and let Σ be the corresponding manifold with ends. The diffeomorphism $x \in (0, 1] \simeq r := \frac{1}{x} \in [1, \infty)$ induces a coordinate system on the ends of Σ : $\Sigma_\infty \simeq M \times [1, \infty)$.

Notice that $dr = -\frac{dx}{x^2}, \partial_r = -x^2\partial_x$ so that

$$\frac{dx^2}{x^4}(1 + xa) + \frac{h}{x^2} = dr^2 + r^2g_M + R$$

where $g_M := h|_{T(\partial X)}, R := \frac{a}{x^3}dx^2 + \frac{h-g_M}{x^2}$. This expression of the metric justifies the name ‘‘asymptotically conical’’. In particular,

$$\begin{aligned} R(\partial_r, \partial_r) &= R(x^2\partial_x, x^2\partial_x) = O(r^{-1}) \\ R(r^{-1}\partial_{y^i}, r^{-1}\partial_{y^j}) &= R(x\partial_{y^i}, x\partial_{y^j}) = O(r^{-1}) \\ R(r^{-1}\partial_{y^i}, \partial_r) &= -R(x\partial_{y^i}, x^2\partial_x) = O(r^{-1}) \end{aligned}$$

so $\|R\| = O(r^{-1})$, where $\|\cdot\|$ is calculated with respect to the metric $dr^2 + r^2g_M$.

The primary example of scattering manifold is $(\mathbb{R}^n, g_{\text{std}})$, compactified via stereographic projection to the half-sphere. See [M2] for details. It is simple to show that the standard symplectic structure ω_{std} on \mathbb{R}^{2n} is an element of ${}^{sc}\Lambda^2X$.

- 4) Let $(X, \frac{dx^2}{x^2} + h)$ be asymptotically cylindrical and let Σ be the corresponding manifold with ends. The diffeomorphism $x \in (0, 1] \simeq r := -\log x \in [0, \infty)$ induces a coordinate system $\Sigma_\infty \simeq M \times [0, \infty)$ with $dr = -\frac{dx}{x}, \partial_r = -x\partial_x$. Thus

$$\frac{dx^2}{x^2} + h = dr^2 + g_M + R$$

where $g_M := h|_{T(\partial X)}, R := h - g_M$. This expression of the metric justifies the name ‘‘asymptotically cylindrical’’.

- 5) It is interesting to compare these definitions of ‘‘asymptotically conical’’ and ‘‘asymptotically cylindrical’’ metrics with other definitions available in the literature.

All of them require some form of decay of the ‘‘perturbation term’’ R , defined in Remarks 3 and 4 above, and of its covariant derivatives; our definitions are however slightly stronger, as they require the tensor R (up to renormalization) to have a smooth extension up to ∂X . This allows for a much greater control of the metrics’ consequent properties.

Most of the properties relevant to this article would, however, continue to hold for weaker definitions (see, e.g., [MO]).

4. The Laplace operator on asymptotically conical manifolds

For any oriented Riemannian manifold (Σ^n, g) , we will use the following standard notation:

- $*_g : \Lambda^i \Sigma \longrightarrow \Lambda^{n-i} \Sigma$ denotes the usual Hodge star operator.
- $d_g^* : \Lambda^i \Sigma \longrightarrow \Lambda^{i-1} \Sigma$ denotes the formal adjoint of d .
- $\Lambda^* \Sigma := \bigoplus \Lambda^i \Sigma$.

As usual, $d_g^* = (-1)^{n(i+1)+1} *_g d *_g$, so $d_g^* \in \text{Diff}^1(\Lambda^i \Sigma; \Lambda^{i-1} \Sigma)$.

Let X denote an oriented compact manifold with boundary and $\Sigma := X \setminus \partial X$. For any fixed b -metric on X , with volume form vol_b , we may define:

- $L_b^2(X) := \{f \in L_{\text{loc}}^2(\Sigma) \text{ such that } \|f\|_b := (\int_{\Sigma} f^2 \text{vol}_b)^{1/2} < \infty\}$.
- $H_b^m(X) := \{f \in L_{\text{loc}}^2(\Sigma) : Pf \in L_b^2(X), \forall P \in \text{Diff}_b^m(X)\}$ ($m \geq 0$).
- $\|f\|_{m,b} := (\sum_{i+|J| \leq m} \|(x\partial_x)^i \partial_y^J f\|_b^2)^{1/2}$ defines a norm on $H_b^m(X)$.

More generally, if E is a metric vector bundle over X , we may define:

- $L_b^2(E) := \{f \in L_{\text{loc}}^2(E|_{\Sigma}) \text{ such that } \int_{\Sigma} |f|^2 \text{vol}_b < \infty\}$.
- $H_b^m(E) := \{f \in L_{\text{loc}}^2(E|_{\Sigma}) : Pf \in L_b^2(E), \forall P \in \text{Diff}_b^m(E)\}$ ($m \geq 0$).

Finally, for $\delta \in \mathbb{R}$, we define “weighted Sobolev spaces” as follows:

- $x^\delta H_b^m := \{f \in L_{\text{loc}}^2 : f = x^\delta u, u \in H_b^m\}$.
- $\|f\|_{\delta,m,b} := \|x^{-\delta} f\|_{m,b}$ defines a norm on $x^\delta H_b^m$, isometric to H_b^m .

Analogously, if we endow X with a fixed ac -metric, we may define L_{sc}^2 , H_{sc}^m and $x^\delta H_{sc}^m$, spaces of functions and sections, using the induced volume form vol_{ac} and operators $P \in \text{Diff}_{sc}^m$.

Since $\Lambda^i(X)|_{\Sigma} = {}^b\Lambda^i(X)|_{\Sigma} = {}^{sc}\Lambda^i(X)|_{\Sigma} = \Lambda^i(\Sigma)$, when E is one of these bundles one can use the notation $H_b^k(\Lambda^i)$, $H_{sc}^k(\Lambda^i)$. In this case, however, it is important to specify which metric is being used on the bundle.

All the above are Hilbert spaces and contain the space $\Lambda_c^0(E|_{\Sigma})$ as a dense subset. Both ac - and b -metrics have the right properties (bounded curvature and positive injectivity radius) for the standard Sobolev immersion theorems to hold (see [A]). For example, if $f \in x^\delta H_b^k(X)$ ($k > \frac{n}{2}$), then $x^{-\delta} f$ is continuous and bounded, so $f = O(x^\delta)$.

Notice that, for any constant $c \in \mathbb{R}$,

$$c \in x^\delta L_b^2(X) \Leftrightarrow x^{-\delta} c \in L_b^2 \Leftrightarrow \int_0^1 x^{-2\delta-1} dx < \infty \Leftrightarrow \delta < 0.$$

Notice also that, if $g_b = x^2 g_{ac}$, then $\text{vol}_b = x^n \text{vol}_{ac}$, so that $L_{sc}^2 = x^{\frac{n}{2}} L_b^2$.

It is a simple consequence of the definitions that any $P \in \text{Diff}_b^m$ has a continuous extension $P : H_b^{k+m} \rightarrow H_b^k$. The analogous fact holds for $P \in \text{Diff}_{sc}^m$. The fact that $\text{Diff}_b^m x^\delta = x^\delta \text{Diff}_b^m$ shows however that, for b -metrics (and analogously for ac -metrics), a continuous extension exists also between weighted Sobolev spaces:

Lemma 3. *Let $P \in \text{Diff}_b^m(E)$. Then, $\forall \delta \in \mathbb{R}, \forall k \geq 0$, P has a continuous extension $P : x^\delta H_b^{k+m}(E) \rightarrow x^\delta H_b^k(E)$.*

Analogously, let $P \in \text{Diff}_{sc}^m(E)$. Then $\forall \delta \in \mathbb{R}, \forall k \geq 0$, P has a continuous extension $P : x^\delta H_{sc}^{k+m}(E) \rightarrow x^\delta H_{sc}^k(E)$.

Proof. Consider, for example, $P \in \text{Diff}_b^m$. Then

$$\|P f\|_{\delta,k,b} = \|x^{-\delta} P f\|_{k,b} = \|\tilde{P}(x^{-\delta} f)\|_{k,b} \leq C \|x^{-\delta} f\|_{k+m,b} = C \|f\|_{\delta,k+m,b}$$

for some $\tilde{P} \in \text{Diff}_b^m$, $C > 0$. □

Analogously to the compact case, there is a well-developed theory of b - and sc -elliptic operators. We will only need the former.

Definition 9. Let $P \in \text{Diff}_b^m(X)$, $P = \sum_{i+|I| \leq m} p_{i,I}(x, y)(x D_x)^i D_y^I$.
 P is b -elliptic iff

$$\sigma_m(P)(\xi, \eta) := \sum_{i+|I|=m} p_{i,I}(x, y) \xi^i \eta^I \neq 0, \quad \forall (x, y) \in X, \forall (\xi, \eta) \neq 0.$$

It turns out that the properties of a b -elliptic operator P are strongly related to $P|_{\partial X}$. We need the following definition:

Definition 10. Let $P \in \text{Diff}_b^m(X)$ be b -elliptic.

For $\lambda \in \mathbb{C}$, let $\hat{P}(\lambda) : C^\infty(\partial X) \rightarrow C^\infty(\partial X)$ denote the operator locally given by $\hat{P}(\lambda) := \sum_{i+|I| \leq m} p_{i,I}(0, y) \lambda^i D_y^I$. This is called the “indicial operator” associated to P .

Let $\text{spec}(P) := \{\lambda \in \mathbb{C} : \hat{P}(\lambda) \text{ is not invertible on } C^\infty(\partial X)\}$.

The above definitions generalize to $P \in \text{Diff}_b^m(E; F)$.

In [M1], Melrose constructs a class of pseudo-differential operators which lead to the following result:

Theorem 1. Let $P \in \text{Diff}_b^m(E; F)$ be b -elliptic. Then:

1. $P : x^\delta H_b^{k+m}(E) \rightarrow x^\delta H_b^k(F)$ is Fredholm iff $\delta \notin -\text{Im spec}(P)$, where Im denotes the imaginary part of a complex number.
2. $u \in x^\delta H_b^{k+m}(E)$, $Pu \in x^\delta H_b^{k+1}(F) \Rightarrow u \in x^\delta H_b^{k+m+1}(E)$.

The set $-\text{Im spec}(P) \subseteq \mathbb{R}$ turns out to be discrete. We call these the “exceptional weights (of P)”. Given a non-exceptional weight $\delta \in \mathbb{R}$, $[\delta]_P$ will denote the connected component of $\mathbb{R} \setminus -\text{Im spec}(P)$ containing δ .

Having laid out the relevant foundations, we may now focus on the specific operators that will be important further on in this paper.

Let us thus fix an asymptotically conical manifold (Σ^n, g) . Let (X, g) be its “scattering compactification”.

Since ${}^{sc}\Lambda^* X$ is generated by forms of length 1, the definition of $*_g$ shows that $*_g({}^{sc}\Lambda^i X) \subseteq {}^{sc}\Lambda^{n-i} X$. Thus d_g^* restricts to an operator in $\text{Diff}_{sc}^1({}^{sc}\Lambda^i X; {}^{sc}\Lambda^{i-1} X)$. Actually, if we set $\tilde{\delta} := (-1)^{n(i+1)+1} *_g \tilde{d} *_g$ and $\bar{\delta} := (-1)^{n(i+1)+1} *_g \bar{d} *_g$, we find that

$$d_g^* = x \cdot \tilde{\delta} = \bar{\delta} \cdot x \in x \cdot \text{Diff}_b^1({}^{sc}\Lambda^i X; {}^{sc}\Lambda^{i-1} X) \subseteq \text{Diff}_{sc}^1.$$

We now define:

- $D_g := d \oplus d_g^* \in \text{Diff}_{sc}^1({}^{sc}\Lambda^* X; {}^{sc}\Lambda^* X)$.
- $\Delta_g := D_g \circ D_g = d d_g^* + d_g^* d \in \text{Diff}_{sc}^2({}^{sc}\Lambda^i X; {}^{sc}\Lambda^i X)$.

Clearly, D_g, Δ_g are the restrictions to ${}^{sc}\Lambda^* X$ of the usual operators defined on $\Lambda^* \Sigma$.

Lemma 4 ([M2]). *Let (Σ, g) be asymptotically conical. Then:*

1. $\Delta_g = x \cdot (\tilde{d} \oplus \tilde{\delta}) \circ (\bar{d} \oplus \bar{\delta}) \cdot x \in x^2 \cdot \text{Diff}_b^2(\text{sc}\Lambda^i X; \text{sc}\Lambda^i X)$.
2. *On functions, one finds the following expressions:*
 - $\Delta_g = x^2 \Delta_M + \sqrt{-1}x(n-1)x^2 D_x + (x^2 D_x)(x^2 D_x) + x^3 \text{Diff}_b^2$.
 - $(\tilde{d} \oplus \tilde{\delta}) \circ (\bar{d} \oplus \bar{\delta}) = \Delta_M + (xD_x)(xD_x) + \sqrt{-1}n(xD_x) + 1 - n + x \text{Diff}_b^2$.

In particular, the operator $(\tilde{d} \oplus \tilde{\delta}) \circ (\bar{d} \oplus \bar{\delta})$ is b -elliptic: For example, on $C^\infty(X)$, the previous lemma shows that

$$\sigma_2((\tilde{d} \oplus \tilde{\delta}) \circ (\bar{d} \oplus \bar{\delta}))(\xi, \eta) = \xi^2 + g_M^{ij} \eta^i \eta^j = \xi^2 + |\eta|^2.$$

We now introduce one last piece of notation: Given an asymptotically conical manifold (Σ, g) , we let $H_b^k(\Lambda^i, \text{sc})$ denote the Sobolev spaces defined above, with respect to the following choices:

- On $\Lambda^i \Sigma$ we use the metric induced by the a -metric g ;
- the volume form on Σ is the one given by the b -metric $x^2 g$.

The following result shows that these are the correct spaces in which to study Δ_g, D_g :

Corollary 1. *Let (Σ, g) be an asymptotically conical manifold. Then:*

1. $\Delta_g : x^\delta H_b^{k+2}(\Lambda^i, \text{sc}) \longrightarrow x^{\delta+2} H_b^k(\Lambda^i, \text{sc})$ ($k \geq 0$) is Fredholm, except for a discrete set of “exceptional weights”.
2. $D_g : x^\delta H_b^{k+1}(\Lambda^*, \text{sc}) \longrightarrow x^{\delta+1} H_b^k(\Lambda^*, \text{sc})$ ($k \geq 1$) is Fredholm, except for a discrete set of “exceptional weights”.
3. All elements in $\text{Ker}(\Delta_g), \text{Ker}(D_g)$ are smooth.

Proof. Consider $\Delta_g = x \cdot (\tilde{d} \oplus \tilde{\delta}) \circ (\bar{d} \oplus \bar{\delta}) \cdot x :$

$$\begin{aligned} x^\delta H_b^{k+2}(\Lambda^*, \text{sc}) &\longrightarrow x^{\delta+1} H_b^{k+2}(\Lambda^*, \text{sc}) \longrightarrow \\ &x^{\delta+1} H_b^k(\Lambda^*, \text{sc}) \longrightarrow x^{\delta+2} H_b^k(\Lambda^*, \text{sc}). \end{aligned}$$

Since multiplication by x is an isometry, by the previous theorem we see that Δ_g is Fredholm, except for a discrete set of weights.

Since $D_g = x(\tilde{d} \oplus \tilde{\delta}) = (\bar{d} \oplus \bar{\delta})x$, this shows that $D_g : x^\delta H_b^{k+2} \longrightarrow x^{\delta+1} H_b^{k+1}$ satisfies $\dim \text{Ker}(D_g) < \infty$ and that $D_g : x^{\delta+1} H_b^{k+1} \longrightarrow x^{\delta+2} H_b^k$ has closed image and finite-dimensional cokernel. Thus D_g is Fredholm if $\delta, \delta - 1$ are non-exceptional for Δ_g .

Finally, to prove (3) notice that, by the above theorem, $\text{Ker}(\Delta_g) \subseteq \bigcap_{m \geq 2} x^\delta H_b^m$. Applying the standard Sobolev immersion theorems to H_b^m , we get smoothness for the elements of $\text{Ker}(\Delta_g)$; (3) then follows from $\text{Ker}(D_g) \subseteq \text{Ker}(\Delta_g)$. □

We may thus define:

$$\begin{aligned} \mathcal{H}_\delta^i(\Sigma) &:= \text{Ker}(\Delta_g), \quad \text{where } \Delta_g : x^\delta H_b^{k+2}(\Lambda^i, sc) \longrightarrow x^{\delta+2} H_b^k(\Lambda^i, sc), \\ \mathcal{K}_\delta^i(\Sigma) &:= \text{Ker}(D_g), \quad \text{where } D_g : x^\delta H_b^{k+1}(\Lambda^i, sc) \longrightarrow x^{\delta+1} H_b^k(\Lambda^*, sc). \end{aligned}$$

From the corollary, we see that these spaces are independent of k . When Δ_g and D_g are Fredholm, they have Fredholm indices $i_{\Delta_g, \delta}$ and $i_{D_g, \delta}$. The following lemma investigates the dependence of these spaces and indices on δ :

Lemma 5. *Let (Σ, g) be an asymptotically conical manifold. Then:*

1. $i_{\Delta_g, \delta}$ and $\mathcal{H}_\delta^i(\Sigma)$ depend only on $[\delta]_\Delta$.
2. $i_{D_g, \delta}$ and $\mathcal{K}_\delta^i(\Sigma)$ depend only on $[\delta]_D$.

The proof is based on the invariance of the Fredholm index of curves of Fredholm operators.

4.1. Δ_g on functions. When studying Δ_g on functions, one gets extra information thanks to the elliptic maximum principle.

Proposition 2. *Let (Σ, g) be an asymptotically conical manifold. Consider $\Delta_g : x^\delta H_b^{k+2}(\Sigma) \longrightarrow x^{\delta+2} H_b^k(\Sigma)$, for δ non-exceptional. Then:*

1. $\delta > 0 \Rightarrow \mathcal{H}_\delta^0 = 0$. Thus:
 - Δ_g is injective.
 - $d : x^\delta H_b^{k+2}(\Sigma) \longrightarrow x^{\delta+1} H_b^{k+1}(\Lambda^1, sc)$ is injective.
2. $\delta < n - 2 \Rightarrow \text{Coker}(\Delta_g) = 0$. Thus:
 - Δ_g is surjective.
 - $d_g^* : x^\delta H_b^{k+1}(\Lambda^1, sc) \longrightarrow x^{\delta+1} H_b^k(\Sigma)$ is surjective for $\delta < n - 1$ non-exceptional.
3. $\text{Dim } \mathcal{H}_\delta^0(\Sigma)$ is independent of the particular choice of asymptotically conical metric on Σ (with respect to a fixed link (M, g_M)).

The proof is based on the elliptic maximum principle and the self-adjointness of Δ_g on $C_c^\infty(\Sigma)$. Cf. [CZ] for similar statements and techniques. Notice that, in particular, Δ_g is an isomorphism for $0 < \delta < n - 2$.

The above proposition has an interesting consequence:

Lemma 6 (Gluing principle for harmonic functions). *Let (Σ, g) be asymptotically conical. For $\delta < n - 2$, let f be a smooth function in $x^\delta H_b^k(\Sigma)$ such that $\Delta_g f|_{\Sigma_\infty^i} = 0$.*

Then there exists a unique $F \in x^\delta H_b^k(\Sigma)$ such that $|F(x) - f(x)| \rightarrow 0$ (as $x \rightarrow 0$) and $\Delta_g F \equiv 0$.

Proof. Since $\Delta_g f$ is smooth and has compact support, it is clear that, $\forall \eta \in \mathbb{R}, \forall s \geq 0, \Delta_g f \in x^{\eta+2} H_b^s(\Sigma)$.

Fix any $\eta : \max\{0, \delta\} < \eta < n - 2$. Since $\Delta_g : x^\eta H_b^{s+2} \rightarrow x^{\eta+2} H_b^s$ is surjective, there exists $\tilde{f} \in x^\eta H_b^{s+2} : \Delta_g \tilde{f} = \Delta_g f$. In particular, $\tilde{f} \in x^\delta H_b^{s+2}$. Choosing s large enough, we get $|\tilde{f}| = O(x^\eta)$, so $\tilde{f} \rightarrow 0$.

It is now enough to set $F := f - \tilde{f}$. Uniqueness is a consequence of the elliptic maximum principle. □

In other words, any collection of harmonic functions (e.g., constants) on Σ_∞^i may, up to a slight perturbation, be “glued together” to get a harmonic function on Σ with the same asymptotic behaviour.

Going a step further, [M1] provides a complete asymptotic expansion of the harmonic functions on any asymptotically conical (Σ, g) in terms of the metric on the link. In particular, Christiansen and Zworski [CZ] show how this can be used to relate the harmonic functions on (Σ, g) to the eigenfunctions on the link. We may summarize their results as follows:

- Recall that, on $(M^{n-1} \times (1, \infty), g := dr^2 + r^2 g_M)$,

$$\Delta_g = \frac{\Delta_{g_M}}{r^2} - \frac{n-1}{r} \partial_r - \partial_r^2.$$

Let $f \in C^\infty(M^{n-1})$ be any eigenfunction of Δ_{g_M} , relative to any eigenvalue λ_i . Then it is easy to check that the function $f r^{a_i}$ is harmonic on $M \times (1, \infty)$, where $a_i := \frac{2-n+\sqrt{(2-n)^2+4\lambda_i}}{2}$.

Notice that, when $(M, g_M) = (S^{n-1}, g_{std})$, these harmonic functions on $\mathbb{R}^n \setminus \{x : |x| \leq 1\}$ are exactly the homogeneous harmonic polynomials.

- If (Σ, g) has link (M, g_M) , these functions are “asymptotic models” for the harmonic functions (with polynomial growth) on Σ , in the following sense: Given any $f \in \mathcal{H}_\delta^0(\Sigma)$, on each end f converges to a linear combination of the above.
- Conversely, assigning an “asymptotic model” to each end determines a unique $f \in \mathcal{H}_\delta^0$ with that asymptotic behaviour.

This allows us to express $\dim \mathcal{H}_\delta^0$ in terms of the number of ends and the dimension of the space of eigenfunctions on the components (M_i, g_{M_i}) of the link (M, g_M) .

In particular, we will be interested in the following conclusion:

Definition 11. We say that a harmonic function f has “strictly sub-linear growth” if it is asymptotic to a linear combination of models $f_i r^{a_i}$ with $0 \leq a_i < 1$ and at least one $a_i > 0$. In other words, if there exist $\delta > -1, \epsilon > 0$ such that $f \in \mathcal{H}_\delta^0(\Sigma), f \notin \mathcal{H}_{-\epsilon}^0(\Sigma)$.

Corollary 2. *Let (Σ, g) be asymptotically conical. Then there exist harmonic functions with strictly sub-linear growth iff at least one end has an eigenvalue in the interval $(0, n - 1)$.*

Proof. Notice that $0 < a_i < 1 \Leftrightarrow 0 < \lambda_i < n-1$. We may use any asymptotic model of the type $f_i r^{a_i} + \sum_{j \neq i} c_j$ where $c_j \in \mathbb{R}$. \square

4.2. D_g on 1-forms. We are now going to calculate $\dim \mathcal{K}_\delta^1 = \dim \text{Ker}(D_g)$, where D_g is acting on weighted 1-forms, in two cases of particular interest.

In this section, i will denote the canonical map $i : H_c^1(\Sigma) \rightarrow H^1(\Sigma)$.

We start with the following:

Lemma 7. *Let (Σ, g) be asymptotically conical. For any $\epsilon > 0$, there exists an injective map*

$$q : H^1(\Sigma) \rightarrow \mathcal{K}_{1-\epsilon}^1$$

which is independent of ϵ . With respect to any $\delta \in [1, n-1)$, it has the following property:

$$q([\alpha]) \in \mathcal{K}_\delta^1 \Leftrightarrow [\alpha] \in i(H_c^1).$$

Proof. Begin by considering any $[\alpha] \in H^1(\Sigma)$. We first show that it has a representative in $\mathcal{K}_{1-\epsilon}^1$.

Recall the isomorphism

$$\begin{aligned} H^1(M) &\simeq H^1(M \times [1, \infty)) \\ [\beta_M] &\mapsto [\beta_\infty] \end{aligned}$$

where β_∞ is defined by $\beta_\infty[\omega, r] := \beta_M[\omega]$. We call such a form “translation-invariant”.

Applying this to $[\alpha|_{\Sigma_\infty}] \in H^1(M \times [1, \infty))$ gives a 1-form α_M on M and a translation-invariant 1-form α_∞ on $M \times [1, \infty)$ such that $[\alpha_\infty] = [\alpha|_{\Sigma_\infty}]$. In other words,

$$\alpha|_{\Sigma_\infty} = \alpha_\infty + d(f_\infty)$$

for some $f_\infty \in C^\infty(\Sigma_\infty)$ and some translation-invariant α_∞ .

We may now extend f_∞ to a function $f \in C^\infty(\Sigma)$; this gives an extension of α_∞ to a global 1-form $\tilde{\alpha}$ on Σ , defined as $\tilde{\alpha} := \alpha - df$.

The form $\tilde{\alpha}$ has the following properties:

- $d\tilde{\alpha} = 0$: This is clear.
- $[\tilde{\alpha}] = [\alpha]$: This is clear.
- $\tilde{\alpha} \in x^\eta H_b^1(\Lambda^1, sc), \forall \eta < 1$: Locally near ∂X , $\tilde{\alpha} = b_i(y) dy^i = x b_i(y) \frac{dy^i}{x}$. Since $x b_i(y) \in C^\infty(X)$, this shows that $\tilde{\alpha} \in {}^{sc}\Lambda^1(X)$.

By definition, $\tilde{\alpha} \in x^\eta H_b^1 \Leftrightarrow \|x^{-\eta} \tilde{\alpha}\|_{1,b} < \infty$. Recall:

$$\begin{aligned} \|x^{-\eta} \tilde{\alpha}\|_{1,b} &= \|x^{-\eta} \tilde{\alpha}\|_b + \sum_i \left\| x \partial_x (x^{-\eta+1} b_i(y)) \frac{dy^i}{x} \right\|_b \\ &\quad + \sum_{ij} \left\| \partial_{y^j} (x^{-\eta+1} b_i(y)) \frac{dy^i}{x} \right\|_b. \end{aligned}$$

It is now enough to examine these terms one by one, to get the result. For example:

$$\begin{aligned} \|x^{-\eta}\tilde{\alpha}\|_b^2 &= \int_{\Sigma} \|x^{-\eta}\tilde{\alpha}\|_{ac}^2 \text{vol}_b \leq c_1 + c_2 \int_0^1 \left\| x^{-\eta+1} \frac{dy^i}{x} \right\|_{ac}^2 x^{-1} dx \\ &\leq c_1 + c_3 \int_0^1 x^{-2\eta+1} dx \end{aligned}$$

so $\|x^{-\eta}\tilde{\alpha}\|_b < \infty$ for $\eta < 1$.

- For $\delta \geq 1$, $\tilde{\alpha} \in x^\delta H_b^1(\Lambda^1, sc) \Leftrightarrow \tilde{\alpha} \in \Lambda_c^1(\Sigma)$: This should be clear from the above.

Consider now, for $\eta < n - 1$, the sequence:

$$x^{\eta-1} H_b^2(\Sigma) \xrightarrow{d} x^\eta H_b^1 \xrightarrow{d_g^*} x^{\eta+1} L_b^2(\Sigma).$$

By surjectivity, setting $\eta = 1 - \epsilon$, there exists a function $k \in x^{-\epsilon} H_b^2 : \Delta_g k = d_g^* \tilde{\alpha}$. Since Δ is not injective, the choice of k is not canonical; however, k is unique up to $\text{Ker}(\Delta)$ and this is independent of ϵ , so k also is. Notice now that $\tilde{\alpha} - dk \in \mathcal{K}_{1-\epsilon}^1$ and that $[\tilde{\alpha} - dk] = [\alpha]$. When $\tilde{\alpha} \in \Lambda_c^1(\Sigma)$, we can act in a similar way, choosing however k with respect to $\eta \in [1, n - 1)$.

The above process thus shows how to build the required linear map q , defined by $q([\alpha]) := \tilde{\alpha} - dk$. To show that q is injective, let $[\alpha_1], \dots, [\alpha_p]$ be a basis of $H^1(\Sigma)$. Assume $\exists \lambda_i \in \mathbb{R} : \sum \lambda_i (\tilde{\alpha}_i - dk_i) = 0$. Then

$$\sum \lambda_i [\alpha_i] = \sum \lambda_i [\tilde{\alpha}_i - dk_i] = 0$$

so $\lambda_i = 0$. □

For all $\eta \in \mathbb{R}$, let $E_\eta := \{\text{exact 1-forms in } \mathcal{K}_\eta^1\}$.

Lemma 8.

1. For $\eta < 1$, $E_\eta = d(\mathcal{H}_{\eta-1}^0)$. In particular, $\dim E_\eta = \dim \mathcal{H}_{\eta-1}^0 - 1$.
2. For $\epsilon > 0$ (sufficiently small) and $1 - \epsilon < \delta < n - 1$, $E_\delta = d(\mathcal{H}_{-\epsilon}^0)$. In particular, $\dim E_\delta = s - 1$, where s is the number of ends of Σ .

Proof. It is clear that, if $f \in \mathcal{H}_{\eta-1}^0$, then $df \in E_\eta$. Conversely, assume $df \in E_\eta$. Then $\Delta_g f \equiv 0$, so the asymptotic expansion of f shows that $f \in \mathcal{H}_{\eta-1}^0$.

The proof of (2) is similar: By the “gluing principle”, if $f \in \mathcal{H}_{-\epsilon}^0$, then $f = c + \tilde{f}$, where c is a function constant on each end and $\tilde{f} \in x^{n-2-\tilde{\epsilon}} H_b^s$. Thus $df \in E_{n-1-\tilde{\epsilon}} \subseteq E_\delta$. Conversely, if $df \in E_\delta$, then $f \in d(\mathcal{H}_{-\epsilon}^0)$.

Notice that, in calculating $\dim E_\eta (\forall \eta \in \mathbb{R})$, one must take into account the fact that any $c \in \mathbb{R}$ is a harmonic function but $d(c) \equiv 0$, so it does not contribute to E_η . □

We now have a good picture of the structure of $\mathcal{K}_\delta^1(\Sigma)$:

Proposition 3.

1. For $\eta < 1$, $\mathcal{K}_\eta^1 = E_\eta \oplus q(H^1(\Sigma))$.
2. For $\delta \in [1, n - 1)$, $\mathcal{K}_\delta^1 = E_\delta \oplus q(i(H_c^1(\Sigma)))$.

Proof. It is clear from the definitions of q and E_η that $E_\eta \cap q(H^1(\Sigma)) = \{0\}$.

Let H_η be any subspace of \mathcal{K}_η^1 containing $q(H^1(\Sigma))$, such that $\mathcal{K}_\eta^1 = E_\eta \oplus H_\eta$.

The map $H_\eta \rightarrow H^1(\Sigma)$, $\alpha \mapsto [\alpha]$ is clearly injective. Thus $\dim H_\eta \leq \dim H^1(\Sigma)$. Since $\dim H^1(\Sigma) = \dim q(H^1(\Sigma))$, we get $H_\eta \leq q(H^1(\Sigma))$, hence the equality.

To prove (2), notice that

$$E_\delta \oplus q(i(H_c^1)) \leq \mathcal{K}_\delta^1 \leq E_{1-\epsilon} \oplus q(H^1).$$

Any $\alpha \in \mathcal{K}_\delta^1$ may be written as $\alpha = df + \alpha'$ where $df \in E_{1-\epsilon}$, $\alpha' = q([\alpha]) \in \mathcal{K}_{1-\epsilon}^1$. Recall that, for small $\epsilon \geq 0$, $E_{1-\epsilon} = E_\delta$. Thus $df \in E_\delta$, which implies that $\alpha' \in \mathcal{K}_\delta^1$ and thus $[\alpha] \in i(H_c^1)$. This proves that $\mathcal{K}_\delta^1 \leq E_\delta \oplus q(i(H_c^1))$. \square

We will be particularly interested in the following conclusion:

Corollary 3. *Let (Σ, g) be asymptotically conical. Then:*

1. $\dim \mathcal{K}_\epsilon^1 = \dim H^1(\Sigma) + \dim \mathcal{H}_{-1+\epsilon}^0 - 1$. *This is the dimension of the space of all 1-forms $\alpha \in \text{Ker}(D_g)$ such that $\|\alpha\|_g$ decays.*
2. $\forall \delta \in (1, n - 1)$, $\dim \mathcal{K}_\delta^1 = \dim H_c^1(\Sigma)$. *In particular, this shows that the space $\text{Ker}(D_g|_{L^2_{sc}(\Lambda^1)}) = \mathcal{K}_{\frac{1}{2}}^1$ of closed and co-closed 1-forms in L^2 has dimension $H_c^1(\Sigma)$.*

Proof. (1) follows directly from the above.

To prove (2), notice that the previous proposition shows that

$$\dim \mathcal{K}_\delta^1 = \dim i(H_c^1) + s - 1$$

where s is the number of ends of Σ .

Let M be the link of $\Sigma = \Sigma_0 \cup_M \Sigma_\infty$. The long exact sequence

$$\begin{array}{ccccccccc} H_c^0(\Sigma_0) & \longrightarrow & H^0(\Sigma_0) & \longrightarrow & H^0(M) & \longrightarrow & H_c^1(\Sigma_0) & \xrightarrow{i} & H^1(\Sigma_0) \\ \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ 0 & & \mathbb{R} & & \mathbb{R}^s & & H_c^1(\Sigma) & & H^1(\Sigma) \end{array}$$

now shows that $H_c^1 \simeq i(H_c^1) \oplus \mathbb{R}^{s-1}$. \square

5. Deformations of asymptotically conical submanifolds

Having developed the theory of asymptotically conical manifolds, we now turn to studying submanifolds. The goal of this section is to present a notion of immersion in the category of manifolds with boundary and to study the stability of this condition under deformations. For the purposes of Section 6,

the main result is Corollary 4, which shows that asymptotically conical submanifolds remain asymptotically conical under “deformations which keep infinity fixed”.

X, X' will denote compact manifolds with boundary.

$\Sigma := X \setminus \partial X, N := X' \setminus \partial X'$ will be their interior.

Definition 12. An immersion $\phi : X \rightarrow X'$ is a “ b -immersion” if:

1. $\phi(\partial X) \subseteq \partial X'$.
2. $\forall p \in \partial X$, consider $d\phi[p] : T_p X \rightarrow T_{\phi(p)} X'$ and, using Condition (1), consider the quotient map

$$d\phi[p] : T_p X / T_p \partial X \rightarrow T_{\phi(p)} X' / T_{\phi(p)} \partial X'$$

The condition is that this quotient map be injective.

Let $\phi : X \rightarrow X'$ be a b -immersion. Let x be a boundary-defining function for X' , $p \in \partial X$ and $V \in T_p X : V \notin T_p \partial X$. Then, by hypothesis,

$$d(x \circ \phi)[p](V) = dx[\phi(p)]d\phi[p](V) \neq 0$$

so $x \circ \phi$ is a boundary-defining function for X . From now on, we will often identify X with its image in X' , omitting ϕ from the notation.

Clearly, ${}^bTX \leq {}^bTX'|_X$ and ${}^{sc}TX \leq {}^{sc}TX'|_X$. We may define:

$$\begin{aligned} \nu_b(X, X') &:= \{Z|_X : Z \in \nu_b(X')\}, \\ \nu_{sc}(X, X') &:= \{Z|_X : Z \in \nu_{sc}(X')\}. \end{aligned}$$

Notice that

$$\begin{aligned} \nu_b(X) &= \{Z \in \nu_b(X, X') : Z \in {}^bTX\}, \\ \nu_{sc}(X) &= \{Z \in \nu_{sc}(X, X') : Z \in {}^{sc}TX\}. \end{aligned}$$

Remark. Analogously to the standard Riemannian case, any b -metric on X' can be seen as a smooth section of a symmetric product of the bundle ${}^bT^*X'$. It thus restricts to a b -metric on X . Regarding sc - and ac -metrics, pointwise there is no difference between them: Both come from global sections of the same symmetric product of ${}^{sc}T^*X'$. The difference arises from the section itself, i.e., from its Taylor expansion at the boundary. Thus, depending on the immersion, a sc -metric on X' may restrict to either a sc - or a ac -metric on X .

Let us fix an asymptotically conical manifold (X', g) . From now on, ∇, R, \exp will denote, respectively, the Levi-Civita connection, curvature tensor and exponential map of (N, g) . If ϕ is a b -immersion and $p \in X$, we will let \perp, T denote the normal and tangential components of any $V \in T_p X'$ with respect to $T_p X$.

We now want to deform $\Sigma \subseteq N$, using the exponential map of N . In particular, we now want to prove that, for any $V \in \nu_{sc}(X')$, $\exp V \circ \phi : X \rightarrow X'$ is also a b -immersion. The main difficulty lies in the fact that,

since g “blows up” on $\partial X'$, $\exp V$ is, a priori, defined only on N . We thus need the following:

Proposition 4. *Let (X', g) be an asymptotically conical manifold.*

1. *Let $V \in \nu_b(X')$. Then $\exp V : N \rightarrow N$ has a smooth extension $\exp V : X' \rightarrow X'$ such that $\exp V(\partial X') \subseteq \partial X'$.*
2. *Let $V \in \nu_{sc}(X')$. Then the extension of $\exp V$ satisfies $\exp V|_{\partial X'} \equiv \text{id}$ and is an immersion on $\partial X'$.*
3. *Let $\|\cdot\|_{1,b}$ denote the norm on $H_b^1(TX', sc)$. Then $\exists \delta > 0 : \forall V \in \nu_{sc}(X'), \|V\|_{1,b} \leq \delta, \exp V : X' \rightarrow X'$ is an immersion.*
4. *Let $V \in \nu_{sc}(X')$. Then $\exp V_*(\nu_{sc}) \subseteq \nu_{sc}$.*

Proof. Let x, y^1, \dots, y^N be coordinates on X' and let $V = V^x x \partial_x + V^i \partial_{y^i} \in \nu_b(X')$. Consider the system of ODE's

$$\begin{cases} \ddot{c}^x + c^x \left(\frac{\Gamma_{ij}^x}{x} \right) (\gamma) \dot{c}^i \dot{c}^j + \Gamma_{xj}^x(\gamma) \dot{c}^x \dot{c}^j + \Gamma_{ix}^x(\gamma) \dot{c}^i \dot{c}^x + \frac{(x\Gamma_{xx}^x)(\gamma)}{c^x} \dot{c}^x \dot{c}^x = 0 \\ \ddot{c}^k + \Gamma_{ij}^k(\gamma) \dot{c}^i \dot{c}^j + \frac{(x\Gamma_{xj}^k)(\gamma)}{c^x} \dot{c}^x \dot{c}^j + \frac{(x\Gamma_{ix}^k)(\gamma)}{c^x} \dot{c}^i \dot{c}^x + \frac{(x^2\Gamma_{xx}^k)(\gamma)}{(c^x)^2} \dot{c}^x \dot{c}^x = 0 \\ c(0) = (1, y_0) \\ \dot{c}(0) = (V^x[x_0, y_0], V^1[x_0, y_0], \dots, V^N[x_0, y_0]) \end{cases}$$

where:

- $c^x(s), c^1(s), \dots, c^N(s)$ are the unknown functions;
- $\Gamma_{\mu\nu}^\sigma$ are the Christoffel symbols of g with respect to $\{\partial_x, \partial_{y^i}\}$;
- $\gamma(s) := (x_0 c^x(s), c^1(s), \dots, c^N(s))$;
- the system depends on the parameters $(x_0, y_0) \in [0, 1] \times \partial X'$.

By Lemma 2, all coefficients in the system are smooth (except when $c^x = 0$); so, with our initial conditions, there exists a unique solution $c(s)$ and it is smooth with respect to s, x_0, y_0 . Thus $\gamma(s)$ is well-defined for all (x_0, y_0) and is smooth.

Notice that, for example, $\left(\frac{\Gamma_{ij}^x}{x} \right) (\gamma) = \frac{\Gamma_{ij}^x(\gamma)}{x_0 c^x}$. Thus, multiplying the first equation by x_0 ($x_0 \neq 0$), the system can be re-written in terms of γ , showing that $\gamma(s)$ is the geodesic of (N, g) with initial conditions

$$\gamma(0) = (x_0, y_0), \quad \dot{\gamma}(0) = (x_0 V^x[x_0, y_0], V^1[x_0, y_0], \dots, V^N[x_0, y_0]) = V[x_0, y_0].$$

This shows that the geodesics generated by any b -vector field, a priori defined only on N , extend smoothly, as curves, to $\partial X'$. Since N is complete, it also shows, for $x_0 \neq 0$, that $c(s)$ is defined $\forall s \in \mathbb{R}$ and that $c(s) \subseteq N$, since this is true for $\gamma(s)$.

On the other hand, notice that, for initial conditions $(0, y_0) \in \partial X'$, the corresponding curve $\gamma(s)$ is completely contained in $\partial X'$. This proves (1).

Now assume $V \in \nu_{sc}(X')$. Then, for $x_0 = 0, \dot{c}(0) = 0$. Thus $c(s) \equiv (1, y_0)$ is the solution, so $\gamma(s) \equiv (0, y_0)$. This proves that, when $V \in \nu_{sc}$, $\exp V|_{\partial X'} \equiv \text{id}$.

Let $p \in \partial X'$. Since $d(\exp V)[p]|_{T_p \partial X'} \equiv \text{id}$, to finish proving (2) it remains only to prove that $d(\exp V)[p](\partial_x)$ has a component in the ∂_x -direction.

In general, let $p \in N$ and $Z \in T_p X'$. Let $c(t)$ be a curve in $N : c(0) = p, \dot{c}(0) = Z$. Let $t \mapsto \gamma(t, s)$ denote the 1-parameter family of geodesics defined by $\gamma(t, 0) = c(t), \frac{\partial}{\partial s} \gamma(t, 0) = V[c(t)]$.

Let $J(s)$ be the Jacobi vector field solution of

$$\nabla_s \nabla_s J(s) + R(J(s), \dot{\gamma})\dot{\gamma} = 0, \quad J(0) = Z(p), \quad \nabla_s J(0) = \nabla_Z V(p)$$

where $\dot{\gamma} := \frac{\partial}{\partial s} \gamma(0, s)$. Notice that:

- $\frac{\partial}{\partial t} \gamma(0, 0) = Z(p)$
- $\nabla_s \frac{\partial}{\partial t} \gamma(0, 0) = \nabla_s \nabla_t \gamma(0, 0) = \nabla_t \nabla_s \gamma(0, 0) = \nabla_t \frac{\partial}{\partial s} \gamma(0, 0) = \nabla_Z V(p)$
- $\nabla_s \nabla_s \frac{\partial}{\partial t} \gamma(0, s) = \nabla_s \nabla_t \frac{\partial}{\partial s} \gamma(0, s) = \nabla_t \nabla_s \frac{\partial}{\partial s} \gamma + R(\frac{\partial}{\partial s} \gamma, \frac{\partial}{\partial t} \gamma) \frac{\partial}{\partial s} \gamma = -R(\frac{\partial}{\partial t} \gamma(0, s), \dot{\gamma})\dot{\gamma}$.

This proves that $J(s) \equiv \frac{\partial}{\partial t} \gamma(t, s)|_{t=0}$. Thus, in general, $\exp V_*$ can be expressed in terms of Jacobi vector fields:

$$\exp V_* Z = \exp V_* \left(\frac{\partial}{\partial t} \gamma(0, 0) \right) = \frac{\partial}{\partial t} \gamma(0, 1) = J(1).$$

Now, for example, let $Z = x^2 \partial_x$. Using the following lemma, we find that

$$J(s) = Z(p) + s \nabla_Z V(p) + Q(s), \quad \|Q(s)\| = O(x^2)$$

where $Z(p), \nabla_Z V(p)$ are extended along γ by parallel transport. Again, we have to prove that this formula has meaning up to $\partial X'$.

Since $\exp V : X' \rightarrow X'$ is smooth, $J(1)$ has a smooth extension up to $\partial X'$. Notice also that $Z, \nabla_Z V$ have smooth extensions up to $\partial X'$ (using Lemma 2).

Recall that the parallel transport of, for example, Z along γ is defined by

$$\nabla_s Z(s) = 0, \quad Z(0) = Z[x_0, y_0].$$

Writing $Z = Z^x \partial_x + Z^i \partial_{y^i}$, we find

$$\begin{aligned} \nabla_s Z(s) &= \dot{Z}^x \partial_x + \dot{Z}^i \partial_{y^i} + Z^x \dot{\gamma}^x (\nabla_{\partial_x} \partial_x)(\gamma) + Z^x \dot{\gamma}^k (\nabla_{\partial_{y^k}} \partial_x)(\gamma) \\ &\quad + Z^i \dot{\gamma}^x (\nabla_{\partial_x} \partial_{y^i})(\gamma) + Z^i \dot{\gamma}^k (\nabla_{\partial_{y^k}} \partial_{y^i})(\gamma). \end{aligned}$$

A proof similar to the one above shows that parallel transport extends smoothly up to the boundary, so $Q(1)$ is also smooth up to $\partial X'$.

Writing $Q(1) = a \partial_x + b_i \partial_{y^i} = \frac{a}{x^2} (x^2 \partial_x) + \frac{b_i}{x} (x \partial_{y^i})$, the estimate on $\|Q(1)\|$ shows that $\frac{a}{x^2} = O(x^2), \frac{b_i}{x} = O(x^2)$. A Taylor expansion of a, b_i based at $p \in \partial X'$ thus shows that $Q(1) \in x^2 \cdot \nu_{sc}$, so

$$\exp V_*(Z) = Z + x \cdot \nu_{sc} + x^2 \cdot \nu_{sc}.$$

Finally:

$$\exp V_*(\partial_x) = \frac{\exp V_*(x^2\partial_x)}{x^2} = \frac{x^2\partial_x + x^2 \cdot \nu_b + x^2 \cdot \nu_{sc}}{x^2} = \partial_x + \nu_b + \nu_{sc}$$

which concludes the proof of (2).

Notice that, in a similar way, one can study the case $Z = x\partial_{y^i}$, showing that $\exp V_*(\nu_{sc}) \subseteq \nu_{sc}$. This proves (4).

To prove (3), we start from the formula

$$\exp V_*(\partial_x) = \partial_x + \frac{\nabla_{x^2\partial_x} V}{x^2} + \frac{Q(1)}{x^2}.$$

From the above, we see that we may write

$$\frac{\nabla_{x^2\partial_x} V}{x^2} = a(x\partial_x) + b_i\partial_{y^i}, \quad \frac{Q(1)}{x^2} = c(x^2\partial_x) + d_i(x\partial_{y^i}).$$

Since $\nabla_{x\partial_x} \in \text{Diff}_b^1$, we find $a \leq \|\frac{\nabla_{x^2\partial_x} V}{x^2}\|_b = \|\nabla_{x\partial_x} V\|_{ac} \leq \|V\|_{1,b}$.

Analogously, one can deduce from Lemma 9 that $\|Q(1)\|_{ac} \leq x^2\|V\|_{1,b}$, so $c \leq \|\frac{Q(1)}{x^2}\|_{ac} \leq \|V\|_{1,b}$.

In other words, a $\|\cdot\|_{1,b}$ -bound yields bounds on a, c which are independent of the particular V . This proves that there exists a neighborhood U of $\partial X'$, independent of V , such that $\exp V_*(\partial_x)$ has a nonzero component in the ∂_x -direction.

In a similar way, one can study $\exp V_*(\partial_{y^i})$. The end result is that any $\|\cdot\|_{1,b}$ -bound gives a neighborhood U of $\partial X'$, independent of V , such that $\exp V$ is an immersion on U .

Since the complement K of U in X' is compact, there exists $\delta > 0$ such that $\|V\|_{ac} \leq \delta$ implies that $\exp V$ is an immersion on K . This proves that, for any V such that $\|V\|_{1,b} \leq \delta$, $\exp V$ is an immersion on X' . \square

Corollary 4. *Let (X', g) be an asymptotically conical manifold and let $\phi : X \rightarrow X'$ be a b -immersion, so that (X, ϕ^*g) is asymptotically conical.*

Let $\|\cdot\|_{1,b}$ denote the norm on $H_b^1(TX', sc)$. Then $\exists \delta > 0 : \forall V \in \nu_{sc}(X'), \|V\|_{1,b} \leq \delta$, $\exp V \circ \phi$ is a b -immersion and thus also induces an asymptotically conical metric on X .

Remark. More generally, if we let $\nu_{1,sc}(X')$ denote the C^1 -analogue of $\nu_{sc}(X')$, the same methods show that

$$V \in \nu_{1,sc}(X'), \|V\|_{1,b} \leq \delta \Rightarrow \exp V \text{ is a } C^1\text{-immersion up to } \partial X'.$$

Thus, if $\phi : X \rightarrow X'$ is a b -immersion, $\exp V \circ \phi$ is a C^1 b -immersion.

The following lemma, used in the proof of the preceding proposition, provides an estimate on certain Jacobi vector fields, with explicit dependence on $\|R\|$:

Lemma 9. *Assume given the following data:*

- (X', g) asymptotically conical, a boundary-defining function x and $V \in \nu_{sc}(X')$.
- An estimate on the curvature tensor

$$\|R(x, \xi)\| \leq \rho^2(x) \leq cx^2$$

where $\rho(x)$ is strictly positive for $x > 0$ and is independent of $\xi \in \partial X'$. Notice that, by Lemma 2, such an estimate always exists.

- $\phi : X \rightarrow X'$ a b-immersion, $y \in \partial X$ and $Z \in \nu_{sc}(X)$ such that Z is nonzero along the curve $(x, y) \subseteq \Sigma$.

Let $J(s)$ denote the Jacobi vector field (depending on the parameter x) solution to

$$\nabla_s \nabla_s J(s) + R(J(s), \dot{\gamma})\dot{\gamma} = 0, \quad J(0) = Z[x, y], \quad \nabla_s J(0) = \nabla_Z V[x, y]$$

where $x \mapsto \gamma(x, s)$ denotes the curve of geodesics defined ($\forall s \in \mathbb{R}$) by $\gamma(x, 0) = (x, y)$, $\frac{\partial}{\partial s} \gamma(x, 0) = V[x, y]$ and with the notation $\dot{\gamma} := \frac{\partial}{\partial s} \gamma$.

Then

$$J(s) = Z[x, y] + s \nabla_Z V[x, y] + Q(s) \quad (s \leq 1)$$

where $Z, \nabla_Z V$ are extended along $\gamma(s)$ by parallel transport and $\|Q(s)\| = O(\rho(x)x)$.

Proof. Since $\|R(J(s), \dot{\gamma})\dot{\gamma}\| \leq \|R\| \cdot \|J(s)\| \cdot \|V\|^2 \leq cx^2 \|J(s)\| \cdot \|V\|^2$, intuitively we expect $J(s) \rightarrow A(s)$, where $A(s)$ is the solution of the equation

$$\nabla_s \nabla_s A(s) = 0, \quad A(0) = Z[x, y], \quad \nabla_s A(0) = \nabla_Z V[x, y].$$

The proof can be divided in several steps and partially follows [J] (Theorem 4.5.2) and [BK] (Lemma 6.3.7).

For simplicity, we will assume $\|V\| \leq 1, c \leq 1$.

Step 1. Assume $g(s)$ is a C^2 function which, for some fixed $\eta \in \mathbb{R}$, satisfies

$$\ddot{g} + \eta g \leq 0, \quad g(0) = \dot{g}(0) = 0.$$

Then:

- If $\eta \leq 0$: $g(s) \leq 0, \forall s \geq 0$.
- If $\eta > 0$: $g(s) \leq 0, 0 \leq s \leq \frac{\pi}{\sqrt{\eta}}$.

Proof. Set

$$f_\eta(s) := \begin{cases} \frac{1}{\sqrt{\eta}} \sin(\sqrt{\eta} s) & \eta > 0 \\ s & \eta = 0 \\ \frac{1}{\sqrt{-\eta}} \sinh(\sqrt{-\eta} s) & \eta < 0, \end{cases}$$

i.e., f_η is the solution of the ODE

$$\ddot{f} + \eta f = 0, \quad f(0) = 0, \quad \dot{f}(0) = 1.$$

Since $g(s)$ has a second-order zero in $s = 0$ while $f_\eta(s)$ has a first-order zero, we find $\frac{g}{f_\eta}|_{s=0} = 0$.

Now set $d := g' f_\eta - g f'_\eta$, so that $\left(\frac{g}{f_\eta}\right)' = \frac{d}{f_\eta^2}$.

Notice that $d(0) = 0$ and that

$$d'(s) = g'' f_\eta - g f''_\eta = (g'' + \eta g) f_\eta \leq 0 \Leftrightarrow f_\eta \geq 0.$$

Assume $\eta \leq 0$. Then $f_\eta \geq 0 \Leftrightarrow s \geq 0$, so $d \leq 0$ for these s . This shows that $\left(\frac{g}{f_\eta}\right)' \leq 0$, so $g \leq 0$.

The case $\eta > 0$ is similar.

Step 2. Consider the solution $b(s)$ of the ODE

$$\ddot{b}(s) = \|R(s)\| \cdot \|J(s)\|, \quad b(0) = \dot{b}(0) = 0.$$

We want to prove that $\|J - A\|(s) \leq b(s)$. Clearly, it is enough to prove that, for all P unit parallel vector fields along γ ,

$$(J - A, P) - b \leq 0.$$

Notice that, setting $\beta := (J - A, P)$, β is C^2 and satisfies

$$\ddot{\beta} \leq \|R(s)\| \cdot \|J(s)\|, \quad \beta(0) = \dot{\beta}(0) = 0$$

so that $g(s) := \beta - b$ satisfies the hypothesis of Step 1 with $\eta = 0$. Thus $\beta(s) \leq b(s)$, for $s \geq 0$.

Step 3. From Step 2, we also get

$$\ddot{b} \leq \|R(s)\| \cdot \|J - A\| + \|R(s)\| \cdot \|A(s)\| \leq \|R(s)\| b(s) + \|R(s)\| \cdot \|A(s)\|.$$

Using the fact that $\|V\|$ is bounded and the uniformity of our estimate on $\|R\|$, we find

$$\|R(s)\| \leq c \rho^2(x) \quad (s \leq 1).$$

For simplicity, we will assume $c \leq 1$.

Notice that, since $\dot{b}(s) \geq 0$, b is nonnegative for $s > 0$. Thus

$$\ddot{b} \leq \rho^2(x) b(s) + \rho^2(x) \|A(s)\| \quad (0 \leq s \leq 1).$$

Let $a(s)$ be the solution to the ODE

$$\ddot{a}(s) = \rho^2(x) a(s) + \rho^2(x) \|A(s)\| \quad a(0) = \dot{a}(0) = 0.$$

Then $g(s) := b - a$ satisfies the hypothesis of Step 1 with $\eta = -\rho^2$, so $b(s) \leq a(s)$ and $\|J - A\|(s) \leq a(s)$, for $0 \leq s \leq 1$.

Step 4. Notice that $A(s) = Z[x, y] + s \nabla_Z V[x, y]$ where $Z[x, y]$ and $\nabla_Z V[x, y]$ are defined along γ by parallel transport.

To finish the proof, we now need to estimate $a(s)$.

Notice that a is C^2 and is defined on some maximal interval (α, β) .

Since $\ddot{a}(0) = \rho^2(x) \cdot \|Z\| > 0$, $\dot{a}(s)$ is positive on some maximal connected interval $(0, \delta)$.

Assume $\delta < \beta$; in particular, this implies $\dot{a}(\delta) = 0$. When $s \in (0, \delta)$, a is monotone so $a(s) > 0$ and $\ddot{a}(s) > 0$. Thus $\dot{a}(s)$ is non-decreasing, which contradicts $\dot{a}(\delta) = 0$.

This shows that $\delta = \beta$, i.e., $\dot{a}(s)$ and $a(s)$ are positive for all $0 < s < \beta$.

By hypothesis (and using $\|\nabla_Z V\| = O(x)$), $\ddot{a} \leq c(\rho^2 a + \rho^2 + s\rho^2 x)$. Thus

$$2\ddot{a} \leq 2c\rho^2 a\dot{a} + 2c\rho^2 \dot{a} + 2c\rho^2 x s \dot{a}.$$

Integrating in ds , we find

$$\dot{a}^2 \leq c\rho^2 a^2 + 2c\rho^2 a + 2c\rho^2 x s a - 2c\rho^2 x \int a ds.$$

By definition, $-c\rho^2 a \leq -\ddot{a} + c\rho^2 + c\rho^2 x s$, so

$$-2c\rho^2 x \int a ds \leq -2x\dot{a} + 2c\rho^2 x s + c\rho^2 x^2 s^2.$$

Applying this above and using $c \leq c^2$, $\rho \leq x$, we find

$$\begin{aligned} \dot{a}^2 &\leq c^2 \rho^2 a^2 + 2c\rho^2 a + 2c^2 \rho^2 x s a - 2x\dot{a} + 2c\rho^2 x s + c^2 \rho^2 x^2 s^2 \\ (\dot{a} + x)^2 &\leq (c\rho a + c\rho x s)^2 + 2c\rho x a + 2c\rho x^2 s + x^2 \\ \dot{a} + x &\leq c\rho a + c\rho x s + x. \end{aligned}$$

When $s \leq 1$, this shows that $\dot{a} \leq c\rho a + c\rho x$ so, using Gronwall’s inequality,

$$a(s) \leq c\rho x e^{c\rho s} \leq c_1 \rho(x) x \quad (s \leq 1).$$

In particular, $a(s)$ is defined up to $s = 1$. □

6. Deformations of asymptotically conical SL submanifolds

Let (X', g) be a fixed, $2n$ -dimensional, asymptotically conical manifold with a CY structure $(g, J, \omega, \Omega = \alpha + i\beta)$ on its interior $N := X' \setminus \partial X'$. We will assume that $\omega \in {}^{sc}\Lambda^2 X', \beta \in {}^{sc}\Lambda^n X'$; all these conditions are, for example, verified by $N = \mathbb{C}^n$ with its standard structures.

Let Σ be a manifold with ends.

Definition 13. An “asymptotically conical special Lagrangian immersion” (AC SL) of Σ into N is a (smooth) b -immersion $\phi : X \rightarrow X'$ such that $\phi|_\Sigma$ is SL, where X is some compactification of Σ .

We refer to Section 2 for examples of AC SL immersions.

The goal of this section is to study the integrability of “infinitesimal SL deformations” of a given AC SL (Σ, ϕ) ; as we will see further on, these are the normal vector fields on Σ corresponding, via the isomorphism $(T\Sigma)^\perp \simeq \Lambda^1(\Sigma)$, to 1-forms ν on Σ such that $(d \oplus d_g^*)\nu = 0$.

Of course, it is important to specify “boundary conditions” for the allowed deformations. Using weighted Sobolev spaces, we will impose that our vector fields decay at infinity; i.e., we study deformations that “keep infinity

fixed”. Notice that the results of Section 5 show that such deformations automatically preserve the “AC” condition.

Theorem 2. *Let $\phi : \Sigma \hookrightarrow N$ be an AC SL immersion. Then, for large k , small $\epsilon > 0$ and using the isometry $\Lambda^1(\Sigma) \simeq T\Sigma^\perp$, the set*

$$\text{Def}_{\text{SL}}(\Sigma, \phi) := \{V \in x^\epsilon H_b^k(\Lambda^1, sc) : \exp V \circ \phi \text{ is SL}\}$$

*is smooth near the origin and has dimension $\dim \mathcal{K}_\epsilon^1(\Sigma, \phi^*g) = \dim H^1(\Sigma) + \dim \mathcal{H}_{-1+\epsilon}^0(\Sigma) - 1$: It thus depends both on the topology of Σ and on the analytic properties of the link of (Σ, ϕ^*g) .*

Proof. To simplify the notation, we will identify Σ with its image in M , omitting the dependence on the fixed immersion ϕ . Consider the map

$$\begin{aligned} F_c : \Lambda_c^1(\Sigma) &\simeq \Lambda_c^0(T\Sigma^\perp) &\longrightarrow & \Lambda_c^2(\Sigma) \oplus C_c^\infty(\Sigma) \\ \nu &\simeq V &\mapsto & (\exp V)^*\omega|_\Sigma \oplus *_g[(\exp V)^*\beta|_\Sigma] \end{aligned}$$

where \exp is the exponential map of (N, g) and $*_g$ is the Hodge star operator of (Σ, ϕ^*g) .

Notice that, if $V \in x^\epsilon H_b^k(T\Sigma^\perp, sc)$ and using Lemma 9,

$$\begin{aligned} (\exp V^*\omega)[p](e_i, e_j) &= \omega[\exp V](\exp V_*e_i, \exp V_*e_j) \\ &= \omega[\exp V](e_i + \nabla_{e_i}V + Q_i(1), e_j + \nabla_{e_j}V + Q_j(1)) \end{aligned}$$

where $\{e_i\}$ is a local orthonormal basis for (Σ, ϕ^*g) .

Using $\nabla\omega = 0$, $\omega \in {}^{sc}\Lambda^2 X'$ and the fact that Σ is Lagrangian, the above terms may be examined one by one. For example:

- $\omega[\exp V](e_i, e_j) = \omega[p](e_i, e_j) = 0$
- $\omega[\exp V](e_i, \nabla_{e_j}^T V) = \omega[p](e_i, \nabla_{e_j}^T V) = 0$
- $|\omega[\exp V](e_i, \nabla_{e_j}^\perp V)| = |\omega[p](e_i, \nabla_{e_j}^\perp V)| \leq |\nabla_{e_j}^\perp V|$
- $|\omega[\exp V](e_i, Q_j(1))| = O(x^2)$.

This proves that $\exp V^*\omega|_\Sigma \in x^{\epsilon+1}L_b^2(\Lambda^2\Sigma, sc)$.

More generally, one can prove that F_c has an extension

$$F : x^\epsilon H_b^k(\Lambda^1, sc) \longrightarrow x^{\epsilon+1}H_b^{k-1}(\Lambda^2, sc) \oplus x^{\epsilon+1}H_b^{k-1}(\Sigma).$$

We want to prove that $F^{-1}(0)$ is smooth using the implicit function theorem. To be able to apply the Banach-space version of this theorem, it is sufficient to prove that $dF[0]$ is surjective and has finite-dimensional kernel.

Notice that

$$\frac{d}{dt}\Big|_{t=0} \exp(tV)^*\omega|_\Sigma = \mathcal{L}_V\omega|_\Sigma = (i_V d + d i_V)\omega|_\Sigma = d\nu.$$

Linear algebra shows that $*_g i_V \beta|_\Sigma = (-1)^{n-1}\nu$; thus,

$$\frac{d}{dt}\Big|_{t=0} *_g [\exp(tV)^*\beta|_\Sigma] = *_g[\mathcal{L}_V\beta|_\Sigma] = (-1)^{n-1}d_g^*(*_g i_V \beta|_\Sigma) = d_g^*\nu$$

so that $dF[0] = D_g$. It is thus Fredholm, as seen above. Surjectivity onto the whole space $x^{\epsilon+1}H_b^{k-1}(\Lambda^2, sc) \oplus x^{\epsilon+1}H_b^{k-1}(\Sigma)$ is actually false, but the following steps show that we may restrict our map to a smaller range and obtain surjectivity:

Step 1. $\text{Im}(F_c) \subseteq \text{Im}(d_c) \oplus \text{Im}(d_c^*)$, i.e.,

$$\exists \sigma \in \Lambda_c^1(\Sigma) : (\exp V)^* \omega|_\Sigma = d_c \sigma, \quad \exists \tau \in \Lambda_c^1(\Sigma) : *_g[(\exp V)^* \beta|_\Sigma] = d_c^* \tau$$

where d_c, d_c^* denote the restrictions of d, d^* to Λ_c^* .

Proof. Extend V to a compactly-supported vector field on N . Then $t \mapsto \phi_t := \exp(tV)$ is a 1-parameter group of diffeomorphisms and

$$\begin{aligned} \phi_t^* \omega &= \phi_0^* \omega + \int_0^t \frac{d}{ds} (\phi_s^* \omega) ds \\ &= \phi_0^* \omega + \int_0^t \phi_s^* (\mathcal{L}_V \omega) ds \\ &= \phi_0^* \omega + \int_0^t d(\phi_s^* i_V \omega) ds \\ &= \phi_0^* \omega + d \left(\int_0^t \phi_s^* i_V \omega ds \right). \end{aligned}$$

If we set $\sigma := \int_0^1 \phi_s^* i_V \omega ds$, it is thus clear that $\sigma|_\Sigma \in \Lambda_c^1(\Sigma)$ and that $\phi^* \omega|_\Sigma = d(\sigma|_\Sigma)$, as claimed.

The second claim may be restated as

$$\exists \tau \in \Lambda_c^{n-1}(\Sigma) : (\exp V)^* \beta|_\Sigma = d_c \tau$$

and can thus be proved as above.

Step 2. $\text{Im}(d) \oplus \text{Im}(d^*) = \text{Im}(D_g)$, i.e.,

$$\forall \alpha, \beta \in x^\epsilon H_b^k(\Lambda^1, sc) \quad \exists \gamma \in x^\epsilon H_b^k(\Lambda^1, sc) : d\alpha \oplus d^* \beta = d\gamma \oplus d^* \gamma.$$

Proof. Let $\gamma := \alpha + df$, for some $f \in x^{\epsilon-1}H_b^{k+1}(\Sigma)$. It is enough to show that f can be determined so that $d^* \beta = d^* \alpha + d^* df$, i.e., $d^*(\beta - \alpha) = \Delta_g f$. So, it is enough to show that

$$\begin{aligned} \text{Im}(d^* : x^\epsilon H_b^k(\Lambda^1, sc) &\longrightarrow x^{\epsilon+1} H_b^{k-1}(\Sigma)) \\ &= \text{Im}(\Delta_g : x^{\epsilon-1} H_b^{k+1} \longrightarrow x^{\epsilon+1} H_b^{k-1}). \end{aligned}$$

Notice that $\text{Im}(d^*) \supseteq \text{Im}(\Delta_g)$. Since $\epsilon < n - 1$, Δ_g is surjective (on functions) so the above is necessarily true.

Step 3. $\text{Im}(F) \subseteq \text{Im}(D_g)$.

Proof.

$$\begin{aligned} \text{Im}(F) &\subseteq \overline{\text{Im}(F_c)} \subseteq \overline{\text{Im}(d_c) \oplus \text{Im}(d_c^*)} \subseteq \overline{\text{Im}(d) \oplus \text{Im}(d^*)} = \overline{\text{Im}(D_g)} \\ &= \text{Im}(D_g). \end{aligned}$$

The above proves that the restriction

$$F : x^\epsilon H_b^k(\Lambda^1, sc) \longrightarrow \text{Im}(D_g)$$

is a well-defined map between Banach spaces, such that $dF[0]$ has finite-dimensional kernel and is surjective. We may thus apply the implicit function theorem, proving that $F^{-1}(0)$ is smooth near the origin and has dimension $\dim \mathcal{K}_\epsilon^1(\Sigma, \phi^*g)$, which we calculated in Section 4.2. \square

Recall that each SL immersion $\exp V \circ \phi$ is, in particular, minimal; thus, standard regularity results prove that $\exp V \circ \phi$ is smooth on Σ ; i.e., $\exp V \circ \phi \in C^\infty(\Sigma, N)$. Since \exp is a diffeomorphism on normal vector fields, this also shows that V is smooth on Σ .

Remarks:

- 1) Given $p \in \Sigma$, recall from Section 2 the isometries

$$(\alpha \in T_p^*\Sigma) \stackrel{g}{\simeq} (W \in T_p\Sigma) \stackrel{J}{\simeq} (V \in T_p\Sigma^\perp).$$

The bundles $T^*\Sigma$, $T\Sigma$ over Σ have natural extensions to bundles ${}^{sc}T^*X$, ${}^{sc}TX$ over X . It is using this fact that we can define operators Diff_b^k and spaces $x^\delta H_b^k(\Lambda^1, sc)$, $x^\delta H_b^k(T\Sigma, sc)$.

Since g is not defined on ∂X , however, it is not clear, a priori, if $T\Sigma^\perp$ has an extension up to ∂X . On the other hand, the Lagrangian condition shows that $T\Sigma^\perp \simeq T\Sigma$; thus, up to this identification, ${}^{sc}TX$ provides an extension of $T\Sigma^\perp$, allowing us to define the spaces $x^\delta H_b^k(T\Sigma^\perp, sc)$, used in the above proof.

- 2) Let $\alpha \in x^\epsilon H_b^k(\Lambda^1, sc)$ for k large and $\epsilon > 0$. By the Sobolev embedding theorem, notice that α is C^1 on Σ and that $\|\alpha\| = O(x^\epsilon)$. Examining the formula for the induced metric on 1-forms shows, however, that this is not enough to ensure that α has a continuous extension up to ∂X ; this corresponds to the fact that, in general, $\alpha \in \nu_{sc}^*(X)$ does not extend up to ∂X .

On the other hand, let $W \in x^\epsilon H_b^k(T\Sigma, sc)$ be the corresponding tangent vector field. Then:

- As above, W is C^1 on Σ .
- If we write $W = a \partial_x + b_i \partial_{y^i} = \frac{a}{x^2}(x^2 \partial_x) + \frac{b_i}{x}(x \partial_{y^i})$, the fact $\|W\|_{ac} = O(x^\epsilon)$ implies that $\frac{a}{x^2} \rightarrow 0$, $\frac{b_i}{x} \rightarrow 0$.

Thus W admits a continuous extension to zero on ∂X .

- Notice that

$$\begin{aligned} \nabla_{x\partial_x}^T W &= x(\partial_x a)\partial_x + xa\Gamma_{xx}^x\partial_x + xa\Gamma_{xx}^j\partial_{y^j} + x(\partial_x b_j)\partial_{y^j} \\ &\quad + xb_i\Gamma_{xi}^x\partial_x + xb_i\Gamma_{xi}^j\partial_{y^j}. \end{aligned}$$

The fact that $\|\nabla_{x\partial_x}^T W\|_{ac} = O(x^\epsilon)$ thus implies that $\partial_x a, \partial_x b_j$ have extensions to zero on ∂X .

In a similar way we can prove that W has a C^1 extension to zero on ∂X .

The same holds for the normal field $V := JW$. If we let $\nu_{1,b}, \nu_{1,sc}$ denote the C^1 analogues of ν_b, ν_{sc} , we thus find that $W \in \nu_{1,sc}(X)$, $V \in \nu_{1,sc}(X, X')$.

- 3) Although from a geometric point of view one is interested only in the smoothness over Σ of the SL deformations $\exp V \circ \phi$, it is important to point out that, under compactification, these deformed submanifolds will not, in general, be smooth up to ∂X . This is a standard situation in b -geometry: b -ellipticity is not sufficient to ensure regularity up to ∂X .

One can however prove the existence of a certain asymptotic expansion of V at ∂X , i.e., “polyhomogeneity” in the sense of [M1]. In our situation, this implies that $\exp V \circ \phi \in C^1(X; X') \cap C^\infty(\Sigma; N)$. In any case, the results of Section 5 apply to show that $\exp V \circ \phi$ is (C^1) asymptotically conical.

- 4) In this article we have given a purely metric characterization of the condition “asymptotically conical”; this has the advantage of allowing for fairly general ambient spaces. When $(N, g) = (\mathbb{C}^n, g_{\text{std}})$ and the submanifold Σ is minimal (as in our case), our definition implies alternative, “set-approximating” definitions, as follows:

Recall that the components of a minimal immersion are harmonic functions of (Σ, g) . The results of Section 4.1 thus apply to show that there exists a minimal cone, lying in \mathbb{C}^n , to which Σ “converges”, with speed $o(r)$. When Σ is Lagrangian, the cone is SL.

- 5) As mentioned in the Acknowledgements below, the results of this paper are very close to those conjectured in [J1] and [J2], and proved in [M]. The main difference between this work and that of Joyce and Marshall is in the set-up of the problem: While we measure our deformations with respect to the given initial AC SL submanifold (this “intrinsic approach” works in any AC CY), Joyce prefers to postulate the existence of a SL cone, then measures the deformations of AC SL submanifolds with respect to that cone. This “extrinsic approach” works only in \mathbb{C}^n (where cones can be defined); in this ambient space, their results could be reconstructed from ours, starting with the observation in remark 4 (above) which (adopting the terminology of [J1] and [J2]) basically

states that, in \mathbb{C}^n , any AC SL submanifold is “weakly asymptotic” to some cone.

As a final step, it is interesting to understand the dependence of the above construction on the weight ϵ : In theory, changing the weight changes the class of allowed deformations and, thus, the dimension of the space of AC SL deformations.

In particular, it is interesting to consider what happens with respect to the spaces $x^{\frac{n}{2}}H_b^k$, corresponding to L^2 -decay of the deformations. As shown in the course of the proof of Theorem 2, to get a smooth structure dependent on this weight one would need an extension of F_c to a map

$$F : x^{\frac{n}{2}}H_b^k(\Lambda^1, sc) \longrightarrow x^{\frac{n}{2}+1}H_b^{k-1}(\Lambda^2, sc) \oplus x^{\frac{n}{2}+1}H_b^{k-1}(\Sigma).$$

However, the existence of such an extension depends on the properties of \exp . According to Lemma 9, the curvature R of N introduces a perturbation Q which depends on the rate of decay of $\|R\|$; if $\|R\|$ does not decay sufficiently fast (with respect to the weights in consideration), the above extension does not exist, because the perturbation is too big with respect to the weights.

Consider, for example, the case $(N, g) = (\mathbb{C}^n, g_{\text{std}})$. In this case, in Lemma 9, $Q \equiv 0$ and the extension exists. The whole proof of the theorem carries through as before, yielding a second, smaller, set of deformations.

We can thus prove the following theorem, which actually holds for any (N, g) with sufficiently fast curvature decay:

Theorem 3. *If $(N, g) = (\mathbb{C}^n, g_{\text{std}})$, the set*

$$\text{Def}_{\text{SL}}^{L^2}(\Sigma, \phi) := \{V \in x^{\frac{n}{2}}H_b^k(\Lambda^1, sc) : \exp V \circ \phi \text{ is SL}\}$$

is a smooth submanifold of $\text{Def}_{\text{SL}}(\Sigma, \phi)$.

*It has dimension $\dim \mathcal{K}_{\frac{1}{2}}^1(\Sigma, \phi^*g) = \dim H_c^1(\Sigma)$ and thus depends only on the topology of Σ .*

Acknowledgements. I wish to thank P. de Bartolomeis and G. Tian for their support, suggestions and interest. I am also grateful to R. Melrose for many useful explanations on scattering geometry and to A. Ghigi and P. Piazza for interesting conversations.

The problem of understanding the deformations of asymptotically conical SL submanifolds was posed by D. Joyce in a previous version of [J1]. This work started as an attempt to understand that article. A similar result to that presented here was more recently conjectured in the updated version of [J1] and in [J2]. After the present work was completed, but independently from it, Joyce’s conjectures were proved by S. Marshall [M]. For further comments, see Remark 5, Section 6.

Finally, I gratefully acknowledge the generous support of the University of Pisa and of GNSAGA, and the hospitality of MIT.

References

- [A] T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer, Berlin, 1998, [MR 1636569](#) (99i:58001), [Zbl 0896.53003](#).
- [B] R. Bartnik, *The mass of an asymptotically flat manifold*, *Comm. Pure and Applied Math.*, **39** (1986), 661–693, [MR 0849427](#) (88b:58144), [Zbl 0598.53045](#).
- [BK] P. Buser and H. Karcher, *Gromov's Almost Flat Manifolds*, *Asterisque*, **81**, Société Mathématique de France, Paris, 1981, [MR 0619537](#) (83m:53070), [Zbl 0459.53031](#).
- [C] E. Calabi, *Métriques Kählériennes et fibrés holomorphes*, *Ann. Scient. Éc. Norm. Sup.* (4), **12** (1979), 269–294, [MR 543218](#) (83m:32033), [Zbl 0431.53056](#).
- [CZ] T. Christiansen and M. Zworski, *Harmonic functions of polynomial growth on certain complete manifolds*, *Geom. Funct. Anal.*, **6**(4) (1996), 619–627, [MR 1406666](#) (98c:58161), [Zbl 0881.58018](#).
- [G] E. Goldstein, *Minimal Lagrangian tori in Kaehler Einstein manifolds*. Preprint, [math.DG/0007135](#).
- [HL] R. Harvey and H.B. Lawson, *Calibrated geometries*, *Acta Math.*, **148** (1982), 47–156, [MR 0666108](#) (85i:53058), [Zbl 0584.53021](#).
- [H] M. Haskins, *Special Lagrangian cones*. Preprint, [math.DG/0005164](#).
- [J] J. Jost, *Riemannian Geometry and Geometric Analysis*, Springer, 1998, [MR 1625976](#) (99g:53025), [Zbl 0997.53500](#).
- [J1] D. Joyce, *On counting special Lagrangian homology 3-spheres*, *Topology and geometry: Commemorating SISTAG*, 125–151, *Contemp. Math.*, **314**, Amer. Math. Soc., Providence, 2002, [MR 1941627](#) (2003i:53076).
- [J2] ———, *Lectures on Calabi–Yau and special Lagrangian geometry*, in ‘Calabi–Yau Manifolds and Related Geometries’ (M. Gross et al., eds.), part I, Springer-Verlag, 2003.
- [M] S. Marshall, *Deformations of Special Lagrangian Submanifolds*, Ph.D. Thesis, Oxford, 2002.
- [ML] R. McLean, *Deformations of calibrated submanifolds*, *Comm. Anal. Geom.*, **6** (1998), 705–747, [MR 1664890](#) (99j:53083), [Zbl 0929.53027](#).
- [MO] R. McOwen, *The Behavior of the Laplacian on Weighted Sobolev Spaces*, *Comm. Pure Appl. Math.*, **32**(6) (1979), 783–795, [MR 0539158](#) (81m:47069), [Zbl 0426.35029](#).
- [M1] R. Melrose, *The Atiyah–Patodi–Singer Index Theorem*, A.K. Peters, Wellesley, MA, 1993, [MR 1348401](#) (96g:58180), [Zbl 0796.58050](#).
- [M2] ———, *Spectral and scattering theory for the Laplacian on asymptotically Euclidean spaces*, in ‘Spectral and Scattering Theory’, Ikawa M. ed., *Lecture notes in pure and applied mathematics*, **161**, Marcel Dekker Inc., 1994, 85–130, [MR 1291640](#) (95k:58168), [Zbl 0837.35107](#).
- [NW] L. Nirenberg and H. Walker, *The null spaces of elliptic partial differential operators in \mathbb{R}^n* , *J. Math. Anal. Appl.*, **42** (1973), 271–301, [MR 0320821](#) (47 #9354), [Zbl 0272.35029](#).
- [Oh] Y.-G. Oh, *Second variation and stabilities of minimal Lagrangian submanifolds in Kaehler manifolds*, *Invent. Math.*, **101**(2) (1990), 501–519, [MR 1062973](#) (91f:58022), [Zbl 0721.53060](#).

- [P] T. Pacini, *Flows and Deformations of Lagrangian Submanifolds in Kaehler-Einstein Geometry*, Ph.D. Thesis, Univ. of Pisa, 2002.
- [Y] S.-T. Yau, *On the Ricci curvature of a compact Kaehler manifold and the complex Monge-Ampère equation*, I, *Comm. Pure Appl. Math.*, **31(3)** (1978), 339–411, [MR 0480350](#) (81d:53045), [Zbl 0369.53059](#).

Received December 5, 2002 and revised June 10, 2003.

SCHOOL OF MATHEMATICS
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA GA 30332-0160
E-mail address: pacini@math.gatech.edu