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For prime rings containing nontrivial idempotents, we describe the bijective additive maps which preserve zero products. Also, we describe the additive maps which behave like derivations when acting on zero products.

1. Introduction

In the last decade considerable works have been done concerning local properties of maps; see [1], [7], and [9]–[32], where other references can be found. The goal of this paper is to show that automorphisms and derivations of prime rings with nontrivial idempotents can be "almost" determined by the action on the zero-product elements.

Our first theorem generalizes a similar result of Wong [34, Corollary D] for simple algebras with nontrivial idempotents, as well as some other results obtained for operator algebras [1, 13, 16, 33].

Let R be a prime ring. The definitions and some basic properties of the maximal right quotient ring $Q(R)$ and extended centroid $C(R)$ of R can be found in [6]. Recall that an element x in $Q(R)$ is said to be algebraic of degree $\leq n$ over $C(R)$ if there exist $c_0, c_1, c_2, \ldots, c_{n-1} \in C(R)$ such that $\sum_{i=0}^{n-1} c_i x^i + x^n = 0$. By $\deg R \geq n$ we mean that there exists an element x in R that is not algebraic of degree $\leq n-1$ over $C(R)$. The condition that $\deg R \geq n$ is equivalent to that R can not be embedded in the ring of $(n{-}1) \times (n{-}1)$ matrices over a field.

Theorem 1. *Let A and B be prime rings and $\theta : A \to B$ a bijective additive map such that $\theta(x)\theta(y) = 0$ for all $x, y \in A$ with $xy = 0$. Suppose that the maximal right quotient ring $Q(A)$ of A contains a nontrivial idempotent e such that $eA \cup Ae \subseteq A$.*

(i) *If $1 \in A$, then $\theta(xy) = \lambda\theta(x)\theta(y)$ for all $x, y \in A$, where $\lambda = 1/\theta(1)$ and $\theta(1) \in Z(B)$, the center of B. In particular, if $\theta(1) = 1$, then θ is a ring isomorphism from A onto B.*
(ii) *If $\deg B \geq 3$, then there exists $\lambda \in C(B)$, the extended centroid of B, such that $\theta(xy) = \lambda\theta(x)\theta(y)$ for all $x, y \in A$.*

It is clear that Theorem 1 can not be extended to arbitrary prime rings, since these rings may not have "enough" zero-divisors. The condition that the idempotent e be an element of $Q(A)$ (instead of A) enables us to consider matrix rings over arbitrary prime rings (not necessarily with units).

Our second result is an analog of Theorem 1 for derivations. In particular, it generalizes [**19**, Theorem 6].

Theorem 2. *Let A be a prime ring with center Z, maximal right quotient ring Q and extended centroid C. Let $\delta : A \to A$ be an additive map such that $\delta(x)y + x\delta(y) = 0$ for all $x, y \in A$ with $xy = 0$. Suppose that Q contains a nontrivial idempotent e such that $eA \cup Ae \subseteq A$.*

(i) *If $1 \in A$, then $\delta(xy) = \delta(x)y + x\delta(y) - \lambda xy$ for all $x, y \in A$, where $\lambda = \delta(1) \in Z$. In particular, if $\delta(1) = 0$, then δ is a derivation on A.*
(ii) *If $\deg A \geq 3$, there exists $\lambda \in C$ such that $\delta(xy) = \delta(x)y + x\delta(y) - \lambda xy$ for all $x, y \in A$.*

2. Isomorphisms

We start with a key result of the paper.

Theorem 3. *Let A and B be prime rings and $\theta : A \to B$ a bijective additive map such that $\theta(x)\theta(y) = 0$ for all $x, y \in A$ with $xy = 0$. Suppose that the maximal right quotient ring $Q(A)$ of A contains a nontrivial idempotent e such that $eA \cup Ae \subseteq A$. Then $\theta(x)\theta(yz) = \theta(xy)\theta(z)$ for all $x, y, z \in A$. Moreover, if A contains the unity 1, then:*

(i) *$\theta(1)$ lies in the center $Z(B)$ of B.*
(ii) *$\theta(1)\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in A$. In particular, if $\theta(1) = 1$, then θ is a ring isomorphism from A onto B.*
(iii) *θ preserves commutativity, that is, $\theta(x)\theta(y) = \theta(y)\theta(x)$ for all $x, y \in A$ with $xy = yx$.*

Proof. Set $f = 1 - e$. Then f is a nontrivial idempotent in $Q(A)$ such that $e + f = 1$, $ef = fe = 0$ and $fA \cup Af \subseteq A$. Since θ is additive and $y = eye + eyf + fye + fyf$ for all $y \in A$, it suffices to show that the identity $\theta(x)\theta(yz) = \theta(xy)\theta(z)$ holds for y in eAe, eAf, fAe and fAf, respectively.

Let $x, z \in A$. Since θ preserves zero products, $(xe)(z - ez) = 0$ implies that $\theta(xe)\theta(z - ez) = 0$ and hence

$$\theta(xe)\theta(z) = \theta(xe)\theta(ez).$$

Similarly, it follows from $(x - ex)(ez) = 0$ that

$$\theta(x)\theta(ez) = \theta(xe)\theta(ez).$$

Thus we have

$$\theta(x)\theta(ez) = \theta(xe)\theta(ez) = \theta(xe)\theta(z) \quad \text{for all } x, z \in A. \tag{1}$$

By the symmetry of e and f, we also have

$$\theta(x)\theta(fz) = \theta(xf)\theta(fz) = \theta(xf)\theta(z) \quad \text{for all } x, z \in A. \tag{2}$$

Note that

$$(xe + xeyf)(eyfz - fz) = 0 \quad \text{for all } x, y, z \in A,$$

so

$$\theta(xe + xeyf)\theta(eyfz - fz) = 0 \quad \text{for all } x, y, z \in A.$$

Since $\theta(xe)\theta(fz) = 0$ and $\theta(xeyf)\theta(eyfz) = 0$, this results in

$$\theta(xe)\theta(eyfz) = \theta(xeyf)\theta(fz) \quad \text{for all } x, y, z \in A,$$

and hence

$$\theta(x)\theta(eyfz) = \theta(xeyf)\theta(z) \quad \text{for all } x, y, z \in A \tag{3}$$

in light of (1) and (2). By the symmetry of e and f, we also have

$$\theta(x)\theta(fyez) = \theta(xfye)\theta(z) \quad \text{for all } x, y, z \in A. \tag{4}$$

Thus it remains to show that

$$\theta(x)\theta(eyez) = \theta(xeye)\theta(z) \quad \text{for all } x, y, z \in A \tag{5}$$

and

$$\theta(x)\theta(fyfz) = \theta(xfyf)\theta(z) \quad \text{for all } x, y, z \in A. \tag{6}$$

Applying (1), (2), (3) and (4) we shall rewrite the product

$$\theta(xeyezeufv_1)\theta(fv_2)\theta(fv_3)\theta(ew)$$

in two ways, via the following sequences of steps (read down each column; each entry can be seen to be equal to the one immediately above):

$\theta(xeyezeufv_1)\theta(fv_2)\theta(fv_3)\theta(ew)$	$\theta(xeyezeufv_1)\theta(fv_2)\theta(fv_3)\theta(ew)$
$\theta(x(eyezeufv_1f))\theta(fv_2)\theta(fv_3)\theta(ew)$	$\theta(xeye(ezeufv_1f))\theta(fv_2)\theta(fv_3)\theta(ew)$
$\theta(x)\theta(eyezeufv_1fv_2)\theta(fv_3)\theta(ew)$	$\theta(xeye)\theta(ezeufv_1fv_2)\theta(fv_3)\theta(ew)$
$\theta(x)\theta(eyeze(eufv_1fv_2f))\theta(fv_3)\theta(ew)$	$\theta(xeye)\theta(ze(eufv_1fv_2f))\theta(fv_3)\theta(ew)$
$\theta(x)\theta(eyeze)\theta(eufv_1fv_2fv_3)\theta(ew)$	$\theta(xeye)\theta(ze)\theta(eufv_1fv_2fv_3)\theta(ew)$
$\theta(x)\theta(eyeze)\theta(u(fv_1fv_2fv_3e))\theta(ew)$	$\theta(xeye)\theta(ze)\theta(u(fv_1fv_2fv_3e))\theta(ew)$
$\theta(x)\theta(eyeze)\theta(u)\theta(fv_1fv_2fv_3ew)$	$\theta(xeye)\theta(ze)\theta(u)\theta(fv_1fv_2fv_3ew)$

Comparing the two expressions on the last line, we get

$$\big(\theta(x)\theta(eyeze) - \theta(xeye)\theta(ze)\big)\theta(u)\theta(fv_1fv_2fv_3ew) = 0$$

for all $x, y, z, u, v_i, w \in A$. Since A and B are prime and θ is bijective, we obtain

$$\theta(x)\theta(eyeze) = \theta(xeye)\theta(ze) \quad \text{for all } x, y, z \in A. \tag{7}$$

Similarly we express the product

$$\theta(xeyezfufv_1)\theta(fv_2)\theta(ev_3)\theta(ew)$$

in two other ways:

$$\begin{array}{ll}
\theta(xeyezfufv_1)\theta(fv_2)\theta(ev_3)\theta(ew) & \theta(xeyezfufv_1)\theta(fv_2)\theta(ev_3)\theta(ew) \\
\theta(x(eyezfufv_1f))\theta(fv_2)\theta(ev_3)\theta(ew) & \theta(xeye(ezfufv_1f))\theta(fv_2)\theta(ev_3)\theta(ew) \\
\theta(x)\theta(eyezfufv_1fv_2)\theta(ev_3)\theta(ew) & \theta(xeye)\theta(ezfufv_1fv_2)\theta(ev_3)\theta(ew) \\
\theta(x)\theta(eyezf(fufv_1fv_2e))\theta(ev_3)\theta(ew) & \theta(xeye)\theta(zf(fufv_1fv_2e))\theta(ev_3)\theta(ew) \\
\theta(x)\theta(eyezf)\theta(fufv_1fv_2ev_3)\theta(ew) & \theta(xeye)\theta(zf)\theta(fufv_1fv_2ev_3)\theta(ew) \\
\theta(x)\theta(eyezf)\theta(u(fv_1fv_2ev_3e))\theta(ew) & \theta(xeye)\theta(zf)\theta(u(fv_1fv_2ev_3e))\theta(ew) \\
\theta(x)\theta(eyezf)\theta(u)\theta(fv_1fv_2ev_3ew) & \theta(xeye)\theta(zf)\theta(u)\theta(fv_1fv_2ev_3ew)
\end{array}$$

Comparing both expressions, we get

$$\big(\theta(x)\theta(eyezf) - \theta(xeye)\theta(zf)\big)\theta(u)\theta(fv_1fv_2ev_3ew) = 0$$

for all $x, y, z, u, v_i, w \in A$. Since A and B are prime and θ is bijective, we obtain

$$\theta(x)\theta(eyezf) = \theta(xeye)\theta(zf) \quad \text{for all } x, y, z \in A. \tag{8}$$

Then (5) follows immediately from the identities (7) and (8). By the symmetry of e and f, we obtain (6) too. Therefore,

$$\theta(x)\theta(yz) = \theta(xy)\theta(z) \qquad \text{for all } x, y, z \in A. \tag{9}$$

Suppose that A contains the unity 1. Setting $x = z = 1$ in (9), we have

$$\theta(1)\theta(y) = \theta(y)\theta(1)$$

for all $y \in A$. Since θ is surjective, $\theta(1)$ lies in the center of B. This establishes statement (i) of Theorem 3.

Setting $z = 1$ in (9), we get

$$\theta(x)\theta(y) = \theta(xy)\theta(1) = \theta(1)\theta(xy)$$

for all $x, y \in A$. In particular, if $\theta(1) = 1$, then θ is a ring isomorphism from A onto B. This establishes (ii).

Finally, by (ii) we have

$$\theta(x)\theta(y) - \theta(y)\theta(x) = \theta(1)(\theta(xy) - \theta(yx)) = \theta(1)\theta(xy - yx)$$

for all $x, y \in A$. Then (iii) follows immediately. □

In view of the preceding theorem, we see that the zero-product preserving map θ satisfies the functional identity

$$\theta(x)\theta(yz) = \theta(xy)\theta(z) \qquad \text{for all } x, y, z \in A.$$

This enables us to apply the recently developed theory of functional identities. Instead of introducing complicated definitions and notations, we shall present some special cases of the results in [**3**, **4**]. The first one follows from [**3**, Theorem 2.4] and [**4**, Theorem 1.2].

Lemma 4. *Let R be a prime ring with maximal right quotient ring Q and extended centroid C such that $\deg R \geq 3$. Let S be a set, $\theta : S \to R$ a surjective map and $M : S \times S \to Q$ a map. Suppose that*

$$\alpha_1\theta(x)M(y,z) + \alpha_2\theta(y)M(x,z) + \alpha_3\theta(z)M(x,y) + \beta_1 M(y,z)\theta(x) + \beta_2 M(x,z)\theta(y) + \beta_3 M(x,y)\theta(z) = 0$$

for all $x, y, z \in S$, where the α_i and β_i are constants in C, not all zero. Then there exist $\lambda_1, \lambda_2 \in C$, $\mu_1, \mu_2 : S \to C$ and $\nu : S \times S \to C$ such that

$$M(x,y) = \lambda_1\theta(x)\theta(y) + \lambda_2\theta(y)\theta(x) + \mu_1(x)\theta(y) + \mu_2(y)\theta(x) + \nu(x,y)$$

for all $x, y \in S$.

The second one follows from [**3**, Theorem 2.4] and [**4**, Theorem 1.1].

Lemma 5. *Let R be a prime ring with maximal right quotient ring Q and extended centroid C, such that $\deg R \geq 3$. Let S be a set and $\theta : S \to R$ a surjective map. Suppose that*

$$\sum_{\sigma\in \mathrm{Sym}(3)} \alpha_\sigma \theta(x_{\sigma(1)})\theta(x_{\sigma(2)})\theta(x_{\sigma(3)}) + \sum_{\sigma\in \mathrm{Sym}(3)} \beta_\sigma(x_{\sigma(1)})\theta(x_{\sigma(2)})\theta(x_{\sigma(3)}) + \gamma_1(x_2,x_3)\theta(x_1) + \gamma_2(x_1,x_3)\theta(x_2) + \gamma_3(x_1,x_2)\theta(x_3) = 0$$

for all $x_1, x_2, x_3 \in S$, where $\mathrm{Sym}(3)$ is the symmetric group on 3 letters, the α_σ are constants in C, the β_σ are maps $S \to C$ and the γ_i are maps $S \times S \to C$. Then the constants α_σ and the maps β_σ and γ_i are all zero.

With these results at hand, we are ready to prove our first main theorem.

Proof of Theorem 1. Since (i) follows from Theorem 3, it remains to prove (ii).

Define a map $M : A \times A \to B$ by $M(x,y) = \theta(xy)$ for $x, y \in A$. By Theorem 3, we have

$$\theta(x)M(y,z) - M(x,y)\theta(z) = 0 \quad \text{for all } x, y, z \in A. \tag{10}$$

Then it follows from Lemma 4 that there exist $\lambda_1, \lambda_2 \in C$, $\mu_1, \mu_2 : A \to C$ and $\nu : A \times A \to C$ such that

$$M(x,y) = \lambda_1\theta(x)\theta(y) + \lambda_2\theta(y)\theta(x) + \mu_1(x)\theta(y) + \mu_2(y)\theta(x) + \nu(x,y)$$

for all $x, y \in A$. Substituting this into (10), we obtain

$$\lambda_2\theta(x)\theta(z)\theta(y) - \lambda_2\theta(y)\theta(x)\theta(z) + \mu_2(z)\theta(x)\theta(y) + \big(\mu_1(y) - \mu_2(y)\big)\theta(x)\theta(z) - \mu_1(x)\theta(y)\theta(z) + \nu(y,z)\theta(x) - \nu(x,y)\theta(z) = 0,$$

for all $x, y, z \in A$. By Lemma 5, the constant λ_2 and the maps μ_1, μ_2 and ν are all zero. In other words, $M(x, y) = \theta(xy) = \lambda_1\theta(x)\theta(y)$ for all $x, y \in A$. Thus the proof is complete. □

3. Derivations

Next we prove a result analogous to Theorem 3. The idea is essentially the same as in the proof of Theorem 3, although the computations are a bit more complicated.

Theorem 6. *Let A be a prime ring with maximal right quotient ring Q and $\delta : A \to A$ an additive map such that $\delta(x)y + x\delta(y) = 0$ for all $x, y \in A$ with $xy = 0$. Suppose that Q contains a nontrivial idempotent e such that $eA \cup Ae \subseteq A$. Then $\delta(x)yz + x\delta(yz) = \delta(xy)z + xy\delta(z)$ for all $x, y, z \in A$. Moreover, if A contains the unity 1, then*

$$\delta(xy) = \delta(x)y + x\delta(y) - \lambda xy,$$

where $\lambda = \delta(1)$ is a central element of A. In particular, if $\delta(1) = 0$, then δ is a derivation on A.

Proof. As before, we set $f = 1 - e$. Then f is a nontrivial idempotent in Q such that $e + f = 1$, $ef = fe = 0$ and $fA \cup Af \subseteq A$. Since δ is additive and $y = eye + eyf + fye + fyf$ for all $y \in A$, it suffices to show that the identity $\delta(x)yz + x\delta(yz) = \delta(xy)z + xy\delta(z)$ holds for y in eAe, eAf, fAe and fAf respectively.

Let $x, z \in A$. Since $(xe)(z-ez) = 0$, we have $\delta(xe)(z-ez)+xe\delta(z-ez) = 0$ by assumption and hence

$$\delta(xe)z + xe\delta(z) = \delta(xe)ez + xe\delta(ez).$$

Similarly, it follows from $(x - xe)(ez) = 0$ that

$$\delta(x)ez + x\delta(ez) = \delta(xe)ez + xe\delta(ez).$$

Thus

$$\delta(x)ez + x\delta(ez) = \delta(xe)z + xe\delta(z) = \delta(xe)ez + xe\delta(ez) \quad \text{for all } x, z \in A. \tag{11}$$

By the symmetry of e and f, we also have

$$\delta(x)fz + x\delta(fz) = \delta(xf)z + xf\delta(z) = \delta(xf)fz + xf\delta(fz) \quad \text{for all } x, z \in A. \tag{12}$$

Note that for $x, y, z \in A$, we have

$$\begin{aligned} (xe)(fz) &= 0, \\ (xeyf)(eyfz) &= 0, \\ (xe + xeyf)(eyfz - fz) &= 0, \end{aligned}$$

so

$$\delta(xe)fz + xe\delta(fz) = 0,$$
$$\delta(xeyf)eyfz + xeyf\delta(eyfz) = 0,$$
$$\delta(xe + xeyf)(eyfz - fz) + (xe + xeyf)\delta(eyfz - fz) = 0.$$

Combining these three identities, we get

$$\delta(xe)eyfz + xe\delta(eyfz) = \delta(xeyf)fz + xeyf\delta(fz),$$

and hence

$$\delta(x)eyfz + x\delta(eyfz) = \delta(xeyf)z + xeyf\delta(z) \quad \text{for all } x, y, z \in A, \tag{13}$$

in light of (11) and (12). By the symmetry of e and f, we also have

$$\delta(x)fyez + x\delta(fyez) = \delta(xfye)z + xfye\delta(z) \quad \text{for all } x, y, z \in A. \tag{14}$$

Thus it remains to show that

$$\delta(x)eyez + x\delta(eyez) = \delta(xeye)z + xeye\delta(z) \quad \text{for all } x, y, z \in A \tag{15}$$

and

$$\delta(x)fyfz + x\delta(fyfz) = \delta(xfyf)z + xfyf\delta(z) \quad \text{for all } x, y, z \in A. \tag{16}$$

Applying (11), (12), (13) and (14), we shall express the sum

$$\delta(x)eyezfufvew + x\delta(eyezfufv)ew + xeyezfufv\delta(ew)$$

in two other ways. On the one hand, we have

$$\begin{aligned}
&\delta(x)eyezfufvew + x\delta(eyezfufv)ew + xeyezfufv\delta(ew)\\
&\qquad = \delta(x)eyezfufvew + x\delta(eyez(fufve))ew + xeyez(fufve)\delta(ew)\\
&\qquad = \delta(x)eyezfufvew + x\delta(eyez)fufvew + xeyez\delta(fufvew).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\delta(x)eyezfufvew + x\delta(eyezfufv)ew + xeyezfufv\delta(ew)\\
&\qquad = \delta(x)(eyezfuf)fvew + x\delta((eyezfuf)fv)ew + xeyezfufv\delta(ew)\\
&\qquad = \delta(xeyezfuf)fvew + xeyezfuf\delta(fv)ew + xeyezfufv\delta(ew)\\
&\qquad = \delta(xey(ezfuf))fvew + xey(ezfuf)\delta(fv)ew + xeyezfufv\delta(ew)\\
&\qquad = \delta(xey)ezfufvew + xey\delta(ezfufv)ew + xeyezfufv\delta(ew)\\
&\qquad = \delta(xey)ezfufvew + xey\delta(ez(fufve))ew + xeyez(fufve)\delta(ew)\\
&\qquad = \delta(xey)ezfufvew + xey\delta(ez)fufvew + xeyez\delta(fufvew)\\
&\qquad = \delta(xeye)zfufvew + xeye\delta(z)fufvew + xeyez\delta(fufvew).
\end{aligned}$$

Comparing both expressions, we get

$$\big(\delta(x)eyez + x\delta(eyez) - \delta(xeye)z - xeye\delta(z)\big)fufvew = 0$$

for all $x, y, z, u, v, w \in A$. Since A is prime, we obtain

$$\big(\delta(x)eyez + x\delta(eyez)\big)f = \big(\delta(xeye)z + xeye\delta(z)\big)f. \tag{17}$$

for all $x, y, z \in A$. Similarly we express the sum

$$\delta(x)eyezeufvfwft + x\delta(eyezeufvfw)ft + xeyezeufvfw\delta(ft)$$

in two other ways. On the one hand, we have

$$\begin{aligned}
&\delta(x)eyezeufvfwft + x\delta(eyezeufvfw)ft + xeyezeufvfw\delta(ft)\\
&= \delta(x)eyezeufvfwft + x\delta(eyez(eufvfwf))ft + xeyez(eufvfwf)\delta(ft)\\
&= \delta(x)eyezeufvfwft + x\delta(eyez)eufvfwft + xeyez\delta(eufvfwft).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\delta(x)eyezeufvfwft + x\delta(eyezeufvfw)ft + xeyezeufvfw\delta(ft)\\
&= \delta(x)(eyezeuf)vfwft + x\delta((eyezeuf)vfw)ft + xeyezeufvfw\delta(ft)\\
&= \delta(xeyezeuf)vfwft + xeyezeuf\delta(vfw)ft + xeyezeufvfw\delta(ft)\\
&= \delta(xey(ezeuf))vfwft + xey(ezeuf)\delta(vfw)ft + xeyezeufvfw\delta(ft)\\
&= \delta(xey)ezeufvfwft + xey\delta(ezeufvfw)ft + xeyezeufvfw\delta(ft)\\
&= \delta(xey)ezeufvfwft + xey\delta(ez(eufvfwf))ft + xeyez(eufvfwf)\delta(ft)\\
&= \delta(xey)ezeufvfwft + xey\delta(ez)eufvfwft + xeyez\delta(eufvfwft)\\
&= \delta(xeye)zeufvfwft + xeye\delta(z)eufvfwft + xeyez\delta(eufvfwft).
\end{aligned}$$

Comparing both expressions, we get

$$\big(\delta(x)eyez + x\delta(eyez) - \delta(xeye)z - xeye\delta(z)\big)eufvfwft = 0$$

for all $x, y, z, u, v, w, t \in A$. Since A is prime, we obtain

$$\big(\delta(x)eyez + x\delta(eyez)\big)e = \big(\delta(xeye)z + xeye\delta(z)\big)e \tag{18}$$

for all $x, y, z \in A$. Then (15) follows immediately from the identities (17) and (18). By the symmetry of e and f, we have (16) too. Therefore,

$$\delta(x)yz + x\delta(yz) = \delta(xy)z + xy\delta(z) \quad \text{for all } x, y, z \in A. \tag{19}$$

Suppose that A contains the unity 1. Setting $x = z = 1$ in (19), we get $\delta(1)y = y\delta(1)$ for all $y \in A$. That is, $\lambda = \delta(1)$ is a central element of A. Setting $z = 1$ in (19) we get

$$\delta(xy) = \delta(x)y + x\delta(y) - \lambda xy \quad \text{for all } x, y \in A.$$

Clearly, if $\delta(1) = 0$, then δ is a derivation. □

Now we need some lemmas to deal with the functional identity $\delta(x)yz + x\delta(yz) = \delta(xy)z + xy\delta(z)$. The following two results are special cases of [**5**, Corollary 2.9]. The first one also appears in [**2**, Theorem 1.2], where bi-additivity of the maps F_i and G_i is assumed.

Lemma 7. *Let R be a prime ring with maximal right quotient ring Q and extended centroid C, such that $\deg R \geq 3$. Let $F_i, G_i : R\times R \to Q$, $i = 1, 2, 3$, be maps. Suppose that*

$$x_1F_1(x_2, x_3) + x_2F_2(x_1, x_3) + x_3F_3(x_1, x_2) + G_1(x_2, x_3)x_1 + G_2(x_1, x_3)x_2 + G_3(x_1, x_2)x_3 = 0$$

for all $x_1, x_2, x_3 \in R$. Then there exist unique maps $p_{i,j} : R \to Q$, for $1 \leq i \neq j \leq 3$, with $p_{i,j} = p_{j,i}$, and maps $\lambda_i : R \times R \to C$, for $i = 1, 2, 3$, such that

$$F_i(x_j, x_k) = p_{i,j}(x_k)x_j + p_{i,k}(x_j)x_k + \lambda_i(x_j, x_k),$$
$$G_j(x_i, x_k) = -x_ip_{i,j}(x_k) - x_kp_{j,k}(x_i) - \lambda_j(x_i, x_k)$$

for all $x_i, x_j, x_k \in R$, where i, j, k are distinct, and $\lambda_i = 0$ if either $F_i = 0$ or $G_i = 0$.

The next result also appears in [8, Lemma 4.5], where additivity of the maps f_i and g_i is assumed.

Lemma 8. *Let R be a prime ring with maximal right quotient ring Q and extended centroid C. Let $f_i, g_i : R \to R$, for $i = 1, 2$, be maps. Suppose that*

$$f_1(x)y + f_2(y)x + xg_1(y) + yg_2(x) = 0$$

for all $x, y \in R$. Then there exist unique constants $c_1, c_2 \in Q$ and maps $\lambda_1, \lambda_2 : R \to C$ such that

$$f_1(x) = xc_1 + \lambda_1(x), \qquad f_2(y) = yc_2 + \lambda_2(y),$$
$$g_1(y) = -c_1y - \lambda_2(y), \qquad g_2(x) = -c_2x - \lambda_1(x)$$

for all $x, y \in R$, where $\lambda_i = 0$ if either $f_i = 0$ or $g_i = 0$.

Now we are ready to conclude the paper by proving our second main result.

Proof of Theorem 2. Since (i) follows from Theorem 6, it remains to prove (ii).

Define two maps $F, G : A \times A \to A$ by $F(x, y) = \delta(xy) - x\delta(y)$ and $G(x, y) = \delta(xy) - \delta(x)y$ for $x, y \in A$. By Theorem 6, we have

$$xF(y, z) - G(x, y)z = 0 \quad \text{for all } x, y, z \in A.$$

Then it follows from Lemma 7 that there exists a map $p : R \to Q$ such that

$$F(x, y) = \delta(xy) - x\delta(y) = p(x)y,$$
$$G(x, y) = \delta(xy) - \delta(y)x = xp(y)$$

for all $x, y \in A$. Thus,

$$\delta(xy) = x\delta(y) + p(x)y = \delta(x)y + xp(y), \tag{20}$$

and hence

$$\big(\delta(x) - p(x)\big)y - x\big(\delta(y) - p(y)\big) = 0,$$

for all $x, y \in A$. By Lemma 8, there exist constants $c_1, c_2 \in Q$ such that $\delta(x) - p(x) = xc_1$ and $\delta(y) - p(y) = c_2 y$ for all $x, y \in R$. Then $c_1 = c_2$ is an element in C. Denote this element by λ. Thus

$$p(x) = \delta(x) - \lambda x$$

for all $x \in R$. Substituting this into (20), we get

$$\delta(xy) = x\delta(y) + \delta(x)y - \lambda xy$$

for all $x, y \in R$. The proof is complete. □

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Department of Mechanics and Mathematics
Tula State University
Tula
Russia
E-mail address: mchebotar@tula.net

Department of Mathematics
National Cheng Kung University
Tainan
Taiwan
E-mail address: wfke@mail.ncku.edu.tw

Department of Mathematics
National Taiwan University
Taipei
Taiwan
E-mail address: phlee@math.ntu.edu.tw