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A recent problem in dynamics is to determine whether an attractor Λ of a C^r flow X is C^r robust transitive. By an *attractor* we mean a transitive set to which all positive orbits close to it converge. An attractor is C^r robust transitive (or C^r *robust* for short) if it has a neighborhood U such that the set $\bigcap_{t>0} Y_t(U)$ is transitive for every flow Y that is C^r close to X. We give sufficient conditions for robustness of attractors based on the following definitions: an attractor is *singular-hyperbolic* if it has singularities, all of which are hyperbolic, and is partially hyperbolic with volume expanding central direction (Morales, Pacifico and Pujals, 1998). An attractor is C^r *critically robust* if it has a neighborhood U such that $\bigcap_{t>0} Y_t(U)$ is in the closure of the closed orbits of every flow Y C^r close to X. We show that on compact 3-manifolds all C^r critically robust singular-hyperbolic attractors with only one singularity are C^r robust.

1. Introduction

A recent problem in dynamics is to determine whether an attractor Λ of a C^r flow X is C^r robust transitive or not. By an *attractor* we mean a transitive set to which all positive orbits close to it converge. An attractor is C^r robust transitive (or C^r *robust* for short) if it has a neighborhood U such that the set $\bigcap_{t>0} Y_t(U)$ is transitive for every flow Y that is C^r close to X. We give sufficient conditions for robustness of attractors based on the following definitions: an attractor is *singular-hyperbolic* if it has singularities, all of which are hyperbolic, and is partially hyperbolic with volume expanding central direction [**MPP**]. An attractor is C^r *critically robust* if it has a neighborhood U such that $\bigcap_{t>0} Y_t(U)$ is in the closure of the closed orbits is every flow Y C^r close to X. We show that, for flows on compact 3-manifolds, all C^r critically robust singular-hyperbolic attractors with only one singularity are C^r robust.

Let us state our result in a precise way. Hereafter X_t is a flow induced by a C^r vector field X on a compact 3-manifold M. The ω-limit set of $p \in M$ is the accumulation point set $\omega_X(p)$ of the positive orbit of p. An invariant set is *transitive* if it equals $\omega_X(p)$ for some point p on it.

Definition 1.1. A compact set in M is an *attracting set* of X if it can be written in the form $\bigcap_{t>0} X_t(U)$ for some neighborhood U. An *attractor* is a transitive attracting set.

See [**Mi**] where several definitions of attractors are discussed. The central definition of this paper is the following:

Definition 1.2. An attractor of a C^r flow X is C^r robust transitive (or C^r *robust* for short) if it has a neighborhood U such that $\bigcap_{t>0} Y_t(U)$ is a transitive set of Y for every flow Y that is C^r close to X.

Recently the problem of finding sufficient conditions for robustness of attractors was introduced in [**B**] and [**P**]. To deal with it we introduce the following definitions: a compact invariant set Λ of X is *partially hyperbolic* if there are an invariant splitting $T\Lambda = E^s \oplus E^c$ and positive constants K, λ such that:

1. E^s *is contracting*, namely
$$\|DX_t/E^s_x\| \le Ke^{-\lambda t}, \qquad \forall x \in \Lambda,\ \forall t > 0.$$
2. E^s *dominates* E^c, namely
$$\|DX_t/E^s_x\| \cdot \left\|DX_{-t}/E^c_{X_t(x)}\right\| \le Ke^{-\lambda t}, \qquad \forall x \in \Lambda,\ \forall t > 0.$$

The central direction E^c of Λ is said to be *volume expanding* if the additional condition
$$\left|J(DX_t/E^c_x)\right| \ge Ke^{\lambda t}$$
holds for all $x \in \Lambda$ and $t > 0$, where $J(\,\cdot\,)$ means the Jacobian.

Definition 1.3 ([**MPP**])**.** An attractor is *singular-hyperbolic* if it has singularities, all of which are hyperbolic, and is partially hyperbolic with volume expanding central direction.

The most representative example of a C^r robust singular-hyperbolic attractor is the geometric Lorenz attractor [**GW**]. The main result in [**MPP**] claims that C^1 robust nontrivial attractors on compact 3-manifolds are singular-hyperbolic. The converse is false: there are singular-hyperbolic attractors on compact 3-manifolds that are not C^r robust [**MPu**]. The following definition gives a further sufficient condition for robustness of singular-hyperbolic attractors:

Definition 1.4. An attractor of a C^r flow X is C^r *critically robust* if it has a neighborhood U such that $\bigcap_{t>0} Y_t(U)$ is in the closure of the closed orbits of Y, for every flow Y that is C^r close to X.

Hyperbolic attractors on compact manifolds are C^r robust and C^r critically robust for all r. The geometric Lorenz attractor [**GW**] is an example of a singular-hyperbolic attractor with only one singularity which is also C^r robust and C^r critically robust. In general singular-hyperbolic attractors with only one singularity may be neither C^r robust nor C^r critically robust [**MPu**]. Nevertheless we shall prove that on compact 3-manifolds C^r critically robustness implies C^r robustness among singular-hyperbolic attractors with only one singularity. More precisely:

Theorem A. *A C^r critically robust singular-hyperbolic attractor with only one singularity on compact 3-manifolds is C^r robust.*

This gives explicit sufficient conditions for the robustness of attractors but they *depend on the perturbed flow*. E. Pujals is interested in conditions *depending on the unperturbed flow only*. It would also be interesting to determine whether the conclusion of Theorem A holds if we interchange the roles of robust and critically robust in the statement.

The proof of Theorem A relies on recent work [**MP2**]. We reproduce the necessary results in Section 2 for completeness. The proof of Theorem A is in Section 3.

2. Singular-hyperbolic attracting sets

In this section we describe the results in [**MP2**], omitting some proofs; see [**MP2**] for details. Hereafter X is a C^r flow on a closed 3-manifold M. The closure of B will be denoted by $\mathrm{Cl}(B)$. If A is a compact invariant set of X we denote by $\mathrm{Sing}_X(A)$ the set of singularities of X in A, and by $\mathrm{Per}_X(A)$ the union of the periodic orbits of X in A. A compact invariant set H of X is *hyperbolic* if the tangent bundle over H has an invariant decomposition $E^s \oplus E^X \oplus E^u$ such that E^s is contracting, E^u is expanding and E^X is generated by the direction of X [**PT**]. Stable Manifold Theory [**HPS**] asserts the existence of the stable manifold $W^s_X(p)$ and the unstable manifold $W^u_X(p)$ associated to $p \in H$. These manifolds are respectively tangent to the subspaces $E^s_p \oplus E^X_p$ and $E^X_p \oplus E^u_p$ of T_pM. In particular, $W^s_X(p)$ and $W^u_X(p)$ are well-defined if p belongs to a hyperbolic periodic orbit of X. If O is an orbit of X we write $W^s_X(O) = W^s_X(p)$ and $W^u_X(O) = W^u_X(p)$ for some $p \in O$. We observe that $W^{s(u)}_X(O)$ does not depend on $p \in O$. When $\dim E^s = \dim E^u = 1$ we say that H is of *saddle type*. In this case $W^s_X(p)$ and $W^u_X(p)$ are two-dimensional submanifolds of M. The maps $p \in H \to W^s_X(p)$ and $p \in H \to W^u_X(p)$ are continuous (on compact parts). Moreover, a compact, singular, invariant set Λ of X is *singular-hyperbolic* if all its singularities are hyperbolic and the tangent bundle over Λ has an invariant decomposition $E^s \oplus E^c$ such that E^s is contracting, E^s dominates E^c and E^c is volume expanding (i.e., the Jacobian of DX_t/E^c grows exponentially as $t \to \infty$). Again, Stable Manifold Theory asserts the existence of the strong stable manifold $W^{ss}_X(p)$ associated to $p \in \Lambda$. This manifold is tangent to the subspace E^s_p of T_pM. For all $p \in \Lambda$ we define $W^s_X(p) = \bigcup_{t\in\mathbf{R}} W^{ss}_X(X_t(p))$. If p is regular (i.e., $X(p) \neq 0$) then $W^s_X(p)$ is a well-defined two-dimensional submanifold of M. The map $p \in \Lambda \to W^s_X(p)$ is continuous (on compact parts) at the regular points p of Λ. A singularity σ of X is *Lorenz-like* if its eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are real and satisfy

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$$

up to some reordering of the eigenvalues. A Lorenz-like singularity σ is hyperbolic, so $W^s_X(\sigma)$ and $W^u_X(\sigma)$ do exist. Moreover, the eigenspace of λ_2 is tangent to a one-dimensional invariant manifold $W^{ss}_X(\sigma)$. This manifold is called the *strong stable manifold* of σ. Clearly $W^{ss}_X(\sigma)$ splits $W^s_X(\sigma)$ into two connected components. We denote by $W^{s,+}_X(\sigma)$ and $W^{s,-}_X(\sigma)$ the two connected components of $W^s_X(\sigma) \setminus W^{ss}_X(\sigma)$.

Let Λ be a singular-hyperbolic set with dense periodic orbits of a three-dimensional flow. It follows from [**MPP**] that every $\sigma \in \mathrm{Sing}_X(\Lambda)$ is Lorenz-like and satisfies $\Lambda \cap W^{ss}_X(\sigma) = \{\sigma\}$. It follows also from [**MPP**] that any compact invariant subset without singularities of Λ is hyperbolic of saddle type. If in addition Λ is attracting, there is for every $p \in \mathrm{Per}_X(\Lambda)$ a $\sigma \in \mathrm{Sing}_X(\Lambda)$ such that

$$W^u_X(p) \cap W^s_X(\sigma) \neq \emptyset.$$

This follows from the methods in [**MP1**].

For every singular-hyperbolic set Λ of a three-dimensional flow X and every Lorenz-like singularity $\sigma \in \mathrm{Sing}_X(\Lambda)$ we define

$$\begin{aligned} P^+ &= \{p \in \mathrm{Per}_X(\Lambda) : W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset\}, \\ P^- &= \{p \in \mathrm{Per}_X(\Lambda) : W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset\}, \\ H^+_X &= \mathrm{Cl}(P^+), \\ H^-_X &= \mathrm{Cl}(P^-). \end{aligned}$$

These sets will play an important role.

Lemma 2.1. *Let Λ be a connected, singular-hyperbolic, attracting set with dense periodic orbits and only one singularity σ. Then $\Lambda = H^+_X \cup H^-_X$.*

Next we state a technical lemma to be used later. If S is a submanifold we denote by T_xS the tangent space at $x \in S$. A *cross-section* of X is a compact submanifold Σ transverse to X and diffeomorphic to the two-dimensional square $[0,1]^2$. If Λ is a singular-hyperbolic set of X and $x \in \Sigma \cap \Lambda$, then x is regular and so $W^s_X(x)$ is a two-dimensional submanifold transverse to Σ. In this case we denote by $W^s_X(x, \Sigma)$ the connected component of $W^s_X(x) \cap \Sigma$ containing x. We shall be interested in a special cross-section described as follows: let Λ be a singular-hyperbolic set of a three-dimensional flow X and let $\sigma \in \mathrm{Sing}_X(\Lambda)$. Suppose that the closed orbits contained in Λ are dense in Λ. Then σ is Lorenz-like [**MPP**], and therefore one can describe the flow using the Grobman–Hartman Theorem [**dMP**]. Indeed, we can assume that the flow of X around σ is the linear flow $\lambda_1\partial_{x_1} + \lambda_2\partial_{x_2} + \lambda_3\partial_{x_3}$

in a suitable coordinate system $(x_1, x_2, x_3) \in [-1, 1]^3$ around $\sigma = (0, 0, 0)$. A cross-section Σ of X is *singular* if it corresponds to the submanifolds $\Sigma^+ = \{x_3 = 1\}$ or $\Sigma^- = \{x_3 = -1\}$ in the coordinate system (x_1, x_2, x_3). We denote by l^+ and l^- the curves obtained by intersecting $\{x_2 = 0\}$ with Σ^+ and Σ^-, respectively; these curves are contained in $W_X^{s,+}(\sigma)$ and $W_X^{s,-}(\sigma)$ respectively. We state without proof the following straightforward lemma:

Lemma 2.2. *Let Λ a singular-hyperbolic set with dense periodic orbits of a three-dimensional flow X, and fix $\sigma \in \mathrm{Sing}_X(\Lambda)$. There are singular cross-sections Σ^+, Σ^- as above such that every orbit of Λ passing close to some point in $W_X^{s,+}(\sigma)$ (resp. $W_X^{s,-}(\sigma)$) intersects Σ^+ (resp. Σ^-). If $p \in \Lambda \cap \Sigma^+$ is close to l^+, then $W_X^s(p, \Sigma^+)$ is a vertical curve crossing Σ^+. If $p \in \mathrm{Per}_X(\Lambda)$ and $W_X^u(p) \cap W_X^{s,+}(\sigma) \neq \emptyset$, then $W_X^u(p)$ contains an interval $J = J_p$ intersecting l^+ transversally; and the same is true if we replace $+$ by $-$.*

To prove transitivity we shall use two lemmas:

Lemma 2.3 (Birkhoff's criterion)**.** *Let T be a compact, invariant set of X such that for all open sets U, V intersecting T there is $s > 0$ such that $X_s(U \cap T) \cap V \neq \emptyset$. Then T is transitive.*

Lemma 2.4. *Let Λ be a connected, singular-hyperbolic, attracting set with dense periodic orbits and only one singularity σ. Let U, V be open sets, $p \in U \cap \mathrm{Per}_X(\Lambda)$ and $q \in V \cap \mathrm{Per}_X(\Lambda)$. If $W_X^u(p) \cap W_X^{s,+}(\sigma) \neq \emptyset$ and $W_X^u(q) \cap W_X^{s,+}(\sigma) \neq \emptyset$, there exist $t > 0$ and $z \in W_X^u(p)$ arbitrarily close to $W_X^u(p) \cap W_X^{s,+}(\sigma)$ such that $X_t(z) \in V$. The same is true if we replace $+$ by $-$.*

Let Λ be a singular-hyperbolic set of $X \in \mathcal{X}^r$ satisfying:

1. Λ is connected.
2. Λ is attracting.
3. The closed orbits contained in Λ are dense in Λ.
4. Λ has only one singularity σ.

We note that condition 3 implies

(H1) $\qquad \Lambda = \mathrm{Cl}(\mathrm{Per}_X(\Lambda))$.

Proposition 2.5. *Suppose that, for any given $p, q \in \mathrm{Per}_X(\Lambda)$, either*

1. $W_X^u(p) \cap W_X^{s,+}(\sigma) \neq \emptyset$ *and* $W_X^u(q) \cap W_X^{s,+}(\sigma) \neq \emptyset$, *or*
2. $W_X^u(p) \cap W_X^{s,-}(\sigma) \neq \emptyset$ *and* $W_X^u(q) \cap W_X^{s,-}(\sigma) \neq \emptyset$.

Then Λ is transitive.

Proof. By Birkhoff's criterion we only need prove that for all open sets U, V intersecting Λ there exists $s > 0$ such that $X_s(U \cap \Lambda) \cap V \neq \emptyset$. For this we proceed as follows: by (H1) there are $p \in \mathrm{Per}_X(\Lambda) \cap U$ and $q \in \mathrm{Per}_X(\Lambda) \cap V$.

First suppose that alternative 1 holds. By Lemma 2.4, there are $z \in W^u_X(p)$ and $t > 0$ such that $X_t(z) \in V$. Since $z \in W^u_X(p)$, we have $w = X_{-t'}(z) \in U$ for some $t' > 0$. Since Λ is an attracting set, $w \in \Lambda$. If $s = t + t' > 0$ we conclude that $w \in (U \cap \Lambda) \cap X_{-s}(V)$ and so $X_s(U \cap \Lambda) \cap V \neq \emptyset$. If alternative 2 of the corollary holds we can find $s > 0$ such that $X_s(U \cap \Lambda) \cap V \neq \emptyset$ in a similar way (replacing $+$ by $-$). The result follows. □

Proposition 2.6. *If there is a sequence $p_n \in \mathrm{Per}_X(\Lambda)$ converging to some point in $W^{s,+}_X(\sigma)$ such that $W^u_X(p_n) \cap W^{s,-}_X(\sigma) \neq \emptyset$ for all n, then Λ is transitive. The same is true interchanging $+$ and $-$.*

Proof. Let $p, q \in \mathrm{Per}_X(\Lambda)$ be fixed. Suppose that $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$ and $W^u_X(q) \cap W^{u,-}_X(\sigma) \neq \emptyset$. By Lemma 2.2 we can fix a cross-section $\Sigma = \Sigma^+$ through $W^{s,+}_X(\sigma)$ and an open arc $J \subset \Sigma \cap W^u_X(p)$ intersecting $W^{s,+}_X(\sigma)$ transversally. Again by Lemma 2.2 we can assume that $p_n \in \Sigma$ for every n. Because the direction E^s of Λ is contracting, the size of $W^s_X(p_n)$ is uniformly bounded away from zero. It follows that there is n large so that J intersects $W^s_X(p_n)$ transversally. Applying the Inclination Lemma [**dMP**] to the saturation of $J \subset W^u_X(p)$, and the assumption $W^u_X(p_n) \cap W^{s,-}_X(\sigma) \neq \emptyset$, we conclude that $W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset$. So alternative 2 of Proposition 2.5 holds; it follows from that proposition that Λ is transitive. □

Proposition 2.7. *If there is $a \in W^u_X(\sigma) \setminus \{\sigma\}$ such that $\sigma \in \omega_X(a)$, then Λ is transitive.*

Proof. Without loss of generality we can assume that there exists z in $\omega_X(a) \cap W^+_X(\sigma)$. If $W^u_X(q) \cap W^{s,-}_X(\sigma) = \emptyset$ for all $q \in \mathrm{Per}_X(\Lambda)$, then $W^u_X(q) \cap W^{s,+}_X(\sigma) \neq \emptyset$ for all $q \in \mathrm{Per}_X(\Lambda)$; see [**MP1**]. Then Λ is transitive by Proposition 2.5 since alternative 1 holds for all $p, q \in \mathrm{Per}_X(\Lambda)$. So we can assume that there is $q \in \mathrm{Per}_X(\Lambda)$ such that $W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset$. It follows from the dominating condition of the singular-hyperbolic splitting of Λ that the intersection $W^u_X(q) \cap W^{s,-}_X(\sigma)$ is transversal. This allows us to choose a point in $W^u_X(q)$ arbitrarily close to $W^{s,-}_X(\sigma)$ on the side of $W^u_X(q) \cap W^{s,-}_X(\sigma)$ accumulating a. Since Λ is attracting and satisfies (H1), one can find a sequence $p_n \in \mathrm{Per}_X(\Lambda)$ converging to $z \in W^{s,+}_X(\sigma)$ such that for all n there is p'_n in the orbit of p_n such that the sequence p'_n converges to some point in $W^{s,-}_X(\sigma)$. Now suppose for a contradiction that Λ is not transitive. Then Proposition 2.6 implies

$$W^u_X(p'_n) \cap W^{s,+}_X(\sigma) = \emptyset \quad \text{and} \quad W^u_X(p_n) \cap W^{s,-}_X(\sigma) = \emptyset$$

for n large. But $W^u_X(p_n) = W^u_X(p'_n)$ since p'_n and p_n are in the same orbit of X. So $W^u_X(p_n) \cap (W^{s,+}_X(\sigma) \cup W^{s,-}_X(\sigma)) = \emptyset$. However

$$W^u_X(p_n) \cap W^s_X(\sigma) = \emptyset,$$

a contradiction since $\mathrm{Sing}_X(\Lambda) = \{\sigma\}$. We conclude that Λ is transitive. □

Theorem 2.8. *If Λ is not transitive, there is for all $a \in W^u_X(\sigma)\setminus\{\sigma\}$ a periodic orbit O of X with positive expanding eigenvalues such that $a \in W^s_X(O)$.*

Proof. Fix $a \in W^u_X(\sigma)\setminus\{\sigma\}$, and assume that $\omega_X(a)$ is *not* a periodic orbit. We will obtain a contradiction once we prove that if $p, q \in \mathrm{Per}_X(\Lambda)$ then p, q satisfy one of the two alternatives in Proposition 2.5. To prove this we proceed as follows: as noted before, both $W^u_X(p)$ and $W^u_X(q)$ intersect $W^s_X(\sigma)$ (see [**MP1**]). Then we can assume

$$W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset \quad \text{and} \quad W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset.$$

By using this and the linear coordinate around σ, it is easy to construct an open interval $I = I_a$, contained in a suitable cross-section $\Sigma = \Sigma_a$ of X containing a, and such that $I \setminus \{a\}$ is formed by two intervals $I^+ \subset W^u_X(p)$ and $I^- \subset W^u_X(q)$. Observe that the tangent vector of I is contained in $E^c \cap T\Sigma_a$. Proposition 2.7 implies that $\sigma \notin \omega_X(a)$, since Λ is not transitive. It follows that $H = \omega_X(a)$ is a hyperbolic set of saddle type; see [**MPP**]. As in [**M**] one proves that H is one-dimensional, so Bowen's Theory of hyperbolic one-dimensional sets [**Bw**] applies. In particular we can choose a family of cross-sections $\mathcal{S} = \{S_1, \dots, S_r\}$ of small diameter such that H is the flow-saturate of $H \cap \mathrm{int}\,\mathcal{S}'$, where $\mathcal{S}' = \bigcup S_i$ and $\mathrm{int}\,\mathcal{S}'$ is the interior of $\mathcal{S}'$. Also, $I \subset \Lambda$ since Λ is attracting. Recall that the tangent direction of I is contained in E^c. Since E^c is volume expanding and H is nonsingular, the Poincaré map induced by X on $\mathcal{S}'$ is expanding along I. As in [**MP1**, p. 371] we can find $\delta > 0$ and a open arc sequence $J_n \subset \mathcal{S}'$ in the positive orbit of I with length bounded away from 0 such that there is a_n *in the positive orbit of* a contained in the interior of J_n. We can fix $S = S_i \in \mathcal{S}$ in order to assume that $J_n \subset S$ for every n. Let $x \in S$ be a limit point of a_n. Then $x \in H \cap \mathrm{int}\,\mathcal{S}'$. Because I is tangent to E^c, the interval sequence J_n converges to an interval $J \subset W^u_X(x)$ in the C^1 topology ($W^u_X(x)$ exists since $x \in H$ and H is hyperbolic). J is not trivial since the length of J_n is bounded away from 0. If a_n were in $W^s_X(x)$ for n large we would conclude that x is periodic by [**MP1**, Lemma 5.6], a contradiction since $\omega_X(a)$ is not periodic. We conclude that for infinitely many values of n, $a_n \notin W^s_X(x)$. Since $J_n \to J$ and Λ has strong stable manifolds of uniformly size, there exists

$$z_n \in \big(W^s_X(a_{n+1}) \cap S\big) \cap \big(J_n \setminus \{a_n\}\big)$$

for all n large. For every n let J^+_n and J^-_n be the two connected components of $J_n \setminus \{a_n\}$, with J^+_n in the positive orbit of I^+ and J^-_n in the positive orbit of I^-. Clearly, either $z_n \in J^+_n$ or $z_n \in J^-_n$.

If $z_n \in J^+_n$ there is $v_{n+1} \in \mathrm{Per}_X(\Lambda) \cap S$ close to a_{n+1} such that

$$W^s_X(v_{n+1}) \cap J^+_n \neq \emptyset \quad \text{and} \quad W^s_X(v_{n+1}) \cap J^-_{n+1} \neq \emptyset.$$

Since v_{n+1} is periodic, it follows from [**MPP**] that $W^u_X(v_{n+1})$ intersects $W^{s,+}_X(\sigma)$ or $W^{s,-}_X(\sigma)$. The choice of v_{n+1} implies that its orbit passes close to a point in $W^{s,-}_X(\sigma)$. Since Λ is not transitive we conclude that $W^u_X(v_{n+1})$ intersects $W^{s,-}_X(\sigma)$. Since $W^s_X(v_{n+1})\cap J^+_n$ is transversal, the Inclination Lemma then implies $W^u_X(p)\cap W^{s,-}_X(\sigma)\neq\emptyset$. Hence

$$W^u_X(p)\cap W^{s,-}_X(\sigma)\neq\emptyset \quad \text{and} \quad W^u_X(q)\cap W^{s,-}_X(\sigma)\neq\emptyset.$$

If $z_n\in J^-_n$ we can prove by similar arguments that

$$W^u_X(p)\cap W^{s,+}_X(\sigma)\neq\emptyset \quad \text{and} \quad W^u_X(q)\cap W^{s,+}_X(\sigma)\neq\emptyset.$$

These alternatives yield the desired contradiction. Therefore $\omega_X(a)=O$ for some periodic orbit O of X. To finish we prove that the expanding eigenvalue of O is positive. Suppose by contradiction that it is not. Fix a cross-section Σ intersecting O in a single point p_0. This section defines a Poincaré map $\Pi:\operatorname{Dom}\Pi\subset\Sigma\to\Sigma$ of which p_0 is a hyperbolic fixed-point. The assumption implies that $D\Pi(p_0)$ has negative expanding eigenvalue. Because $p_0\in \operatorname{Per}_X(\Lambda)$, it follows from [**MPP**] that $W^u_X(p_0)$ intersects $W^{s,+}_X(\sigma)$ or $W^{s,-}_X(\sigma)$. We shall assume the former case since the proof in the latter is similar. We claim that $W^u_X(p)\cap W^{s,+}_X(\sigma)\neq\emptyset$ for all $p\in\operatorname{Per}_X(\Lambda)$. Indeed, let $p\in\operatorname{Per}_X(\Lambda)$ be fixed. Again $W^u_X(p)$ intersects $W^{s,+}_X(\sigma)$ or $W^{s,-}_X(\sigma)$. In the first case we are done. So we can assume that $W^u_X(p)\cap W^{s,-}_X(\sigma)\neq\emptyset$. By flow-saturating this intersection we obtain an interval $K\subset W^u_X(p)\cap\Sigma$ intersecting $W^s_X(p_0,\Sigma)$ transversally. At the same time, there is an interval $J\subset W^{s,+}_X(\sigma)\cap\Sigma$ intersecting $W^u_X(p_0,\Sigma)$ transversally. Since the expanding eigenvalue of $D\Pi(p_0)$ is negative the Inclination Lemma implies that the backward iterates $\Pi^{-n}(J)$ of J accumulate on $W^s_X(p_0,\Sigma)$ *in both sides*. Because K has transversal intersection with $W^s_X(p_0\Sigma)$ we conclude that one such backward iterate intersects K, and this yields $W^u_X(p)\cap W^{s,+}_X(\sigma)\neq\emptyset$ as desired, proving the claim. The claim together with Proposition 2.5 implies that Λ is transitive, yielding the contradiction needed to complete the proof of theorem. □

Hereafter we shall assume that Λ is not transitive. Let $a\in W^s_X(\sigma)\setminus\{\sigma\}$ be fixed. By Theorem 2.8, $a\in W^s_X(O)$ for some periodic orbit O with positive expanding eigenvalue. This last property implies that the unstable manifold $W^u_X(O)$ of O is a cylinder with generating curve O. Then O separates $W^u_X(O)$ into two connected components, which we denote by $W^{u,+},W^{u,-}$ according to the following convention (see Figure 1): there is an interval $I=I_a$, contained in a suitable cross-section of X and containing a, such that if I^+,I^- are the connected components of $I\setminus\{a\}$ then $I^+\subset W^u_X(p)$ and $I^-\subset W^u_X(q)$ for some periodic points $p,q\in\Lambda$ (recall that Λ is not transitive). In addition I is tangent to the central direction E^c of Λ (see Figure 1). Since $a\in W^s_X(O)$ and I is tangent to E^c, the flow of X carries I to an interval I' transverse to $W^s_X(O)$ at a. Note that the flow carries I^+ and I^- into I^+_0 and I^-_0 respectively.

Definition 2.9. We denote by $W^{u,+}$ the connected component of $W^u \setminus O$ that is accumulated (via the Inclination Lemma and the Strong λ Lemma [**dMP**, **D**]) by the positive orbit of I_0^+. We denote $W^{u,-}$ the connected component of $W^u_X(O) \setminus O$ accumulated by the positive orbit of I_0^-.

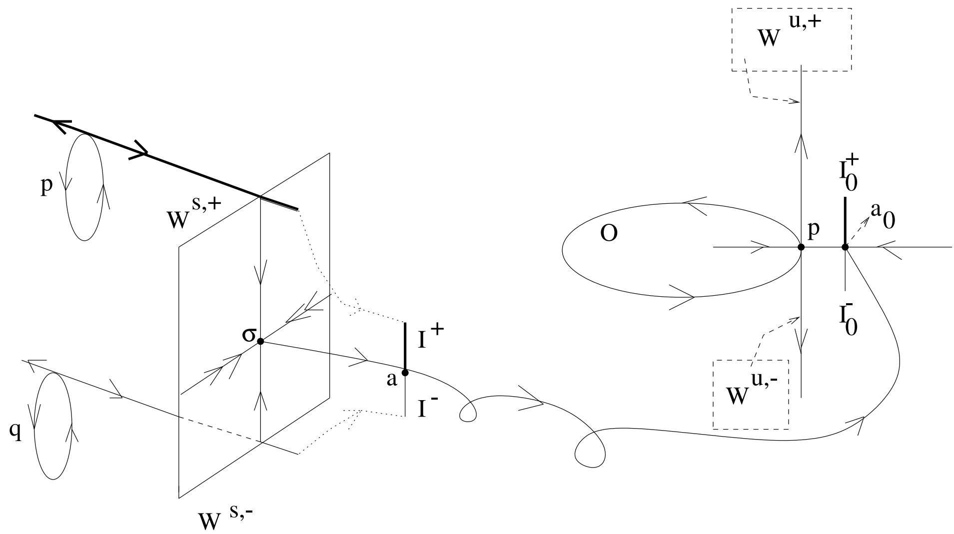

Figure 1. Definition of $W^{u,+}$ and $W^{u,-}$.

It can easily be proved using the Strong λ Lemma [**D**] that this definition does not depend on p, q, J_p, J_q.

Proposition 2.10. *$W^{u,+} \cap W^{s,-}_X(\sigma) = \emptyset$ and $W^{u,+} \cap W^{s,+}_X(\sigma) \neq \emptyset$. The same is true interchanging $+$ and $-$.*

Proof. For simplicity set $W = W^{u,+}$. First we prove that $W \cap W^{s,-}_X(\sigma) = \emptyset$. Suppose for a contradiction that $W \cap W^{s,-}_X(\sigma) \neq \emptyset$. Since this last intersection is transversal, there is an interval $J \subset W^{s,-}_X(\sigma)$ intersecting W transversally. Now, fix a cross-section $\Sigma = \Sigma^+$ as in Lemma 2.2 and let $p \in \mathrm{Per}_X(\Lambda)$ be such that $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$. Then there is an small interval $I \subset W^u_X(p) \cap \Sigma$ transversal to $\Sigma \cap W^{s,+}_X(\sigma)$. By the definition of $W = W^{u,+}$ (Definition 2.9), the positive orbit of I accumulates on W. Since J is transversal to W the Inclination Lemma implies that the positive orbit of I intersects J. This proves $W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset$ for all $p \in \mathrm{Per}_X(\Lambda)$. It follows that alternative 2 of Proposition 2.5 holds for all p, q, which contradicts the nontransitivity of Λ and proves that $W \cap W^{s,-}_X(\sigma) = \emptyset$, as desired. Now suppose for a contradiction that $W \cap W^{s,+}_X(\sigma) = \emptyset$. Since $W \cap W^{s,-}_X(\sigma) = \emptyset$ we obtain $W \cap W^s_X(\sigma) = \emptyset$ (see [**MPP**]). But the denseness of periodic orbits and the Inclination Lemma imply that $W \cap W^s_X(\sigma) \neq \emptyset$.

This is a contradiction, which proves that $W \cap W^{s,+}_X(\sigma) \neq \emptyset$. The result follows. ☐

Proposition 2.11. *$H^+ = \mathrm{Cl}(W^{u,+})$ and $H^- = \mathrm{Cl}(W^{u,-})$.*

Proof. Fix $q \in P^+$, i.e., $W^u_X(q) \cap W^{s,+}_X(\sigma) \neq \emptyset$. Note that $W^{u,+} \cap W^{s,+}_X(\sigma)$ is nonempty by Lemma 2.10. Using (H1) and the Inclination Lemma it is not hard to prove that $W^{u,+}$ accumulates on q; therefore $H^+ \subset \mathrm{Cl}(W^{u,+})$. Conversely let $x \in W^{u,+}$ be fixed. By (H1) and $W^{u,+} \subset \Lambda$ there is z in $\mathrm{Per}_X(\Lambda)$ near x. Choosing z close to x we ensure that $W^s_X(z) \cap W^{u,+} \neq \emptyset$, because stable manifolds have size uniformly bounded away from zero. If $W^u_X(z) \cap W^{s,-}_X(\sigma) \neq \emptyset$, the Inclination Lemma and the fact that $W^s_X(z)$ intersects $W^{u,+}$ imply that $W^u \cap W^{s,-}_X(\sigma) \neq \emptyset$. This contradicts Proposition 2.10, so $W^u_X(z) \cap W^{s,-}_X(\sigma) = \emptyset$. By [**MP1**] we obtain $z \in P^+$, proving that $x \in H^+$ and the lemma. ☐

Proposition 2.12. *If $z \in \mathrm{Per}_X(\Lambda)$ and $W^s_X(z) \cap W^{u,+} \neq \emptyset$, then*

$$\mathrm{Cl}(W^s_X(z) \cap W^{u,+}) = \mathrm{Cl}(W^{u,+}).$$

The same is true if we replace $+$ by $-$.

Proof. We have shown that $\mathrm{Cl}(W^{s,+}_X(\sigma) \cap W^{u,+}) = \mathrm{Cl}(W^{u,+})$. Fix $x \in W^{u,+}$. By (H1) there is $w \in \mathrm{Per}_X(\Lambda)$ close to x. In particular $W^s_X(w)$ intersects $W^{u,+}$. If $W^u_X(w) \cap W^{s,+}_X(\sigma) = \emptyset$ then $W^u_X(w) \cap W^{s,-}_X(\sigma) \neq \emptyset$ by [**MP1**]. It follows from the Inclination Lemma that $W^{u,+} \cap W^{s,-}_X(\sigma) \neq \emptyset$, contradicting Proposition 2.10. We conclude that $W^u_X(w) \cap W^{s,+}_X(\sigma) \neq \emptyset$. Note that $W^s_X(w) \cap W^{u,+} \neq \emptyset$ gets close to x as $w \to x$. Since the intersection $W^u_X(w) \cap W^{s,+}_X(\sigma) \neq \emptyset$ is transversal we can apply the Inclination Lemma to find a transverse intersection $W^{u,+} \cap W^{s,+}_X(\sigma)$ close to x. This proves $\mathrm{Cl}(W^{s,+}_X(\sigma) \cap W^{u,+}) = \mathrm{Cl}(W^{u,+})$. Finally we prove $\mathrm{Cl}(W^s_X(z) \cap W^{u,+}) = \mathrm{Cl}(W^{u,+})$. Choose $x \in W^{u,+}$. Since $\mathrm{Cl}(W^{s,+}_X(\sigma) \cap W^{u,+}) = \mathrm{Cl}(W^{u,+})$ there is an interval $I_x \subset W^{u,+}$ arbitrarily close to x such that $I_x \cap W^{s,+}_X(\sigma) \neq \emptyset$. The positive orbit of I_x first passes through a and then accumulates on $W^{u,+}$. But $W^s_X(z) \cap W^{u,+} \neq \emptyset$ by assumption. Since this intersection is transversal the Inclination Lemma implies that the positive orbit of I_x intersects $W^s_X(z)$. By taking the backward flow of the last intersection we get $W^s_X(z) \cap I_x \neq \emptyset$. This proves the lemma. ☐

Given $z \in \mathrm{Per}_X(\Lambda)$, let $H_X(z)$ be the homoclinic class associated to z.

Proposition 2.13. *If $z \in \mathrm{Per}_X(\Lambda)$ is close to a point in $W^{u,+}$, then*

$$H_X(z) = \mathrm{Cl}(W^{u,+}).$$

The same is true if we replace $+$ by $-$.

Proof. Let $z \in \mathrm{Per}_X(\Lambda)$ be a point close to one in $W^{u,+}$. It follows from the continuity of the stable manifolds that $W^s_X(z) \cap W^{u,+} \neq \emptyset$. We claim that $H_X(z) = \mathrm{Cl}(W^{u,+})$. Indeed $W^{u,+} \cap W^{s,-}_X(\sigma) = \emptyset$ by Proposition 2.10. This equality and the Inclination Lemma imply that $W^u_X(z) \cap W^s_X(\sigma) \subset W^{s,+}_X(\sigma)$. By Proposition 2.12, $W^s_X(z) \cap W^{u,+}$ is dense in $W^{u,+}$ since $W^s_X(z) \cap W^{u,+} \neq \emptyset$. Let Σ be a cross-section containing p_0 and fix $x \in W^{u,+}$. We can assume $x, z \in \Sigma$. Since $W^u_X(z) \cap W^s_X(\sigma) \neq \emptyset$ and $W^u_X(z) \cap W^s_X(\sigma) \subset W^{s,+}_X(\sigma)$ there is an interval $I \subset W^u_X(z)$ intersecting $W^{s,+}_X(\sigma)$. Then the positive orbit of I yields an interval J close to σ intersecting $W^{s,+}_X(\sigma)$. In addition, the positive orbit of J yields an interval K whose positive orbit *accumulates* $W^{u,+}$ (recall Definition 2.9). Since $W^s_X(z) \cap W^{u,+}$ is dense in $W^{u,+}$ and $x \in W^{u,+}$, the orbit $W^s_X(z)$ passes close to x. The Inclination Lemma applied to the positive orbit of K yields a homoclinic point z' associated to z which is close to x. This proves that $x \in H_X(z)$, so $\mathrm{Cl}(W^{u,+}) \subset H_X(z)$. The opposite inclusion is a direct consequence of the Inclination Lemma applied to $W^s_X(z) \cap W^{u,+} \neq \emptyset$. We conclude that $\mathrm{Cl}(W^{u,+}) = H_X(z)$ as desired. □

Theorem 2.14. *Let Λ be a singular-hyperbolic set of a C^r flow X on a closed three-manifold, where $r \geq 1$. Suppose that the following properties hold:*

1. *Λ is connected.*
2. *Λ is attracting.*
3. *The closed orbits contained in Λ are dense in Λ.*
4. *Λ has a unique singularity σ.*
5. *Λ is not transitive.*

Then H^+ and H^- are homoclinic classes of X.

Proof. Let Λ be a singular-hyperbolic set of X satisfying the theorem's conditions. To prove that H^+ is a homoclinic class it suffices by Proposition 2.11 to prove that $\mathrm{Cl}(W^{u,+})$ is a homoclinic class. By condition 3 of the Theorem we can choose $z \in \mathrm{Per}_X(\Lambda)$ arbitrarily close to a point in $W^{u,+}$. Then $\mathrm{Cl}(W^{u,+}) = H_X(z)$ by Proposition 2.13 and the result follows. □

3. Proof of Theorem A

First we introduce some notations. Hereafter M is a compact 3-manifold and $\mathcal{X}^r$ is the space of C^r flows in M equipped with the C^r topology, $r \geq 1$. The *nonwandering set* of $X \in \mathcal{X}^r$ is the set $\Omega(X)$ of points $p \in M$ such that for all neighborhood U of p and $T > 0$ there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. An attracting Λ with isolating block U has a continuation $\Lambda(Y)$ for Y C^r close to X defined by $\Lambda(Y) = \bigcap_{t>0} Y_t(U)$. This continuation is then defined when Λ is an attractor. A compact invariant set is *nontrivial* if it is not a closed orbit of X. Transitive sets for flows are always connected. The proof of Theorem A is based on the following result:

Theorem 3.1. *Let Λ be a singular-hyperbolic set of $X \in \mathcal{X}^r$, $r \geq 1$. Suppose that the following properties hold:*

1. *Λ is connected.*
2. *Λ is attracting.*
3. *The closed orbits contained in Λ are dense in Λ.*
4. *Λ has only one singularity.*
5. *Λ is not transitive.*

Then for every neighborhood U of Λ there is a flow Y that is C^r close to X and such that

$$\Lambda(Y) \not\subset \Omega(Y).$$

To prove this theorem we shall use the following definitions and facts: let $X \in \mathcal{X}^r$ and let Λ be a singular-hyperbolic set of X satisfying the conditions of the theorem. Let σ be the unique singularity of Λ. As mentioned on page 330, σ is Lorenz-like. As in Section 2, $W^{ss}_X(\sigma)$ divides $W^s_X(\sigma)$ into two connected components, which we denote by $W^{s,+}_X(\sigma)$ and $W^{s,-}_X(\sigma)$, or $W^{s,+}, W^{s,-}$ for short. Recall that $\mathrm{Per}_X(\Lambda)$ denotes the union of the periodic orbits of X in Λ. Fix such $a \in W^u_X(\sigma)\setminus\{\sigma\}$. By Theorem 2.8, $\omega_X(a) = O$ for some periodic orbit with positive expanding eigenvalues of X. In particular, $W^{u,+}$ and $W^{u,-}$ are defined (Definition 2.9).

Lemma 3.2. $\mathrm{Cl}(W^{u,+}) \cap W^{s,-} = \emptyset$.

Proof. Suppose for a contradiction that $\mathrm{Cl}(W^{u,+}) \cap W^{s,-} \neq \emptyset$. By Lemma 2.2 there is a singular cross-section Σ^- such that every orbit of Λ passing close to some point in $W^{s,-}_X(\sigma)$ intersects Σ^-. Let $q \in \Lambda$ be periodic such that $W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset$. Since $\mathrm{Cl}(W^{u,+}) \cap W^{s,-} \neq \emptyset$, we have $\mathrm{Cl}(W^{u,+}) \cap \Sigma^- \neq \emptyset$. Because closed orbits are dense we can prove that $q \in \mathrm{Cl}(W^{u,+})$. It follows that $H^- \subset \mathrm{Cl}(W^{u,+})$, so $\Lambda = \mathrm{Cl}(W^{u,+})$ by Lemma 2.1. Also, since Λ is not transitive, $H^+ = \mathrm{Cl}(W^{u,+})$ (Proposition 2.11) and H^+ is a homoclinic class (Theorem 2.14). Since homoclinic classes are transitive sets we conclude that Λ is transitive, a contradiction. This proves the result. ◻

Lemma 3.3. *Let D be a fundamental domain of $W^{uu}_X(p_0)$ contained in $W^{u,+}$. There exist a neighborhood V of D and a cross-section Σ^- of X intersecting $W^{s,-}$ satisfying the following properties:*

1. *Every X-orbit's sequence in Λ converging to a point in $W^{s,-}$ intersects Σ^-.*
2. *No positive X-orbit with initial point in V intersects Σ^-.*

Proof. Fix a fundamental domain F of $W^s_X(\sigma)$ and define $F^- = F \cap W^{s,-}_X(\sigma)$ Then there is a compact interval $F' \subset F^-$ such that $\Lambda \cap F^- \subset F'$. By

Lemma 3.2 there is $\epsilon > 0$ such that $B_\epsilon(\mathrm{Cl}(W^{u,+})) \cap B_\epsilon(F') = \emptyset$. Clearly we can choose a cross-section Σ^- of X inside $B_\epsilon(F')$ such that every X-orbit's sequence in Λ converging to some point in $W^{s,-}$ intersects Σ^-. Since $W^{u,+}$ is invariant and $\mathrm{Cl}(W^{u,+}) \cap B_\epsilon(F') = \emptyset$, every positive orbit with initial point in D cannot intersect Σ^-. By using the contracting foliation of Λ we have the same property for every positive trajectory with initial point in a neighborhood V of D. This proves the result. □

Now we define a perturbation (pushing) close to a point $a \in W^u_X(\sigma) \setminus \{\sigma\}$. To this end we fix the following cross-sections:

1. Σ_a, containing a in its interior.
2. $\Sigma' = X_1(\Sigma)$.
3. Σ_0, intersecting O in a single interior point.
4. Σ^+, Σ^-, which intersect $W^{s,+}, W^{s,-}$, respectively, and point toward the side of a.

Every X-orbit intersecting $\Sigma^+ \cup \Sigma^-$ will intersect Σ. Note that there is a well-defined neighborhood $\mathcal{O}$ given by

$$\mathcal{O} = X_{[0,1]}(\Sigma_a).$$

This neighborhood will be the support of the pushing described in the Figures 2 and 3. The pushing in $\mathcal{O}$ yielding the perturbed flow Y of X is obtained in the standard way (see [**dMP**]).

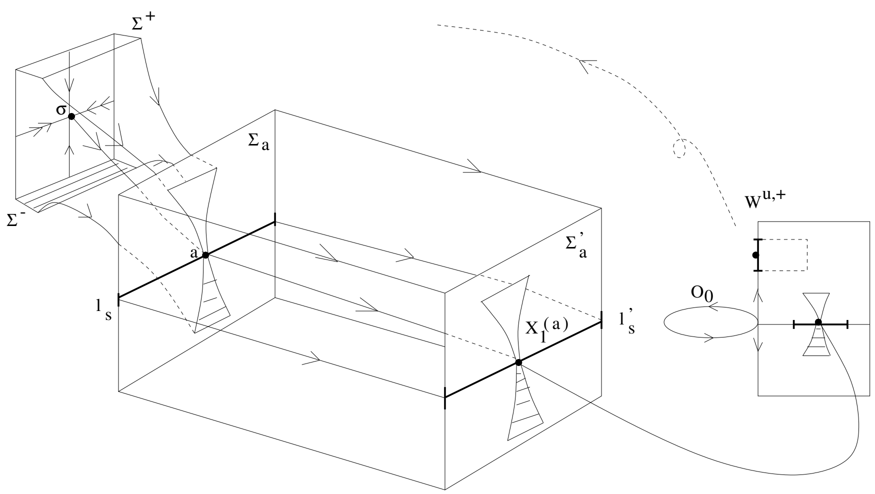

Figure 2. Unperturbed flow X.

We have to prove that $\Lambda(Y) \not\subset \Omega(Y)$ for the perturbed flow Y. For this purpose we observe that by 5 of Theorem 3.1 and Proposition 2.5 we can assume that there q periodic in U such that $W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset$. We obtain in this way an interval K in $\Sigma^- \cap W^u_X(q)$ crossing Σ^- as in

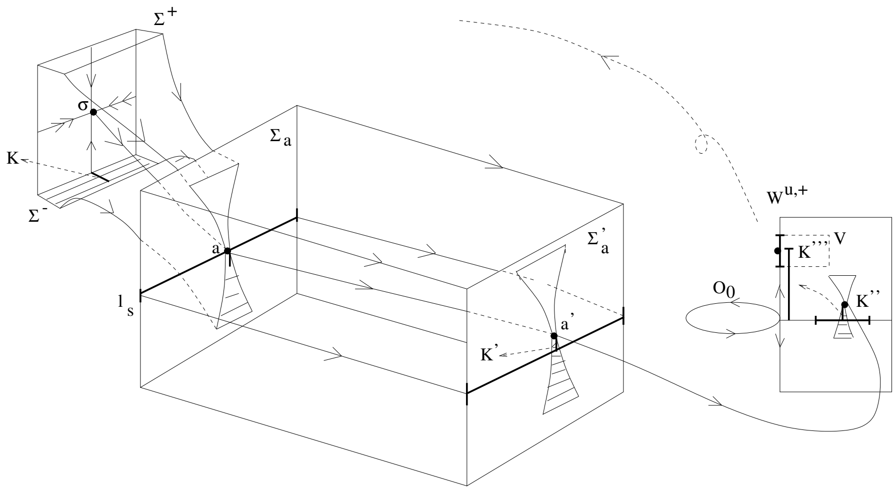

Figure 3. Perturbed flow Y.

Figure 3. The Y-flow carries K to an interval K'' as in Figure 3. Let $q(Y), W^u_Y(q(Y)), K''(Y), \sigma(Y)$ denote the continuation of these objects for the perturbed flow Y. We observe that $K(Y) \subset \Lambda(Y)$ since $\Lambda(Y)$ is an attracting set, $q(Y) \in \Lambda(Y)$ and $K \subset W^u_Y(q(Y))$. Then Theorem 3.1 will follow from the lemma below:

Lemma 3.4. *$K(Y) \not\subset \Omega(Y)$.*

Proof. Suppose for a contradiction that $K(Y) \subset \Omega(Y)$ and pick $x \in \mathrm{Int}\, K(Y)$, the interior of the interval $K(Y)$. The flow of Y carries the points near x to the neighborhood V depicted in Figure 3. This neighborhood is obtained by saturating a fundamental domain in $W^{u,+}$ by the strong stable manifolds [**HPS**]. Note that there are points x' near x that back up close to x under the forward flow of Y ($x \in K(Y) \subset \Omega(Y)$). In particular, *the positive Y-orbit of x' returns to Σ^-*. At the same time, Lemma 3.3-2 implies that no X-orbit starting in V intersects Σ^-. Since $X = Y$ outside $\mathcal{O}$ we conclude that the positive Y-orbit of x' intersects Σ^+. Afterward this positive orbit passes through the box $\mathcal{O}$ and arrives to V. By repeating the argument we conclude that *the positive Y-orbit of x' never returns to* Σ^-, a contradiction. The lemma is proved. □

Proof of Theorem A. Let Λ be a singular-hyperbolic attractor of a C^r flow X on a compact 3-manifold M. Assume that Λ is C^r critically robust and has a unique singularity σ. Denote by $\Lambda(Y) = \bigcap_{t>0} Y_t(U)$ the continuation of Λ in a neighborhood U of Λ for Y close to X. Denote by $C(Y)$ the union of the closed orbits of a flow Y. Since Λ is C^r critically robust, there is a neighborhood U of Λ such that $\Lambda(Y) \cap C(Y)$ is dense in $\Lambda(Y)$ for every flow

Y that is C^r close to X. Clearly $\Lambda(Y)$ is a singular-hyperbolic set of Y for all Y close to X. Because Λ has a unique singularity, so does $\Lambda(Y)$. Now recall that Λ is an attractor by assumption. In particular, Λ is transitive (recall Definition 1.1). It follows that Λ is connected and so the neighborhood U above can be arranged to be connected. Then $\Lambda(Y)$ is connected as well. Summarizing, $\Lambda(Y)$ is a singular-hyperbolic set of Y satisfying conditions 1–4 of Theorem 3.1. If Λ is not C^r robust, we can find a Y that is is C^r close to X and such that $\Lambda(Y)$ is not transitive. Then $\Lambda(Y)$ satisfies all conditions of Theorem 3.1, and we can find a Y' that is C^r close to Y and such that $\Lambda(Y') \not\subset \Omega(Y')$. This is a contradiction, since $\Lambda(Y') \subset \Omega(Y')$ (recall that $\Lambda(Y') \cap C(Y')$ is dense in $\Lambda(Y')$). This contradiction proves the result. □

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