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We show that the Seiberg–Witten invariant is zero for all smooth 4-manifolds with $b_+>1$ that admit circle actions having at least one fixed point. We also show that all symplectic 4-manifolds that admit (possibly nonsymplectic) circle actions with fixed points are rational or ruled, and thus admit a symplectic circle action.

1. Introduction

This paper addresses two problems in 4-manifold theory concerning S^1-actions. The first is the computation of 4-dimensional diffeomorphism invariants. Ever since the introduction of Donaldson invariants in the early 1980's, efforts to calculate diffeomorphism invariants centered upon large classes of smooth manifolds that have some additional structure. One such class of manifolds thought to have promise was 4-manifolds with effective circle actions, but the extra structure given by such manifolds turned out to be insufficient for calculating Donaldson invariants.

With the introduction of Seiberg–Witten invariants in 1994, old problems were revisited with new hope. Donaldson showed how to calculate the Seiberg–Witten invariants in the simplest case where the 4-manifold was a product of a 3-manifold and a circle [**7**]. In 1997 Mrowka, Ozvath, and Yu [**16**], and simultaneously Nicolaescu [**17**], studied the 3-dimensional Seiberg–Witten equations of Seifert-fibered spaces. In 2001, Seiberg–Witten invariants for manifolds with free and fixed-point-free circle actions were calculated in [**4**], [**5**]. In this paper we finish this line of research by calculating the Seiberg–Witten invariants for S^1-manifolds with fixed points. We prove:

Theorem 1.1. *If X is a smooth, closed, oriented 4-manifold with $b_+>1$ and admits a circle action having at least one fixed point, then the Seiberg–Witten invariant vanishes for all* Spin^c *structures.*

When the action on X^4 is free, the quotient of the S^1-action is a smooth, closed 3-manifold Y. In this case X can be thought of as a unit circle bundle of a complex line bundle over Y with Euler class $\chi \in H^2(Y;\mathbb{Z})$.

A fixed-point-free circle action will have nontrivial isotropy groups and the orbit space will be an orbifold rather than a manifold. As in the free case, a manifold with a fixed-point-free S^1-action can still be considered a unit circle bundle, but it is also a principal S^1-bundle of an orbifold line bundle over a 3-dimensional orbifold. In this setup, $H^2(Y;\mathbb{Z})$ is replaced by the group $\mathrm{Pic}^t(Y)$, which records local data around the singular set (see [5]).

Manifolds that admit circle actions with fixed points have more complicated local data, yet this extra structure gives more control when computing the Seiberg–Witten invariants.

Theorem 1.1 combined with the formula derived in [5] gives the means for calculating the Seiberg–Witten invariants of *any* S^1-manifold X with $b_+{>}1$. See [12] for an introduction to Seiberg–Witten theory or [18] for a more detailed analysis.

Theorem 1.2 (General Formula)**.** *Let X be a smooth, closed, oriented 4-manifold with $b_+{>}1$ and a smooth circle action.*

(1) *If the action has fixed points, $\mathrm{SW}_X(\xi)=0$ for any Spin^c structure ξ.*
(2) *If X has a fixed-point-free S^1-action, let Y^3 be the orbifold quotient space and suppose that $\chi\in\mathrm{Pic}^t(Y)$ is the orbifold Euler class of the circle action. If ξ is a Spin^c structure over X with $\mathrm{SW}^4_X(\xi)\neq 0$, then $\xi=\pi^*(\xi_0)$ for some Spin^c structure on Y and*

$$\mathrm{SW}^4_X(\xi)=\sum_{\xi'\equiv\xi_0\,\mathrm{mod}\,\chi}\mathrm{SW}^3_Y(\xi'),$$

where $\xi'-\xi_0$ is a well-defined element of $\mathrm{Pic}^t(Y)$. (See [5] with respect to the $b_+{=}1$ fixed-point-free case.)

As an application of Theorem 1.1 we illustrate the usefulness of Seiberg–Witten invariants even when they vanish. A theorem of Taubes [20] says that the Spin^c structure associated with the first Chern class of a symplectic 4-manifold must have Seiberg–Witten invariant ±1. Symplectic 4-manifolds always have $b_+{>}0$ because the wedge product of the symplectic form with itself is the volume form. Putting these facts together with Theorem 1.1 we conclude:

Corollary 1.3. *A 4-manifold that admits a symplectic form and admits a circle action with fixed points must have $b_+{=}1$.*

This result is unexpected because it is easy to imagine that such manifolds with $b_+>1$ can exist. For instance, build a symplectic 4-manifold with a free circle action (they exist for any $b_+>1$). There are then several cut-and-paste methods available to produce a new 4-manifold with a circle action that has fixed points. The proof of Theorem 1.1 gives a visual reason why the new 4-manifold cannot also carry a symplectic form.

We need to distinguish the manifolds in Corollary 1.3 from similar manifolds that have been intensely studied. If the circle action preserves the symplectic form in the sense that the generating vector field of the action

T satisfies $\mathcal{L}_{\mathrm{T}}\omega = d\iota_{\mathrm{T}}\omega = 0$ then the action is called *symplectic*. The manifolds in Corollary 1.3 do not have this restrictive pointwise condition and therefore they are not necessarily symplectic circle actions.

In 1990 Audin [**2**] and Ahara–Hattori [**1**] classified 4-manifolds with symplectic circle actions having fixed points up to equivariant diffeomorphism (see Karshon's classification [**13**] for equivariant symplectomorphism). But the classification up to diffeomorphism of 4-manifolds that admit both a symplectic form and a circle action with fixed points remained unknown. We can settle this issue using the proof of Theorem 1.1 together with T.J. Li's analysis of symplectic manifolds with b_+=1 [**14**]. We prove that such manifolds are rational or ruled.

Theorem 1.4. *Every* 4*-manifold that admits a symplectic form and that admits a circle action with at least one fixed point is diffeomorphic to* $\mathbb{CP}^2$*, an* S^2*-bundle over a surface, or is obtained by a sequence of blowups from* $\mathbb{CP}^2$ *or from an* S^2*-bundle over a surface.*

It is easy to put symplectic circle actions on rational or ruled surfaces. This implies the next corollary and proves the conjecture below in the case of circle actions with fixed points.

Corollary 1.5. *Every* 4*-manifold that admits a symplectic form and that admits a circle action with at least one fixed point also admits a symplectic circle action* (*with respect to a possibly different symplectic form*)*.*

Taubes asked the following interesting and deep question in [**21**]: if $Y^3 \times S^1$ is symplectic, does Y fiber over the circle? Partial positive results have been posted in [**6**], [**8**]. One can ask a much more general question for any S^1-manifold (cf. [**4**]):

Conjecture 1.6. Every symplectic 4-manifold that admits a circle action also admits (possibly a different) symplectic form and circle action that are symplectic with respect to each other.

If the conjecture holds it effectively answers Taubes' question. While the full proof of this conjecture seems out of reach at the moment, Corollary 1.5 does lend support to the hypothesis that symplectic 4-manifolds with S^1-actions are very special.

2. Proofs

The theorem below proves both Theorem 1.1 and Theorem 1.4:

Theorem 2.1. *Suppose* X *is a smooth, closed, oriented,* 4*-manifold with* $b_+(X) > 0$ *that admits a smooth* S^1*-action with at least one fixed point. Then* X *contains an essential embedded sphere of nonnegative self-intersection.*

If a 4-manifold X has an essential embedded sphere of nonnegative self-intersection and $b_+(X)>1$, the Seiberg–Witten invariant vanishes (see [**11**]), proving Theorem 1.1. Likewise, the existence of such a sphere in a symplectic 4-manifold implies that the manifold is rational or ruled [**14**].

The proof of Theorem 2.1 requires Fintushel's foundational work on 4-manifolds with S^1-actions [**9**], [**10**]. Every such 4-manifold $\pi : X \to Y$ has a quotient 3-manifold Y together with the following data (altogether called a legally weighted 3-manifold Y):

(1) A finite collection of weighted arcs and circles in int Y.
(2) A finite set of isolated fixed points in int Y disjoint from the sets of (1).
(3) A class (the "Euler class") $\chi \in H_1(Y, S)$ where S is the union of ∂Y, points of (2), and arcs of (1).

The endpoints of weighted arcs are fixed points and there may be a finite set of fixed points in the interior of a weighted arc or circle. To each component of an arc or circle that runs between two fixed points (without passing through another fixed point) one assigns a weight that records the local isotropy data of the circle action. If a weighted circle contains no fixed points, it is called *simply weighted*; otherwise it is *multiply weighted*.

Each component of (1) and (2) has an associated index, which is simply the Euler number of the principal S^1-bundle over the boundary of a tubular neighborhood of the component. Similarly, an index can be given to each boundary component. The sum of the indices of all of the components is zero [**10**, §9].

Fintushel used this weighted space to distinguish simply connected 4-manifolds with circle actions [**9**], [**10**]. We will need this result also.

Theorem 2.2 (Fintushel)**.** *Let S^1 act smoothly on a simply connected 4-manifold X, and suppose the quotient space $Y \simeq S^3$ is not a counterexample to the 3-dimensional Poincaré conjecture. Then X is a connected sum of copies of S^4, $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$, and $S^2 \times S^2$.*

Proof of Theorem 2.1. Let X be a smooth, closed 4-manifold with a smooth S^1-action whose quotient is a legally weighted 3-manifold Y. First we show how to reduce to the case where there are no multiply weighted circles in Y.

Suppose Y contains a multiply weighted circle C with 3 or more fixed points. Note that C could represent a nontrivial class in $H_1(Y;\mathbb{Z})$ or be embedded in Y nontrivially as in Figure 1.

In this situation X can be decomposed into an equivariant connect sum of two 4-manifolds $X = X_0 \# N_1$, both with circle actions. The weighted orbit space of X_0 is the same as before except the weighted circle C has exactly two fixed points; the weighted orbit space of N_1 is S^3 with a trivially embedded multiply weighted circle with the original weights (Figure 2).

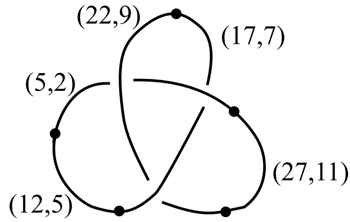

Figure 1. Example of a multiply weighted circle.

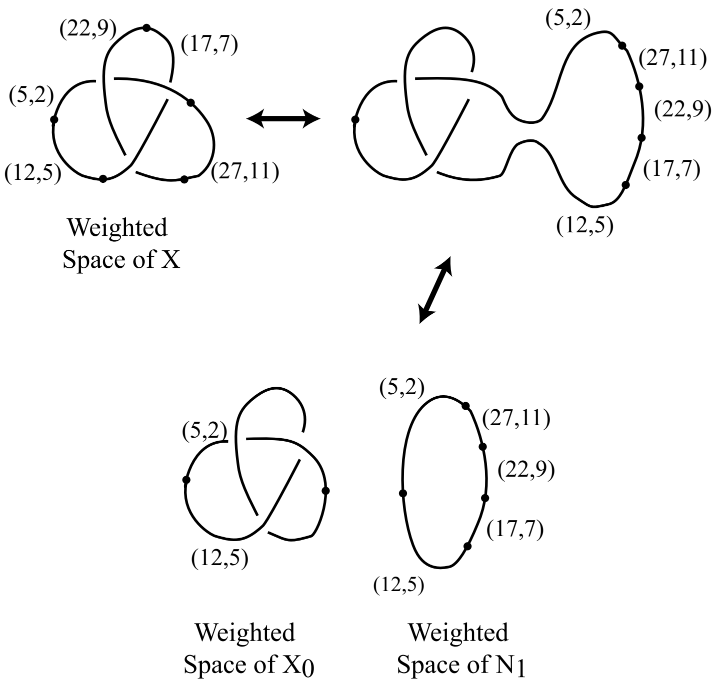

Figure 2. Decomposing X into $X_0 \# N_1$.

The equivariant connect sum in this example is performed by removing the preimage of a D^3 neighborhood of the fixed point between the weights $(5, 2)$ and $(12, 5)$ from both X_0 and N_1 and then gluing equivariantly along the S^3 boundary.

If we repeat this for all multiply weighted circles in X with 3 or more fixed points, we can decompose X into

$$X = X_0 \# N_1 \# N_2 \cdots \# N_k$$

and we can further assume each N_i decomposes into connect sums of $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$, $S^2 \times S^2$, or S^4 by Theorem 2.2. If any of the N_i's have a $\mathbb{CP}^2$ or $S^2 \times S^2$ factor, then X has an essential embedded sphere of nonnegative square. So we may assume that X decomposes into $X_0 \# k\,\overline{\mathbb{CP}}^2$, where $b_+(X_0) > 0$ and $k \geq 0$. If there is an essential sphere of nonnegative square in X_0, such a sphere survives in any of its blowups. Hence it is enough to find a sphere of nonnegative square in X_0.

This leaves us with multiply weighted circles containing two fixed points each. If these circles could be embedded in a 3-ball contained in X_0, then we could deal with them in a similar manner as we do with fixed points and weighted arcs below. So while we have simplified multiply weighted circles considerably, we have not addressed the issue that they might represent nontrivial cycles in homology.

Fortunately we are in a situation where we can change the circle action on X_0 and obtain a new weighted orbit space — one without multiply weighted circles. We briefly describe this trick from [**19**]. Let C be a multiply weighted circle in the weighted orbit space of X_0 and let α be an arc in C running from one fixed point to the other. Let N be a tubular neighborhood of α. The 4-manifold with boundary whose weighted orbit space is N can be identified with $D^2 \times S^2$, with $\pi^{-1}(\alpha) = \mathrm{pt} \times S^2$.

The manifold $D^2 \times S^2$ has many other circle actions that restrict to the same circle action on $\partial(D^2 \times S^2)$. In particular, there is a circle action on $D^2 \times S^2$ with an isotropy group whose order is smaller than the order of the original isotropy group along α. This new circle action depends upon the isotropy data of α and the data from the other arc of C. Remove $\pi^{-1}(N)$ from X_0 and equivariantly glue $D^2 \times S^2$ with this new circle action into $X_0 \setminus \pi^{-1}(N)$ instead. It can be shown that this changes the circle action and the weighted orbit space, but not X_0.

One can then do the same trick for the other arc of C and reduce the order of its isotropy group without changing X_0. We can repeat this procedure several times, continually reducing the order of the isotropy groups. The algorithm ends with a circle action on X_0 whose weighted orbit space has the same data as before except it has two isolated fixed points instead of C or it has a 3-ball removed instead of C.

This algorithm can be carried out for each multiply weighted circle locally. Hence X_0 admits a different S^1-action whose weighted orbit space is the same as the original orbit space except that the weighted circles have been replaced with isolated fixed points or it has had a few 3-balls removed.

Thus we can work with X_0, rename it X, and assume it has only boundary components, weighted arcs, isolated fixed points, or simply weighted circles. If there are two or more boundary components in the quotient, the preimage of an arc that runs from one boundary component to another boundary component is an essential sphere of square zero; so we may reduce to the case where Y has only one boundary component. This case can be eliminated using a short argument in the lemma below.

Therefore we need only consider an S^1-manifold X with weighted arcs, isolated fixed points, and simply weighted circles. Isotope the arcs and fixed points into a smooth ball $D^3 \subset Y$ and enclose them by a sphere $\partial D^3 \subset Y$. Since the sum of the indices of the components is 0, the Euler class of the preimage of the sphere in X is zero, i.e., we can realize X as the fiber sum of two manifolds X_1 and N by

$$X = \big(X_1 \setminus (D^3 \times S^1)\big) \cup_{S^2 \times S^1} \big(N \setminus (D^3 \times S^1)\big),$$

where the orbit space of N contains all of the weighted arcs and isolated fixed points. The quotient of X_1 contains no fixed points but it could still have complicated topology. N is a simply connected 4-manifold with an S^1 action and quotient S^3, so it is diffeomorphic to a connect sum of S^4's, $\mathbb{CP}^2$'s, $\overline{\mathbb{CP}}^2$'s, and $S^2 \times S^2$'s by Theorem 2.2. We can in fact build N by starting with a circle action on S^4 with two fixed points and equivariantly connect summing the other factors. This may be yet a different S^1-action, but the resulting manifold is still diffeomorphic to N. Since N is simply connected, any two embedded circles are isotopic; hence the fiber sum of X_1 with N along $S^2 \times S^1$ using the new S^1-action will still be diffeomorphic to X. Once again we can eliminate all connect sum factors of N except for the S^4 we started with.

Thus we have reduced the problem to finding an essential sphere of non-negative square in

$$X = \big(X_1 \setminus (D^3 \times S^1)\big) \cup_{S^2 \times S^1} \big(S^4 \setminus (D^3 \times S^1)\big),$$

where the quotient of S^4 has 2 fixed points. Note that X is a 4-manifold with an S^1-action, $b_+ > 0$, and 2 isolated fixed points $F \subset Y$. Because the sum of the indices is zero, one of the fixed points comes with a ${}^+1$ index and the other comes with a ${}^-1$ index. In this situation, $b_+(X) > 0$ forces $b_1(X) > 0$ by the formula $\chi(F) = \chi(X) = 2 - 2b_1(X) + b_2(X)$ derived from the Smith–Gysin sequence. A calculation in homology shows that $H^1(X) \cong H^1(Y)$, implying $b_1(Y) = b_1(X) > 0$. Hence $H_1(Y;\mathbb{Z})$ in the long exact sequence of the pair (Y, F) is nonzero:

$$0 \to H_1(Y;\mathbb{Z}) \to H_1(Y,F;\mathbb{Z}) \xrightarrow{\partial} H_0(F;\mathbb{Z}) \to H_0(Y;\mathbb{Z}) \to 0.$$

Since $\partial\chi = (1,-1) \in H_0(F;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ there exists a 1-cycle in $H_1(X,F;\mathbb{Z})$ represented by a closed loop β that is not a multiple of the Euler class χ of the action. The preimage $\pi^{-1}(\beta)$ is an essential torus of self-intersection 0. Isotope β to the fixed point set such that $\beta = \beta_1 + \beta_2$, where β_1 and β_2 are arcs running from one fixed point to the other. The preimage of both of these arcs is a sphere of self-intersection 0. Since the preimage of β is essential, one of these spheres must be essential as Theorem 2.1 demands.

We finish the proof of Theorem 2.1 with the following lemma to cover the case when there is one boundary component of fixed points.

Lemma 2.3. *Let X be a smooth, closed oriented $b_+>1$ 4-manifold with a smooth circle action whose orbit space Y has weighted circles and arcs, isolated fixed points, and one boundary component. Then there exists an essential sphere of nonnegative square in X.*

Proof. As before, we eliminate cases that have spheres of nonnegative square by using techniques in the proof above. Thus we can assume that the quotient space of X contains only simply weighted circles, n isolated fixed points $\{x_1, x_2, \ldots, x_n\}$ each with a $^{+}1$ index, and one boundary component with index $-n$. Denote the fixed point set by F.

A linearly independent subset of $H_2(X;\mathbb{Z})$ can be constructed as follows: for $i = 1$ to n, let γ_i be an arc that runs from x_i to a point on ∂Y such that all arcs are mutually disjoint. The preimage $\pi^{-1}(\gamma_i)$ of each of these arcs is an essential sphere S_i representing a 2-cycle in $H_2(X;\mathbb{Z})$. These linearly independent classes have an intersection matrix with respect to each other given by

$$S_i \cdot S_j = \begin{cases} -1, & i = j, \\ \ \ 0, & \text{otherwise.} \end{cases}$$

Because the intersection form of X is not negative definite, we must have $b_2(X) > n$. Let g be the genus of ∂Y. Using the fact $\chi(X) = \chi(F)$, we get

$$b_1(X) = \tfrac{1}{2}(b_2(X) - n + 2g).$$

In particular, $b_1(Y) > g$ because $b_1(Y) = b_1(X)$. The long exact sequence for the pair (Y, F) implies that there is a 1-cycle in $H_1(Y, F; \mathbb{Z})$ represented by a closed loop γ that is not a multiple of the Euler class χ. The preimage $\pi^{-1}(\gamma)$ is an essential torus of self-intersection zero. The loop γ is homologous to an arc that starts and ends on ∂Y but is otherwise disjoint from F; and the preimage of the arc is a sphere homologous to the torus. This is an essential sphere of nonnegative square, proving Lemma 2.3. □

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