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The W_3 algebra of central charge 6/5 is realized as a subalgebra of the vertex operator algebra $V_{\sqrt{2}A_2}$ associated with a lattice of type $\sqrt{2}A_2$ by using both coset construction and orbifold theory. It is proved that W_3 is rational. Its irreducible modules are classified and constructed explicitly. The characters of those irreducible modules are also computed.

1. Introduction

The vertex operator algebras associated with positive definite even lattices afford a large family of known examples of vertex operator algebras. An isometry of the lattice induces an automorphism of the lattice vertex operator algebra. The subalgebra of fixed points of an automorphism is the so-called orbifold vertex operator algebra. In this paper we deal with the case where the lattice $L = \sqrt{2}A_2$ is $\sqrt{2}$ times an ordinary root lattice of type A_2 and the isometry τ is an element of the Weyl group of order 3. We use this algebra to study the W_3 algebra of central charge 6/5. In fact, by using both coset construction and orbifold theory we construct the W_3 algebra of central charge 6/5 inside V_L and classify its irreducible modules. We also prove that the W_3 algebra is rational and compute the characters of the irreducible modules.

The vertex operator algebra V_L associated with $L = \sqrt{2}A_2$ contains three mutually orthogonal conformal vectors ω^1, ω^2, ω^3 with central charge $c = 1/2$, $7/10$, or $4/5$ respectively [**10**]. The subalgebra $\mathrm{Vir}(\omega^i)$ generated by ω^i is the Virasoro vertex operator algebra $L(c, 0)$, which is the irreducible unitary highest weight module for the Virasoro algebra with central charge c and highest weight 0. The structure of V_L as a module for $\mathrm{Vir}(\omega^1) \otimes \mathrm{Vir}(\omega^2) \otimes \mathrm{Vir}(\omega^3)$ was discussed in [**23**]. Among other things it was shown that V_L contains a subalgebra of the form $L(4/5, 0) \oplus L(4/5, 3)$. Such a vertex operator algebra is called a 3-state Potts model. This subalgebra is contained in the subalgebra $(V_L)^\tau$ of fixed points of τ. There is another subalgebra M in V_L, which is of the form

$$L\left(\frac{1}{2},0\right)\otimes L\left(\frac{7}{10},0\right)\oplus L\left(\frac{1}{2},\frac{1}{2}\right)\otimes L\left(\frac{7}{10},\frac{3}{2}\right)$$

and is invariant under τ. The representation theory of M was studied in [21] and [24].

We are interested in the subalgebra M^τ of fixed points of τ in M. Its Virasoro element is $\omega=\omega^1+\omega^2$. The central charge of ω is $1/2+7/10=6/5$. We find an element J of weight 3 in M^τ such that the component operators $L(n)=\omega_{n+1}$ and $J(n)=J_{n+2}$ satisfy the same commutation relations as in [3, (2.1), (2.2)] for W_3. Thus the vertex operator subalgebra $\mathcal{W}$ generated by ω and J is a W_3 algebra with central charge $6/5$.

We construct 20 irreducible M^τ-modules. 8 of them are inside irreducible untwisted M-modules, while 6 of them are inside irreducible τ-twisted M-modules and the remaining 6 are inside irreducible τ^2-twisted M-modules. There are exactly two inequivalent irreducible τ^i-twisted M-modules $M_T(\tau^i)$ and $W_T(\tau^i)$, $i=1,2$. We investigate the irreducible τ^i-twisted V_L-modules constructed in [7] and obtain $M_T(\tau^i)$ and $W_T(\tau^i)$ inside them.

We classify the irreducible modules for $\mathcal{W}$ by determining the Zhu algebra $A(\mathcal{W})$ (cf. [36]). The method used here is similar to that in [35], where the Zhu algebra of a W_3 algebra with central charge -2 is studied. We can define a map of the polynomial algebra $\mathbb{C}[x,y]$ with two variables x,y to $A(\mathcal{W})$ by $x\mapsto[\omega]$ and $y\mapsto[J]$, which is a surjective algebra homomorphism. Thus it is sufficient to determine its kernel $\mathcal{I}$. The key point is the existence of a singular vector $\mathbf{v}$ for the W_3 algebra $\mathcal{W}$ of weight 12. A positive definite invariant Hermitian form on V_L implies that $\mathbf{v}$ is in fact 0. Thus $[\mathbf{v}]=0$. Moreover, $[J(-1)\mathbf{v}]=[J(-2)\mathbf{v}]=[J(-1)^2\mathbf{v}]=0$. Hence the corresponding polynomials in $\mathbb{C}[x,y]$ must be contained in the ideal $\mathcal{I}$. It turns out that $\mathcal{I}$ is generated by those four polynomials and the classification of irreducible $\mathcal{W}$-modules is established by Zhu's theory ([36]). That is, there are exactly 20 inequivalent irreducible $\mathcal{W}$-modules. The calculation of explicit form of the singular vector $\mathbf{v}$ and the calculation of the ideal $\mathcal{I}$ were done by a computer algebra system Risa/Asir.

By the classification of irreducible $\mathcal{W}$-modules and a positive definite invariant Hermitian form, we can show that $M^\tau=\mathcal{W}$. The eigenvalues of the action of weight preserving operators $L(0)=\omega_1$ and $J(0)=J_2$ on the top levels of those 20 irreducible M^τ-modules coincide with the values $\Delta\begin{pmatrix} n & m\\ n' & m'\end{pmatrix}$ and $w\begin{pmatrix} n & m\\ n' & m'\end{pmatrix}$ of [14, (1.2), (5.6)] with $p=5$. Hence our M^τ is an algebra denoted by $[Z_3^{(5)}]$ in [14].

We prove that $\mathcal{W}$ is C_2-cofinite and rational by using the singular vector $\mathbf{v}$ of weight 12 and the irreducible modules for $\mathcal{W}$. In the course of the proof we use a result about a general vertex operator algebra V. It says that if V

is C_2-cofinite, then V is rational if and only if $A(V)$ is semisimple and any simple $A(V)$-module generates an irreducible V-module. This result will certainly be useful in the future study of relationship between rationality and C_2-cofiniteness.

We also study the characters of those irreducible M^τ-modules. Using the modular invariance of trace functions in orbifold theory (cf. [9]), we describe the characters of the 20 irreducible M^τ-modules in terms of the characters of irreducible unitary highest weight modules for the Virasoro algebras.

The results in this paper have applications to the Monster simple group. Recently, it was shown in [22] that the $\mathbb{Z}_3$ symmetry of a 3-state Potts model in $(V_L)^\tau$ affords $3A$ elements of the Monster simple group. Such a result has been suggested by [28]. It is expected that the $\mathbb{Z}_3$ symmetry of M^τ affords $3B$ elements.

The organization of the paper is as follows: In Section 2 we review some properties of M for later use. In Section 3 we define the vector J and compute the commutation relations among the component operators $L(n) = \omega_{n+1}$ and $J(n) = J_{n+2}$. In Section 4 we construct 20 irreducible M^τ-modules and discuss their properties. In Section 5 we determine the Zhu algebra of the vertex operator subalgebra $\mathcal{W}$ generated by ω and J and show that $M^\tau = \mathcal{W}$. Thus we conclude that M^τ has exactly 20 inequivalent irreducible modules. Finally, in Section 6 we study the characters of those irreducible M^τ-modules.

2. Subalgebra M of $V_{\sqrt{2}A_2}$

In this section we fix notation. For basic definitions concerning lattice vertex operator algebras we refer to [7] and [17]. We also recall certain properties of the vertex operator algebra $V_{\sqrt{2}A_2}$ (cf. [23]).

Let α_1, α_2 be the simple roots of type A_2 and set $\alpha_0 = -(\alpha_1 + \alpha_2)$. Then $\langle \alpha_i, \alpha_i \rangle = 2$ and $\langle \alpha_i, \alpha_j \rangle = -1$ if $i \neq j$. Set $\beta_i = \sqrt{2}\alpha_i$ and let $L = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2$ be the lattice spanned by β_1 and β_2. We usually denote L by $\sqrt{2}A_2$.

We follow Sections 2 and 3 of [7] with $L = \sqrt{2}A_2$, $p = 3$, and $q = 6$. In our case $\langle \alpha, \beta \rangle \in 2\mathbb{Z}$ for all $\alpha, \beta \in L$, so that the alternating $\mathbb{Z}$-bilinear map $c_0 : L \times L \to \mathbb{Z}/6\mathbb{Z}$ defined by [7, (2.9)] is trivial. Thus the central extension

$$1 \longrightarrow \langle \kappa_6 \rangle \longrightarrow \widehat{L} \xrightarrow{\ \bar{\ }\ } L \longrightarrow 1 \tag{2.1}$$

determined by the commutator condition $aba^{-1}b^{-1} = \kappa_6^{c_0(\bar{a},\bar{b})}$ splits. Then for each $\alpha \in L$, we can choose an element e^α of $\hat{L}$ so that $e^\alpha e^\beta = e^{\alpha+\beta}$. The twisted group algebra $\mathbb{C}\{L\}$ is isomorphic to the ordinary group algebra $\mathbb{C}[L]$.

We adopt the same notation as in [21] to denote cosets of L in the dual lattice $L^{\perp} = \{\alpha \in \mathbb{Q} \otimes_{\mathbb{Z}} L \,|\, \langle \alpha, L\rangle \subset \mathbb{Z}\}$, namely,

$$L^0 = L, \quad L^1 = \frac{-\beta_1 + \beta_2}{3} + L, \quad L^2 = \frac{\beta_1 - \beta_2}{3} + L,$$
$$L_0 = L, \quad L_a = \frac{\beta_2}{2} + L, \quad L_b = \frac{\beta_0}{2} + L, \quad L_c = \frac{\beta_1}{2} + L,$$

and

$$L^{(i,j)} = L_i + L^j$$

for $i = 0, a, b, c$ and $j = 0, 1, 2$, where $\{0, a, b, c\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then, $L^{(i,j)}, i \in \{0, a, b, c\}, j \in \{0, 1, 2\}$ are all the cosets of L in $L^{\perp}$ and $L^{\perp}/L \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

Our notation for the vertex operator algebra $(V_L, Y(\,\cdot\,, z))$ associated with L is standard [17]. In particular, $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ is an abelian Lie algebra, $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ is the corresponding affine Lie algebra, $M(1) = \mathbb{C}[\alpha(n)\,;\, \alpha \in \mathfrak{h}, n < 0]$, where $\alpha(n) = \alpha \otimes t^n$, is the unique irreducible $\hat{\mathfrak{h}}$-module such that $\alpha(n)1 = 0$ for all $\alpha \in \mathfrak{h}$ and $n > 0$, and $c = 1$. As a vector space $V_L = M(1) \otimes \mathbb{C}[L]$ and for each $v \in V_L$, a vertex operator $Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \in \mathrm{End}(V_L)[[z, z^{-1}]]$ is defined. The coefficient v_n of z^{-n-1} is called a component operator. The vector $\mathbf{1} = 1 \otimes 1$ is called the vacuum vector.

By Dong [5], there are exactly 12 isomorphism classes of irreducible V_L-modules, which are represented by $V_{L^{(i,j)}}$, $i = 0, a, b, c$ and $j = 0, 1, 2$. We use the symbol $e^{\alpha}, \alpha \in L^{\perp}$ to denote a basis of $\mathbb{C}\{L^{\perp}\}$.

To describe certain weight 2 elements in V_L, we introduce the following notation:

$$\begin{aligned} x(\alpha) &= e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}, \\ y(\alpha) &= e^{\sqrt{2}\alpha} - e^{-\sqrt{2}\alpha}, \\ w(\alpha) &= \frac{1}{2}\alpha(-1)^2 - x(\alpha) \end{aligned}$$

for $\alpha \in \{\pm\alpha_0, \pm\alpha_1, \pm\alpha_2\}$. We have

$$w(\alpha_i)_1 w(\alpha_j) = \begin{cases} 8w(\alpha_i) & \text{if } i = j \\ w(\alpha_i) + w(\alpha_j) - w(\alpha_k) & \text{if } i \neq j, \end{cases} \tag{2.2}$$

where k is such that $\{i, j, k\} = \{0, 1, 2\}$. Moreover, $w(\alpha_i)_2 w(\alpha_j) = 0$ and

$$w(\alpha_i)_3 w(\alpha_j) = \begin{cases} 4\mathbf{1} & \text{if } i = j \\ \frac{1}{2}\mathbf{1} & \text{if } i \neq j. \end{cases} \tag{2.3}$$

Let

$$\begin{aligned}
\omega &= \frac{1}{5}\big(w(\alpha_1)+w(\alpha_2)+w(\alpha_0)\big),\\
\widetilde{\omega} &= \frac{1}{6}\big(\alpha_1(-1)^2+\alpha_2(-1)^2+\alpha_0(-1)^2\big),\\
\omega^1 &= \frac{1}{4}w(\alpha_1), \qquad \omega^2=\omega-\omega^1, \qquad \omega^3=\widetilde{\omega}-\omega.
\end{aligned}$$

Then $\widetilde{\omega}$ is the Virasoro element of V_L and $\omega^1,\omega^2,\omega^3$ are mutually orthogonal conformal vectors of central charge $1/2, 7/10, 4/5$ respectively (cf. [**10**]). The subalgebra $\mathrm{Vir}(\omega^i)$ generated by ω^i is isomorphic to the Virasoro vertex operator algebra of given central charge, and ω^1, ω^2, and ω^3 generate

$$\mathrm{Vir}(\omega^1)\otimes\mathrm{Vir}(\omega^2)\otimes\mathrm{Vir}(\omega^3)\cong L\left(\frac{1}{2},0\right)\otimes L\left(\frac{7}{10},0\right)\otimes L\left(\frac{4}{5},0\right).$$

We study certain subalgebras, and also submodules for them in $V_{L_i}, i=0,a,b,c$ and $V_{L^j}, j=0,1,2$. Set

$$\begin{aligned}
M_k^i &= \{v\in V_{L_i}\mid(\omega^3)_1v=0\},\\
W_k^i &= \left\{v\in V_{L_i}\mid(\omega^3)_1v=\frac{2}{5}v\right\}, \qquad \text{for } i=0,a,b,c,
\end{aligned}$$

and

$$\begin{aligned}
M_t^j &= \{v\in V_{L^j}\mid(\omega^1)_1v=(\omega^2)_1v=0\},\\
W_t^j &= \left\{v\in V_{L^j}\mid(\omega^1)_1v=0,\quad(\omega^2)_1v=\frac{3}{5}v\right\}, \qquad \text{for } j=0,1,2.
\end{aligned}$$

Then M_k^0 and M_t^0 are simple vertex operator algebras. Furthermore, $\{M_k^i,\, W_k^i,\, i=0,a,b,c\}$ and $\{M_t^j,\, W_t^j,\, j=0,1,2\}$ are the sets of all inequivalent irreducible modules for M_k^0 and M_t^0, respectively ([**21**], [**23**] and [**24**]). We also have

$$\begin{aligned}
M_k^0 &\cong L\left(\frac{1}{2},0\right)\otimes L\left(\frac{7}{10},0\right)\oplus L\left(\frac{1}{2},\frac{1}{2}\right)\otimes L\left(\frac{7}{10},\frac{3}{2}\right),\\
W_k^0 &\cong L\left(\frac{1}{2},0\right)\otimes L\left(\frac{7}{10},\frac{3}{5}\right)\oplus L\left(\frac{1}{2},\frac{1}{2}\right)\otimes L\left(\frac{7}{10},\frac{1}{10}\right),\\
M_k^a &\cong M_k^b\cong L\left(\frac{1}{2},\frac{1}{16}\right)\otimes L\left(\frac{7}{10},\frac{7}{16}\right),\\
W_k^a &\cong W_k^b\cong L\left(\frac{1}{2},\frac{1}{16}\right)\otimes L\left(\frac{7}{10},\frac{3}{80}\right),\\
M_k^c &\cong L\left(\frac{1}{2},\frac{1}{2}\right)\otimes L\left(\frac{7}{10},0\right)\oplus L\left(\frac{1}{2},0\right)\otimes L\left(\frac{7}{10},\frac{3}{2}\right),
\end{aligned}$$

$$W_k^c \cong L\left(\frac{1}{2},\frac{1}{2}\right)\otimes L\left(\frac{7}{10},\frac{3}{5}\right)\oplus L\left(\frac{1}{2},0\right)\otimes L\left(\frac{7}{10},\frac{1}{10}\right)$$

as $L(1/2,0)\otimes L(7/10,0)$-modules and

$$M_t^0 \cong L\left(\frac{4}{5},0\right)\oplus L\left(\frac{4}{5},3\right), \qquad W_t^0 \cong L\left(\frac{4}{5},\frac{2}{5}\right)\oplus L\left(\frac{4}{5},\frac{7}{5}\right),$$
$$M_t^1 \cong M_t^2 \cong L\left(\frac{4}{5},\frac{2}{3}\right), \qquad W_t^1 \cong W_t^2 \cong L\left(\frac{4}{5},\frac{1}{15}\right)$$

as $L(4/5,0)$-modules.

Note also that

$$V_{L^{(i,j)}} \cong \left(M_k^i \otimes M_t^j\right)\oplus\left(W_k^i\otimes W_t^j\right)$$

as an $M_k^0\otimes M_t^0$-module.

We consider the following three isometries of $(L,\langle\cdot,\cdot\rangle)$:

$$\begin{aligned}&\tau:\beta_1\to\beta_2\to\beta_0\to\beta_1,\\ &\sigma:\beta_1\to\beta_2,\qquad \beta_2\to\beta_1,\\ &\theta:\beta_i\to-\beta_i,\quad i=1,2.\end{aligned}$$

Note that τ is fixed-point-free and of order 3. Note also that $\sigma\tau\sigma=\tau^{-1}$. The isometries τ,σ, and θ of L can be extended to isometries of $L^\perp$. Then they induce permutations on $L^\perp/L$. Since $\hat{L}$ is a split extension, the isometry τ of L lifts naturally to an automorphism of $\hat{L}$. Then it induces an automorphism of V_L:

$$\alpha^1(-n_1)\cdots\alpha^k(-n_k)e^\beta \longmapsto (\tau\alpha^1)(-n_1)\cdots(\tau\alpha^k)(-n_k)e^{\tau\beta}.$$

By abuse of notation, we denote it by τ also. Moreover, we can consider the action of τ on $V_{L^{(i,j)}}$ in a similar way. We apply the same argument to σ and θ.

Set $M=M_k^0$. The vertex operator algebra M plays an important role in this paper. Recall that

$$M\cong L\left(\frac{1}{2},0\right)\otimes L\left(\frac{7}{10},0\right)\oplus L\left(\frac{1}{2},\frac{1}{2}\right)\otimes L\left(\frac{7}{10},\frac{3}{2}\right)$$

as $\mathrm{Vir}(\omega^1)\otimes\mathrm{Vir}(\omega^2)$-modules. Note that ω is the Virasoro element of M whose central charge is $6/5$. For $u\in M$, we have $\omega_1 u = hu$ for some $h\in\mathbb{Z}$ if and only if $\widetilde{\omega}_1 u = hu$. In such a case h is called the weight of u. Note also that M is generated by $w(\alpha_1)$, $w(\alpha_2)$, and $w(\alpha_0)$. In particular, M is invariant under τ, σ, and θ. In fact, θ acts on M as the identity.

We next show that the automorphism group $\mathrm{Aut}(M)$ of M is generated by σ and τ.

Theorem 2.1.

(1) *There are exactly three conformal vectors of central charge* $1/2$ *in* M, *which are* $\frac{1}{4}w(\alpha_i)$, $i = 0, 1, 2$.
(2) $\mathrm{Aut}(M) = \langle \sigma, \tau \rangle$ *is isomorphic to a symmetric group of degree* 3.

Proof. We first consider conformal vectors in M. By [**27**, Lemma 5.1], a weight 2 vector v is a conformal vector of central charge 1/2 if and only if $v_1 v = 2v$ and $v_3 v = \frac{1}{4}\mathbf{1}$. Since $\{w(\alpha_0), w(\alpha_1), w(\alpha_2)\}$ is a basis of the weight 2 subspace of M, we may write $v = \sum_{i=0}^{2} a_i w(\alpha_i)$ for some $a_i \in \mathbb{C}$. From (2.2) and (2.3) we see that $v_1 v = 2v$ and $v_3 v = \frac{1}{4}\mathbf{1}$ hold only if $(a_0, a_1, a_2) = (1/4, 0, 0)$, $(0, 1/4, 0)$, or $(0, 0, 1/4)$. This proves (1). Then any automorphism of M induces a permutation on $\{w(\alpha_0), w(\alpha_1), w(\alpha_2)\}$. If an automorphism induces the identity permutation on the set, it must be the identity since M is generated by $w(\alpha_1)$, $w(\alpha_2)$, and $w(\alpha_0)$. Now

$$\tau : w(\alpha_1) \to w(\alpha_2) \to w(\alpha_0) \to w(\alpha_1),$$

and

$$\sigma : w(\alpha_1) \to w(\alpha_2), \quad w(\alpha_2) \to w(\alpha_1), \quad w(\alpha_0) \to w(\alpha_0).$$

Hence (2) holds. □

Let $v_h = w(\alpha_2) - w(\alpha_0)$. This vector is a highest weight vector of highest weight $(1/2, 3/2)$ for $\mathrm{Vir}(\omega^1) \otimes \mathrm{Vir}(\omega^2)$, that is, $(\omega^1)_1 v_h = (1/2)v_h$, $(\omega^2)_1 v_h = (3/2)v_h$, and $(\omega^1)_n v_h = (\omega^2)_n v_h = 0$ for $n \geq 2$. Thus the $\mathrm{Vir}(\omega^1) \otimes \mathrm{Vir}(\omega^2)$-submodule in M generated by v_h is isomorphic to $L(1/2, 1/2) \otimes L(7/10, 3/2)$. In particular, M is generated by ω^1, ω^2, and v_h.

We can choose another generator of M. Let

$$\begin{aligned} u^1 &= w(\alpha_1) + \xi^2 w(\alpha_2) + \xi w(\alpha_0), \\ u^2 &= w(\alpha_1) + \xi w(\alpha_2) + \xi^2 w(\alpha_0), \end{aligned} \tag{2.4}$$

where $\xi = \exp(2\pi\sqrt{-1}/3)$ is a primitive cubic root of unity. Then $\tau u^1 = \xi u^1$, $\tau u^2 = \xi^2 u^2$, and $\sigma u^1 = \xi^2 u^2$. We also have $(u^1)_1 u^1 = 4u^2$ and $((u^1)_1 u^1)_1 u^1 = 140\omega$. Thus u^1, $(u^1)_1 u^1$, and $((u^1)_1 u^1)_1 u^1$ span the weight 2 subspace of M. This implies that M is generated by a single vector u^1. A similar assertion holds for u^2.

The subalgebra $M_t^0 \cong L(4/5, 0) \oplus L(4/5, 3)$ is called a 3-state Potts model. It plays an important role in Subsection 4.2. The irreducible M_t^0-modules and their fusion rules are determined in [**23**] and [**28**]. The Virasoro element of M_t^0 is ω^3. Let

$$v_t = \frac{1}{9}(\alpha_1 - \alpha_2)(-1)(\alpha_2 - \alpha_0)(-1)(\alpha_0 - \alpha_1)(-1) - \frac{1}{2}(\alpha_1 - \alpha_2)(-1)x(\alpha_0) - \frac{1}{2}(\alpha_2 - \alpha_0)(-1)x(\alpha_1) - \frac{1}{2}(\alpha_0 - \alpha_1)(-1)x(\alpha_2), \tag{2.5}$$

which is denoted by q in [**23**]. The vector v_t is a highest weight vector in M_t^0 of highest weight 3 for $\mathrm{Vir}(\omega^3)$. Clearly, $\tau v_t = v_t$ and thus τ fixes every element in M_t^0. Moreover, $\sigma v_t = -v_t$ and $\theta v_t = -v_t$. Hence σ and θ induce the same automorphism of M_t^0, namely, 1 on $\mathrm{Vir}(\omega^3) \cong L(4/5, 0)$ and -1 on the $\mathrm{Vir}(\omega^3)$-submodule generated by v_t, which is isomorphic to $L(4/5, 3)$. The automorphism group $\mathrm{Aut}(M_t^0)$ is of order 2 generated by θ.

3. Subalgebra $\mathcal{W}$ generated by ω and J in M^τ

For any τ-invariant space U, set $U(\epsilon) = \{u \in U \,|\, \tau u = \xi^\epsilon u\}$, $\epsilon = 0, 1, 2$, where $\xi = \exp(2\pi\sqrt{-1}/3)$. We usually denote the subspace $U(0)$ of fixed points by U^τ also.

We are interested in the subalgebra M^τ. The weight 2 subspace of M^τ is spanned by ω. In fact, ω is the Virasoro element of M with central charge $6/5$. This means that the subalgebra $\mathrm{Vir}(\omega)$ generated by ω is isomorphic to $L(6/5, 0)$. Note that M and M^τ are completely reducible as modules for $\mathrm{Vir}(\omega)$, since V_L possesses a positive definite invariant Hermitian form (see Subsection 5.3). Every irreducible direct summand in M or M^τ is isomorphic to $L(6/5, h)$ for some nonnegative integer h. Note also that σ leaves $M^\tau = M(0)$ invariant and interchanges $M(1)$ and $M(2)$. Since σ fixes ω, σ acts on $\mathrm{Vir}(\omega)$ as the identity. Thus $M(1)$ and $M(2)$ are equivalent $\mathrm{Vir}(\omega)$-modules.

We now count dimensions of homogeneous subspaces of M of small weights. The characters of $L(1/2, h)$, $L(7/10, h)$, and $L(6/5, h)$ are well-known (cf. [**19**] and [**32**]). Using them, we have the first several terms of the character of M:

$$\begin{aligned}\mathrm{ch}\, M &= \mathrm{ch}\, L\left(\frac{1}{2}, 0\right) \mathrm{ch}\, L\left(\frac{7}{10}, 0\right) + \mathrm{ch}\, L\left(\frac{1}{2}, \frac{1}{2}\right) \mathrm{ch}\, L\left(\frac{7}{10}, \frac{3}{2}\right) \\ &= 1 + 3q^2 + 4q^3 + 9q^4 + 12q^5 + 22q^6 + \cdots .\end{aligned}$$

Comparing $\mathrm{ch}\, M$ with the character of $L(6/5, h)$, we see that

$$M \cong L\left(\frac{6}{5}, 0\right) + 2L\left(\frac{6}{5}, 2\right) + L\left(\frac{6}{5}, 3\right) + 2L\left(\frac{6}{5}, 4\right) + L\left(\frac{6}{5}, 6\right) + \cdots$$

as $\mathrm{Vir}(\omega)$-modules.

The vectors u^1 and u^2 of (2.4) are highest weight vectors for $\mathrm{Vir}(\omega)$ of weight 2. Hence the $\mathrm{Vir}(\omega)$-submodule generated by u^ϵ in $M(\epsilon)$ is isomorphic to $L(6/5, 2)$, $\epsilon = 1, 2$.

Next, we study the weight 3 subspace. The weight 3 subspace of M is of dimension 4 and so there are nontrivial relations among $w(\alpha_i)_0 w(\alpha_j)$, $i, j \in \{0, 1, 2\}$. For example,

$$\begin{aligned} &w(\alpha_1)_0 w(\alpha_2) - w(\alpha_2)_0 w(\alpha_1) \\ &= w(\alpha_2)_0 w(\alpha_0) - w(\alpha_0)_0 w(\alpha_2) \\ &= w(\alpha_0)_0 w(\alpha_1) - w(\alpha_1)_0 w(\alpha_0). \end{aligned}$$

Set $J = w(\alpha_1)_0 w(\alpha_2) - w(\alpha_2)_0 w(\alpha_1)$. In terms of the lattice vertex operator algebra V_L, J can be written as

$$\begin{aligned} J = \frac{1}{3}\Big(&\alpha_1(-2)\big(\alpha_0(-1) - \alpha_2(-1)\big) \\ &+ \alpha_2(-2)\big(\alpha_1(-1) - \alpha_0(-1)\big) + \alpha_0(-2)\big(\alpha_2(-1) - \alpha_1(-1)\big)\Big) \\ + \sqrt{2}\Big(&\big(\alpha_0(-1) - \alpha_2(-1)\big)y(\alpha_1) \\ &+ \big(\alpha_1(-1) - \alpha_0(-1)\big)y(\alpha_2) + \big(\alpha_2(-1) - \alpha_1(-1)\big)y(\alpha_0)\Big). \end{aligned}$$

Note that $(u^1)_1 u^2 - (u^2)_1 u^1 = 3\sqrt{-3}J$. Note also that $\tau J = J$, $\sigma J = -J$ and $\theta J = J$. The weight 3 subspace of M^τ is of dimension 2 and it is spanned by $\omega_0\omega$ and J. Furthermore, we have $\omega_1 J = 3J$ and $\omega_n J = 0$ for $n \geq 2$. Hence:

Lemma 3.1. *J is a highest weight vector for $\mathrm{Vir}(\omega)$ of highest weight 3 in M^τ.*

The weight 4 subspace of M is of dimension 9. By a direct calculation, we can verify that $w(\alpha_i)_{-1} w(\alpha_j)$, $0 \leq i, j \leq 2$ are linearly independent. Hence $w(\alpha_i)_{-1} w(\alpha_j)$'s form a basis of the weight 4 subspace of M. From this it follows that the weight 4 subspace of M^τ is of dimension 3. Since the weight 4 subspace of $\mathrm{Vir}(\omega) \cong L(6/5, 0)$ is of dimension 2 and since the weight 4 subspace of the $\mathrm{Vir}(\omega)$-submodule generated by J, which is isomorphic to $L(6/5, 3)$, is of dimension 1, we conclude that there is no highest weight vector for $\mathrm{Vir}(\omega)$ in the weight 4 subspace of M^τ. We have shown that:

Lemma 3.2.

(1) *$\{w(\alpha_i)_{-1} w(\alpha_j) \mid 0 \leq i, j \leq 2\}$ is a basis of the weight 4 subspace of M.*
(2) *There is no highest weight vector for $\mathrm{Vir}(\omega)$ of weight 4 in M^τ.*

By the above argument, we know all the irreducible direct summands $L(6/5, h)$ with $h \leq 6$ in the decomposition of $M(\epsilon)$ into a direct sum of irreducible $\mathrm{Vir}(\omega)$-modules. Namely,

$$\begin{aligned} M^\tau = M(0) &\cong L\left(\frac{6}{5}, 0\right) + L\left(\frac{6}{5}, 3\right) + L\left(\frac{6}{5}, 6\right) + \cdots, \\ M(1) \cong M(2) &\cong L\left(\frac{6}{5}, 2\right) + L\left(\frac{6}{5}, 4\right) + \cdots. \end{aligned}$$

We now consider the vertex operator algebra $\mathcal{W}$ generated by ω and J in M^τ. Of course $\mathcal{W}$ is a subalgebra of M^τ. We shall show that $\mathcal{W}$ is, in fact, equal to M^τ. The basic data are the commutation relations of the component operators ω_m and J_n. For the determination of the commutation relation $[J_m, J_n]$, it is sufficient to express $J_n J$, $0 \le n \le 5$, by using ω. First of all we note that the weight $\mathrm{wt}\, J_n J = 5 - n$ is at most 5 for $0 \le n \le 5$. Thus $J_n J$ is contained in $L(6/5, 0) + L(6/5, 3)$, where $L(6/5, 0)$ and $L(6/5, 3)$ stand for $\mathrm{Vir}(\omega)$ and the $\mathrm{Vir}(\omega)$-submodule generated by J respectively. Since σ fixes every element in $\mathrm{Vir}(\omega)$ and $\sigma J = -J$, σ acts as -1 on the $\mathrm{Vir}(\omega)$-submodule generated by J. Hence $J_n J$ is in fact contained in $\mathrm{Vir}(\omega)$.

By a direct calculation, we have

$$\begin{aligned} J_5 J &= -84\mathbf{1}, \\ J_4 J &= 0, \\ J_3 J &= -420\omega, \\ J_2 J &= -210\omega_0\omega, \\ J_1 J &= 9\omega_0\omega_0\omega - 240\omega_{-1}\omega, \\ J_0 J &= 22\omega_0\omega_0\omega_0\omega - 120\omega_0\omega_{-1}\omega. \end{aligned} \tag{3.1}$$

Note that $\{\omega_0\omega,\ J\}$, $\{\omega_0\omega_0\omega,\ \omega_{-1}\omega,\ \omega_0 J\}$, and $\{\omega_0\omega_0\omega_0\omega, \omega_0\omega_{-1}\omega, \omega_0\omega_0 J, \omega_{-1} J\}$ are bases of weight 3, 4, and 5 subspaces of M^τ respectively.

In terms of the lattice vertex operator algebra V_L, the vectors $J_2 J, J_1 J$, and $J_0 J$ can be written as follows:

$$\begin{aligned} J_2 J &= -42\sum_{i=0}^{2} \alpha_i(-2)\alpha_i(-1) + 42\sqrt{2}\sum_{i=0}^{2} \alpha_i(-1)y(\alpha_i), \\ J_1 J &= -38\sum_{i=0}^{2} \alpha_i(-3)\alpha_i(-1) - 3\sum_{i=0}^{2} \alpha_i(-2)^2 - 8\sum_{i=0}^{2} \alpha_i(-1)^4 \\ &\quad + 6\sum_{i=0}^{2} \alpha_i(-1)^2 x(\alpha_i) + 51\sqrt{2}\sum_{i=0}^{2} \alpha_i(-2)y(\alpha_i), \end{aligned}$$

$$J_0J = -36\sum_{i=0}^{2}\alpha_i(-4)\alpha_i(-1) - 4\sum_{i=0}^{2}\alpha_i(-3)\alpha_i(-2) - 16\sum_{i=0}^{2}\alpha_i(-2)\alpha_i(-1)^3 + 36\sum_{i=0}^{2}\alpha_i(-2)\alpha_i(-1)x(\alpha_i) + \sum_{i=0}^{2}\big(44\sqrt{2}\alpha_i(-3) - 4\sqrt{2}\alpha_i(-1)^3\big)y(\alpha_i).$$

We need some formulas for vertex operator algebras (cf. [17]), namely,

$$[u_m, v_n] = \sum_{k=0}^{\infty}\binom{m}{k}(u_kv)_{m+n-k}, \tag{3.2}$$

$$(u_mv)_n = \sum_{k=0}^{\infty}(-1)^k\binom{m}{k}\big(u_{m-k}v_{n+k} - (-1)^m v_{m+n-k}u_k\big), \tag{3.3}$$

$$(\omega_0v)_n = -nv_{n-1}. \tag{3.4}$$

Using them we can obtain the commutation relations of the component operators ω_m and J_n.

Theorem 3.3. *Let* $L(n) = \omega_{n+1}$ *and* $J(n) = J_{n+2}$ *for* $n \in \mathbb{Z}$*, so that the weights of these operators are* $\operatorname{wt} L(n) = \operatorname{wt} J(n) = -n$*. Then*

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\cdot\frac{6}{5}\cdot\delta_{m+n,0}, \tag{3.5}$$

$$[L(m), J(n)] = (2m-n)J(m+n), \tag{3.6}$$

$$\begin{aligned}[J(m), J(n)] = (m-n)\Big(&22(m+n+2)(m+n+3) \\ &+ 35(m+2)(n+2)\Big)L(m+n) \\ &- 120(m-n)\Big(\sum_{k\le -2}L(k)L(m+n-k) \\ &+ \sum_{k\ge -1}L(m+n-k)L(k)\Big) \\ &- \frac{7}{10}m(m^2-1)(m^2-4)\delta_{m+n,0}.\end{aligned} \tag{3.7}$$

Proof. The first equation holds since ω is the Virasoro element of central charge $6/5$. We know that $\omega_1J = 3J$ and $\omega_nJ = 0$ for $n \ge 2$. Hence the second equation holds. Now

$$
\begin{aligned}
(\omega_{-1}\omega)_{n+3} &= \sum_{k=0}^{\infty}(-1)^k\binom{-1}{k}\Big(\omega_{-1-k}\omega_{n+3+k} - (-1)^{-1}\omega_{n+2-k}\omega_k\Big) \\
&= \sum_{k=0}^{\infty}\Big(L(-k-2)L(n+k+2) + L(n-k+1)L(k-1)\Big) \\
&= \sum_{k\le -2} L(k)L(n-k) + \sum_{k\ge -1} L(n-k)L(k).
\end{aligned}
$$

Thus the last equation follows from (3.1). □

Remark 3.4. Let $L_n = L(n)$ and $W_n = \sqrt{-1/210}J(n)$. Then the commutation relations in the above theorem coincide with the commutation relations (2.1) and (2.2) of [**3**]. Thus $\mathcal{W}$ is a W_3 algebra of central charge 6/5.

Let $\lambda(m) = i(i+1)$ if $m = 2i+1$ is odd and $\lambda(m) = i^2$ if $m = 2i$ is even. Let $: L(n_1)L(n_2) :$ be the normal ordered product, so that it is equal to $L(n_1)L(n_2)$ if $n_1 \le n_2$ and $L(n_2)L(n_1)$ if $n_1 \ge n_2$. Then we have another expression of $(\omega_{-1}\omega)_{n+3}$. That is (cf. [**14**]),

$$
(\omega_{-1}\omega)_{n+3} = \lambda(n+3)L(n) + \sum_{k\in\mathbb{Z}} : L(k)L(n-k) : .
$$

4. 20 irreducible modules for M^τ

In this section we construct 20 irreducible modules for M^τ. Furthermore, we calculate the action of the weight preserving component operators $L(0) = \omega_1$ and $J(0) = J_2$ on the top levels of those irreducible modules for M^τ. Recall that M has exactly 8 inequivalent irreducible modules M_k^i, W_k^i, $i = 0, a, b, c$. Let (U, Y_U) be one of those irreducible M-modules. Following [**9**], we consider a new M-module $(U \circ \tau, Y_{U\circ\tau})$ such that $U \circ \tau = U$ as vector spaces and

$$
Y_{U\circ\tau}(v, z) = Y_U(\tau v, z) \quad \text{for } v \in M.
$$

Then $U \mapsto U \circ \tau$ induces a permutation on the set of irreducible M-modules. If U and $U \circ \tau$ are equivalent M-modules, U is said to be τ-stable. By the definition, we have $U \circ \tau^2 = (U \circ \tau) \circ \tau$. The following lemma is a straightforward consequence of the definition of M_k^i and W_k^i:

Lemma 4.1.

(1) $M_k^0 \circ \tau = M_k^0$ *and* $W_k^0 \circ \tau = W_k^0$.
(2) $M_k^a \circ \tau = M_k^c$, $M_k^c \circ \tau = M_k^b$, *and* $M_k^b \circ \tau = M_k^a$.
(3) $W_k^a \circ \tau = W_k^c$, $W_k^c \circ \tau = W_k^b$, *and* $W_k^b \circ \tau = W_k^a$.

Here $W_k^0 \circ \tau = W_k^0$ means that there exists a linear isomorphism $\phi(\tau) : W_k^0 \longrightarrow W_k^0$ such that $\phi(\tau) Y_{W_k^0}(v, z) \phi(\tau)^{-1} = Y_{W_k^0}(\tau v, z)$ for all $v \in M$. The automorphism τ of V_L fixes ω^3 and so W_k^0 is invariant under τ. Hence we can take τ as $\phi(\tau)$. Note also that $\tau Y(v, z)\tau^{-1} = Y(\tau v, z)$ for all $v \in M = M_k^0$ since $\tau \in \mathrm{Aut}(M)$.

4.1. Irreducible M^τ-modules in untwisted M-modules. We first find 8 irreducible M^τ-modules inside the 8 irreducible modules for M. Recall that $M(\epsilon) = \{v \in M_k^0 \mid \tau v = \xi^\epsilon v\}$. Likewise, set $W(\epsilon) = \{v \in W_k^0 \mid \tau v = \xi^\epsilon v\}$. From Lemma 4.1, [**11**, Theorem 4.4] and [**13**, Theorem 6.14], we see that $M(\epsilon)$ and $W(\epsilon)$ are inequivalent irreducible M^τ-modules for $\epsilon = 0, 1, 2$. Note that M_k^i, $i = a, b, c$ are equivalent irreducible M^τ-modules and that W_k^i, $i = a, b, c$ are also equivalent irreducible M^τ-modules by [**13**, Theorem 6.14]. Hence we obtain 8 inequivalent irreducible M^τ-modules.

The top levels, that is, the weight subspaces of the smallest weights of $M(0)$, $M(1)$, and $M(2)$ are $\mathbb{C}\mathbf{1}$, $\mathbb{C}u^1$, and $\mathbb{C}u^2$ respectively. The top levels of $W(0)$, $W(1)$, and $W(2)$ are

$$\mathbb{C}(y(\alpha_1) + y(\alpha_2) + y(\alpha_0)), \quad \mathbb{C}(\alpha_1(-1) - \xi\alpha_2(-1)), \quad \text{and}$$
$$\mathbb{C}(\alpha_1(-1) - \xi^2\alpha_2(-1))$$

respectively. Moreover, the top levels of M_k^c and W_k^c are

$$\mathbb{C}(e^{\beta_1/2} - e^{-\beta_1/2}) \quad \text{and} \quad \mathbb{C}(e^{\beta_1/2} + e^{-\beta_1/2})$$

respectively. All of those top levels are of dimension one.

Next, we deal with the action of $L(0)$ and $J(0)$ on those top levels. The operator $L(0)$ acts as multiplication by the weight of each top level. For the calculation of the action of $J(0)$, we first notice that

$$[w(\alpha_i)_1, w(\alpha_j)_1] = (w(\alpha_i)_0 w(\alpha_j))_2 + (w(\alpha_i)_1 w(\alpha_j))_1$$

by (3.2). Since $w(\alpha_i)_1 w(\alpha_j) = w(\alpha_j)_1 w(\alpha_i)$, it follows that

$$\begin{aligned} J(0) &= (w(\alpha_1)_0 w(\alpha_2))_2 - (w(\alpha_2)_0 w(\alpha_1))_2 \\ &= [w(\alpha_1)_1, w(\alpha_2)_1] - [w(\alpha_2)_1, w(\alpha_1)_1]. \end{aligned}$$

Using this formula it is relatively easy to calculate the eigenvalue for the action of $J(0)$ on each of the 8 top levels. The results are collected in Table 1.

4.2. Irreducible M^τ-modules in τ-twisted M-modules. Using [**9**], we show that there are exactly two inequivalent irreducible τ-twisted (resp. τ^2-twisted) M-modules. Moreover, we find 3 inequivalent irreducible M^τ-modules in each of the irreducible τ-twisted (resp. τ^2-twisted) M-modules. Those irreducible τ-twisted (resp. τ^2-twisted) M-modules will in turn be constructed inside irreducible τ-twisted (resp. τ^2-twisted) V_L-modules. Basic references to twisted modules for lattice vertex operator algebras are [**6**], [**7**] and [**25**]. The argument here is similar to that in [**22**, Section 6].

Table 1. Irreducible M^τ-modules in M_k^i and W_k^i.

irred. module	top level	$L(0)$	$J(0)$
$M(0)$	$\mathbb{C}\mathbf{1}$	0	0
$M(1)$	$\mathbb{C}u^1$	2	$-12\sqrt{-3}$
$M(2)$	$\mathbb{C}u^2$	2	$12\sqrt{-3}$
$W(0)$	$\mathbb{C}(y(\alpha_1)+y(\alpha_2)+y(\alpha_0))$	$\frac{8}{5}$	0
$W(1)$	$\mathbb{C}(\alpha_1(-1)-\xi\alpha_2(-1))$	$\frac{3}{5}$	$2\sqrt{-3}$
$W(2)$	$\mathbb{C}(\alpha_1(-1)-\xi^2\alpha_2(-1))$	$\frac{3}{5}$	$-2\sqrt{-3}$
M_k^c	$\mathbb{C}(e^{\beta_1/2}-e^{-\beta_1/2})$	$\frac{1}{2}$	0
W_k^c	$\mathbb{C}(e^{\beta_1/2}+e^{-\beta_1/2})$	$\frac{1}{10}$	0

We follow [7] with $L=\sqrt{2}A_2$, $p=3$, $q=6$, and $\nu=\tau$. Let $\mathfrak{h}=\mathbb{C}\otimes_{\mathbb{Z}}L$ and extend the $\mathbb{Z}$-bilinear form $\langle\cdot,\cdot\rangle$ on L to $\mathfrak{h}$ linearly. Set

$$h_1=\frac{1}{3}(\beta_1+\xi^2\beta_2+\xi\beta_0),\qquad h_2=\frac{1}{3}(\beta_1+\xi\beta_2+\xi^2\beta_0).$$

Then $\tau h_j=\xi^j h_j$, $\langle h_1,h_1\rangle=\langle h_2,h_2\rangle=0$, and $\langle h_1,h_2\rangle=2$. Moreover, $\beta_i=\xi^{i-1}h_1+\xi^{2(i-1)}h_2$, $i=0,1,2$. For $n\in\mathbb{Z}$, set

$$\mathfrak{h}_{(n)}=\{\alpha\in\mathfrak{h}\mid\tau\alpha=\xi^n\alpha\}.$$

Since τ is fixed-point-free on L, it follows that $\mathfrak{h}_{(0)}=0$. Furthermore, $\mathfrak{h}_{(1)}=\mathbb{C}h_1$ and $\mathfrak{h}_{(2)}=\mathbb{C}h_2$. For $\alpha\in\mathfrak{h}$, we denote by $\alpha_{(n)}$ the component of α in $\mathfrak{h}_{(n)}$. Thus $(\beta_i)_{(1)}=\xi^{i-1}h_1$ and $(\beta_i)_{(2)}=\xi^{2(i-1)}h_2$ for $i=0,1,2$.

Define the τ-twisted affine Lie algebra to be

$$\hat{\mathfrak{h}}[\tau]=\left(\bigoplus_{n\in\mathbb{Z}}\mathfrak{h}_{(n)}\otimes t^{n/3}\right)\oplus\mathbb{C}c$$

with the bracket

$$[x\otimes t^m,y\otimes t^n]=m\langle x,y\rangle\delta_{m+n,0}c$$

for $x\in\mathfrak{h}_{(3m)}$, $y\in\mathfrak{h}_{(3n)}$, $m,n\in(1/3)\mathbb{Z}$, and $[c,\hat{\mathfrak{h}}[g]]=0$. The isometry τ acts on $\hat{\mathfrak{h}}[\tau]$ by $\tau(x\otimes t^{m/3})=\xi^m x\otimes t^{m/3}$ and $\tau(c)=c$. Set

$$\hat{\mathfrak{h}}[\tau]^+=\bigoplus_{n>0}\mathfrak{h}_{(n)}\otimes t^{n/3},\quad \hat{\mathfrak{h}}[\tau]^-=\bigoplus_{n<0}\mathfrak{h}_{(n)}\otimes t^{n/3},\quad\text{and}\quad \hat{\mathfrak{h}}[\tau]^0=\mathbb{C}c$$

and consider the $\hat{\mathfrak{h}}[\tau]$-module

$$S[\tau]=U(\hat{\mathfrak{h}}[\tau])\otimes_{U(\hat{\mathfrak{h}}[\tau]^+\oplus\hat{\mathfrak{h}}[\tau]^0)}\mathbb{C}$$

induced from the $\hat{\mathfrak{h}}[\tau]^+ \oplus \hat{\mathfrak{h}}[\tau]^0$-module $\mathbb{C}$, where $\hat{\mathfrak{h}}[\tau]^+$ acts trivially on $\mathbb{C}$ and c acts as 1 on $\mathbb{C}$.

We define the weight in $S[\tau]$ by

$$\operatorname{wt}(x \otimes t^n) = -n \quad \text{and} \quad \operatorname{wt} 1 = \frac{1}{9},$$

where $n \in (1/3)\mathbb{Z}$ and $x \in \mathfrak{h}_{(3n)}$ (cf. [7, (4.6), (4.10)]). By the weight gradation $S[\tau]$ becomes a $(1/3)\mathbb{Z}$-graded space. Its character is

$$\operatorname{ch} S[\tau] = q^{1/9} \prod_{n=1}^{\infty} (1 - q^n) \Big/ \prod_{n=1}^{\infty} (1 - q^{n/3}). \tag{4.1}$$

For $\alpha \in \mathfrak{h}$ and $n \in (1/3)\mathbb{Z}$, denote by $\alpha(n)$ the operator on $S[\tau]$ induced by $\alpha_{(3n)} \otimes t^n$. Then, as a vector space $S[\tau]$ can be identified with a polynomial algebra with variables $h_1(1/3 + n)$ and $h_2(2/3 + n)$, $n \in \mathbb{Z}$. The weight of the operator $h_j(j/3 + n)$ is $-j/3 - n$.

The alternating $\mathbb{Z}$-bilinear map $c_0^\tau : L \times L \to \mathbb{Z}/6\mathbb{Z}$ defined by [7, (2.10)] is such that

$$c_0^\tau(\alpha, \beta) = \sum_{r=0}^{2} (3 + 2r) \langle \tau^r \alpha, \beta \rangle + 6\mathbb{Z}.$$

In our case $\sum_{r=0}^{2} \tau^r \alpha = 0$, since τ is fixed-point-free on L. Moreover, we can verify that

$$\sum_{r=0}^{2} r \langle \tau^r \beta_i, \beta_j \rangle = \begin{cases} \pm 6 & \text{if } \tau\beta_i \neq \beta_j \\ 0 & \text{if } \tau\beta_i = \beta_j. \end{cases}$$

Hence $c_0^\tau(\alpha, \beta) = 0$ for all $\alpha, \beta \in L$. This means that the central extension

$$1 \longrightarrow \langle \kappa_6 \rangle \longrightarrow \hat{L}_\tau \xrightarrow{\ -\ } L \longrightarrow 1 \tag{4.2}$$

determined by the commutator condition $aba^{-1}b^{-1} = \kappa_6^{c_0^\tau(\bar{a}, \bar{b})}$ splits.

We consider the relation between two central extensions $\hat{L}$ of (2.1) and $\hat{L}_\tau$ of (4.2). Since both of $\hat{L}$ and $\hat{L}_\tau$ are split extensions, we use the same symbol e^α to denote both of an element in $\hat{L}$ and an element in $\hat{L}_\tau$ which correspond naturally to $\alpha \in L$. Actually, in Section 2 we choose $e^\alpha \in \hat{L}$ so that the multiplication in $\hat{L}$ is $e^\alpha \times e^\beta = e^{\alpha+\beta}$. Also we can choose $e^\alpha \in \hat{L}_\tau$ such that the multiplication $e^\alpha \times_\tau e^\beta$ in $\hat{L}_\tau$ is related to the multiplication in $\hat{L}$ by (cf. [7, (2.4)])

$$e^\alpha \times e^\beta = \kappa_6^{\varepsilon_0(\alpha, \beta)} e^\alpha \times_\tau e^\beta, \tag{4.3}$$

where the $\mathbb{Z}$-linear map $\varepsilon_0 : L \times L \to \mathbb{Z}/6\mathbb{Z}$ is defined by [7, (2.13)]. In our case

$$\varepsilon_0(\alpha, \beta) = -\langle \tau^{-1}\alpha, \beta \rangle + 6\mathbb{Z}. \tag{4.4}$$

As in Section 2, we usually write $e^\alpha e^\beta = e^{\alpha+\beta}$ to denote the product of e^α and e^β in $\hat{L}$. Note, for example, that the inverse of e^{β_1} in $\hat{L}$ is $e^{-\beta_1}$, while the inverse of e^{β_1} in $\hat{L}_\tau$ is $\kappa_3^2 e^{-\beta_1}$.

The automorphism τ of L lifts to an automorphism $\hat{\tau}$ of $\hat{L}$ such that $\hat{\tau}(e^\alpha) = e^{\tau\alpha}$ and $\hat{\tau}(\kappa_6) = \kappa_6$. Since ε_0 is τ-invariant, we can also think $\hat{\tau}$ to be an automorphism of $\hat{L}_\tau$ in a similar way. By abuse of notation we shall denote $\hat{\tau}$ by simply τ also.

We have $(1-\tau)L = \mathrm{span}_{\mathbb{Z}}\{\beta_1-\beta_2, \beta_1+2\beta_2\}$. The quotient group $L/(1-\tau)L$ is of order 3 and generated by $\beta_1+(1-\tau)L$. Now $K = \{a^{-1}\tau(a) \,|\, a \in \hat{L}_\tau\}$ is a central subgroup of $\hat{L}_\tau$ with $\overline{K} = (1-\tau)L$ and $K \cap \langle\kappa_6\rangle = 1$. Here note that a^{-1} is the inverse of a in $\hat{L}_\tau$ and $a^{-1}\tau(a)$ is the product $a^{-1} \times_\tau \tau(a)$ in $\hat{L}_\tau$. In $\hat{L}_\tau$ we can verify that

$$e^{3\beta_1} = (e^{\beta_0-\beta_1})^{-1} \times_\tau \tau(e^{\beta_0-\beta_1}) \in K.$$

Since

$$\kappa_3 e^{\beta_1} \times_\tau \kappa_3 e^{\beta_1} \times_\tau \kappa_3 e^{\beta_1} = e^{3\beta_1} \qquad \text{and} \qquad \kappa_3 e^{\beta_1} \times_\tau \kappa_3 e^{-\beta_1} = 1,$$

it follows that

$$\hat{L}_\tau/K = \{K, \kappa_3 e^{\beta_1}K, \kappa_3 e^{-\beta_1}K\} \times \langle\kappa_6\rangle K/K \cong \mathbb{Z}_3 \times \mathbb{Z}_6.$$

For $j = 0,1,2$, define a linear character $\chi_j : \hat{L}_\tau/K \to \mathbb{C}^\times$ by

$$\chi_j(\kappa_6) = \xi_6, \quad \chi_j(\kappa_3 e^{\beta_1}K) = \xi^j, \quad \text{and} \quad \chi_j(\kappa_3 e^{-\beta_1}K) = \xi^{-j},$$

where $\xi_6 = \exp(2\pi\sqrt{-1}/6)$. Let T_{χ_j} be the one-dimensional $\hat{L}_\tau/K$-module affording the character χ_j. As an $\hat{L}_\tau$-module, K acts trivially on T_{χ_j}. Since $\sum_{r=0}^2 \tau^r\alpha = 0$ for $\alpha \in L$, those T_{χ_j}, $j = 0,1,2$, are the irreducible $\hat{L}_\tau$-modules constructed in [**25**, Section 6].

Let

$$V_L^{T_{\chi_j}} = V_L^{T_{\chi_j}}(\tau) = S[\tau] \otimes T_{\chi_j}$$

and define the τ-twisted vertex operator $Y^\tau(\cdot, z) : V_L \to \mathrm{End}(V_L^{T_{\chi_j}})\{z\}$ as in [**7**]. For $a \in \hat{L}$, define

$$Y^\tau(a,z) = 3^{-\langle\bar{a},\bar{a}\rangle/2}\phi(\bar{a})E^-(-\bar{a},z)E^+(-\bar{a},z)az^{-\langle\bar{a},\bar{a}\rangle/2},$$

where

$$E^\pm(\alpha,z) = \exp\left(\sum_{n\in(1/3)\mathbb{Z}_\pm} \frac{\alpha(n)}{n} z^{-n}\right), \tag{4.5}$$

$$\phi(\alpha) = (1-\xi^2)^{\langle\tau\alpha,\alpha\rangle}, \tag{4.6}$$

and $a \in \hat{L}$ acts on T_{χ_j} through the set theoretic identification between $\hat{L}$ and $\hat{L}_\tau$. Here we denote $\sigma(\alpha)$ of [7, (4.35)] by $\phi(\alpha)$. For $v = \alpha^1(-n_1)\cdots\alpha^k(-n_k)\cdot\iota(a) \in V_L$ with $\alpha^1,\ldots,\alpha^k \in \mathfrak{h}$ and $n_1,\ldots,n_k \in \mathbb{Z}_{>0}$, set

$$W(v,z) = {}^{\circ}_{\circ}\left(\frac{1}{(n_1-1)!}\left(\frac{d}{dz}\right)^{n_1-1}\alpha^1(z)\right)\cdots \left(\frac{1}{(n_k-1)!}\left(\frac{d}{dz}\right)^{n_k-1}\alpha^k(z)\right)Y^\tau(a,z)^{\circ}_{\circ},$$

where $\alpha(z) = \sum_{n\in(1/3)\mathbb{Z}}\alpha(n)z^{-n-1}$. Define constants $c^i_{mn} \in \mathbb{C}$ for $m,n \geq 0$ and $i = 0,1,2$ by

$$\sum_{m,n\geq 0} c^0_{mn}x^m y^n = -\frac{1}{2}\sum_{r=1}^{2}\log\left(\frac{(1+x)^{1/3}-\xi^{-r}(1+y)^{1/3}}{1-\xi^{-r}}\right),$$
$$\sum_{m,n\geq 0} c^i_{mn}x^m y^n = \frac{1}{2}\log\left(\frac{(1+x)^{1/3}-\xi^{-i}(1+y)^{1/3}}{1-\xi^{-i}}\right) \quad \text{for} \quad i \neq 0.$$

Let $\{\gamma_1,\gamma_2\}$ be an orthonormal basis of $\mathfrak{h}$ and set

$$\Delta_z = \sum_{m,n\geq 0}\sum_{i=0}^{2}\sum_{j=1}^{2} c^i_{mn}(\tau^{-i}\gamma_j)(m)\gamma_j(n)z^{-m-n}.$$

Then for $v \in V_L$, $Y^\tau(v,z)$ is defined by

$$Y^\tau(v,z) = W(e^{\Delta_z}v,z).$$

We extend the action of τ to $V_L^{T_{\chi_j}}$ so that τ is the identity on T_{χ_j}. The weight of every element in T_{χ_j} is defined to be 0. Then the character of $V_L^{T_{\chi_j}}$ is identical with that of $S[\tau]$.

By [7, Theorem 7.1], $(V_L^{T_{\chi_j}}(\tau), Y^\tau(\cdot,z))$, $j = 0,1,2$ are inequivalent irreducible τ-twisted V_L-modules. Now among the 12 irreducible V_L-modules $V_{L^{(i,j)}}$, $i \in \{0,a,b,c\}$ and $j \in \{0,1,2\}$, the τ-stable irreducible modules are $V_{L^{(0,j)}}$, $j \in \{0,1,2\}$. Hence by [9, Theorem 10.2], we conclude that $(V_L^{T_{\chi_j}}(\tau), Y^\tau(\cdot,z))$, $j = 0,1,2$, are all the inequivalent irreducible τ-twisted V_L-modules. The isometry θ of $(L,\langle\cdot,\cdot\rangle)$ induces a permutation on $V_L^{T_{\chi_j}}(\tau)$, $j = 0,1,2$. In fact, the permutation leaves $V_L^{T_{\chi_0}}(\tau)$ invariant and interchanges $V_L^{T_{\chi_1}}(\tau)$ and $V_L^{T_{\chi_2}}(\tau)$.

Since $M^\tau \otimes M^0_t$ is contained in the subalgebra $(V_L)^\tau$ of fixed points of τ in V_L, we can deal with $(V_L^{T_{\chi_j}}(\tau), Y^\tau(\cdot,z))$ as an $M^\tau \otimes M^0_t$-module. We shall find 6 irreducible M^τ-modules inside $V_L^{T_{\chi_j}}(\tau)$. Recall that ω, ω^3, and

$\widetilde{\omega} = \omega + \omega^3$ are the Virasoro element of M^τ, M_t^0, and V_L respectively. Our main tool is a careful study of the action of ω_1 on homogeneous subspaces of $V_L^{T_{\chi_j}}(\tau)$ of small weights. Here we denote by u_n the coefficient of z^{-n-1} in the twisted vertex operator $Y^\tau(u,z) = \sum u_n z^{-n-1}$ associated with a vector u in V_L. The weight in $V_L^{T_{\chi_j}}(\tau)$ defined above is exactly the eigenvalue for $\widetilde{\omega}_1$ (cf. [7, (6.10), (6.28)]).

The character of $V_L^{T_{\chi_j}}(\tau)$ is equal to the character of $S[\tau]$ (cf. (4.1)). Its first several terms are

$$\begin{aligned}\operatorname{ch} V_L^{T_{\chi_j}}(\tau) &= \operatorname{ch} S[\tau] \\ &= q^{1/9} + q^{1/9+1/3} + 2q^{1/9+2/3} + 2q^{1/9+1} + 4q^{1/9+4/3} + \cdots .\end{aligned}$$

Fix a nonzero vector $v \in T_{\chi_j}$. We can choose a basis of each homogeneous subspace of $V_L^{T_{\chi_j}}(\tau)$ of weight at most $1/9 + 4/3$ as in Table 2.

Table 2. Basis of homogeneous subspace in $V_L^{T_{\chi_j}}(\tau)$.

weight	basis
$\frac{1}{9}$	$1 \otimes v$
$\frac{1}{9}+\frac{1}{3}$	$h_2(-\frac{1}{3}) \otimes v$
$\frac{1}{9}+\frac{2}{3}$	$h_1(-\frac{2}{3}) \otimes v, \quad h_2(-\frac{1}{3})^2 \otimes v$
$\frac{1}{9}+1$	$h_1(-\frac{2}{3})h_2(-\frac{1}{3}) \otimes v, \quad h_2(-\frac{1}{3})^3 \otimes v$
$\frac{1}{9}+\frac{4}{3}$	$h_2(-\frac{4}{3}) \otimes v, \quad h_1(-\frac{2}{3})^2 \otimes v, \quad h_1(-\frac{2}{3})h_2(-\frac{1}{3})^2 \otimes v, \quad h_2(-\frac{1}{3})^4 \otimes v$

We need to know the action of ω_1 on those bases. For this purpose, notice that

$$Y^\tau(e^{\pm\beta_i}, z) = -\frac{1}{27} E^-(\mp\beta_i, z) E^+(\mp\beta_i, z) \xi^{\pm j} z^{-2}, \qquad i = 0, 1, 2,$$

since $\phi(\pm\beta_i) = -\xi/3$ and since $e^{\pm\beta_i}$ acts on T_{χ_j} as a multiplication by $\chi_j(e^{\pm\beta_i}) = \xi^{\pm j-1}$ for $i, j = 0, 1, 2$. The image of the vectors in Table 2 under the operator ω_1 are calculated as follows:

$$\begin{aligned}
\omega_1(1\otimes v) &= \left(\frac{1}{15}+\frac{1}{45}(\xi^j+\xi^{-j})\right)1\otimes v,\\
\omega_1\left(h_2\left(-\frac{1}{3}\right)\otimes v\right) &= \left(\frac{4}{15}-\frac{1}{9}(\xi^j+\xi^{-j})\right)h_2\left(-\frac{1}{3}\right)\otimes v,\\
\omega_1\left(h_1\left(-\frac{2}{3}\right)\otimes v\right) &= \left(\frac{7}{15}-\frac{2}{45}(\xi^j+\xi^{-j})\right)h_1\left(-\frac{2}{3}\right)\otimes v\\
&\quad -\frac{1}{5}(\xi^j-\xi^{-j})h_2\left(-\frac{1}{3}\right)^2\otimes v,\\
\omega_1\left(h_2\left(-\frac{1}{3}\right)^2\otimes v\right) &= \left(\frac{7}{15}+\frac{7}{45}(\xi^j+\xi^{-j})\right)h_2\left(-\frac{1}{3}\right)^2\otimes v\\
&\quad +\frac{2}{15}(\xi^j-\xi^{-j})h_1\left(-\frac{2}{3}\right)^2\otimes v,\\
\omega_1\left(h_1\left(-\frac{2}{3}\right)h_2\left(-\frac{1}{3}\right)\otimes v\right) &= \left(\frac{2}{3}+\frac{2}{9}(\xi^j+\xi^{-j})\right)h_1\left(-\frac{2}{3}\right)h_2\left(-\frac{1}{3}\right)\otimes v\\
&\quad +\frac{1}{5}(\xi^j-\xi^{-j})h_2\left(-\frac{1}{3}\right)^3\otimes v,\\
\omega_1\left(h_2\left(-\frac{1}{3}\right)^3\otimes v\right) &= \left(\frac{2}{3}+\frac{1}{45}(\xi^j+\xi^{-j})\right)h_2\left(-\frac{1}{3}\right)^3\otimes v\\
&\quad -\frac{2}{5}(\xi^j-\xi^{-j})h_1\left(-\frac{2}{3}\right)h_2\left(-\frac{1}{3}\right)\otimes v,
\end{aligned}$$

$$\begin{aligned}
&\omega_1\left(h_2\left(-\frac{4}{3}\right)\otimes v\right)\\
&= \frac{13}{15}h_2\left(-\frac{4}{3}\right)\otimes v\\
&\quad + (\xi^j+\xi^{-j})\left(-\frac{1}{90}h_2\left(-\frac{4}{3}\right)-\frac{3}{10}h_1\left(-\frac{2}{3}\right)h_2\left(-\frac{1}{3}\right)^2\right)\otimes v\\
&\quad + (\xi^j-\xi^{-j})\left(-\frac{1}{20}h_1\left(-\frac{2}{3}\right)^2-\frac{3}{20}h_2\left(-\frac{1}{3}\right)^4\right)\otimes v,
\end{aligned}$$

$$\begin{aligned}
&\omega_1\left(h_1\left(-\frac{2}{3}\right)^2\otimes v\right)\\
&=\frac{13}{15}h_1\left(-\frac{2}{3}\right)^2\otimes v\\
&\quad+(\xi^j+\xi^{-j})\left(-\frac{1}{90}h_1\left(-\frac{2}{3}\right)^2+\frac{3}{10}h_2\left(-\frac{1}{3}\right)^4\right)\otimes v\\
&\quad+(\xi^j-\xi^{-j})\left(\frac{1}{15}h_2\left(-\frac{4}{3}\right)+\frac{1}{5}h_1\left(-\frac{2}{3}\right)h_2\left(-\frac{1}{3}\right)^2\right)\otimes v,
\end{aligned}$$

$$\begin{aligned}
&\omega_1\left(h_1\left(-\frac{2}{3}\right)h_2\left(-\frac{1}{3}\right)^2\otimes v\right)\\
&=\frac{13}{15}h_1\left(-\frac{2}{3}\right)h_2\left(-\frac{1}{3}\right)^2\otimes v\\
&\quad+(\xi^j+\xi^{-j})\left(-\frac{2}{15}h_2\left(-\frac{4}{3}\right)-\frac{14}{45}h_1\left(-\frac{2}{3}\right)h_2\left(-\frac{1}{3}\right)^2\right)\otimes v\\
&\quad-\frac{1}{15}(\xi^j-\xi^{-j})h_1\left(-\frac{2}{3}\right)^2\otimes v,
\end{aligned}$$

$$\begin{aligned}
\omega_1\left(h_2\left(-\frac{1}{3}\right)^4\otimes v\right)&=\frac{13}{15}h_2\left(-\frac{1}{3}\right)^4\otimes v\\
&\quad+(\xi^j+\xi^{-j})\left(\frac{2}{5}h_1\left(-\frac{2}{3}\right)^2-\frac{1}{9}h_2\left(-\frac{1}{3}\right)^4\right)\otimes v\\
&\quad+\frac{4}{15}(\xi^j-\xi^{-j})h_2\left(-\frac{4}{3}\right)\otimes v.
\end{aligned}$$

The decomposition of $V_L^{T_{\chi_j}}(\tau)$ as a τ-twisted $M\otimes M_t^0$-module was studied in [**22**]. The outline of the argument is as follows: For $j=0,1,2$, the vectors

$$1\otimes v,\quad h_1\left(-\frac{2}{3}\right)\otimes v+(\xi^j-\xi^{-j})h_2\left(-\frac{1}{3}\right)^2\otimes v,$$
$$h_2\left(-\frac{1}{3}\right)^2\otimes v+\frac{2}{3}(\xi^j-\xi^{-j})h_1\left(-\frac{2}{3}\right)\otimes v$$

are simultaneous eigenvectors for ω_1 and $(\omega^3)_1$. Denote by k_1 and k_2 the eigenvalues for ω_1 and $(\omega^3)_1$ respectively. Then the pairs (k_1,k_2) are

j	0	1	2
$1 \otimes v$	$(\frac{1}{9}, 0)$	$(\frac{2}{45}, \frac{1}{15})$	$(\frac{2}{45}, \frac{1}{15})$
$h_1(-\frac{2}{3}) \otimes v + (\xi^j - \xi^{-j})h_2(-\frac{1}{3})^2 \otimes v$	$(\frac{17}{45}, \frac{2}{5})$	$(\frac{1}{9}, \frac{2}{3})$	$(\frac{1}{9}, \frac{2}{3})$
$h_2(-\frac{1}{3})^2 \otimes v + \frac{2}{3}(\xi^j - \xi^{-j})h_1(-\frac{2}{3}) \otimes v$	$(\frac{7}{9}, 0)$	$(\frac{32}{45}, \frac{1}{15})$	$(\frac{32}{45}, \frac{1}{15})$

We first discuss the decomposition of $V_L^{T_{\chi_j}}(\tau)$ into a direct sum of irreducible M_t^0-modules. We use the classification of irreducible M_t^0-modules [23] and their fusion rules [28]. Note also that the vector $y(\alpha_1) + y(\alpha_2) + y(\alpha_0)$ in $(V_L)^\tau$ is an eigenvector for ω_1 of eigenvalue $8/5$. Hence $(V_L)^\tau$ contains the $\mathrm{Vir}(\omega) \otimes M_t^0$-submodule generated by the vector, which is isomorphic to

$$L\left(\frac{6}{5}, \frac{8}{5}\right) \otimes \left(L\left(\frac{4}{5}, \frac{2}{5}\right) + L\left(\frac{4}{5}, \frac{7}{5}\right)\right).$$

Set

$$M_T^0(\tau) = \left\{u \in V_L^{T_{\chi_0}}(\tau) \,|\, (\omega^3)_1 u = 0\right\},$$
$$W_T^0(\tau) = \left\{u \in V_L^{T_{\chi_0}}(\tau) \,|\, (\omega^3)_1 u = \frac{2}{5}u\right\}.$$

Moreover, for $j = 1, 2$ set

$$M_T^j(\tau) = \left\{u \in V_L^{T_{\chi_j}}(\tau) \,|\, (\omega^3)_1 u = \frac{2}{3}u\right\},$$
$$W_T^j(\tau) = \left\{u \in V_L^{T_{\chi_j}}(\tau) \,|\, (\omega^3)_1 u = \frac{1}{15}u\right\}.$$

Then, by [22, Proposition 6.8], $M_T^j(\tau)$ and $W_T^j(\tau)$, $j = 0, 1, 2$, are irreducible τ-twisted M-modules. Furthermore, for $j = 0, 1, 2$,

$$V_L^{T_{\chi_j}}(\tau) \cong M_T^j(\tau) \otimes M_t^j \oplus W_T^j(\tau) \otimes W_t^j$$

as τ-twisted $M \otimes M_t^0$-modules.

There are at most two inequivalent irreducible τ-twisted M-modules by Lemma 4.1 and [9, Theorem 10.2]. Then, looking at the smallest weight of $M_T^j(\tau)$ and $W_T^j(\tau)$, we have that $M_T^0(\tau) \cong M_T^1(\tau) \cong M_T^2(\tau)$ and $W_T^0(\tau) \cong W_T^1(\tau) \cong W_T^2(\tau)$ and that $M_T^0(\tau) \not\cong W_T^0(\tau)$ as τ-twisted M-modules. We denote $M_T^0(\tau)$ by $M_T(\tau)$ and $W_T^0(\tau)$ by $W_T(\tau)$. We conclude that there are exactly two inequivalent irreducible τ-twisted M-modules, which are represented by $M_T(\tau)$ and $W_T(\tau)$. As τ-twisted $M \otimes \mathrm{Vir}(\omega^3)$-modules, we have

$$V_L^{T_{\chi_0}}(\tau) \cong M_T(\tau) \otimes \left(L\left(\frac{4}{5}, 0\right) + L\left(\frac{4}{5}, 3\right) \right) \oplus W_T(\tau) \otimes \left(L\left(\frac{4}{5}, \frac{2}{5}\right) + L\left(\frac{4}{5}, \frac{7}{5}\right) \right), \tag{4.7}$$

$$V_L^{T_{\chi_1}}(\tau) \cong V_L^{T_{\chi_2}}(\tau) \cong M_T(\tau) \otimes L\left(\frac{4}{5}, \frac{2}{3}\right) \oplus W_T(\tau) \otimes L\left(\frac{4}{5}, \frac{1}{15}\right). \tag{4.8}$$

The first several terms of the characters of $M_T(\tau)$ and $W_T(\tau)$ are

$$\begin{aligned} \operatorname{ch} M_T(\tau) &= q^{\frac{1}{9}} + q^{\frac{1}{9}+\frac{2}{3}} + q^{\frac{1}{9}+1} + q^{\frac{1}{9}+\frac{4}{3}} + \cdots, \\ \operatorname{ch} W_T(\tau) &= q^{\frac{2}{45}} + q^{\frac{2}{45}+\frac{1}{3}} + q^{\frac{2}{45}+\frac{2}{3}} + q^{\frac{2}{45}+1} + 2q^{\frac{2}{45}+\frac{4}{3}} + \cdots. \end{aligned}$$

For $\epsilon = 0, 1, 2$, let

$$\begin{aligned} M_T(\tau)(\epsilon) &= \{u \in M_T(\tau) \mid \tau u = \xi^\epsilon u\}, \\ W_T(\tau)(\epsilon) &= \{u \in W_T(\tau) \mid \tau u = \xi^\epsilon u\}. \end{aligned}$$

Those 6 modules for M^τ are inequivalent irreducible modules by [**30**, Theorem 2]. Their top levels are of dimension one. Those top levels and the eigenvalues for the action of $L^\tau(0) = \omega_1$ and $J^\tau(0) = J_2$ are collected in Table 3.

Table 3. Irreducible M^τ-modules in $M_T(\tau)$ and $W_T(\tau)$.

irred. module	top level	$L^\tau(0)$	$J^\tau(0)$
$M_T(\tau)(0)$	$\mathbb{C}1 \otimes v$	$\frac{1}{9}$	$\frac{14}{81}\sqrt{-3}$
$M_T(\tau)(1)$	$\mathbb{C}h_2(-\frac{1}{3})^2 \otimes v$	$\frac{1}{9} + \frac{2}{3}$	$-\frac{238}{81}\sqrt{-3}$
$M_T(\tau)(2)$	$\mathbb{C}(\frac{4}{3}h_1(-\frac{2}{3})^2 \otimes v + h_2(-\frac{1}{3})^4 \otimes v)$	$\frac{1}{9} + \frac{4}{3}$	$\frac{374}{81}\sqrt{-3}$
$W_T(\tau)(0)$	$\mathbb{C}h_2(-\frac{1}{3})^3 \otimes v$	$\frac{2}{45} + \frac{2}{3}$	$\frac{176}{81}\sqrt{-3}$
$W_T(\tau)(1)$	$\mathbb{C}h_1(-\frac{2}{3}) \otimes v$	$\frac{2}{45} + \frac{1}{3}$	$-\frac{22}{81}\sqrt{-3}$
$W_T(\tau)(2)$	$\mathbb{C}h_2(-\frac{1}{3}) \otimes v$	$\frac{2}{45}$	$-\frac{4}{81}\sqrt{-3}$

4.3. Irreducible M^τ-modules in τ^2-twisted M-modules. Finally, we find 6 irreducible M^τ-modules in τ^2-twisted M-modules. The argument is parallel to that in Subsection 4.2. Instead of τ, we take τ^2. Thus we follow [**7**] with $\nu = \tau^2$. Set $h_1' = h_2$, $h_2' = h_1$, and

$$\mathfrak{h}'_{(n)} = \{\alpha \in \mathfrak{h} \mid \tau^2\alpha = \xi^n\alpha\}.$$

Then $\mathfrak{h}'_{(0)} = 0$, $\mathfrak{h}'_{(1)} = \mathbb{C}h'_1$, and $\mathfrak{h}'_{(2)} = \mathbb{C}h'_2$. Consider a split central extension

$$1 \longrightarrow \langle \kappa_6 \rangle \longrightarrow \hat{L}_{\tau^2} \stackrel{-}{\longrightarrow} L \longrightarrow 1$$

and choose linear characters $\chi'_j : \hat{L}_{\tau^2}/K \to \mathbb{C}^\times$, $j = 0, 1, 2$, such that

$$\chi'_j(\kappa_6) = \xi_6, \quad \chi'_j(\kappa_3 e^{\beta_1} K) = \xi^j, \quad \text{and} \quad \chi'_j(\kappa_3 e^{-\beta_1} K) = \xi^{-j},$$

where $K = \{a^{-1}\tau^2(a) \,|\, a \in \hat{L}_{\tau^2}\}$. Let $T_{\chi'_j}$ be the one-dimensional $\hat{L}_{\tau^2}/K$-module affording the character χ'_j. Then the irreducible τ^2-twisted V_L-module associated with $T_{\chi'_j}$ is

$$V_L^{T_{\chi'_j}}(\tau^2) = S[\tau^2] \otimes T_{\chi'_j}.$$

As a vector space $S[\tau^2]$ is isomorphic to a polynomial algebra with variables $h'_1(1/3 + n)$ and $h'_2(2/3 + n)$, $n \in \mathbb{Z}$. The weight on $S[\tau^2]$ is given by $\operatorname{wt} 1 = 1/9$ and $\operatorname{wt} h'_j(j/3 + n) = -j/3 - n$. Moreover, $\operatorname{wt} v = 0$ for $v \in T_{\chi'_j}$. Set

$$\begin{aligned} M_T(\tau^2) &= \left\{ u \in V_L^{T_{\chi'_0}}(\tau^2) \,|\, (\omega^3)_1 u = 0 \right\}, \\ W_T(\tau^2) &= \left\{ u \in V_L^{T_{\chi'_0}}(\tau^2) \,|\, (\omega^3)_1 u = \frac{2}{5} u \right\}. \end{aligned}$$

Then $M_T(\tau^2)$ and $W_T(\tau^2)$ are the inequivalent irreducible τ^2-twisted M-modules. Furthermore, we have

$$\begin{aligned} V_L^{T_{\chi'_0}}(\tau^2) &\cong M_T(\tau^2) \otimes \left(L\left(\frac{4}{5}, 0\right) + L\left(\frac{4}{5}, 3\right) \right) \\ &\qquad \oplus W_T(\tau^2) \otimes \left(L\left(\frac{4}{5}, \frac{2}{5}\right) + L\left(\frac{4}{5}, \frac{7}{5}\right) \right), \\ V_L^{T_{\chi'_1}}(\tau^2) &\cong V_L^{T_{\chi'_2}}(\tau^2) \cong M_T(\tau^2) \otimes L\left(\frac{4}{5}, \frac{2}{3}\right) \oplus W_T(\tau^2) \otimes L\left(\frac{4}{5}, \frac{1}{15}\right) \end{aligned}$$

as τ^2-twisted $M \otimes \operatorname{Vir}(\omega^3)$-modules. The character of $M_T(\tau^2)$ or $W_T(\tau^2)$ is equal to that of $M_T(\tau)$ or $W_T(\tau)$ respectively. For $\epsilon = 0, 1, 2$, let

$$\begin{aligned} M_T(\tau^2)(\epsilon) &= \{u \in M_T(\tau^2) \,|\, \tau^2 u = \xi^\epsilon u\}, \\ W_T(\tau^2)(\epsilon) &= \{u \in W_T(\tau^2) \,|\, \tau^2 u = \xi^\epsilon u\}. \end{aligned}$$

Those 6 modules for M^τ are inequivalent irreducible modules by [**30**, Theorem 2]. Their top levels and the eigenvalues for the action of $L^{\tau^2}(0) = \omega_1$ and $J^{\tau^2}(0) = J_2$ are collected in Table 4.

Table 4. Irreducible M^τ-modules in $M_T(\tau^2)$ and $W_T(\tau^2)$.

irred. module	top level	$L^{\tau^2}(0)$	$J^{\tau^2}(0)$
$M_T(\tau^2)(0)$	$\mathbb{C}1 \otimes v$	$\frac{1}{9}$	$-\frac{14}{81}\sqrt{-3}$
$M_T(\tau^2)(1)$	$\mathbb{C}h_2'(-\frac{1}{3})^2 \otimes v$	$\frac{1}{9}+\frac{2}{3}$	$\frac{238}{81}\sqrt{-3}$
$M_T(\tau^2)(2)$	$\mathbb{C}(\frac{4}{3}h_1'(-\frac{2}{3})^2 \otimes v + h_2'(-\frac{1}{3})^4 \otimes v)$	$\frac{1}{9}+\frac{4}{3}$	$-\frac{374}{81}\sqrt{-3}$
$W_T(\tau^2)(0)$	$\mathbb{C}h_2'(-\frac{1}{3})^3 \otimes v$	$\frac{2}{45}+\frac{2}{3}$	$-\frac{176}{81}\sqrt{-3}$
$W_T(\tau^2)(1)$	$\mathbb{C}h_1'(-\frac{2}{3}) \otimes v$	$\frac{2}{45}+\frac{1}{3}$	$\frac{22}{81}\sqrt{-3}$
$W_T(\tau^2)(2)$	$\mathbb{C}h_2'(-\frac{1}{3}) \otimes v$	$\frac{2}{45}$	$\frac{4}{81}\sqrt{-3}$

4.4. Remarks on 20 irreducible M^τ-modules. We have obtained 20 irreducible M^τ-modules in Subsections 4.1, 4.2, and 4.3. Note that the top levels of them are of dimension one and they can be distinguished by the eigenvalues for ω_1 and J_2.

The isometry σ of the lattice $(L, \langle\cdot,\cdot\rangle)$ induces a permutation of order 2 on those 20 irreducible M^τ-modules. Clearly, σ leaves $M(0)$ and $W(0)$ invariant and transforms M_k^c (resp. W_k^c) into an irreducible M^τ-module equivalent to M_k^c (resp. W_k^c). Moreover, σ interchanges irreducible M^τ-modules as follows:

$$\begin{array}{rclcrcl} M(1) & \longleftrightarrow & M(2), & \qquad & W(1) & \longleftrightarrow & W(2), \\ M_T(\tau)(\epsilon) & \longleftrightarrow & M_T(\tau^2)(\epsilon), & & W_T(\tau)(\epsilon) & \longleftrightarrow & W_T(\tau^2)(\epsilon) \end{array} \tag{4.9}$$

for $\epsilon = 0,1,2$. The top level of $M_T(\tau^2)(\epsilon)$ can be obtained by replacing $h_j(j/3+n)$ with $h_j'(j/3+n)$ for $j=1,2$ in the top level of $M_T(\tau)(\epsilon)$. Similar symmetry holds for $W_T(\tau^2)(\epsilon)$ and $W_T(\tau)(\epsilon)$. The action of $J(0)$ on the top level of $M_T(\tau^2)(\epsilon)$ (resp. $W_T(\tau^2)(\epsilon)$) is negative of the action on the top level of $M_T(\tau)(\epsilon)$ (resp. $W_T(\tau)(\epsilon)$). These symmetries are consequences of the fact that $\sigma\tau\sigma = \tau^2$ and $\sigma J = -J$.

In [**14**] an infinite series of $2D$ conformal field theory models with $\mathbb{Z}_3$ symmetry was studied. In the case $p=5$ of [**14**], 20 irreducible representations are discussed [**14**, (5.5)]. If we multiply the values $w\begin{pmatrix} n & m \\ n' & m' \end{pmatrix}$ of [**14**, (5.6)] by $\sqrt{-105/2}$, then the pairs

$$\left(\Delta\begin{pmatrix} n & m \\ n' & m' \end{pmatrix}, \sqrt{-105/2}\, w\begin{pmatrix} n & m \\ n' & m' \end{pmatrix}\right)$$

coincide with the pairs of the eigenvalues for ω_1 and J_2 of the top levels of the 20 irreducible M^τ-modules listed in Tables 1, 3 and 4. Here $\Delta\begin{pmatrix} n & m \\ n' & m' \end{pmatrix}$ is given by [**14**, (1.3)].

5. Classification of irreducible modules for M^τ

We show in this section that the 20 irreducible modules discussed in Section 4 are all the inequivalent irreducible modules for M^τ. This is achieved by determining the Zhu algebra $A(\mathcal{W})$ of the vertex operator subalgebra $\mathcal{W}$ in M^τ generated by ω and J. It turns out that $A(\mathcal{W})$ is isomorphic to a quotient algebra of the polynomial algebra $\mathbb{C}[x,y]$ with two variables x and y by a certain ideal $\mathcal{I}$ and that $A(\mathcal{W})$ is of dimension 20. We shall also prove that $M^\tau = \mathcal{W}$ and $\mathcal{W}$ is rational.

As in Theorem 3.3, let $L(n) = \omega_{n+1}$ and $J(n) = J_{n+2}$ for $n \in \mathbb{Z}$. The action of those operators on the vacuum vector $\mathbf{1}$ is such that

$$L(n)\mathbf{1} = 0 \quad \text{for} \quad n \geq -1, \qquad L(-2)\mathbf{1} = \omega, \tag{5.1}$$

$$J(n)\mathbf{1} = 0 \quad \text{for} \quad n \geq -2, \qquad J(-3)\mathbf{1} = J. \tag{5.2}$$

5.1. A spanning set for $\mathcal{W}$. For a vector expressed in the form $u^1_{n_1}\cdots u^k_{n_k}\mathbf{1}$ with $u^i \in \{\omega, J\}$ and $n_i \in \mathbb{Z}$, we denote by $l_\omega(u^1_{n_1}\cdots u^k_{n_k}\mathbf{1})$ or $l_J(u^1_{n_1}\cdots u^k_{n_k}\mathbf{1})$ the number of i, $1 \leq i \leq k$ such that $u^i = \omega$ or $u^i = J$ respectively. We shall call these numbers the ω-length or the J-length of the expression $u^1_{n_1}\cdots u^k_{n_k}\mathbf{1}$. Since each vector in $\mathcal{W}$ is not necessarily expressed uniquely in such a form, the ω-length and the J-length are not defined for a vector. They depend on a specific expression in the form $u^1_{n_1}\cdots u^k_{n_k}\mathbf{1}$.

Lemma 5.1. *Let the ω-length and the J-length of $u^1_{n_1}\cdots u^k_{n_k}\mathbf{1}$ be s and t respectively. Then $u^1_{n_1}\cdots u^k_{n_k}\mathbf{1}$ can be written as a linear combination of vectors of the form*

$$L(-m_1)\cdots L(-m_p)J(-n_1)\cdots J(-n_q)\mathbf{1}$$

such that:

(1) $m_1 \geq \cdots \geq m_p \geq 2, \quad n_1 \geq \cdots \geq n_q \geq 3,$
(2) $q \leq t,$
(3) $p + q \leq s + t,$
(4) $m_1 + \cdots + m_p + n_1 + \cdots + n_q = \mathrm{wt}(u^1_{n_1}\cdots u^k_{n_k}\mathbf{1}).$

Proof. We proceed by induction on t. If $t = 0$, the assertion follows from the commutation relation (3.5) and the action of $L(n)$ on the vacuum vector (5.1).

Suppose the assertion holds for the case where the J-length of $u^1_{n_1}\cdots u^k_{n_k}\mathbf{1}$ is at most $t-1$ and consider the case where the J-length is t. By (3.6), we can replace $J(-n)L(-m)$ with $L(-m)J(-n)$ or $J(-m-n)$. Hence we may assume that $u^1_{n_1}\cdots u^k_{n_k}\mathbf{1}$ is of the form

$$L(-m_1)\cdots L(-m_s)J(-n_1)\cdots J(-n_t)\mathbf{1} \tag{5.3}$$

for some $m_i, n_j \in \mathbb{Z}$.

By (5.2), we may assume that $n_t \geq 3$. Suppose $n_i < n_{i+1}$ for some i. Then by the commutation relation (3.7), the vector (5.3) can be written as a linear combination of the vectors which are obtained by replacing $J(-n_i)J(-n_{i+1})$ with:

(i) $J(-n_{i+1})J(-n_i)$,
(ii) $L(-n_i - n_{i+1})$,
(iii) $L(k)L(-n_i - n_{i+1} - k)$ or $L(-n_i - n_{i+1} - k)L(k)$ for some $k \in \mathbb{Z}$, or
(iv) a constant.

In Cases (ii), (iii), or (iv), we get an expression whose J-length is at most $t-2$, and so we can apply the induction hypothesis. Therefore, in (5.3) we may assume that $n_1 \geq \cdots \geq n_t \geq 3$.

Now we argue by induction on the ω-length s of the expression (5.3). If $s = 0$, the assertion holds. Suppose the assertion holds for the case where the ω-length is at most $s-1$. By (3.6), we can replace $L(-m_s)J(-n_1)$ with:

(i) $J(-n_1)L(-m_s)$ or
(ii) $J(-m_s - n_1)$.

In Case (ii), we get an expression of ω-length at most $s-1$, so that we can apply the induction hypothesis. Arguing similarly, we can reach

$$L(-m_1)\cdots L(-m_{s-1})J(-n_1)\cdots J(-n_t)L(-m_s)\mathbf{1}.$$

Hence we may assume that $m_s \geq 2$ by (5.1). Suppose $m_i < m_{i+1}$ for some i. Then by (3.5), the vector (5.3) can be written as a linear combination of the vectors which are obtained by replacing $L(-m_i)L(-m_{i+1})$ with:

(i) $L(-m_{i+1})L(-m_i)$,
(ii) $L(-m_i - m_{i+1})$, or
(iii) a constant.

Since Case (ii) or (iii) yields an expression whose ω-length is at most $s-1$, we can apply the induction hypothesis. This completes the proof. □

A vector of the form

$$L(-m_1)\cdots L(-m_p)J(-n_1)\cdots J(-n_q)\mathbf{1} \tag{5.4}$$

with $m_1 \geq \cdots \geq m_p \geq 2$ and $n_1 \geq \cdots \geq n_q \geq 3$ will be called of normal form.

Corollary 5.2. *$\mathcal{W}$ is spanned by the vectors of normal form*

$$L(-m_1)\cdots L(-m_p)J(-n_1)\cdots J(-n_q)\mathbf{1}$$

with $m_1 \geq \cdots \geq m_p \geq 2$, $n_1 \geq \cdots \geq n_q \geq 3$, $p = 0, 1, 2, \ldots,$ and $q = 0, 1, 2, \ldots.$

Proof. As a vector space $\mathcal{W}$ is spanned by the vectors $u^1_{n_1}\cdots u^k_{n_k}\mathbf{1}$ with $u^i \in \{\omega, J\}$, $n_i \in \mathbb{Z}$, and $k = 0, 1, 2, \ldots$. Hence the assertion follows from Lemma 5.1. □

A similar argument for a spanning set can be found in [**12**, Section 3]. See also [**3**, Section 2.2].

Remark 5.3. Let U be an admissible $\mathcal{W}$-module generated by $u \in U$ such that $L(n)u = J(n)u = 0$ for $n > 0$ and $L(0)u = hu$, $J(0)u = ku$ for some $h, k \in \mathbb{C}$. It can be proved in a same way that U is spanned by

$$L(-m_1)\cdots L(-m_p)J(-n_1)\cdots J(-n_q)u$$

with $m_1 \geq \cdots \geq m_p \geq 1$, $n_1 \geq \cdots \geq n_q \geq 1$, $p = 0, 1, 2, \ldots$, and $q = 0, 1, 2, \ldots$.

5.2. A singular vector $\mathbf{v}^{12}$. A singular vector v of weight h for $\mathcal{W}$ is by definition a vector v which satisfies

(1) $L(0)v = hv$,
(2) $L(n)v = 0$ and $J(n)v = 0$ for $n \geq 1$.

Note that v is not necessarily an eigenvector for $J(0)$. By commutation relations (3.5) and (3.6), it is easy to show that the condition (2) holds if v satisfies

(2′) $L(1)v = L(2)v = J(1)v = 0$.

We consider $\mathcal{W}$ as a space spanned by the vectors of the form (5.4) . The weight of such a vector is $m_1 + \cdots + m_p + n_1 + \cdots + n_q$. Let v be a linear combination of the vectors of the form (5.4) of weight h. For example, there are 76 vectors of the form (5.4) of weight 12. We use the conditions (5.1) and (5.2) and the commutation relations (3.5), (3.6), and (3.7) to compute $L(1)v$, $L(2)v$, and $J(1)v$. This computation was done by a computer algebra system Risa/Asir. The result is as follows:

Lemma 5.4. *Let v be a linear combination of the vectors of the form* (5.4) *of weight h. Under the conditions* (5.1) *and* (5.2) *and the commutation relations* (3.5), (3.6) *and* (3.7), *we have $L(1)v = L(2)v = J(1)v = 0$ only if $v = 0$ in the case $h \leq 11$. In the case $h = 12$, there exists a unique, up to scalar multiple, linear combination $\mathbf{v}^{12}$ which satisfies $L(1)\mathbf{v}^{12} = L(2)\mathbf{v}^{12} = J(1)\mathbf{v}^{12} = 0$. The explicit form of $\mathbf{v}^{12}$ is given in Appendix A. We also have $J(0)\mathbf{v}^{12} = 0$.*

We only use conditions (5.1) and (5.2) and the commutation relations (3.5), (3.6) and (3.7) to obtain $\mathbf{v}^{12}$ in the above computation. Since we consider $\mathcal{W}$ inside the lattice vertex operator algebra V_L, there might exist some nontrivial relations among the vectors of the form (5.4) which are not known so far. This ambiguity will be removed in Subsection 5.3.

5.3. A positive definite invariant Hermitian form on V_L. It is well-known that the vertex operator algebra constructed from any positive definite even lattice as in [**17**] possesses a positive definite Hermitian form which is invariant in a certain sense ([**15**], [**17**], [**26**] and [**29**]). Following [**29**, Section 2.5], we review it for our V_L.

Set $\widetilde{L}(n) = \widetilde{\omega}_{n+1}, n \in \mathbb{Z}$, where $\widetilde{\omega}$ is the Virasoro element of V_L. Then $\widetilde{L}(1)(V_L)_{(1)} = 0$ and $(V_L)_{(0)}$ is one dimensional. Thus by [**26**, Theorem 3.1], there is a unique symmetric invariant bilinear form $(\,\cdot\,,\,\cdot\,)$ on V_L such that $(\mathbf{1}, \mathbf{1}) = 1$. That the form is invariant means

$$(Y(u,z)v, w) = (v, Y(e^{z\widetilde{L}(1)}(-z^{-2})^{\widetilde{L}(0)}u, z^{-1})w) \tag{5.5}$$

for $u, v, w \in V_L$. The value (u, v) is determined by

$$(\mathbf{1}, \mathbf{1}) = 1, \tag{5.6}$$

$$(u, v) = \mathrm{Res}_z\, z^{-1}(\mathbf{1}, Y(e^{z\widetilde{L}(1)}(-z^{-2})^{\widetilde{L}(0)}u, z^{-1})v). \tag{5.7}$$

From (5.5), we see that $(\widetilde{L}(n)u, v) = (u, \widetilde{L}(-n)v)$. In case of $n = 0$, this implies $((V_L)_{(m)}, (V_L)_{(n)}) = 0$ if $m \neq n$. For $\alpha \in L$ and $u, v \in V_L$,

$$\begin{aligned} (\alpha(n)u, v) &= \mathrm{Res}_z\, z^n (Y(\alpha(-1), z)u, v) \\ &= -(u, \alpha(-n)v). \end{aligned} \tag{5.8}$$

Furthermore, for $\alpha, \beta \in L$ we have

$$(e^{\alpha}, e^{\beta}) = \delta_{\alpha+\beta,0}. \tag{5.9}$$

Note that $(-1)^{\langle\alpha,\alpha\rangle/2} = 1$ since $\alpha \in L$. Consider an $\mathbb{R}$-form $V_{L,\mathbb{R}}$ of V_L as in [**17**, Section 12.4]. That is, let $M(1)_{\mathbb{R}} = \mathbb{R}[\alpha(n);\, \alpha \in L,\, n < 0]$ and $V_{L,\mathbb{R}} = M(1)_{\mathbb{R}} \otimes \mathbb{R}[L]$. Then $\mathbb{C} \otimes_{\mathbb{R}} V_{L,\mathbb{R}} = V_L$. Moreover, $V_{L,\mathbb{R}}$ is invariant under the automorphism θ. Let $V_{L,\mathbb{R}}^{\pm} = \{v \in V_{L,\mathbb{R}} \,|\, \theta v = \pm v\}$. We shall show that the form $(\,\cdot\,,\,\cdot\,)$ is positive definite on $V_{L,\mathbb{R}}^{+}$ and negative definite on $V_{L,\mathbb{R}}^{-}$. Indeed, let $\{\gamma_1, \gamma_2\}$ be an orthonormal basis of $\mathbb{R} \otimes_{\mathbb{Z}} L$. Then using (5.8) and (5.9) we can verify that

$$(\gamma_{i_1}(-m_1) \cdots \gamma_{i_p}(-m_p)e^{\alpha}, \gamma_{j_1}(-n_1) \cdots \gamma_{j_q}(-n_q)e^{\beta}) \neq 0$$

only if $\gamma_{i_1}(-m_1) \cdots \gamma_{i_p}(-m_p) = \gamma_{j_1}(-n_1) \cdots \gamma_{j_q}(-n_q)$ in $M(1)_{\mathbb{R}}$ and $\alpha + \beta = 0$. Furthermore,

$$\begin{aligned} &(\gamma_{i_1}(-m_1) \cdots \gamma_{i_p}(-m_p)e^{\alpha}, \gamma_{i_1}(-m_1) \cdots \gamma_{i_p}(-m_p)e^{-\alpha}) \\ &\quad = (-1)^p \cdot (\text{a positive integer}). \end{aligned} \tag{5.10}$$

We can choose a basis of $V_{L,\mathbb{R}}^{+}$ consisting of vectors of the form

$$\begin{aligned} &\gamma_{i_1}(-m_1) \cdots \gamma_{i_p}(-m_p)(e^{\alpha} + e^{-\alpha}), \quad p \text{ even}, \quad \alpha \in L, \\ &\gamma_{i_1}(-m_1) \cdots \gamma_{i_p}(-m_p)(e^{\alpha} - e^{-\alpha}), \quad p \text{ odd}, \quad 0 \neq \alpha \in L. \end{aligned}$$

By (5.9) and (5.10), these vectors are mutually orthogonal and the square length of each of them is a positive integer. Hence the form $(\,\cdot\,,\,\cdot\,)$ is positive definite on $V_{L,\mathbb{R}}^+$. Likewise, we see that the form $(\,\cdot\,,\,\cdot\,)$ is negative definite on $V_{L,\mathbb{R}}^-$.

We also have $(V_{L,\mathbb{R}}^+, V_{L,\mathbb{R}}^-) = 0$. Thus the form $(\,\cdot\,,\,\cdot\,)$ is positive definite on $V_{L,\mathbb{R}}^+ + \sqrt{-1}V_{L,\mathbb{R}}^-$. The $\mathbb{R}$-vector space $V_{L,\mathbb{R}}^+ + \sqrt{-1}V_{L,\mathbb{R}}^-$ is an $\mathbb{R}$-form of V_L since $V_L = \mathbb{C} \otimes_{\mathbb{R}} (V_{L,\mathbb{R}}^+ + \sqrt{-1}V_{L,\mathbb{R}}^-)$. Note that it is invariant under the component operators u_n of $Y(u,z)$ for $u \in V_{L,\mathbb{R}}^+$.

Define a Hermitian form $((\,\cdot\,,\,\cdot\,))$ on V_L by $((\lambda u, \mu v)) = \lambda\overline{\mu}(u,v)$ for $\lambda, \mu \in \mathbb{C}$ and $u, v \in V_{L,\mathbb{R}}^+ + \sqrt{-1}V_{L,\mathbb{R}}^-$. Then the Hermitian form $((\,\cdot\,,\,\cdot\,))$ is positive definite on V_L and invariant under $V_{L,\mathbb{R}}^+$, that is,

$$((Y(u,z)v,w)) = \left(\left(v, Y\left(e^{z\widetilde{L}(1)}(-z^{-2})^{\widetilde{L}(0)}u, z^{-1}\right)w\right)\right) \tag{5.11}$$

for $u \in V_{L,\mathbb{R}}^+$ and $v, w \in V_L$.

Using the Hermitian form $((\,\cdot\,,\,\cdot\,))$, we can show that V_L is semisimple as a $\mathcal{W}$-module and that $\mathcal{W}$ is a simple vertex operator algebra. Note that $\widetilde{L}(n)v = L(n)v$ for $v \in M$. Note also that $V_{L,\mathbb{R}}^+$ contains ω and J. Then by (5.11),

$$((L(n)u,v)) = ((u, L(-n)v)), \tag{5.12}$$

$$((J(n)u,v)) = -((u, J(-n)v)) \tag{5.13}$$

for $n \in \mathbb{Z}$ and $u, v \in V_L$.

Let U be a $\mathcal{W}$-submodule. Denote by $U^\perp$ the orthogonal complement of U in V_L with respect to $((\,\cdot\,,\,\cdot\,))$. Then $V_L = U \oplus U^\perp$ since $((\,\cdot\,,\,\cdot\,))$ is positive definite. Moreover, $U^\perp$ is also a $\mathcal{W}$-submodule by (5.12) and (5.13). Thus we conclude that:

Theorem 5.5. *V_L is semisimple as a $\mathcal{W}$-module.*

Since the weight 0 subspace $\mathbb{C}\mathbf{1}$ of $\mathcal{W}$ is one dimensional and since $\mathcal{W}$ is generated by $\mathbf{1}$ as a $\mathcal{W}$-module, we have:

Theorem 5.6. *$\mathcal{W}$ is a simple vertex operator algebra.*

Then there is no singular vector in $\mathcal{W}$ of positive weight. Hence:

Corollary 5.7. *The singular vector $\mathbf{v}^{12} = 0$.*

5.4. The Zhu algebra $A(\mathcal{W})$. Based on the properties of $\mathcal{W}$ we have obtained so far, we shall determine the Zhu algebra $A(\mathcal{W})$ of $\mathcal{W}$. First we review some notations and formulas for the Zhu algebra $A(V)$ of an arbitrary vertex operator algebra $(V, Y, \mathbf{1}, \omega)$. The standard reference is [**36**, Section 2].

For $u, v \in V$ with u being homogeneous, define two binary operations

$$u * v = \mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}\, u}}{z} Y(u,z)v \right) = \sum_{i=0}^{\infty} \binom{\mathrm{wt}\, u}{i} u_{i-1}v, \tag{5.14}$$

$$u \circ v = \mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}\, u}}{z^2} Y(u,z)v \right) = \sum_{i=0}^{\infty} \binom{\mathrm{wt}\, u}{i} u_{i-2}v. \tag{5.15}$$

We extend $*$ and $\circ$ for arbitrary $u, v \in V$ by linearity. Let $O(V)$ be the subspace of V spanned by all $u \circ v$ for $u, v \in V$. By a theorem of Zhu [**36**], $O(V)$ is a two-sided ideal with respect to the operation $*$. Thus it induces an operation on $A(V) = V/O(V)$. Denote by $[v]$ the image of $v \in V$ in $A(V)$. Then $[u] * [v] = [u * v]$ and $A(V)$ is an associative algebra by this operation. Moreover, $[\mathbf{1}]$ is the identity and $[\omega]$ is in the center of $A(V)$. We denote by $[u]^{*p}$ the product of p copies of $[u]$ in $A(V)$. For $u, v \in V$, we write $u \sim v$ if $[u] = [v]$. For $f, g \in \mathrm{End}\, V$, we write $f \sim g$ if $fv \sim gv$ for all $v \in V$. We need some formulas from [**36**].

(5.16)

$$\mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(u)+m}}{z^{2+n}} Y(u,z)v \right) = \sum_{i=0}^{\infty} \binom{\mathrm{wt}(u)+m}{i} u_{i-n-2}v \in O(V)$$

for $n \geq m \geq 0$ and

$$v * u \sim \mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(u)-1}}{z} Y(u,z)v \right) = \sum_{i=0}^{\infty} \binom{\mathrm{wt}(u)-1}{i} u_{i-1}v. \tag{5.17}$$

Moreover (see [**34**]),

$$L(-n) \sim (-1)^n \big\{ (n-1)\big(L(-2) + L(-1)\big) + L(0) \big\} \tag{5.18}$$

for $n \geq 1$ and

$$[\omega] * [v] = [(L(-2) + L(-1))v]. \tag{5.19}$$

It follows from (5.18) and (5.19) that

$$[L(-n)u] = (-1)^n (n-1)[\omega] * [u] + (-1)^n [L(0)u] \tag{5.20}$$

for $n \geq 1$.

For a homogeneous $u \in V$, set $o(u) = u_{\mathrm{wt}(u)-1}$, which is the weight zero component operator of $Y(u,z)$. Extend $o(u)$ for arbitrary $u \in V$ by linearity. We call a module in the sense of [**36**] as an admissible module as in [**9**]. If $U = \oplus_{n=0}^{\infty} U(n)$ is an admissible V-module with $U(0) \neq 0$, then $o(u)$ acts on its top level $U(0)$. Zhu's theory [**36**] says:

(1) $o(u)o(v) = o(u * v)$ as operators on the top level $U(0)$ and $o(u)$ acts as 0 on $U(0)$ if $u \in O(V)$. Thus $U(0)$ is an $A(V)$-module, where $[u]$ acts on $U(0)$ as $o(u)$.

(2) The map $U \mapsto U(0)$ is a bijection between the set of equivalence classes of irreducible admissible V-modules and the set of equivalence classes of irreducible $A(V)$-modules.

We now return to $\mathcal{W}$. Since $\mathrm{wt}\, J = 3$, we have

$$[J(-n-4)v] = -3[J(-n-3)v] - 3[J(-n-2)v] - [J(-n-1)v] \tag{5.21}$$

for $v \in \mathcal{W}$ and $n \geq 0$ by (5.16).

Lemma 5.8. *The image $[L(-m_1)\cdots L(-m_p)J(-n_1)\cdots J(-n_q)\mathbf{1}]$ of the vector (5.4) with $m_1 \geq \cdots \geq m_p \geq 2$ and $n_1 \geq \cdots \geq n_q \geq 3$ in $A(\mathcal{W})$ is contained in*

$$\mathrm{span}\Big\{[\omega]^{*s} * [J]^{*t} \,|\, 0 \leq s,\, 0 \leq t \leq q,\, 2s + 3t \leq m_1 + \cdots + m_p + n_1 + \cdots + n_q\Big\}.$$

In particular, $A(\mathcal{W})$ is commutative and every element of $A(\mathcal{W})$ is a polynomial in $[\omega]$ and $[J]$.

Proof. We proceed by induction on the J-length q. By a repeated use of (5.20), we see that $[L(-m_1)\cdots L(-m_p)J(-n_1)\cdots J(-n_q)\mathbf{1}]$ is a linear combination of $[\omega]^{*s} * [J(-n_1)\cdots J(-n_q)\mathbf{1}]$, $0 \leq s \leq p$. Thus the assertion holds if $q = 0$.

Suppose the assertion holds for vectors of normal form with J-length at most $q-1$ and consider $[J(-n_1)\cdots J(-n_q)\mathbf{1}]$. Let $v = J(-n_1)\cdots J(-n_q)\mathbf{1}$ and $u = J(-n_2)\cdots J(-n_q)\mathbf{1}$, so that $v = J(-n_1)u$. We proceed by induction on the weight. The vector of the smallest weight is the case $n_1 = 3$. In this case $v = J(-3)^q\mathbf{1}$ and $u = J(-3)^{q-1}\mathbf{1}$. Since $v = J_{-1}u$, it follows from (5.14) that

$$[v] = [J] * [u] - 3[J(-2)u] - 3[J(-1)u] - [J(0)u].$$

The weight of $J(-n)u$, $0 \leq n \leq 2$, is less than $\mathrm{wt}\, v$. By Lemma 5.1, each of these three vectors is a linear combination of vectors of normal form with J-length at most $q-1$. Then we can apply the induction hypothesis on J-length and the assertion holds if $n_1 = 3$. Assume that $n_1 \geq 4$. By (5.21), $[v] = [J(-n_1)u]$ is a linear combination of $[J(-n)u]$, $n_1 - 3 \leq n \leq n_1 - 1$. The weight of $J(-n)u$, $n_1 - 3 \leq n \leq n_1 - 1$, is less than $\mathrm{wt}\, v$. Hence by Lemma 5.1, these three vectors are linear combinations of vectors of normal form with J-length at most q and weight less than $\mathrm{wt}\, v$. The induction is complete. □

The image $[L(-m_1)\cdots L(-m_p)J(-n_1)\cdots J(-n_q)\mathbf{1}]$ of the vector of normal form (5.4) with $m_1 \geq \cdots \geq m_p \geq 2$ and $n_1 \geq \cdots \geq n_q \geq 3$ in $A(\mathcal{W})$ can be written explicitly as a polynomial in $[\omega]$ and $[J]$ by the following algorithm:

Since $A(\mathcal{W})$ is commutative, it follows from (5.17) that

$$[J(-3)v] = [J] * [v] - 2[J(-2)v] - [J(-1)v] \tag{5.22}$$

for $v \in \mathcal{W}$. Now we use (5.20), (5.21) and (5.22). Although $J(-n-4)v$ is of normal form, the vectors $J(-n-3)v$, $J(-n-2)v$, and $J(-n-1)v$ in (5.21) may not be of normal form. However, the weight of any of these three vectors is less than the weight of $J(-n-4)v$, and so we can apply the argument in the proof of Lemma 5.1. A similar discussion is also needed for the formula (5.22). Thus the algorithm is by induction on the weight. We use formulas (5.20), (5.21), (5.22) and apply Lemma 5.1, that is, use the commutation relations (3.5), (3.6), (3.7) and the conditions (5.1) and (5.2). By induction on the weight and a repeated use of those formulas and conditions, we can write explicitly the image of the vector (5.4) in $A(\mathcal{W})$ as a polynomial in $[\omega]$ and $[J]$.

Consider the algebra homomorphism

$$\mathbb{C}[x, y] \longrightarrow A(\mathcal{W}); \quad x \longmapsto [\omega], \quad y \longmapsto [J]$$

of the polynomial algebra $\mathbb{C}[x, y]$ with two variables x, y onto $A(\mathcal{W})$. Denote its kernel by $\mathcal{I}$. Then $\mathbb{C}[x, y]/\mathcal{I} \cong A(\mathcal{W})$. We shall consider $\mathbf{v}^{12}$, $J(-1)\mathbf{v}^{12}$, $J(-2)\mathbf{v}^{12}$, and $J(-1)^2\mathbf{v}^{12}$. These vectors are described explicitly as linear combinations of vectors of normal form in Appendix A. Their images $[\mathbf{v}^{12}]$, $[J(-1)\mathbf{v}^{12}]$, $[J(-2)\mathbf{v}^{12}]$, and $[J(-1)^2\mathbf{v}^{12}]$ can be written as polynomials in $[\omega]$ and $[J]$ by the above mentioned algorithm. The results are given in Appendix B. Let $F_i(x, y) \in \mathbb{C}[x, y]$, $1 \le i \le 4$, be the polynomials which are obtained by replacing $[\omega]$ with x and $[J]$ with y in the polynomials given in Appendix B. Since $\mathbf{v}^{12} = 0$ by Corollary 5.7, $F_i(x, y)$'s are contained in $\mathcal{I}$. Let $\mathcal{I}'$ be the ideal in $\mathbb{C}[x, y]$ generated by $F_i(x, y), 1 \le i \le 4$.

The primary decomposition of $\mathcal{I}'$ is $\mathcal{I}' = \cap_{i=1}^{20} \mathcal{P}_i$, where $\mathcal{P}_i$, $1 \le i \le 20$ are

$$\begin{array}{ll}
\langle x,\, y\rangle, & \langle 5x-8,\, y\rangle,\\
\langle 2x-1,\, y\rangle, & \langle 10x-1,\, y\rangle,\\
\langle x-2,\, y-12\sqrt{-3}\rangle, & \langle x-2,\, y+12\sqrt{-3}\rangle,\\
\langle 5x-3,\, y-2\sqrt{-3}\rangle, & \langle 5x-3,\, y+2\sqrt{-3}\rangle,\\
\langle 9x-1,\, 81y-14\sqrt{-3}\rangle, & \langle 9x-1,\, 81y+14\sqrt{-3}\rangle,\\
\langle 9x-7,\, 81y-238\sqrt{-3}\rangle, & \langle 9x-7,\, 81y+238\sqrt{-3}\rangle,\\
\langle 9x-13,\, 81y-374\sqrt{-3}\rangle, & \langle 9x-13,\, 81y+374\sqrt{-3}\rangle,\\
\langle 45x-2,\, 81y-4\sqrt{-3}\rangle, & \langle 45x-2,\, 81y+4\sqrt{-3}\rangle,\\
\langle 45x-17,\, 81y-22\sqrt{-3}\rangle, & \langle 45x-17,\, 81y+22\sqrt{-3}\rangle,\\
\langle 45x-32,\, 81y-176\sqrt{-3}\rangle, & \langle 45x-32,\, 81y+176\sqrt{-3}\rangle.
\end{array} \tag{5.23}$$

These primary ideals correspond to the 20 irreducible M^τ-modules listed in Tables 1, 3 and 4 in Section 4. The correspondence is given by substituting x and y with the eigenvalues for $L(0)$ and $J(0)$ on the top levels of 20 irreducible modules. The eigenvalues are the zeros of those primary ideals.

Note that the 20 pairs of those eigenvalues for $L(0)$ and $J(0)$ on the top levels are different from each other. Since the top levels of the 20 irreducible M^τ-modules are one dimensional and since $\mathcal{W}$ is contained in M^τ, there are at least 20 inequivalent irreducible $\mathcal{W}$-modules whose top levels are the same as those of irreducible M^τ-modules. Hence by Zhu's theory [**36**], we conclude that $\mathcal{I} = \mathcal{I}'$ and $A(\mathcal{W}) \cong \oplus_{i=1}^{20} \mathbb{C}[x,y]/\mathcal{P}_i$. In particular, $\mathcal{W}$ has exactly 20 inequivalent irreducible modules.

If $\mathcal{W} \neq M^\tau$, then we can take an irreducible $\mathcal{W}$-module U in M^τ such that $\mathcal{W} \cap U = 0$ by Theorem 5.5. From the classification of irreducible $\mathcal{W}$-modules we see that the smallest weight of U is at most 2. But we can verify that the homogeneous subspaces of $\mathcal{W}$ of weight 0, 1, and 2 coincide with those of M^τ. Therefore, $\mathcal{W} = M^\tau$.

We have obtained the following theorem:

Theorem 5.9.

(1) $M^\tau = \mathcal{W}$.
(2) $A(M^\tau) \cong \oplus_{i=1}^{20} \mathbb{C}[x,y]/\mathcal{P}_i$ *is a* 20*-dimensional commutative associative algebra.*
(3) *There are exactly* 20 *inequivalent irreducible M^τ-modules. Their representatives are listed in Tables* 1*,* 3 *and* 4 *in Section* 4*, namely,* $M(\epsilon)$*,* $W(\epsilon)$*,* M_k^c*,* W_k^c*,* $M_T(\tau^i)(\epsilon)$*, and* $W_T(\tau^i)(\epsilon)$ *for* $\epsilon = 0, 1, 2$ *and* $i = 1, 2$.

Remark 5.10. The explicit description of $\mathbf{v}^{12}$, $J(-1)\mathbf{v}^{12}$, $J(-2)\mathbf{v}^{12}$, and $J(-1)^2\mathbf{v}^{12}$ in Appendix A, the images of these four vectors in $A(\mathcal{W})$ in Appendix B, and the primary ideals (5.23) were obtained by a computer algebra system Risa/Asir.

5.5. Rationality of $\mathcal{W}$. Recall that a vertex operator algebra V is called C_2-cofinite if $V/C_2(V)$ is finite dimensional where $C_2(V)$ is the subspace of V spanned by $u_{-2}v$ for $u, v \in V$. The following result about a general vertex operator algebra was essentially proved in [**31**, Theorem 9.0.1]:

Proposition 5.11. *Let $V = \oplus_{n \geq 0} V_n$ be a C_2-cofinite vertex operator algebra such that V_0 is one-dimensional. Assume that $A(V)$ is semisimple and any V-module generated by an irreducible $A(V)$-module is irreducible. Then V is a rational vertex operator algebra.*

Proof. By the definition of rationality (cf. [**8**]), we need to prove that any admissible $\mathcal{W}$-module Z is completely reducible. By [**1**, Lemma 5.5], Z is a direct sum of generalized eigensapces for $L(0)$. So it is enough to prove that any submodule generated by a generalized eigenvector for $L(0)$ is completely reducible. We can assume that Z is generated by a generalized eigenvector for $L(0)$. Then $Z = \oplus_{n \geq 0} Z_{\lambda+n}$ for some $\lambda \in \mathbb{C}$ where $Z_{\lambda+n}$ is the generalized eigenspace for $L(0)$ with eigenvalue $\lambda+n$ and $Z_\lambda \neq 0$. We call λ the minimal weight of Z. By [**4**, Theorem 1], each $Z_{\lambda+n}$ is finite dimensional.

Let X be the submodule of Z generated by Z_λ. Then X is completely reducible by the assumption. So we have an exact sequence

$$0 \to X \to Z \to Z/X \to 0$$

of admissible V-modules. Let $Z' = \oplus_{n\geq 0} Z^*_{\lambda+n}$ be the graded dual of Z. Then Z' is also an admissible V-module (see [**15**]) and we have an exact sequence

$$0 \to (Z/X)' \to Z' \to X' \to 0$$

of admissible V-modules. On the other hand, the V-submodule of Z' generated by Z^*_λ is isomorphic to X'. As a result we have Z' is isomorphic to $X' \oplus (Z/X)'$. This implies that $Z \cong X \oplus Z/X$. Clearly, the minimal weight of Z/X is greater than the minimal weight of Z. Continuing in this way we prove that Z is a direct sum irreducible modules. □

Now we turn our attention to $\mathcal{W}$.

Theorem 5.12. *$\mathcal{W}$ is C_2-cofinite.*

Proof. Note from Corollary 5.2 that $\mathcal{W}$ is spanned by

$$L(-m_1)\cdots L(-m_p)J(-n_1)\cdots J(-n_q)\mathbf{1}$$

with $m_1 \geq \cdots \geq m_p \geq 2$, $n_1 \geq \cdots \geq n_q \geq 3$, $p = 0, 1, 2, \ldots$, and $q = 0, 1, 2, \ldots$. Then $\mathcal{W}$ is spanned by $L(-2)^p J(-3)^q \mathbf{1}$ modulo $C_2(\mathcal{W})$. It is well-known that $\mathcal{W}/C_2(\mathcal{W})$ is a commutative associative algebra under the product $u \cdot v = u_{-1}v$ for $u, v \in \mathcal{W}$ (cf. [**36**]). So $\mathcal{W}$ is spanned by $\omega^p \cdot J^q$ modulo $C_2(\mathcal{W})$ for $p, q \geq 0$.

The key idea to prove that $\mathcal{W}$ is C_2-cofinite is to use the singular vector $\mathbf{v}^{12}$. By the explicit form of $\mathbf{v}^{12}$, $J(-1)\mathbf{v}^{12}$, and $J(-1)^2\mathbf{v}^{12}$ in Appendix A, we have the following relations in $\mathcal{W}/C_2(\mathcal{W})$:

$$\begin{aligned}
&-(59680000/3501)\omega^6 - (184400/1167)\omega^3 \cdot J^2 + J^4 = 0,\\
&-926640\omega^2 \cdot J^3 - 89856000\omega^5 \cdot J = 0,\\
&21565440000\omega^7 - 680659200\omega^4 \cdot J^2 - 5559840\omega \cdot J^4 = 0.
\end{aligned}$$

Multiplying by ω^2, J, ω respectively we get

$$\begin{aligned}
&-(59680000/3501)\omega^8 - (184400/1167)\omega^5 \cdot J^2 + \omega^2 \cdot J^4 = 0,\\
&-926640\omega^2 \cdot J^4 - 89856000\omega^5 \cdot J^2 = 0,\\
&21565440000\omega^8 - 680659200\omega^5 \cdot J^2 - 5559840\omega^2 \cdot J^4 = 0.
\end{aligned}$$

It follows immediately that

$$\omega^8 = \omega^2 \cdot J^4 = \omega^5 \cdot J^2 = 0.$$

Thus

$$J^8 = \left((59680000/3501)\omega^6 + (184400/1167)\omega^3 \cdot J^2\right)^2 = 0.$$

As a result, $\mathcal{W}/C_2(\mathcal{W})$ is spanned by $\omega^p \cdot J^q$ for $0 \leq p, q \leq 7$, as desired. □

Lemma 5.13. *Let U be an irreducible $A(\mathcal{W})$-module. Then any $\mathcal{W}$-module Z generated by U is irreducible.*

Proof. By Theorem 5.9, $A(\mathcal{W})$ has exactly 20 irreducible modules and ω acts on each irreducible module as a constant in the set

$$P = \Big\{0, 2, 8/5, 3/5, 1/2, 1/10, 1/9, 1/9 + 2/3, 1/9 + 4/3, \\ 2/45, 2/45 + 1/3, 2/45 + 2/3\Big\}.$$

Let ω act on U as λ. Assume that $\lambda \neq 0, 3/5$. Then λ is maximal in the set $P \cap (\lambda + \mathbb{Z})$. Let $Z = \oplus_{n\geq 0} Z_{\lambda+n}$ and $Z_\lambda = U$. If Z is not irreducible then Z has a proper submodule $X = \sum_{n\geq 0} X_{\lambda+n_0+n}$ for some $n_0 > 0$ with $X_{\lambda+n_0} \neq 0$ where $X_{\lambda+m} = X \cap Z_{\lambda+m}$. So $X_{\lambda+n_0}$ is an $A(\mathcal{W})$-module on which ω acts on $\lambda+n_0$. Since $\lambda+n_0 \in P\cap(\lambda+\mathbb{Z})$ is greater than λ we have a contradiction. This shows that Z must be irreducible.

It remains to prove the result with $\lambda = 0$ or $\lambda = 3/5$. If $\lambda = 0$, then $U \cong \mathbb{C}\mathbf{1}$ and Z is isomorphic to $\mathcal{W}$ (see [**26**]). Now let $\lambda = 3/5$. By Theorem 5.9, U can be either $W(1)_{3/5}$ or $W(2)_{3/5}$ (see Table 1). We can assume that $U = W(1)_{3/5}$ and the proof for $U = W(2)_{3/5}$ is similar. In this case $J(0)$ acts on U as $2\sqrt{-3}$. Let $U = \mathbb{C}u$. Then Z is spanned by

$$L(-m_1)\cdots L(-m_p)J(-n_1)\cdots J(-n_q)u$$

with $m_1 \geq \cdots \geq m_p \geq 1$, $n_1 \geq \cdots \geq n_q \geq 1$, $p = 0, 1, 2, \ldots$, and $q = 0, 1, 2, \ldots$ (see Remark 5.3). Since $8/5$ is the only number in $P \cap (3/5 + \mathbb{Z})$ greater than $3/5$, Z is irreducible if and only if there is no nonzero vector $v \in Z_{8/5}$ such that $L(1)v = J(1)v = 0$.

Note that $Z_{8/5}$ is spanned by $L(-1)u$ and $J(-1)u$. By formulas (3.5)-(3.7) we see that

$$\begin{aligned}
L(1)L(-1)u &= \frac{6}{5}u,\\
L(1)J(-1)u &= 6\sqrt{-3}u,\\
J(1)L(-1)u &= 6\sqrt{-3}u,\\
J(1)J(-1)u &= \left(\frac{237\times 6}{5} - \frac{48\times 39}{5}\right)u.
\end{aligned}$$

Now let $v = \alpha L(-1)u + \beta J(-1)u \in Z_{8/5}$ such that $L(1)v = J(1)v = 0$. Then we have a system of linear equations

$$\begin{aligned}
\frac{6}{5}\alpha + 6\sqrt{-3}\beta &= 0,\\
6\sqrt{-3}\alpha - 90\beta &= 0.
\end{aligned}$$

Unfortunately, the system is degenerate and has solutions $\alpha = -5\sqrt{-3}\beta$. Thus up to a constant we can assume that $v = -5\sqrt{-3}L(-1)u + J(-1)u$.

We have to prove that $v = 0$. If v is not zero, then $\mathbb{C}v$ is an irreducible module for $A(\mathcal{W})$ on which ω acts as $8/5$. Again by Table 1, J must act on v as 0. Using (3.6) and (3.7), we find out that $J(0)v = -120L(-1)u - 8\sqrt{-3}J(-1)u = -8\sqrt{-3}v$. This implies that $v = 0$. Clearly we have a contradiction. Thus Z is an irreducible $\mathcal{W}$-module. □

Combining Proposition 5.11, Theorem 5.12, and Lemma 5.13 together yields:

Theorem 5.14. *The vertex operator algebra $\mathcal{W}$ is rational.*

It is proved in [**1**] that a rational and C_2-cofinite vertex operator algebra is regular in the sense that any weak module is a direct sum of irreducible admissible modules. Thus we, in fact, have proved that $\mathcal{W}$ is also regular.

6. Characters of irreducible M^τ-modules

We shall describe the characters of the 20 irreducible M^τ-modules by the characters of irreducible modules for the Virasoro vertex operator algebras. Throughout this section z denotes a complex number in the upper half plane $\mathcal{H}$ and $q = \exp(2\pi\sqrt{-1}z)$. First we recall the character of the irreducible module $L(c_m, h^{(m)}_{r,s})$ with highest weight $h^{(m)}_{r,s}$ for the Virasoro vertex operator algebra $L(c_m, 0)$ with central charge c_m, where

$$c_m = 1 - \frac{6}{(m+2)(m+3)}, \quad m = 1, 2, \ldots,$$
$$h^{(m)}_{r,s} = \frac{\big((m+3)r - (m+2)s\big)^2 - 1}{4(m+2)(m+3)}, \quad 1 \le s \le r \le m+1.$$

The character of $L(c_m, h^{(m)}_{r,s})$ is obtained in [**32**] as follows:

$$\operatorname{ch} L(c_m, h^{(m)}_{r,s}) = \frac{\sum_{k\in\mathbb{Z}}(q^{b(k)} - q^{a(k)})}{\prod_{i=1}^{\infty}(1-q^i)}, \tag{6.1}$$

where

$$a(k) = \frac{\big(2(m+2)(m+3)k + (m+3)r + (m+2)s\big)^2 - 1}{4(m+2)(m+3)},$$
$$b(k) = \frac{\big(2(m+2)(m+3)k + (m+3)r - (m+2)s\big)^2 - 1}{4(m+2)(m+3)}.$$

Define $\Xi^{(m)}_{r,s}(z) = q^{-c_m/24} \operatorname{ch} L(c_m, h^{(m)}_{r,s})$. For $1 \le s \le r \le m+1$, the following transformation formula holds (cf. [**18**, Exercise 13.27]):

(6.2)

$$\Xi^{(m)}_{r,s}\left(\frac{-1}{z}\right) = \sqrt{\frac{8}{(m+2)(m+3)}} \cdot \sum_{1\le j\le i\le m+1} (-1)^{(r+s)(i+j)} \sin\frac{\pi r i}{m+2} \sin\frac{\pi s j}{m+3} \Xi^{(m)}_{i,j}(z).$$

Let $\eta(z) = q^{1/24}\prod_{i=1}^{\infty}(1-q^i)$ be the Dedekind η-function. The following transformation formula is well-known (cf. [**2**]):

$$\eta\left(\frac{-1}{z}\right) = \left(-\sqrt{-1}z\right)^{1/2}\eta(z),$$

where we choose the branch of the square root function $x^{1/2}$ so that it is positive when $x > 0$.

We review notations and some properties of the trace function in [**9**]. Let $g, h \in \operatorname{Aut}(M)$ be such that $gh = hg$. Let $\mathcal{C}_1(g,h)$ be the space of (g,h) 1-point functions. Let W be a g-twisted h-stable M-module with conformal weight λ. There is a linear isomorphism $\phi(h) : W \to W$ such that

$$\phi(h)Y_W(u,z) = Y_W(hu,z)\phi(h).$$

Define

$$T_W(u,(g,h),z) = \operatorname{tr}_W u_{\mathrm{wt}(u)-1}\phi(h)q^{L(0)-1/20}$$

for homogeneous $u \in M$ and extend it for arbitrary $u \in M$ linearly. Note that the central charge of M is $6/5$. Then $T_W(\,\cdot\,,(g,h),z) \in \mathcal{C}_1(g,h)$ by [**9**, Theorem 8.1]. Let $F(\,\cdot\,,z) \in \mathcal{C}_1(g,h)$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Define $F|_A$ by

$$F|_A(u,z) = (cz+d)^{-k}F\left(u, \frac{az+b}{cz+d}\right)$$

for $u \in M_{[k]}$ and extend it for arbitrary $u \in M$ linearly. Then $F|_A \in \mathcal{C}_1(g^ah^c, g^bh^d)$ by [**9**, Theorem 5.4]. We denote $T_W(\mathbf{1},(g,h),z)$ by $T_W((g,h),z)$ for simplicity. Recall that the character $\operatorname{ch} W$ of W is defined to be $\operatorname{tr}_W q^{L(0)}$.

We want to determine the characters of the 20 irreducible M^τ-modules $M(\epsilon)$, $W(\epsilon)$, M^c_k, W^c_k, $M_T(\tau^i)(\epsilon)$, and $W_T(\tau^i)(\epsilon)$ for $\epsilon = 0,1,2$ and $i = 1,2$. We have shown in Theorem 2.1 that $\operatorname{Aut}(M)$ is generated by σ and τ. We shall consider the cases where $g = 1$ and $h = \tau$ or $g = \tau$ and $h = 1$. We specify $\phi(h)$ as follows: If $h = 1$, we take $\phi(h) = 1$. We shall deal with the case $g = 1$ and $h = \tau$ for $W = M$ or W^0_k. In such a case we consider the same $\phi(\tau)$ as in Section 4. Thus if $W = M$, we take $\phi(\tau)$ to be the automorphism τ. If $W = W_k^0$, we take $\phi(\tau)$ to be the linear isomorphism which is naturally induced from the isometry τ of the lattice $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$.

Note that $T_W((g,1),z) = q^{-1/20}\,\mathrm{ch}\,W$. Note also that the symmetry (4.9) induced by σ implies $T_{M(1)}((1,1),z) = T_{M(2)}((1,1),z)$. A similar assertion holds for $W(1)$ and $W(2)$.

Proposition 6.1. *For $i = 1,2$,*

$$T_{M_T(\tau^i)}((\tau^i,1),z) = \frac{\eta(z)}{\eta(z/3)}\left(-\Xi^{(3)}_{2,1} - \Xi^{(3)}_{3,1} + \Xi^{(3)}_{3,3}\right),$$
$$T_{W_T(\tau^i)}((\tau^i,1),z) = \frac{\eta(z)}{\eta(z/3)}\left(\Xi^{(3)}_{1,1} + \Xi^{(3)}_{4,1} - \Xi^{(3)}_{4,3}\right).$$

Proof. Since $\mathrm{ch}\,V_L^{T_{\chi_j}}(\tau) = \mathrm{ch}\,S(\tau)$ for $j = 0,1,2$, we have

$$q^{-1/12}\,\mathrm{ch}\,V_L^{T_{\chi_j}}(\tau) = \frac{\eta(z)}{\eta(z/3)}$$

by (4.1). Then (4.7) and (4.8) imply that

$$\frac{\eta(z)}{\eta(z/3)} = T_{M_T(\tau)}((\tau,1),z)\cdot\left(\Xi^{(3)}_{1,1} + \Xi^{(3)}_{4,1}\right) + T_{W_T(\tau)}((\tau,1),z)\cdot\left(\Xi^{(3)}_{2,1} + \Xi^{(3)}_{3,1}\right), \tag{6.3}$$

$$\frac{\eta(z)}{\eta(z/3)} = T_{M_T(\tau)}((\tau,1),z)\cdot\Xi^{(3)}_{4,3} + T_{W_T(\tau)}((\tau,1),z)\cdot\Xi^{(3)}_{3,3}. \tag{6.4}$$

Now consider $\left(\Xi^{(3)}_{1,1}+\Xi^{(3)}_{4,1}\right)\Xi^{(3)}_{3,3}-\left(\Xi^{(3)}_{2,1}+\Xi^{(3)}_{3,1}\right)\Xi^{(3)}_{4,3}$. Using (6.2) we can verify that it is invariant under the action of $SL_2(\mathbb{Z})$. Moreover, its q-expansion is $1 + 0\cdot q + \cdots$. Thus

$$\left(\Xi^{(3)}_{1,1} + \Xi^{(3)}_{4,1}\right)\Xi^{(3)}_{3,3} - \left(\Xi^{(3)}_{2,1} + \Xi^{(3)}_{3,1}\right)\Xi^{(3)}_{4,3} = 1.$$

Hence the assertions for $i = 1$ follow from (6.3) and (6.4). The assertions for $i = 2$ also hold by the symmetry (4.9). $\square$

Theorem 6.2. *The characters of the 20 irreducible M^τ-modules $M(\epsilon)$, $W(\epsilon)$, M_k^c, W_k^c, $M_T(\tau^i)(\epsilon)$, and $W_T(\tau^i)(\epsilon)$ for $\epsilon = 0,1,2$ and $i = 1,2$ are given by the following formulas:*

(1) *For* $\epsilon = 1, 2$ *we have*

$$\begin{aligned}
q^{-1/20} \operatorname{ch} M(0) &= \frac{1}{3}\Big(\Xi^{(1)}_{1,1}\Xi^{(2)}_{1,1} + \Xi^{(1)}_{2,1}\Xi^{(2)}_{3,1} + 2\frac{\eta(z)}{\eta(3z)}\Xi^{(3)}_{3,3}\Big),\\
q^{-1/20} \operatorname{ch} M(\epsilon) &= \frac{1}{3}\Big(\Xi^{(1)}_{1,1}\Xi^{(2)}_{1,1} + \Xi^{(1)}_{2,1}\Xi^{(2)}_{3,1} - \frac{\eta(z)}{\eta(3z)}\Xi^{(3)}_{3,3}\Big),\\
q^{-1/20} \operatorname{ch} W(0) &= \frac{1}{3}\Big(\Xi^{(1)}_{1,1}\Xi^{(2)}_{3,2} + \Xi^{(1)}_{2,1}\Xi^{(2)}_{3,3} - 2\frac{\eta(z)}{\eta(3z)}\Xi^{(3)}_{4,3}\Big),\\
q^{-1/20} \operatorname{ch} W(\epsilon) &= \frac{1}{3}\Big(\Xi^{(1)}_{1,1}\Xi^{(2)}_{3,2} + \Xi^{(1)}_{2,1}\Xi^{(2)}_{3,3} + \frac{\eta(z)}{\eta(3z)}\Xi^{(3)}_{4,3}\Big),\\
q^{-1/20} \operatorname{ch} M^c_k &= \Xi^{(1)}_{22}\Xi^{(2)}_{21},\\
q^{-1/20} \operatorname{ch} W^c_k &= \Xi^{(1)}_{22}\Xi^{(2)}_{22}.
\end{aligned}$$

(2) *For* $i = 1, 2$ *we have*

$$\begin{aligned}
&\begin{pmatrix} q^{-1/20}\operatorname{ch}(M_T(\tau^i)(0)) \\ q^{-1/20}\operatorname{ch}(M_T(\tau^i)(1)) \\ q^{-1/20}\operatorname{ch}(M_T(\tau^i)(2)) \end{pmatrix}\\
&= \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi \end{pmatrix}\begin{pmatrix} T_{M_T(\tau^i)}((\tau^i,1),z) \\ e^{-11\pi\sqrt{-1}/90}T_{M_T(\tau^i)}((\tau^i,1),z+1) \\ e^{-22\pi\sqrt{-1}/90}T_{M_T(\tau^i)}((\tau^i,1),z+2) \end{pmatrix},\\
&\begin{pmatrix} q^{-1/20}\operatorname{ch}(W_T(\tau^i)(0)) \\ q^{-1/20}\operatorname{ch}(W_T(\tau^i)(1)) \\ q^{-1/20}\operatorname{ch}(W_T(\tau^i)(2)) \end{pmatrix}\\
&= \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi \end{pmatrix}\begin{pmatrix} T_{W_T(\tau^i)}((\tau^i,1),z) \\ e^{61\pi\sqrt{-1}/90}T_{W_T(\tau^i)}((\tau^i,1),z+1) \\ e^{122\pi\sqrt{-1}/90}T_{W_T(\tau^i)}((\tau^i,1),z+2) \end{pmatrix},
\end{aligned}$$

where $\xi = \exp(2\pi\sqrt{-1}/3)$.

Proof. Since $M_T(\tau^i) = \oplus_{\epsilon=0}^2 M_T(\tau^i)(\epsilon)$ for $i = 1, 2$, we have

$$T_{M_T(\tau^i)}((\tau^i,1),z) = \sum_{\epsilon=0}^{2} T_{M_T(\tau^i)(\epsilon)}((1,1),z).$$

Replace z with $z + k$, where $k = 0, 1, 2$. Then

$$T_{M_T(\tau^i)(\epsilon)}((1,1),z+k) = \operatorname{tr}_{M_T(\tau^i)(\epsilon)} q^{L(0)-1/20}\exp(2\pi\sqrt{-1}k)^{L(0)-1/20}.$$

Note that $\exp(2\pi\sqrt{-1}k)^{L(0)-1/20} = \exp(11\pi\sqrt{-1}k/90)\xi^{2k\epsilon}$ on $M_T(\tau^i)(\epsilon)$, since the eigenvalues for $L(0)$ on $M_T(\tau^i)(\epsilon)$ are of the form $1/9 + 2\epsilon/3 + n$

with $n \in \mathbb{Z}_{\geq 0}$. Thus

$$T_{M_T(\tau^i)}((\tau^i, 1), z+k) = \exp(11\pi\sqrt{-1}k/90) \sum_{\epsilon=0}^{2} \xi^{2k\epsilon} T_{M_T(\tau^i)(\epsilon)}((1,1), z).$$

We can solve these equations for $k = 0, 1, 2$ with respect to

$$T_{M_T(\tau^i)(\epsilon)}((1,1), z), \qquad \epsilon = 0, 1, 2,$$

and obtain the expressions of $T_{M_T(\tau^i)(\epsilon)}((1,1), z) = q^{-1/20} \operatorname{ch}(M_T(\tau^i)(\epsilon))$ in the theorem.

Similarly, $W_T(\tau^i) = \oplus_{\epsilon=0}^{2} W_T(\tau^i)(\epsilon)$ and the eigenvalues for $L(0)$ on $W_T(\tau^i)(\epsilon)$ are of the form $2/45 + (2-\epsilon)/3 + n$, $n \in \mathbb{Z}_{\geq 0}$. Hence $\exp(2\pi\sqrt{-1}k)^{L(0)-1/20} = \exp(-61\pi\sqrt{-1}k/90)\xi^{2k\epsilon}$ on $W_T(\tau^i)(\epsilon)$ and we obtain the expressions of $q^{-1/20} \operatorname{ch}(W_T(\tau^i)(\epsilon)), \epsilon = 0, 1, 2$.

It is proved in [**24**] that $M = M_k^0$ is a rational vertex operator algebra. Moreover, there are exactly two inequivalent irreducible τ-stable M-modules, namely, M and W_k^0 by Lemma 4.1. Since $M_T(\tau)$ and $W_T(\tau)$ are two inequivalent irreducible τ-twisted M-modules, we have $\dim \mathcal{C}_1(\tau, 1) = \dim \mathcal{C}_1(1, \tau) = 2$ and

$$\{T_{M_T(\tau)}(\,\cdot\,, (\tau, 1), z), T_{W_T(\tau)}(\,\cdot\,, (\tau, 1), z)\}$$

is a basis of $\mathcal{C}_1(\tau, 1)$ by [**9**, Theorems 5.4 and 10.1]. Now $T_M(\,\cdot\,, (1, \tau), z)|_S \in \mathcal{C}_1(\tau, 1)$ for $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ by [**9**, Theorems 5.4 and 8.1]. Thus,

$$T_M((1,\tau), z) = \alpha T_{M_T(\tau)}\left((\tau, 1), \frac{-1}{z}\right) + \beta T_{W_T(\tau)}\left((\tau, 1), \frac{-1}{z}\right)$$

for some $\alpha, \beta \in \mathbb{C}$.

From (6.2) and Proposition 6.1 it follows that

$$\begin{aligned}
\frac{\eta(3z)}{\eta(z)} T_{M_T(\tau)}\left((\tau, 1), \frac{-1}{z}\right) &= \frac{2\sin(\frac{\pi}{5})}{\sqrt{5}} \Xi_{3,3}^{(3)} - \frac{2\sin(\frac{2\pi}{5})}{\sqrt{5}} \Xi_{4,3}^{(3)}, \\
\frac{\eta(3z)}{\eta(z)} T_{W_T(\tau)}\left((\tau, 1), \frac{-1}{z}\right) &= \frac{2\sin(\frac{2\pi}{5})}{\sqrt{5}} \Xi_{3,3}^{(3)} + \frac{2\sin(\frac{\pi}{5})}{\sqrt{5}} \Xi_{4,3}^{(3)}.
\end{aligned}$$

Thus

$$\begin{aligned}
T_M((1,\tau), z) = q^{-1/20}\Bigg(&\left(\alpha \frac{2\sin(\frac{\pi}{5})}{\sqrt{5}} + \beta \frac{2\sin(\frac{2\pi}{5})}{\sqrt{5}}\right) \\
&+ \left(-\alpha \frac{2\sin(\frac{2\pi}{5})}{\sqrt{5}} + \beta \frac{2\sin(\frac{\pi}{5})}{\sqrt{5}}\right) q^{3/5} + \cdots \Bigg).
\end{aligned}$$

Furthermore, we see that $T_M((1,\tau),z) = q^{-1/20}(1+0\cdot q^{3/5}+\cdots)$ by a direct computation. Hence $\alpha = 2\sin(\frac{\pi}{5})/\sqrt{5}$ and $\beta = 2\sin(\frac{2\pi}{5})/\sqrt{5}$. Therefore,

$$T_M((1,\tau),z) = \frac{\eta(z)}{\eta(3z)}\Xi^{(3)}_{3,3}.$$

Note that

$$T_M((1,1),z) = q^{-1/20}\operatorname{ch} M = \Xi^{(1)}_{1,1}\Xi^{(2)}_{1,1} + \Xi^{(1)}_{2,1}\Xi^{(2)}_{3,1}.$$

Now $M = M(0)\oplus M(1)\oplus M(2)$ and $T_{M(1)}((1,1),z) = T_{M(2)}((1,1),z)$ by the symmetry (4.9). Then

$$\begin{aligned}T_M((1,1),z) &= T_{M(0)}((1,1),z) + T_{M(1)}((1,1),z) + T_{M(2)}((1,1),z)\\ &= T_{M(0)}((1,1),z) + 2T_{M(1)}((1,1),z)\end{aligned}$$

and

$$\begin{aligned}T_M((1,\tau),z) &= T_{M(0)}((1,1),z) + \xi T_{M(1)}((1,1),z) + \xi^2 T_{M(2)}((1,1),z)\\ &= T_{M(0)}((1,1),z) - T_{M(1)}((1,1),z)\end{aligned}$$

by the definition of trace functions. Thus $q^{-1/20}\operatorname{ch} M(\epsilon) = T_{M(\epsilon)}((1,1),z)$ can be expressed as

$$\begin{aligned}q^{-1/20}\operatorname{ch} M(0) &= \frac{1}{3}\Big(T_M((1,1),z) + 2T_M((1,\tau),z)\Big)\\ &= \frac{1}{3}\Big(\Xi^{(1)}_{1,1}\Xi^{(2)}_{1,1} + \Xi^{(1)}_{2,1}\Xi^{(2)}_{3,1} + 2\frac{\eta(z)}{\eta(3z)}\Xi^{(3)}_{3,3}\Big),\\ q^{-1/20}\operatorname{ch} M(\epsilon) &= \frac{1}{3}\Big(T_M((1,1),z) - T_M((1,\tau),z)\Big)\\ &= \frac{1}{3}\Big(\Xi^{(1)}_{1,1}\Xi^{(2)}_{1,1} + \Xi^{(1)}_{2,1}\Xi^{(2)}_{3,1} - \frac{\eta(z)}{\eta(3z)}\Xi^{(3)}_{3,3}\Big)\end{aligned}$$

for $\epsilon = 1,2$. The computations for $W(\epsilon), \epsilon = 0,1,2$ are similar.

Since M^i_k, $i = a,b,c$ are equivalent irreducible M^τ-modules by Lemma 4.1, we have $q^{-1/20}\operatorname{ch} M^c_k = q^{-1/20}\operatorname{ch} M^a_k = \Xi^{(1)}_{2,2}\Xi^{(2)}_{2,1}$. Likewise, $q^{-1/20}\operatorname{ch} W^c_k = q^{-1/20}\operatorname{ch} W^a_k = \Xi^{(1)}_{2,2}\Xi^{(2)}_{2,2}$. The proof is complete. □

We now discuss the relation between the characters computed here and those of modules for a W-algebra computed in [**16**]. We use the notation of [**16**] without any comments. We refer to their results in the case that $\bar{\mathfrak{g}}$ is the simple finite dimensional Lie algebra over $\mathbb{C}$ of type A_2 and $(p,p') = (6,5)$.

In this case, we have

$$P_+^{p-h^\vee} = P_+^3 = \left\{ \sum_{i=0}^{2} a_i \Lambda_i \,|\, 0 \le a_i \in \mathbb{Z} \text{ and } \sum_{i=0}^{2} a_i = 3 \right\},$$
$$P_+^{\vee p'-h} = P_+^{\vee 2} = \left\{ \sum_{i=0}^{2} b_i \Lambda_i^\vee \,|\, 0 \le b_i \in \mathbb{Z} \text{ and } \sum_{i=0}^{2} b_i = 2 \right\}.$$

It can be easily shown that $\widetilde{W}_+ = \langle g \rangle$ is the cyclic group of order 3 such that $g(\Lambda_0) = \Lambda_1, g(\Lambda_1) = \Lambda_2$, and $g(\Lambda_2) = \Lambda_0$. The cardinality of $I_{p,p'} = (P_+^3 \times P_+^{\vee 2})/\widetilde{W}_+$ is equal to 20.

For $\lambda \in P_+^3, \mu \in P_+^{\vee 2}$, define

$$\varphi_{\lambda,\mu}(z) = \eta(z)^{-2} \sum_{w \in W} \epsilon(w) q^{\frac{1}{2pp'}|p'w(\lambda+\rho) - p(\mu+\rho^\vee)|^2}.$$

The vector space spanned by $\varphi_{\lambda,\mu}(z), (\lambda,\mu) \in I_{p,p'}$ is invariant under the action of $SL_2(\mathbb{Z})$ and the transformation formula

$$\varphi_{\lambda,\lambda'}\left(\frac{-1}{z}\right) = \sum_{(\mu,\mu') \in I_{p,p'}} S_{(\lambda,\lambda'),(\mu,\mu')} \varphi_{\mu,\mu'}(z)$$

is given by [**16**, (4.2.2)]. Define $\mathcal{F}_1 = \{\varphi_{\lambda,\mu}(z) \mid (\lambda,\mu) \in I_{p,p'}\}$. In [**16**, Section 3], it is shown that each $\varphi_{\lambda,\mu}(z) \in \mathcal{F}_1$ is the character of a module for the W-algebra associated to $\bar{\mathfrak{g}}$ and (p,p') which is conjectured to be irreducible.

We denote by $\mathcal{F}_2$ the set of characters of all irreducible M^τ-modules computed in Theorem 6.2. For any m, there is a congruence subgroup Γ_m such that each $\Xi_{r,s}^{(m)}$ is a modular form for Γ_m (cf. [**33**, (6.11)]). Then there is a congruence subgroup Γ such that each character in $\mathcal{F}_2$ is invariant under the action of Γ. The following transformation formulas hold by the formula (6.2):

$$\begin{pmatrix} T_M((1,1),\frac{-1}{z}) \\ T_{W_k^0}((1,1),\frac{-1}{z}) \\ T_{M_k^c}((1,1),\frac{-1}{z}) \\ T_{W_k^c}((1,1),\frac{-1}{z}) \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\sin(\frac{\pi}{5})}{\sqrt{5}} & \frac{\sin(\frac{2\pi}{5})}{\sqrt{5}} & \frac{3\sin(\frac{\pi}{5})}{\sqrt{5}} & \frac{3\sin(\frac{2\pi}{5})}{\sqrt{5}} \\ \frac{\sin(\frac{2\pi}{5})}{\sqrt{5}} & -\frac{\sin(\frac{\pi}{5})}{\sqrt{5}} & \frac{3\sin(\frac{2\pi}{5})}{\sqrt{5}} & -\frac{3\sin(\frac{\pi}{5})}{\sqrt{5}} \\ \frac{\sin(\frac{\pi}{5})}{\sqrt{5}} & \frac{\sin(\frac{2\pi}{5})}{\sqrt{5}} & -\frac{\sin(\frac{\pi}{5})}{\sqrt{5}} & -\frac{\sin(\frac{2\pi}{5})}{\sqrt{5}} \\ \frac{\sin(\frac{2\pi}{5})}{\sqrt{5}} & -\frac{\sin(\frac{\pi}{5})}{\sqrt{5}} & -\frac{\sin(\frac{2\pi}{5})}{\sqrt{5}} & \frac{\sin(\frac{\pi}{5})}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} T_M((1,1),z) \\ T_{W_k^0}((1,1),z) \\ T_{M_k^c}((1,1),z) \\ T_{W_k^c}((1,1),z) \end{pmatrix}$$

and

$$\begin{pmatrix} T_M((1,\tau),\frac{-1}{z}) \\ T_{W_k^0}((1,\tau),\frac{-1}{z}) \end{pmatrix} = \begin{pmatrix} \frac{2\sin(\frac{\pi}{5})}{\sqrt{5}} & \frac{2\sin(\frac{2\pi}{5})}{\sqrt{5}} \\ \frac{2\sin(\frac{2\pi}{5})}{\sqrt{5}} & \frac{-2\sin(\frac{\pi}{5})}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} T_{M_T(\tau)}((\tau,1),z) \\ T_{W_T(\tau)}((\tau,1),z) \end{pmatrix}.$$

Thus we have the transformation formulas for elements of $\mathcal{F}_2$. Comparing the q-expansions and the coefficients of transformation formulas of elements in $\mathcal{F}_1$ and $\mathcal{F}_2$, it can be shown that $\mathcal{F}_1 = \mathcal{F}_2$ using Lemma 1.7.1 in [20]. In particular, $\varphi_{3\Lambda_0,2\Lambda_0^\vee}(z) = q^{-1/20}\,\mathrm{ch}\,M^\tau$ holds.

Appendix A. $\mathbf{v}^{12}$, $J(-1)\mathbf{v}^{12}$, $J(-2)\mathbf{v}^{12}$, and $J(-1)^2\mathbf{v}^{12}$

$$\begin{aligned}
\mathbf{v}^{12} = &-(5877264800/3501)L(-12)\mathbf{1} + (3404072000/3501)L(-10)L(-2)\mathbf{1}\\
&- (2653990000/3501)L(-9)L(-3)\mathbf{1} - (266376800/3501)L(-8)L(-4)\mathbf{1}\\
&+ (282988000/1167)L(-8)L(-2)^2\mathbf{1} - (23744800/1167)L(-7)L(-5)\mathbf{1}\\
&- (30824000/1167)L(-7)L(-3)L(-2)\mathbf{1} + (1242377600/1167)L(-6)^2\mathbf{1}\\
&- (61947200/3501)L(-6)L(-4)L(-2)\mathbf{1} - (1313806000/1167)L(-6)L(-3)^2\mathbf{1}\\
&- (45496000/1167)L(-6)L(-2)^3\mathbf{1} - (3046768400/3501)L(-5)^2L(-2)\mathbf{1}\\
&+ (299424800/1167)L(-5)L(-4)L(-3)\mathbf{1} + (2347094000/3501)L(-5)L(-3)L(-2)^2\mathbf{1}\\
&- (17280400/1167)L(-4)^3\mathbf{1} - (2036373200/3501)L(-4)^2L(-2)^2\mathbf{1}\\
&+ (82996000/3501)L(-4)L(-3)^2L(-2)\mathbf{1} + (1074512000/3501)L(-4)L(-2)^4\mathbf{1}\\
&+ (511628125/3501)L(-3)^4\mathbf{1} - (418850000/3501)L(-3)^2L(-2)^3\mathbf{1}\\
&- (59680000/3501)L(-2)^6\mathbf{1} - (505200/389)L(-6)J(-3)^2\mathbf{1}\\
&+ (3380480/1167)L(-4)L(-2)J(-3)^2\mathbf{1} + 1150L(-3)^2J(-3)^2\mathbf{1}\\
&- (184400/1167)L(-2)^3J(-3)^2\mathbf{1} + (3788680/1167)L(-5)J(-4)J(-3)\mathbf{1}\\
&- (8788400/3501)L(-3)L(-2)J(-4)J(-3)\mathbf{1} - (12761440/3501)L(-4)J(-5)J(-3)\mathbf{1}\\
&- (5727500/10503)L(-4)J(-4)^2\mathbf{1} + (352400/389)L(-2)^2J(-5)J(-3)\mathbf{1}\\
&+ (5727500/10503)L(-2)^2J(-4)^2\mathbf{1} + (1593900/389)L(-3)J(-6)J(-3)\mathbf{1}\\
&+ (12935800/10503)L(-3)J(-5)J(-4)\mathbf{1} + (4108000/3501)L(-2)J(-7)J(-3)\mathbf{1}\\
&- (2811800/1167)L(-2)J(-6)J(-4)\mathbf{1} - (3131600/10503)L(-2)J(-5)^2\mathbf{1}\\
&- (14904160/3501)J(-9)J(-3)\mathbf{1} + (32677600/10503)J(-8)J(-4)\mathbf{1}\\
&+ (9423200/10503)J(-7)J(-5)\mathbf{1} + (2432375/1167)J(-6)^2\mathbf{1}\\
&+ J(-3)^4\mathbf{1}.
\end{aligned}$$

$$\begin{aligned}
J(-1)\mathbf{v}^{12} = &(47528/389)L(-4)J(-3)^3\mathbf{1} - (53552200/1167)J(-7)J(-3)^2\mathbf{1}\\
&- (14322122880/389)L(-10)J(-3)\mathbf{1} - (7313862400/389)L(-8)L(-2)J(-3)\mathbf{1}\\
&- (2263268800/389)L(-7)L(-3)J(-3)\mathbf{1}\\
&+ (7140323840/1167)L(-6)L(-4)J(-3)\mathbf{1}\\
&- (4870066240/389)L(-5)^2J(-3)\mathbf{1} - (41271174880/3501)L(-9)J(-4)\mathbf{1}
\end{aligned}$$

$$
\begin{aligned}
&+ (87811701120/389)J(-13)\mathbf{1} - (65647722400/389)L(-2)J(-11)\mathbf{1}\\
&+ (195884000/1167)L(-8)J(-5)\mathbf{1} + (18292448200/389)L(-3)J(-10)\mathbf{1}\\
&- (2704504400/389)L(-7)J(-6)\mathbf{1} - (8342231040/389)L(-4)J(-9)\mathbf{1}\\
&- (2134787200/389)L(-6)J(-7)\mathbf{1} + (17270275360/1167)L(-5)J(-8)\mathbf{1}\\
&- (926640/389)L(-2)^2J(-3)^3\mathbf{1} + (24986000/1167)L(-2)J(-5)J(-3)^2\mathbf{1}\\
&- (2833833600/389)L(-6)L(-2)^2J(-3)\mathbf{1}\\
&+ (1692496000/389)L(-5)L(-3)L(-2)J(-3)\mathbf{1}\\
&- (6705813920/1167)L(-4)^2L(-2)J(-3)\mathbf{1}\\
&- (10899200/9)L(-7)L(-2)J(-4)\mathbf{1}\\
&+ (20147275200/389)L(-2)^2J(-9)\mathbf{1}\\
&+ (32842950400/3501)L(-6)L(-2)J(-5)\mathbf{1}\\
&- (8472651200/389)L(-3)L(-2)J(-8)\mathbf{1}\\
&- (10511649200/1167)L(-5)L(-2)J(-6)\mathbf{1}\\
&+ (12944796800/1167)L(-4)L(-2)J(-7)\mathbf{1} - (9963200/389)J(-5)^2J(-3)\mathbf{1}\\
&+ (1607444800/389)L(-5)L(-3)J(-5)\mathbf{1} - (1408915040/1167)L(-4)^2J(-5)\mathbf{1}\\
&+ (1271140/1167)L(-3)J(-4)J(-3)^2\mathbf{1}\\
&- (2312728600/1167)L(-4)L(-3)^2J(-3)\mathbf{1}\\
&- (38446491200/3501)L(-6)L(-3)J(-4)\mathbf{1}\\
&- (1002365000/1167)L(-3)^2J(-7)\mathbf{1}\\
&+ (462527000/389)L(-4)L(-3)J(-6)\mathbf{1}\\
&+ (14283100/389)J(-6)J(-4)J(-3)\mathbf{1}\\
&+ (15075473920/3501)L(-5)L(-4)J(-4)\mathbf{1}\\
&- (11661041600/3501)L(-4)L(-2)^2J(-5)\mathbf{1}\\
&- (7223710000/3501)L(-3)^2L(-2)J(-5)\mathbf{1}\\
&+ (2639390000/389)L(-3)L(-2)^2J(-6)\mathbf{1}\\
&- (30846400/389)L(-5)L(-2)^2J(-4)\mathbf{1}\\
&+ (5365349600/3501)L(-4)L(-3)L(-2)J(-4)\mathbf{1}\\
&- (34590712000/3501)L(-2)^3J(-7)\mathbf{1} + (6230282500/3501)L(-3)^3J(-4)\mathbf{1}\\
&+ (1547000/389)J(-5)J(-4)^2\mathbf{1} - (4562948000/3501)L(-3)L(-2)^3J(-4)\mathbf{1}\\
&- (10829000/1167)L(-2)J(-4)^2J(-3)\mathbf{1}\\
&+ (4340336000/3501)L(-2)^4J(-5)\mathbf{1}\\
&+ (1919403200/1167)L(-4)L(-2)^3J(-3)\mathbf{1}\\
&- (99658000/389)L(-3)^2L(-2)^2J(-3)\mathbf{1}\\
&- (89856000/389)L(-2)^5J(-3)\mathbf{1}.
\end{aligned}
$$

$$
\begin{aligned}
J(-2)\mathbf{v}^{12} = &-4272L(-5)J(-3)^3\mathbf{1} - (21069744/389)J(-8)J(-3)^2\mathbf{1}\\
&- (14150438080/1167)L(-11)J(-3)\mathbf{1}
\end{aligned}
$$

$$
\begin{aligned}
&- (3639849600/389)L(-9)L(-2)J(-3)\mathbf{1}\\
&- (9699222400/1167)L(-8)L(-3)J(-3)\mathbf{1}\\
&+ (2157139840/1167)L(-7)L(-4)J(-3)\mathbf{1}\\
&- (5925448960/1167)L(-6)L(-5)J(-3)\mathbf{1}\\
&+ (3006435200/389)L(-10)J(-4)\mathbf{1}\\
&+ (325064548960/389)J(-14)\mathbf{1} - (176823168000/389)L(-2)J(-12)\mathbf{1}\\
&+ (10174691200/3501)L(-9)J(-5)\mathbf{1} + (38988751200/389)L(-3)J(-11)\mathbf{1}\\
&- (4612321600/1167)L(-8)J(-6)\mathbf{1} - (33023056960/389)L(-4)J(-10)\mathbf{1}\\
&- (13371577600/1167)L(-7)J(-7)\mathbf{1} + (54711326720/1167)L(-5)J(-9)\mathbf{1}\\
&+ (4368409600/1167)L(-6)J(-8)\mathbf{1} - 960L(-3)L(-2)J(-3)^3\mathbf{1}\\
&+ (5080480/389)L(-2)J(-6)J(-3)^2\mathbf{1}\\
&- (1269523200/389)L(-7)L(-2)^2J(-3)\mathbf{1}\\
&+ (1626342400/1167)L(-6)L(-3)L(-2)J(-3)\mathbf{1}\\
&- (3954100480/1167)L(-5)L(-4)L(-2)J(-3)\mathbf{1}\\
&+ (6597673600/1167)L(-8)L(-2)J(-4)\mathbf{1}\\
&+ (382495595200/3501)L(-2)^2J(-10)\mathbf{1}\\
&+ (23379344000/3501)L(-7)L(-2)J(-5)\mathbf{1}\\
&- (41865472000/1167)L(-3)L(-2)J(-9)\mathbf{1}\\
&+ (5662851200/1167)L(-6)L(-2)J(-6)\mathbf{1}\\
&+ (41335582720/1167)L(-4)L(-2)J(-8)\mathbf{1}\\
&- (66974297600/3501)L(-5)L(-2)J(-7)\mathbf{1} + 7760L(-3)J(-5)J(-3)^2\mathbf{1}\\
&- (13118000/1167)J(-6)J(-5)J(-3)\mathbf{1}\\
&- (3036691200/389)L(-6)L(-3)J(-5)\mathbf{1}\\
&+ (2541514240/1167)L(-5)L(-4)J(-5)\mathbf{1}\\
&- (4489884400/1167)L(-5)L(-3)^2J(-3)\mathbf{1}\\
&+ (48898000/389)L(-5)L(-3)J(-6)\mathbf{1}\\
&+ (524720000/389)L(-4)^2L(-3)J(-3)\mathbf{1}\\
&- (478727200/389)L(-4)^2J(-6)\mathbf{1}\\
&- (315678400/3501)L(-7)L(-3)J(-4)\mathbf{1}\\
&- (5656762000/1167)L(-3)^2J(-8)\mathbf{1}\\
&+ (4388915200/1167)L(-4)L(-3)J(-7)\mathbf{1}\\
&+ (5080480/1167)L(-4)J(-4)J(-3)^2\mathbf{1}\\
&+ (7809478400/1167)L(-6)L(-4)J(-4)\mathbf{1}\\
&+ (117493120/3501)J(-7)J(-4)J(-3)\mathbf{1}\\
&- (6924715520/3501)L(-5)^2J(-4)\mathbf{1}\\
&+ (7972739200/3501)L(-5)L(-2)^2J(-5)\mathbf{1}\\
&+ (726208000/3501)L(-4)L(-3)L(-2)J(-5)\mathbf{1}\\
&+ (15160000/3501)J(-5)^2J(-4)\mathbf{1}
\end{aligned}
$$

$$
\begin{aligned}
&- (9229774400/1167)L(-4)L(-2)^2J(-6)\mathbf{1}\\
&- (1273060000/1167)L(-3)^2L(-2)J(-6)\mathbf{1}\\
&- (5021408000/3501)L(-6)L(-2)^2J(-4)\mathbf{1}\\
&+ (4736835200/1167)L(-5)L(-3)L(-2)J(-4)\mathbf{1}\\
&- (4697646400/3501)L(-4)^2L(-2)J(-4)\mathbf{1}\\
&- (28325800/3501)J(-6)J(-4)^2\mathbf{1}\\
&+ (10330016000/1167)L(-3)L(-2)^2J(-7)\mathbf{1}\\
&+ (6184910000/3501)L(-3)^3J(-5)\mathbf{1}\\
&- (2988476000/3501)L(-4)L(-3)^2J(-4)\mathbf{1}\\
&+ (2298688000/1167)L(-4)L(-2)^3J(-4)\mathbf{1}\\
&- (59886716800/3501)L(-2)^3J(-8)\mathbf{1}\\
&- (1320284000/3501)L(-3)^2L(-2)^2J(-4)\mathbf{1}\\
&+ (22910000/10503)L(-2)J(-4)^3\mathbf{1}\\
&- (3979216000/3501)L(-3)L(-2)^3J(-5)\mathbf{1}\\
&- (122435200/389)L(-5)L(-2)^3J(-3)\mathbf{1}\\
&- (977670400/1167)L(-4)L(-3)L(-2)^2J(-3)\mathbf{1}\\
&- (22467200/3501)L(-2)J(-5)J(-4)J(-3)\mathbf{1}\\
&+ (58888000/1167)L(-3)^3L(-2)J(-3)\mathbf{1}\\
&- (17576800/3501)L(-3)J(-4)^2J(-3)\mathbf{1}\\
&+ (29504000/389)L(-3)L(-2)^4J(-3)\mathbf{1} + (2281792000/1167)L(-2)^4J(-6)\mathbf{1}\\
&- (368800/389)L(-2)^2J(-4)J(-3)^2\mathbf{1} - (238720000/1167)L(-2)^5J(-4)\mathbf{1}.
\end{aligned}
$$

$$
\begin{aligned}
J(-1)^2\mathbf{v}^{12} = {}& (28587850894720/389)L(-14)\mathbf{1} + (40679435680000/1167)L(-12)L(-2)\mathbf{1}\\
&- (20370766707200/389)L(-11)L(-3)\mathbf{1}\\
&- (29040708661120/389)L(-10)L(-4)\mathbf{1}\\
&- (1357372140800/389)L(-10)L(-2)^2\mathbf{1}\\
&- (120978369778240/1167)L(-9)L(-5)\mathbf{1}\\
&+ (15046999864000/1167)L(-9)L(-3)L(-2)\mathbf{1}\\
&- (120139236131200/1167)L(-8)L(-6)\mathbf{1}\\
&+ (7353135836800/1167)L(-8)L(-4)L(-2)\mathbf{1}\\
&+ (5914869272000/389)L(-8)L(-3)^2\mathbf{1}\\
&- (9027652192000/1167)L(-8)L(-2)^3\mathbf{1} - (19757556187200/389)L(-7)^2\mathbf{1}\\
&+ (10357377908800/389)L(-7)L(-5)L(-2)\mathbf{1}\\
&+ (6212435174400/389)L(-7)L(-4)L(-3)\mathbf{1}\\
&- (3066391744000/389)L(-7)L(-3)L(-2)^2\mathbf{1}\\
&- (34866323814400/1167)L(-6)^2L(-2)\mathbf{1}
\end{aligned}
$$

$$
\begin{aligned}
&- (1360052761600/389)L(-6)L(-5)L(-3)\mathbf{1}\\
&- (3455809144320/389)L(-6)L(-4)^2\mathbf{1}\\
&+ (8114060115200/1167)L(-6)L(-4)L(-2)^2\mathbf{1}\\
&+ (2356317080000/1167)L(-6)L(-3)^2L(-2)\mathbf{1}\\
&- (4200302912000/1167)L(-6)L(-2)^4\mathbf{1}\\
&+ (2046779720960/389)L(-5)^2L(-4)\mathbf{1}\\
&+ (5012264899200/389)L(-5)^2L(-2)^2\mathbf{1}\\
&+ (5606971697600/1167)L(-5)L(-4)L(-3)L(-2)\mathbf{1}\\
&+ (4546296703000/1167)L(-5)L(-3)^3\mathbf{1}\\
&- (3986231288000/1167)L(-5)L(-3)L(-2)^3\mathbf{1}\\
&- (824891421120/389)L(-4)^3L(-2)\mathbf{1}\\
&+ (129922182000/389)L(-4)^2L(-3)^2\mathbf{1}\\
&+ (9190279446400/1167)L(-4)^2L(-2)^3\mathbf{1}\\
&- (3417631724000/1167)L(-4)L(-3)^2L(-2)^2\mathbf{1}\\
&- (1854416512000/1167)L(-4)L(-2)^5\mathbf{1}\\
&- (339474200000/1167)L(-3)^4L(-2)\mathbf{1}\\
&+ (472407520000/1167)L(-3)^2L(-2)^4\mathbf{1} + (21565440000/389)L(-2)^7\mathbf{1}\\
&- (33906046720/389)L(-8)J(-3)^2\mathbf{1}\\
&- (38547928640/389)L(-6)L(-2)J(-3)^2\mathbf{1}\\
&+ (8889576280/389)L(-5)L(-3)J(-3)^2\mathbf{1} - (52680368/389)L(-4)^2J(-3)^2\mathbf{1}\\
&+ (1681515680/389)L(-4)L(-2)^2J(-3)^2\mathbf{1}\\
&- (4900781600/389)L(-3)^2L(-2)J(-3)^2\mathbf{1}\\
&- (680659200/389)L(-2)^4J(-3)^2\mathbf{1}\\
&- (21316634560/1167)L(-7)J(-4)J(-3)\mathbf{1}\\
&+ (15456968800/389)L(-5)L(-2)J(-4)J(-3)\mathbf{1}\\
&- (57407779520/1167)L(-4)L(-3)J(-4)J(-3)\mathbf{1}\\
&+ (769371200/389)L(-3)L(-2)^2J(-4)J(-3)\mathbf{1}\\
&+ (82018834560/389)L(-6)J(-5)J(-3)\mathbf{1}\\
&- (318755320000/3501)L(-6)J(-4)^2\mathbf{1}\\
&- (62232722240/1167)L(-4)L(-2)J(-5)J(-3)\mathbf{1}\\
&+ (59657182000/3501)L(-4)L(-2)J(-4)^2\mathbf{1}\\
&+ (4384283800/1167)L(-3)^2J(-5)J(-3)\mathbf{1}\\
&+ (28313585300/1167)L(-3)^2J(-4)^2\mathbf{1}\\
&+ (14719931200/1167)L(-2)^3J(-5)J(-3)\mathbf{1}\\
&- (15017860000/3501)L(-2)^3J(-4)^2\mathbf{1}\\
&- (102815580920/389)L(-5)J(-6)J(-3)\mathbf{1}
\end{aligned}
$$

$$
\begin{aligned}
&+ (214806972640/3501)L(-5)J(-5)J(-4)\mathbf{1}\\
&+ (20784972000/389)L(-3)L(-2)J(-6)J(-3)\mathbf{1}\\
&- (133605586400/3501)L(-3)L(-2)J(-5)J(-4)\mathbf{1}\\
&+ (243575438080/1167)L(-4)J(-7)J(-3)\mathbf{1}\\
&+ (7292932400/389)L(-4)J(-6)J(-4)\mathbf{1}\\
&- (12891781760/389)L(-4)J(-5)^2\mathbf{1}\\
&- (49983377600/389)L(-2)^2J(-7)J(-3)\mathbf{1}\\
&+ (10825750000/389)L(-2)^2J(-6)J(-4)\mathbf{1}\\
&- (13957486400/3501)L(-2)^2J(-5)^2\mathbf{1}\\
&- (173848522640/1167)L(-3)J(-8)J(-3)\mathbf{1}\\
&- (65060216000/1167)L(-3)J(-7)J(-4)\mathbf{1}\\
&+ (25622862200/389)L(-3)J(-6)J(-5)\mathbf{1}\\
&+ (174271514560/389)L(-2)J(-9)J(-3)\mathbf{1}\\
&- (232573421600/3501)L(-2)J(-8)J(-4)\mathbf{1}\\
&+ (392430209600/3501)L(-2)J(-7)J(-5)\mathbf{1}\\
&- (31534947600/389)L(-2)J(-6)J(-6)\mathbf{1}\\
&- (5559840/389)L(-2)J(-3)^4\mathbf{1} - (291151720080/389)J(-11)J(-3)\mathbf{1}\\
&+ (257458099600/1167)J(-10)J(-4)\mathbf{1} - (140099797760/389)J(-9)J(-5)\mathbf{1}\\
&+ (83988236280/389)J(-8)J(-6)\mathbf{1} - (44378890400/389)J(-7)^2\mathbf{1}\\
&+ (22538776/389)J(-5)J(-3)^3\mathbf{1} - (26131300/1167)J(-4)^2J(-3)^2\mathbf{1}.
\end{aligned}
$$

Appendix B. The images of four vectors in $A(\mathcal{W})$

For simplicity of notation we omit the symbol $*$ for multiplication in $A(\mathcal{W})$.

$$
\begin{aligned}
[\mathbf{v}^{12}] = &-(59680000/3501)[\omega]^6 + (156040000/3501)[\omega]^5 \\
&- (115878400/3501)[\omega]^4 \\
&+ \big(-(184400/1167)[J]^2 + 32328400/3501\big)[\omega]^3 \\
&+ \big((536500/1167)[J]^2 - 3155968/3501\big)[\omega]^2 \\
&+ \big(-(87812/389)[J]^2 + 93184/3501\big)[\omega] \\
&+ [J]^4 + (75776/3501)[J]^2.
\end{aligned}
$$

$$
\begin{aligned}
[J(-1)\mathbf{v}^{12}] = &-(89856000/389)[J][\omega]^5 + (228945600/389)[J][\omega]^4 \\
&- (555607520/1167)[J][\omega]^3 \\
&+ \big(-(926640/389)[J]^3 + (57790304/389)[J]\big)[\omega]^2 \\
&+ \big((1637064/389)[J]^3 - (19542016/1167)[J]\big)[\omega] \\
&- (668408/389)[J]^3 + (186368/389)[J].
\end{aligned}
$$

$$
\begin{aligned}
[J(-2)\mathbf{v}^{12}] = &(179712000/389)[J][\omega]^5 - (457891200/389)[J][\omega]^4 \\
&+ (1111215040/1167)[J][\omega]^3 \\
&+ \big((1853280/389)[J]^3 - (115580608/389)[J]\big)[\omega]^2 \\
&+ \big(-(3274128/389)[J]^3 + (39084032/1167)[J]\big)[\omega] \\
&+ (1336816/389)[J]^3 - (372736/389)[J].
\end{aligned}
$$

$$
\begin{aligned}
[J(-1)^2\mathbf{v}^{12}] = &(21565440000/389)[\omega]^7 + (513849856000/1167)[\omega]^6 \\
&- (552497504000/389)[\omega]^5 \\
&+ \big(-(680659200/389)[J]^2 + 1285515063040/1167\big)[\omega]^4 \\
&+ \big((3994427840/389)[J]^2 - 121501591744/389\big)[\omega]^3 \\
&+ \big(-(8220864912/389)[J]^2 + 36103315456/1167\big)[\omega]^2 \\
&+ \big(-(5559840/389)[J]^4 + (3836073072/389)[J]^2 \\
&- (363417600/389)\big)[\omega] \\
&- (9879324/389)[J]^4 - (355536896/389)[J]^2.
\end{aligned}
$$

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