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We consider the Euler–Lagrange equation of a functional arising from conformal geometry in four dimensions, a fourth order equation with borderline nonlinearity. We present a short proof of the fact that any $W^{2,2}$-solution is smooth.

1. Introduction

Let (M, g) be a four-dimensional compact Riemannian manifold. Motivated by problems in four-dimensional spectral theory and conformal geometry, Chang and Yang [**CY**] (cf. also Chang [**C**]) introduced the functional $F : W^{2,2}(M) \to \mathbf{R}$:

(1.1)
$$F(w) = \int_M \big\{(\Delta w)^2 + (\alpha\Delta w + \beta|Dw|^2)^2 + T(Dw, Dw) + E(w - \overline{w})\big\}\, dv,$$

where $\alpha, \beta \in \mathbf{R}$, $\overline{w} = \frac{1}{\operatorname{vol} M}\int w$; $E : \mathbf{R} \to \mathbf{R}$ and $T \in \operatorname{sym}^2(T^*M)$ satisfy:

$$\max\big\{|E(x)|, |E'(x)|\big\} \le c_1 e^{c_2|x|}, \quad |T(v, v)| \le c_3|v|^2. \tag{1.2}$$

Direct computations show that the Euler–Lagrange equation associated with critical points of F on $W^{2,2}(M)$ is

$$\begin{aligned} 2(1 + \alpha^2)\Delta^2 w + 2\beta \operatorname{div}\big(\alpha D(|Dw|^2) - (\alpha\Delta w + \beta|Dw|^2)Dw\big) \\ = \operatorname{div}(T(Dw, \cdot)) - \big(E'(w - \overline{w}) - \overline{E}'(w - \overline{w})\big) \end{aligned} \tag{1.3}$$

where $\overline{E}'(w - \overline{w}) = \frac{1}{\operatorname{vol} M}\int E'(w - \overline{w})$. Chang, Gursky and Yang [**CGY**] proved that any F-minimizing solution $u \in W^{2,2}(M)$ to (1.3) is actually smooth. It was asked in [**CGY**] whether any weak solution $u \in W^{2,2}(M)$ is smooth. Indeed, Uhlenbeck and Viaclovsky [**UV**] confirmed this recently and proved the smoothness for any weak solution $u \in W^{2,2}(M)$ to (1.3). The proof in [**CGY**] relied on F-minimality. The idea in [**UV**] is based on some uniqueness properties for small perturbations of Δ^2 in various Sobolev spaces and seems to be an indirect argument. Here we provide an alternative and direct proof of the smoothness for weak solutions to (1.3); namely, we show that under a smallness assumption on the $W^{2,2}$ norm, the normalized L^p-norm of the gradient of u on a ball decays like a positive power of the radius of the ball. This, combined with Morrey's decay lemma and the conformal invariance of the $W^{2,2}$ norm in dimension four, implies the Hölder continuity of u. Higher-order regularity then follows from [**CGY**]. This type of so-called ϵ_0-decay lemma is very common in the context of regularity theory for harmonic maps (cf. Schoen–Uhlenbeck [**SU**]). In fact, this kind of idea was also employed by Chang, Wang and Yang in their study of the regularity problem of biharmonic maps into spheres [**CWY**].

Since regularity is a local result, we assume, for simplicity, that $M = \Omega \subset \mathbf{R}^4$ is a bounded smooth domain, with the Euclidean metric g. Now we state the decay lemma:

Lemma A. *There exist $\epsilon_0 > 0$ and $\theta_0 \in (0, \frac{1}{2})$ such that if $u \in W^{2,2}(\Omega)$ is a weak solution to (1.3) and if for $B_r(x) \subset \Omega$ we have*

$$\int_{B_r(x)} |Du|^4 + |D^2u|^2 \le \epsilon_0^2 \tag{1.4}$$

then, for any $2 < p < 4$,

$$(\theta_0 r)^{p-4} \int_{B_{\theta_0 r}(x)} |Du|^p \le \tfrac{1}{2} r^{p-4} \textstyle\int_{B_r(x)} |Du|^p + C\big(p, \|D^2u\|_{L^2(\Omega)}\big) r^p. \tag{1.5}$$

Since $u \in W^{2,2}(\Omega)$, the absolute continuity of $\int |Du|^4 + |D^2u|^2$ implies that there exists an $r_0 > 0$ such that (1.4) holds for u over any ball $B_r(x) \subset \Omega$ with $0 < r \le r_0$. Therefore, we can apply the lemma repeatedly and conclude that there exists a $\delta_0 \in (0,1)$ such that $r^{p-4} \int_{B_r(x)} |Du|^p$ behaves like $r^{p\delta_0}$ for all $0 < r < r_0$ and $x \in \Omega$. This, combined with Morrey's lemma, implies that $u \in C^{\delta_0}(\Omega)$ and hence $u \in C^\infty(\Omega)$, via [**CGY**]. In particular, one has (cf. also [**UV**]):

Theorem B. *If $u \in W^{2,2}(M)$ is a weak solution to (1.3), then $u \in C^\infty(M)$.*

2. Proof of Lemma A

It follows from Fubini's theorem that there is an $s \in [\frac{r}{2}, r]$ such that

$$\int_{\partial B_s(x)} |Du|^4 + |D^2u|^2 \le 2r^{-1} \int_{B_r(x)} |Du|^4 + |D^2u|^2. \tag{2.1}$$

Let $u_1 \in W^{2,2}(B_s(x))$ satisfy

$$\Delta^2 u_1 = -\frac{\beta}{1+\alpha^2} \operatorname{div}\big(\alpha D(|Du|^2) - (\alpha \Delta u + \beta |Du|^2)\, Du\big), \tag{2.2}$$

$$u_1 = \frac{\partial u_1}{\partial r} = 0 \quad \text{on } \partial B_s(x). \tag{2.3}$$

Let $u_2 \in W^{2,2}(B_s(x))$ satisfy

$$\Delta^2 u_2 = \frac{1}{2(1+\alpha^2)}\big(\mathrm{div}(T(Du,\cdot)) - (E'(u-\bar{u}) - \overline{E}'(u-\bar{u}))\big), \tag{2.4}$$

$$u_2 = \frac{\partial u_2}{\partial r} = 0 \quad \text{on } \partial B_s(x). \tag{2.5}$$

Let $u_3 = u - u_1 - u_2 \in W^{2,2}(B_s(x))$. Then we have

$$\begin{aligned} \Delta^2 u_3 &= 0 && \text{in } B_s(x), \\ u_3 = u \quad \text{and} \quad \frac{\partial u_3}{\partial r} &= \frac{\partial u}{\partial r} && \text{on } \partial B_s(x). \end{aligned} \tag{2.6}$$

For u_1, it follows (see, e.g., Lemma 2.2 of [**CWY**]) that for any $q \in (1, \frac{4}{3})$

$$\begin{aligned} \|D^3 u_1\|_{L^q(B_s(x))} &\le C\||Du||D^2u| + |Du|^2|Du|\|_{L^q(B_s(x))} \\ &\le C\big(\|D^2 u\|_{L^2(B_s(x))} + \|Du\|^2_{L^4(B_s(x))}\big)\|Du\|_{L^{\frac{2q}{2-q}}(B_s(x))} \\ &\le C\epsilon_0 \|Du\|_{L^{\frac{2q}{2-q}}(B_s(x))}. \end{aligned}$$

This, combined with the Sobolev embedding theorem, implies

$$\begin{aligned} \|Du_1\|_{L^{\frac{2q}{2-q}}(B_{\frac{r}{2}}(x))} \le \|Du_1\|_{L^{\frac{2q}{2-q}}(B_s(x))} &\le C\epsilon_0 \|Du\|_{L^{\frac{2q}{2-q}}(B_s(x))} \\ &\le C\epsilon_0 \|Du\|_{L^{\frac{2q}{2-q}}(B_r(x))}. \end{aligned} \tag{2.7}$$

Here we have used the fact that $Du_1 = 0$ on $\partial B_s(x)$. To estimate u_2, observe that (1.2) implies that $|T(Du,\cdot)| \le C|Du| \in L^4(\Omega)$ and

$$\|T(Du,\cdot)\|_{L^4(\Omega)} \le C\|u\|_{W^{2,2}(\Omega)}. \tag{2.8}$$

The Moser–Trudinger inequality and (1.2) imply $E'(u-\bar{u}) - \overline{E}'(u-\bar{u}) \in L^p(\Omega)$ for any $1 < p < \infty$ and

$$\big\|E'(u-\bar{u}) - \overline{E}'(u-\bar{u})\big\|_{L^4(\Omega)} \le C\|u\|_{W^{2,2}(\Omega)}. \tag{2.9}$$

Multiplying (2.4) by u_2 and integrating it over $B_s(x)$, we get

$$\begin{aligned} \int_{B_s(x)} |D^2 u_2|^2 &= \int_{B_s(x)} |\Delta u_2|^2 \\ &\le C \int_{B_s(x)} \big(|Du|\,|Du_2| + |E'(u-\bar{u}) - \overline{E}'(u-\bar{u})|\,|u_2|\big) \\ &\le C\|u\|_{W^{2,2}(\Omega)} \bigg(\int_{B_s(x)} (|u_2|^2 + |Du_2|^2)\bigg)^{\frac{1}{2}} \\ &\le C\|u\|_{W^{2,2}(\Omega)} r \bigg(\int_{B_s(x)} |D^2 u_2|^2\bigg)^{\frac{1}{2}}. \end{aligned}$$

Here we have applied the Poincaré inequality for u_2 in the last step. Thus

$$\int_{B_s(x)} |D^2 u_2|^2 \le C\|u\|^2_{W^{2,2}(\Omega)} r^2. \tag{2.10}$$

This, combined with the Sobolev embedding theorem, gives

$$\int_{B_s(x)} |Du_2|^4 \le C\Big(\int_{B_s(x)} |D^2 u_2|^2\Big)^2 \le Cr^4 \|u\|^2_{W^{2,2}(\Omega)}. \tag{2.11}$$

In particular, for any $q \in (1, \frac{4}{3})$, we have

$$\Big(\frac{r}{2}\Big)^{\frac{2q}{2-q}-4} \int_{B_{\frac{r}{2}}(x)} |Du_2|^{\frac{2q}{2-q}} \le C(\|u\|_{W^{2,2}(\Omega)}) r^{\frac{2q}{2-q}}. \tag{2.12}$$

Since u_3 is a biharmonic function on $B_s(x)$, we know that

$$\int_{B_s(x)} |D^2 u_3|^2 \le \int_{B_s(x)} |D^2 u|^2.$$

A standard Caccipolli-type argument implies that

$$\int_{B_{\frac{r}{3}}(x)} |D^2 u_3|^2 \le Cr^{-2} \int_{B_{\frac{r}{2}}(x)} |Du_3|^2. \tag{2.13}$$

This, combined with the subharmonicity of $|\Delta u_3|^2$, implies

$$r^2 \|Du_3\|^2_{L^\infty(B_{\frac{r}{4}}(x))} \le C \int_{B_{\frac{r}{3}}(x)} |D^2 u_3|^2 \le Cr^{-2} \int_{B_{\frac{r}{2}}(x)} |Du_3|^2. \tag{2.14}$$

In particular, for any $\theta \in (0, \frac{1}{4})$ and $q \in (1, \frac{4}{3})$,

$$(\theta r)^{\frac{2q}{2-q}-4} \int_{B_{\theta r}(x)} |Du_3|^{\frac{2q}{2-q}} \le C\theta^{\frac{2q}{2-q}} r^{\frac{2q}{2-q}-4} \int_{B_{\frac{r}{2}}(x)} |Du_3|^{\frac{2q}{2-q}}. \tag{2.15}$$

Putting (2.7), (2.13), (2.15) together, we obtain, for any $q \in (1, \frac{4}{3})$ and $\theta \in (0, \frac{1}{4})$,

$$\begin{aligned}(\theta r)^{\frac{2q}{2-q}-4}\int_{B_{\theta r}(x)} |Du|^{\frac{2q}{2-q}} &\le (C\epsilon_0 \theta^{\frac{2q}{2-q}-4} + C\theta^{\frac{2q}{2-q}}) r^{\frac{2q}{2-q}-4} \int_{B_r(x)} |Du|^{\frac{2q}{2-q}} \\ &\quad + C\big(\theta, q, \|u\|_{W^{2,2}(\Omega)}\big) r^{\frac{2q}{2-q}}.\end{aligned} \tag{2.16}$$

Therefore, by choosing $\theta_0 = (4C)^{\frac{q-2}{2q}}$ and then choosing ϵ_0 sufficiently small, we have

$$\begin{aligned}(\theta_0 r)^{\frac{2q}{2-q}-4} \int_{B_{\theta_0 r}(x)} |Du|^{\frac{2q}{2-q}} &\le \frac{1}{2} r^{\frac{2q}{2-q}-4} \int_{B_r(x)} |Du|^{\frac{2q}{2-q}} \\ &\quad + C\big(q, \|u\|_{W^{2,2}(\Omega)}\big) r^{\frac{2q}{2-q}}.\end{aligned} \tag{2.17}$$

Set $p = \frac{2q}{2-q}$. Observe that $p \in (2, 4)$ for $q \in (1, \frac{4}{3})$. This completes the proof of Lemma A. $\square$

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