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The main theorem characterizes the Lagrangian homology classes of a compact symplectic $2n$-manifold with integral symplectic form ω. An integral homology n-class α is Lagrangian (i.e., can be represented by a Lagrangian n-cycle) if and only if $\alpha \cap [\omega] = 0$.

1. Introduction

In euclidean space $\mathbb{R}^{2n}$ equipped with the standard symplectic structure $\omega = \sum_i^n dx_i \wedge dy_i$ a Lagrangian submanifold Σ is called *exact* if the integral of the Liouville form $\lambda = 1/2 \sum_i^n (x_i \wedge dy_i - y_i \wedge dx_i)$ along any closed path on Σ is zero. It is well-known that a Lagrangian submanifold is exact if and only if it admits a lift to a Legendrian submanifold of the contact manifold $\mathbb{R}^{2n+1}$. It is interesting that the notion of "exact Lagrangian" can be generalized to symplectic manifolds for which the symplectic form is not an exact form. To do this we use a construction sometimes called *prequantization*. Suppose that (N, ω) is a compact symplectic manifold with *integral* symplectic form ω. There is a complex line bundle L over N with unitary connection η and curvature ω [**K**]. Denote the associated principal S^1-bundle by P. Then (P, η) is a contact manifold. The line bundle L is flat over a Lagrangian submanifold Σ and therefore, locally, Σ admits a Legendrian lift to (P, η). When Σ admits a global lift we call it *exact*. In this note we utilize the contact manifold (P, η) and the Legendrian lift to prove two results on the topology of Lagrangian submanifolds. Before we describe these results we remark on the integrality assumption. The Chern class of the line bundle L is $[\omega]$ and therefore it is necessary that the symplectic form be integral. However if ω is rational then after multiplying by a suitable scalar the form is integral. Since the property of being Lagrangian is independent of taking scalar multiples we can weaken the integrality assumption on ω to a rationality assumption.

Consider an integral homology class $\alpha \in H_n(N, \mathbb{Z})$ in the symplectic manifold (N, ω). An integral n-cycle is called a *Lagrangian cycle* if the n-simplices are Lagrangian (more precisely, if the n-simplices are the images of piecewise C^1 Lagrangian maps $\triangle^n \to N$). The class α is a *Lagrangian class* if α can be represented by a Lagrangian cycle. Note that if C is a Lagrangian

n-cycle and ϕ is a closed $(n-2)$-form on N then $\int_C \omega \wedge \phi = 0$. Therefore, by Poincare duality, a Lagrangian class α satisfies the condition:

$$\alpha \cap [\omega] = 0, \tag{1.1}$$

where $[\omega]$ denotes the cohomology class represented by ω. Is this necessary condition also sufficient? That is, if an integral class α satisfies $\alpha \cap [\omega] = 0$ is it a Lagrangian class? We prove the following answer:

Theorem 1.1. *Let N be a compact symplectic manifold of dimension $2n$ with an* integral *symplectic form ω. An integral homology class $\alpha \in H_n(N, \mathbb{Z})$ is a Lagrangian class if and only if $\alpha \cap [\omega] = 0$.*

When $n = 2$ the condition (1.1) becomes $[\omega](\alpha) = 0$. In [**S-W**] it is shown that this condition implies that α is a Lagrangian class. When $n = 3$ and N is simply connected (1.1) is satisfied for any class α. In [**W2**] it is shown that in this case every class is Lagrangian. In fact, every class can be represented by a Lagrangian immersion.

Next let Σ be a minimal submanifold of S^{2n-1} that is Legendrian for the standard contact structure on S^{2n-1}. It is easy to see that the image of Σ under the Hopf map is a minimal, Lagrangian submanifold $\overline{\Sigma}$ of $\mathbb{CP}^{n-1}$. Conversely, we show:

Theorem 1.2. *Let $\overline{\Sigma}$ be a minimal, Lagrangian submanifold in $\mathbb{CP}^{n-1}$. Then $\overline{\Sigma}$ lifts to a minimal Legendrian submanifold $\Sigma \subset S^{2n-1}$.*

The local version of the theorem is elementary. The global version follows from a consideration of the holonomy of the complex line bundle L described above. We remark that the tangent cone of an isolated special Lagrangian singularity is the cone on a minimal Legendrian submanifold of S^{2n-1} and therefore this theorem has some application to such singularities.

2. Lagrangian homology

Let N be a compact symplectic manifold of dimension $2n$ equipped with an *integral* symplectic form ω. Adopting the notation of the introduction we let (P, η) be the contact manifold such that $d\eta = \omega$. In particular $\pi : P \to N$ is an S^1 bundle with Chern class $[\omega]$.

Proposition 2.1. *Let $\alpha \in H_n(N, \mathbb{Z})$. There is a class $\widetilde{\alpha} \in H_n(P, \mathbb{Z})$ such that $\pi_\star(\widetilde{\alpha}) = \alpha$ if and only if $\alpha \cap [\omega] = 0$.*

Proof. If $\pi_\star(\widetilde{\alpha}) = \alpha$ then

$$\alpha \cap [\omega] = \pi_\star(\widetilde{\alpha}) \cap [\omega] = \pi_\star(\widetilde{\alpha} \cap \pi^\star[\omega]) = \pi_\star(\widetilde{\alpha} \cap [d\eta]) = 0.$$

Conversely, consider the Thom–Gysin sequence [**S**] of the S^1-bundle $\pi : P \to N$:

$$\cdots \longrightarrow H_n(P; \mathbb{Z}) \xrightarrow{\pi_*} H_n(N; \mathbb{Z}) \xrightarrow{\Phi} H_{n-2}(N; \mathbb{Z}) \longrightarrow \cdots,$$

where for $z \in H_n(N;\mathbb{Z}), \Phi(z) = z \cap [\omega]$. By assumption $\Phi(\alpha) = 0$ and therefore there exists $\widetilde{\alpha} \in H_n(P, \mathbb{Z})$ such that $\pi_\star(\widetilde{\alpha}) = \alpha$. □

The next proposition uses successive applications of a horizontal extension lemma due to Gromov [**G**, 3.5]. Let V be a compact contact $2n+1$-manifold and S be an open k-simplex. A piecewise C^1 map $h : S \to V$ is called *horizontal* if the image of the tangent space of S lies in the contact plane at each point. Suppose W is a simplical k-complex with $k \leq n$ and $W_0 \subset W$ is a subcomplex. Let $f_0 : W_0 \to V$ be a piecewise C^1 horizontal map and suppose that there is continuous extension $g : W \to V$ of f_0. Then there is a piecewise C^1 horizontal extension $f : W \to V$ of f_0 that is homotopic to g.

Proposition 2.2. *Any homology class in $H_n(P;\mathbb{Z})$ can be represented by a horizontal n-cycle.*

Proof. Let $b \in H_n(P;\mathbb{Z})$ and suppose B is a simplicial cycle that represents b. Suppose that B is sufficiently fine so that the 0 and 1 simplices have the following property: If v_1^0 and v_2^0 are vertices such that there is a 1-simplex s^1 in B joining them then there is a horizontal 1-simplex σ^1 joining v_1^0 and v_2^0 that is C^0 close to s^1. In particular there is a homotopy joining s^1 and σ^1. Use this property to construct a horizontal 1-cycle approximating the 1-simplices of B (and with the same 0-simplices as B). Consider a 2-simplex t^2 of B with 1-simplices s_i^1, $i = 1, 2, 3$ such that $\partial t^2 = s_1^1 - s_2^1 + s_3^1$. To each simplex s_i^1 there is a corresponding horizontal simplex σ_i^1. The homotopies between s_i^1 and σ_i^1 and the 2-simplex t^2 show that the cycle $\sigma_1^1 - \sigma_2^1 + \sigma_3^1$ has a continuous extension and therefore by the horizontal extension lemma there is a horizontal 2-simplex τ^2 with $\partial\tau^2 = \sigma_1^1 - \sigma_2^1 + \sigma_3^1$. Moreover τ^2 is homotopic to t^2 and the homotopy can be taken to extend with the homotopies between s_i^1 and σ_i^1. Carrying out this construction for each 2-simplex of B there is a horizontal 2-cycle approximating the 2-simplices of B. Successively apply this construction to the 3-simplices, the 4-simplices, ..., the n-simplices to constuct a horizontal n-cycle Λ that is homotopic to B. The image of the homotopy determines an $(n+1)$-chain C with $B - \Lambda = \partial C$, proving the proposition. □

The following proposition and Proposition 2.1 together prove Theorem 1.1.

Proposition 2.3. *Suppose that (N, ω) is a compact symplectic $2n$-manifold with integral symplectic form ω. If $\alpha \in H_n(N;\mathbb{Z})$ has a lift $\widetilde{\alpha} \in H_n(P;\mathbb{Z})$ then α is a Lagrangian class.*

Proof. By Proposition 2.2 there is a Legendrian cycle Λ that represents $\widetilde{\alpha}$. Then $\pi(\Lambda)$ is a Lagrangian cycle representing α. □

Remark. The above proof shows that under the hypothesis of Theorem 1.1 a homology class α satisfying $\alpha \cap [\omega] = 0$ can be represented by a Lagrangian cycle that lifts to a Legendrian cycle in (P, η).

As an application of Theorem 1.1 we prove:

Theorem 2.4. *Let (N, ω) be a compact symplectic $2n$-manifold with integral symplectic form ω. Suppose that $\alpha \in H_n(N; \mathbb{Z})$ satisfies $\alpha \cap [\omega] = 0$. Then α can be represented by a Lagrangian integral current T that minimizes mass (volume) among all Lagrangian n-cycles representing α.*

Proof. By Theorem 1.1 the set of Lagrangian integral currents S with $\partial S = \emptyset$ representing α is nonempty. A mass minimizing sequence of such currents converges in the weak topology to a current T with $\partial T = \emptyset$. It follows easily that T is Lagrangian. Since convergence in the weak topology is equivalent to convergence in the flat norm topology it follows from [**F-F**] that T is an integral current that represents $[\alpha]$. Thus

$$\|T\| = \inf_{\{S \in \mathcal{IL} : [S] = \alpha\}} \|S\|,$$

where $\mathcal{IL}$ denotes the set of Lagrangian integral currents. □

3. Lagrangian cones

Suppose that M is a special Lagrangian submanifold with an isolated singularity at $p \in M$ in the Calabi–Yau manifold N. Let C_p denote the tangent cone of p. Then C_p is a special Lagrangian cone in $\mathbb{C}^n$. Denote the link $C_p \cap S^{2n-1}$ by Σ. Then Σ is an imbedded Legendrian $n-1$-submanifold for the standard contact structure on S^{2n-1} and, in addition, Σ is a minimal submanifold of S^{2n-1}. Let $h : S^{2n-1} \to \mathbb{CP}^{n-1}$ be the Hopf map. Equivalently, let $L \to \mathbb{CP}^{n-1}$ be the tautological bundle so that the total space of the associated S^1 bundle is S^{2n-1}. Then the bundle projection is the Hopf map. Denote the image of Σ under the bundle projection by $\overline{\Sigma}$. It is easy to verify that $\overline{\Sigma}$ is a minimal, Lagrangian submanifold in $\mathbb{CP}^{n-1}$. This (well-known) construction thus associates a minimal Lagrangian submanifold of $\mathbb{CP}^{n-1}$ to an isolated singularity in a special Lagrangian submanifold. The converse is given in the statement of Theorem 1.2. It follows from Theorem 1.2 that isolated special Lagrangian singularities are classified by minimal, Lagrangian submanifolds of $\mathbb{CP}^{n-1}$.

The proof of the theorem proceeds as follows: Consider $\mathbb{CP}^{n-1}$ equipped with the Fubini-Study metric and Kähler form ω. Note that ω is integral. The tautological line bundle L over $\mathbb{CP}^{n-1}$ has a unitary connection η with curvature ω. The total space of the associated S^1 principal bundle is S^{2n-1} and (S^{2n-1}, η) is a contact manifold over $(\mathbb{CP}^{n-1}, \omega)$. Consider a smooth Lagrangian immersion $\imath : \Lambda \to \mathbb{CP}^{n-1}$. Locally $\imath$ has a Legendrian lift to S^{2n-1}. To determine when this lift is global note that the pulled-back connection $\imath^*\eta$ of $\imath^*L$ is flat. Thus there is a homomorphism (the holonomy homomorphism):

$$\mathrm{Hol}(\imath^*\eta) : H_1(\Sigma, \mathbb{Z}) \to S^1.$$

Proposition 3.1. *$\imath$ lifts to a Legendrian immersion if and only if the values of* $\mathrm{Hol}(\imath^*\eta)$ *are integral.*

Suppose next instead of the line bundle L, we consider the canonical line bundle K with connection τ induced by the metric and with curvature $Ric = R\omega$. As above if $\imath : \Lambda \to \mathbb{CP}^{n-1}$ is a smooth Lagrangian immersion (not necessarily minimal) then the pulled-back connection $\imath^*\tau$ of $\imath^*K$ is flat and $\imath$ lifts to a Legendrian immersion if and only if the values of the holonomy homomorphism of $\imath^*\tau$ are integral. On $\imath(\Lambda)$ the mean curvature 1-form $\sigma_H = H \lrcorner\, \omega$ is closed since $\mathbb{CP}^{n-1}$ is Kähler–Einstein.

Proposition 3.2. *Let γ be a closed curve on Λ. Then*

$$\mathrm{Hol}_\gamma(\imath^*\tau) = \exp\left(-2\pi i \int_\gamma \sigma_H\right).$$

Proof. See [**W1**]. □

Proof of Theorem 1.2. Note that $\overline{\Sigma}$ lifts locally to a minimal Legendrian submanifold of S^{2n-1}. The issue is the existence of a global lift. Because $\overline{\Sigma}$ is minimal the periods of σ_H vanish. Since $K = L^n$ and $\tau = n\eta$, it follows from Proposition 3.2 that the holonomy of η is integral. Therefore by Proposition 3.1 there is a global Legendrian lift.

References

[F-F] H. Federer and W. Fleming, *Normal and integral currents*, Ann. of Math., **72** (1960), 458–520, MR 0123260 (23 #A588), Zbl 0187.31301.

[G] M. Gromov, *Carnot-Caratheodory spaces seen from within*, Sub-Riemannian Geometry, 79–323, Progress in Math., **144**, Birkhäuser, Basel, 1996, MR 1421823 (2000f:53034), Zbl 0864.53025.

[K] B. Kostant, *Quantization and unitary representations.* I, Prequantization. Lectures in modern analysis and applications, III, Lecture Notes in Math., **170**, Springer, Berlin, 1970, 87–208, MR 0294568 (45 #3638), Zbl 0223.53028.

[S-W] R. Schoen and J. Wolfson, *Minimizing area among Lagrangian surfaces: The mapping problem*, J. Diff. Geom., **58(1)** (2001), 1–86, MR 1895348 (2003c:53119).

[S] E. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966, MR 0210112 (35 #1007), Zbl 0145.43303.

[W1] J. Wolfson, *Minimal Lagrangian diffeomorphisms and the Monge–Ampère equation*, J. Differential Geom., **46(2)** (1997), 335–373, MR 1484047 (99e:58045), Zbl 0926.53032.

[W2] ———, *Lagrangian cycles and volume*, in 'Conformal, Riemannian and Lagrangian Geometry', The 2000 Barrett Lectures, A. Chang, P. Yang, K. Grove and J. Wolfson, University Lecture Series, **27**, Amer. Math. Soc., 2002, 55–85, MR 1922722 (2003j:53124), Zbl 1019.53001.

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