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TRANSITION DENSITY ESTIMATES FOR STABLE PROCESSES ON SYMMETRIC SPACES

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We present estimates of the transition densities for stable processes on Riemannian symmetric spaces of noncompact type. We show that these processes have a weak scaling property and we address in this way a question of Getoor about the stability properties of pseudostable measures on symmetric spaces.

1. Introduction

Stable Lévy processes on a group, where stability is meant with respect to group automorphisms, can only exist on nilpotent groups (see [**Ku**], [**App1**], [**App2**]). In [**Ge**], using a subordination procedure, Getoor defined stable processes with respect to Brownian motion on hyperbolic spaces of noncompact type. He asked whether stability properties of such a process and its semigroup can be found. In this paper we answer this question positively (see Theorem 4.3 and Remark 1 thereafter), proving a *weak scaling* property of the transition densities of the stable process in the sense of Getoor. We also solve in this way, in the case of symmetric spaces, Open Problem 4 from [**App2**]. The main results of our paper are given in Theorem 4.3 and Corollaries 5.3 and 5.6.

2. Preliminaries

Let G denote a noncompact semisimple Lie group, K a maximal compact subgroup, and $X = G/K$ the associated Riemannian symmetric space with nonpositive curvature. We adopt the notation and conventions from [**AJ**]. In particular, if $\mathfrak{a}$ is the Cartan space and $\lambda \in \mathfrak{a}$, then we denote by ϕ_λ the spherical functions on X.

It is well-known that the heat kernel on $X = G/K$ is given by

$$h_t(x) = C \int_{\mathfrak{a}} \frac{d\lambda}{|c(\lambda)|^2} e^{-t(|\lambda|^2+|\rho|^2)} \phi_\lambda(x)$$

where $C = C(X)$ is a constant, $c(\lambda)$ is the c-function appearing in the inverse spherical Fourier transform formula and $\rho = \frac{1}{2}\sum_{\alpha>0} m_\alpha \alpha$ is the half-sum of the positive roots with multiplicities m_α.

We set $n = \dim X$, $m = \sum_{\alpha>0} m_\alpha$, and we denote by Σ^{++} the set of positive indivisible roots and by $\mathfrak{a}^+ \subset \mathfrak{a}$ the positive Weyl chamber. We have global estimates:

Theorem 2.1 ([**AJ**], Theorem 3.7)**.** *Let $k > 0$. Then*

$$(1)\quad h_t(\exp H) \asymp t^{-\frac{n}{2}}(1+t)^{\frac{m}{2}-|\Sigma^{++}|}\Bigg(\prod_{\alpha\in\Sigma^{++}}(1+\langle\alpha, H\rangle)\Bigg)e^{-|\rho|^2 t-\langle\rho,H\rangle-\frac{|H|^2}{4t}},$$

provided $|H| < k(1+t)$, $H \in \overline{\mathfrak{a}^+}$.

Observe that $\big(\prod_{\alpha\in\Sigma^{++}}(1+\langle\alpha, H\rangle)\big)e^{-\langle\rho,H\rangle} \asymp \phi_0(\exp H)$ when $H \in \overline{\mathfrak{a}^+}$. Thus for any $x \in X$ we have

$$(2)\qquad h_t(x) \asymp t^{-\frac{n}{2}}(1+t)^{\frac{m}{2}-|\Sigma^{++}|}\phi_0(x)\, e^{-|\rho|^2 t-\frac{|x|^2}{4t}},$$

where $|x| = d(x_0, x)$, the Riemannian distance between $x \in X$ and $x_0 = eK$.

From the probabilistic point of view, the $h_t(x)$ are the densities of the Brownian motion on $X = G/K$. If (X_t) is the corresponding diffusion on G with stationary independent (left) increments and the distribution of X_0 given by the Haar measure on K then its transition function is

$$P_t(x, y) = h_t(x^{-1}y), \qquad x, y \in G.$$

Convention: by c without subscripts we denote a positive constant that may vary from term to term, but otherwise depends only on the underlying space and α (see below).

3. Stable semigroups

In the sequel we understand the objects under discussion (processes, transition probabilities, etc.) equivalently on the symmetric space $X = G/K$ or on the group G, without changing notation.

Let $\alpha \in (0, 2)$. The α-stable process on a symmetric space X with transition densities $P_t(x, y) = p_t(x^{-1}y)$ was defined by Getoor ([**Ge**]) by means of a subordination procedure. In particular,

$$(3)\qquad p_t(x) = \int_0^\infty h_u(x)\eta_t(u)\, du,$$

where η_t is the density of the $\alpha/2$-stable subordinator (cf. also [**Be**]).

In [**AJ**] estimates for p_t when $\alpha = 1$ were given. In [**Gruet**] estimates for $p_t(x)$ when $t \to \infty$ were given in the rank-one case. In this paper we obtain estimates for $p_t(x)$ with respect to both variables x, t for any $\alpha \in (0, 2)$ on all Riemannian symmetric spaces of noncompact type.

It is well-known that the symmetric stable densities on Euclidean spaces cannot be written explicitly, except for $\alpha = 1$. We now recall the exact estimates of the densities $\eta_t(u)$, which will be crucial for our estimates of $p_t(x)$.

By Theorem 37.1 of [D],

$$\lim_{u\to\infty} \eta_1(u)u^{1+\alpha/2} = \frac{\alpha}{2\Gamma(1-\frac{\alpha}{2})}. \tag{4}$$

This, together with the boundedness of $\eta_1(\cdot)$ and the scaling property

$$\eta_t(u) = t^{-2/\alpha}\eta_1(t^{-2/\alpha}u), \qquad t,u>0, \tag{5}$$

gives

$$\eta_t(u) \leq ctu^{-1-\alpha/2}, \qquad t,u>0, \tag{6}$$

$$\eta_t(u) \geq \tilde{c}tu^{-1-\alpha/2}, \qquad t>0,\ u>t^{2/\alpha}. \tag{7}$$

Moreover, by [H] and (5) we know that

$$\eta_t(u) \asymp t^{\frac{1}{2-\alpha}} u^{-\frac{4-\alpha}{4-2\alpha}} e^{-c_1 t^{\frac{2}{2-\alpha}} u^{-\frac{\alpha}{2-\alpha}}}, \qquad t^{-2/\alpha}u \in (0,1), \tag{8}$$

where $c_1 = c_1(\alpha) = \frac{2-\alpha}{2}(\frac{\alpha}{2})^{\frac{\alpha}{2-\alpha}}$. Observe that for $u > t^{2/\alpha}$ we have

$$\exp(-c_1 t^{\frac{2}{2-\alpha}} u^{-\frac{\alpha}{2-\alpha}}) \geq \exp(-c_1),$$

so that (6) and (7) give

$$\eta_t(u) \asymp tu^{-1-\alpha/2} e^{-c_1 t^{\frac{2}{2-\alpha}} u^{-\frac{\alpha}{2-\alpha}}}, \qquad t^{-2/\alpha}u > 1, \tag{9}$$

which of course simplifies to $\eta_t(u) \asymp tu^{-1-\alpha/2}$, but we want to make the estimates (8) and (9) as similar as possible.

Consider now the case of an n-dimensional Riemannian symmetric space $X = G/K$ when G is a complex Lie group. We have then

$$h_u(x) = \phi_0(x)(4\pi u)^{-n/2}\, e^{-\frac{|x|^2}{4u} - |\rho|^2 u}$$

and $h_u(\exp H)$ is a probability density with respect to the polar coordinate Jacobian $J(H)\,dH$ $=$vol$(K/M)\prod_{\alpha\in\Sigma^+}\sinh^{m_\alpha}\alpha(H)\,dH$. Thanks to this explicit formula for h_u we are able to compute the 1-stable density in the complex case, in terms of the modified Bessel function of the third kind (or the MacDonald function) $K_\nu(x)$.

Proposition 3.1. *If G is a complex Lie group and $\alpha = 1$ then*

$$p_t(x) = 2\left(\frac{2\pi}{|\rho|}\right)^{-\frac{n+1}{2}} t\phi_0(x)(|x|^2+t^2)^{-\frac{n+1}{4}} K_{\frac{n+1}{2}}\big(|\rho|\sqrt{|x|^2+t^2}\,\big).$$

Proof. For $\alpha = 1$ we have $\eta_t(u) = \frac{t}{2\sqrt{\pi}}u^{-3/2}\exp(-\frac{t^2}{4u})$. By using the subordination formula and an integral representation for MacDonald function [GR], p. 907, (8.432.6),

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-t-\frac{z^2}{4t}} t^{-\nu-1}dt,$$

we get the desired assertion. ∎

4. Estimates of $p_t(x)$

By (8) and (9) we have

$$(10)\quad p_t(x) \asymp t^{\frac{1}{2-\alpha}} \int_0^{t^{2/\alpha}} h_u(x) u^{-\frac{4-\alpha}{4-2\alpha}} e^{-c_1 t^{\frac{2}{2-\alpha}} u^{-\frac{\alpha}{2-\alpha}}} du + t \int_{t^{2/\alpha}}^{\infty} h_u(x) u^{-1-\alpha/2} e^{-c_1 t^{\frac{2}{2-\alpha}} u^{-\frac{\alpha}{2-\alpha}}} du.$$

As in [**AJ**] we will see that the main contribution in (10) will come from the interval where

$$u \sim u_0 = u_0(x,t)$$

with u_0 minimizing the function

$$(11)\quad f(u) = \frac{|x|^2}{4u} + |\rho|^2 u + c_1 t^{\frac{2}{2-\alpha}} u^{-\frac{\alpha}{2-\alpha}}.$$

The exponent equal to $-f$ is then maximal in the exponential appearing in (10), if we replace $h_u(x)$ by its estimate given in Theorem 2.1. More precisely, when $|x| < k(1+u)$ and $u > \delta > 0$ with δ fixed, we have

$$(12)\quad h_u(x)\, \eta_t(u) \asymp \phi_0(x) G(t,u) e^{-\frac{|x|^2}{4u} - |\rho|^2 u - c_1 t^{\frac{2}{2-\alpha}} u^{-\frac{\alpha}{2-\alpha}}},$$

where

$$G(t,u) = \begin{cases} t^{\frac{1}{2-\alpha}} u^{-r(\alpha)}, & u \leqslant t^{2/\alpha}, \\ t u^{-s(\alpha)}, & u \geqslant t^{2/\alpha}, \end{cases}$$

with $r(\alpha) = \frac{l}{2} + |\Sigma^{++}| + \frac{4-\alpha}{4-2\alpha}$ and $s(\alpha) = \frac{l}{2} + |\Sigma^{++}| + 1 + \frac{\alpha}{2}$, where $l = \dim \mathfrak{a}$ is the rank of X (we have used $l = n - m$).

Since

$$f'(u) = -\frac{|x|^2}{4u^2} + |\rho|^2 - \frac{c_1 \alpha}{2-\alpha} \frac{t^{\frac{2}{2-\alpha}}}{u^{\frac{2}{2-\alpha}}},$$

we get

$$(13)\quad |\rho|^2 u_0^2 = \frac{|x|^2}{4} + \frac{c_1 \alpha}{2-\alpha} t^{\frac{2}{2-\alpha}} u_0^{\frac{2(1-\alpha)}{2-\alpha}},$$

or, in more convenient form,

$$(14)\quad |\rho|^2 = \frac{1}{4} \left(\frac{|x|}{u_0} \right)^2 + c_2 \left(\frac{t}{u_0} \right)^{\frac{2}{2-\alpha}},$$

with $c_2 = c_1 \alpha / (2-\alpha)$.

Remark. Equation (13) can be solved explicitly as a (bi)quadratic equation for $\frac{2(1-\alpha)}{2-\alpha} = -2, 0, 1, 2, 4$, corresponding respectively to $\alpha = \frac{3}{2}, 1, 0, \emptyset, 3$, so only for $\alpha = \frac{3}{2}$ or 1. In the case $\alpha = 1$ we obtain $|\rho|^2 u_0^2 = \frac{|x|^2}{4} + c_1 t^2 =$

$\frac{|x|^2+t^2}{4}$ and $u_0 = \sqrt{|x|^2+t^2}/2|\rho|$ as in [**AJ**]. In the case $\alpha = \frac{3}{2}$ we obtain a biquadratic equation and

$$u_0 = \frac{1}{|\rho|\sqrt{2}}\left(\frac{|x|^2}{4} + \sqrt{\frac{|x|^4}{16} + 4c_2 t^4|\rho|^2}\right)^{\frac{1}{2}}.$$

Equation (13) may be solved explicitly for some other particular values of α (see the following lemma). For a generic $\alpha \in (0,2)$ the function u_0 is given only implicitly by (13).

Lemma 4.1. *For $\alpha = 4/3$ and $t > c_3|x|$ with $c_3 = \dfrac{1}{\sqrt{3}(4c_2|\rho|)^{1/3}}$, we have*

$$u_0 = \frac{1}{\sqrt[3]{2|\rho|^2}}\left(\left(c_2t^3 - \sqrt{c_2^2t^6 - \frac{|x|^6}{432|\rho|^2}}\right)^{\frac{1}{3}} + \left(c_2t^3 + \sqrt{c_2^2t^6 - \frac{|x|^6}{432|\rho|^2}}\right)^{\frac{1}{3}}\right).$$

Proof. If $\alpha = \frac{4}{3}$ then (13) becomes by rearrangement an equation of degree 3 with respect to u_0. It can be solved explicitly by Cardano formulas. However, a reasonable expression occurs only when *the determinant* is positive. This is ensured by the second condition in the hypothesis. □

Lemma 4.2. *For each x and t there exists exactly one solution $u_0(x,t)$ of (13) and (14). We have $u_0 \asymp |x| + t$. The functions $(H,t) \to u_0(\exp H, t)$ and $(H,t) \to f(u_0(\exp H, t))$ are homogeneous of degree 1.*

Proof. From (14) we have $(|x|/u_0)^2 < 4|\rho|^2$ and $(t/u_0)^{2/(2-\alpha)} < c_2^{-1}|\rho|^2$, so that $u_0 > |x|/(2|\rho|)$ and $u_0 > tc_2^{(2-\alpha)/2}|\rho|^{-(2-\alpha)}$, which gives the lower bound. On the other hand, for any constant $A > 1$ we have

$$\frac{1}{4}\left(\frac{|x|}{A(|x|+t)}\right)^2 + c_2\left(\frac{t}{A(|x|+t)}\right)^{\frac{2}{2-\alpha}} \leq \frac{1}{4A^2} + c_2\left(\frac{1}{A}\right)^{\frac{2}{2-\alpha}},$$

which is less than $|\rho|^2$ for A sufficiently large. Since the right-hand side of (14) is decreasing in u_0, it follows that $u_0 \leq A(|x|+t)$. By a similar argument the solution of (14) exists and is unique.

Write $u_0(H,t) = u_0(\exp H, t)$. If $u_0 = u_0(H,t)$ is a solution of (14) then for $a > 0$ the solution of (14) for aH and at is au_0, so $au_0 = u_0(aH, at)$. Thus $u_0(H,t)$ is 1-homogeneous. The homogeneity of $f(u_0)$ is now evident by (11). □

We now give the exact bounds of the α-stable density $p_t(x)$ in terms of u_0.

Theorem 4.3.

$$p_t(x) \asymp \begin{cases} t(t^{1/\alpha} + |x|)^{-n-\alpha} & \text{if } t + |x| \leqslant 1, \\ \phi_0(x) t^{\frac{1}{2-\alpha}} (|x|+t)^{-r(\alpha)+\frac{1}{2}} e^{-f(u_0)} & \text{if } t+|x| \geqslant 1 \text{ and } u_0 \leqslant t^{2/\alpha}, \\ \phi_0(x) t(|x|+t)^{-s(\alpha)+\frac{1}{2}} e^{-f(u_0)} & \text{if } t+|x| \geqslant 1 \text{ and } u_0 \geqslant t^{2/\alpha}, \end{cases}$$

with $r(\alpha) = \frac{l}{2} + |\Sigma^{++}| + \frac{4-\alpha}{4-2\alpha}$, $s(\alpha) = \frac{l}{2} + |\Sigma^{++}| + 1 + \frac{\alpha}{2}$, $u_0 = u_0(x,t)$ *given implicitly by* (13) *and the function* f *defined in* (11)*.*

Remark 1. Theorem 4.3 gives a weak scaling property for stable densities on symmetric spaces (the possibility of recovering, at least asymptotically, the density $p_t(x)$ if $p_1(x)$ is known).

More precisely, when $t+|x| \leqslant 1$, then $p_t(\exp H) \asymp t^{-n/\alpha} p_1(\exp(t^{-1/\alpha}H))$ (in the Euclidean case we have equality).

When $t + |x| \geqslant 1$, it is not the function $p_t(x)$ itself but the function $\phi_0(x)^{-1}p_t(x)$ that has a weak scaling property (the role of the function ϕ_0 may be explained as the influence of the non-Euclidean structure of X). It is clear that the factors $t^{\frac{1}{2-\alpha}}\,(|H|+t)^{-r(\alpha)+\frac{1}{2}}$ and $t\,(|H|+t)^{-s(\alpha)+\frac{1}{2}}$ of the bounds of the function $\phi_0(x)^{-1}p_t(\exp H)$ may be obtained from the factors $(|H|+1)^{-r(\alpha)+\frac{1}{2}}$ and $(|H|+1)^{-s(\alpha)+\frac{1}{2}}$ of the bounds of $\phi_0(x)^{-1}p_1(\exp H)$ by transformations of the form $H \mapsto t^{\beta}\, p_1(\exp(t^{-1}H))$, with a convenient value of β. The same is true for the function $f(u_0)$ since, by the 1-homogeneity of $f(u_0)$, we have $tf(u_0(\exp(t^{-1}H,1) = f(u_0(\exp H,t))$.

On structures different from $\mathbb{R}$ stable measures defined by subordination may preserve such a weaker scaling property (cf. [**BSS**] for fractals). We answer in this way a question raised by Getoor in [**Ge**] about looking for "stability" properties of the densities p_t.

Remark 2. After the proof of the theorem we will give a simple criterion concerning the conditions $u_0 \geqslant t^{2/\alpha}$ and $u_0 \leqslant t^{2/\alpha}$. Note that for $u_0 \asymp t^{2/\alpha}$ the second and the third estimates coincide.

Proof of Theorem 4.3*.* The estimates for small t and $|x|$ are the same as on $\mathbb{R}^n$ and may be found in [**BSS**] (cf. also [**Ben**]). From Lemma 4.2 it follows that $u_0(x,t) \to \infty$ when $\max(x,t) \to \infty$. Following [**AJ**], Theorem 4.3.1, we will look for the estimates of $p_t(x)$ in terms of u_0, when $u_0 \to \infty$. We split the integral (3) for $p_t(x)$ into

$$\int_0^{\kappa^{-1}u_0} + \int_{\kappa^{-1}u_0}^{\infty} = I_1 + I_2 \tag{15}$$

for a constant $\kappa > 1$. We will see below that the first integral is essentially smaller than the second one, so $p_t(x) \asymp \int_{\kappa^{-1}u_0}^{\infty} h_u(x)\,\eta_t(u)\,du$. Now we may use the estimate for $h_u(x)$ from the Theorem 2.1. Write $J_2 = \phi_0(x)^{-1}I_2$ and in J_2 apply the change of variables $u = u_0(x,t)v$. Set

$$P = P(x,t) = \frac{|x|^2}{4u_0}/|\rho|^2 u_0 \quad \text{and} \quad Q = Q(x,t) = \frac{c_1\alpha}{2-\alpha} t^{\frac{2}{2-\alpha}} u_0^{\frac{-\alpha}{2-\alpha}}/|\rho|^2 u_0.$$

We have $0 < P, Q < 1$ and $P + Q \equiv 1$ (the functions P and Q measure in a sense the proportion of importance of variables x and t respectively in the function $u_0(x,t)$). We get

$$f(u_0 v) = |\rho|^2 u_0 \Big(v + P\frac{1}{v} + Q\frac{2-\alpha}{\alpha} v^{-\frac{\alpha}{2-\alpha}} \Big). \tag{16}$$

For any values of P and Q, the point v where the function

$$g(v,x,t) = v + P\frac{1}{v} + Q\,\frac{2-\alpha}{\alpha}\,v^{-\frac{\alpha}{2-\alpha}}$$

attains its minimum must satisfy the equation $P\frac{1}{v^2} + Qv^{-\frac{2}{2-\alpha}} = 1$, which admits the only solution $v = 1$. The functions u_0, P and Q depend on x and t. Nevertheless, due to the uniqueness property just mentioned and the fact that $\frac{\partial^2 g}{\partial v^2} \asymp 1$ for all x and t, the proof of the Laplace method (see [**O**], pp. 80–82)) may be adapted to the present situation. The moderate price to pay is obtaining bounds instead of asymptotics in u_0. Consequently we get

$$J_2 \asymp c\,t^{\frac{1}{2-\alpha}} u_0^{-r(\alpha)+\frac{1}{2}} e^{-f(u_0)}, \quad u_0 \to \infty, \qquad u_0 \leqslant t^{2/\alpha}, \tag{17}$$

$$J_2 \asymp c\,t u_0^{-s(\alpha)+\frac{1}{2}} e^{-f(u_0)}, \quad u_0 \to \infty, \qquad u_0 \geqslant t^{2/\alpha}. \tag{18}$$

To complete the proof we need to justify the claim that the first integral in (15), which we denote by I_1, is essentially smaller that the second one. We use the global upper estimate for the heat kernel ([**An**], [**AJ**, (3.3)]), and proceed as with J_2 above, applying the Laplace method. It follows that $I_1/I_2 \to 0$ when $u_0 \to \infty$. □

Lemma 4.4. *There exist positive constants K, M such that if $t \leqslant K|x|^{\alpha/2}$ then $u_0 \geqslant t^{2/\alpha}$ and if $t \geqslant M|x|^{\alpha/2}$ then $u_0 \leqslant t^{2/\alpha}$. If $t \asymp |x|^{\alpha/2}$ then $u_0 \asymp t^{2/\alpha}$.*

Proof. The first two assertions follow easily from Lemma 4.2. If $t \asymp |x|^{\alpha/2}$ then $|x| \asymp t^{2/\alpha}$ and, consequently, $u_0 \asymp |x| + t \asymp t^{2/\alpha} + t$. Since u_0 and t are supposed to be large ($t \to \infty$), we have $u_0 \asymp t^{2/\alpha}$. □

5. Properties of u_0

Note that for general $\alpha \in (0,2)$ we have no explicit formula for u_0 and $f(u_0)$. In this section we study their properties in greater depth.

Proposition 5.1. *$(H,t) \mapsto u_0 = u_0(\exp H, t)$ is a norm on $\mathfrak{a} \times \mathbb{R}^+$.*

Proof. Again set $u_0(H,t) = u_0(\exp H, t)$. We already know that $u_0(H,t)$ is 1-homogeneous. Set $u_0 = u_0(H,t)$, $u_1 = u_0(H_1,t_1)$, $u_2 = u_0(H + H_1, t + t_1)$, $a = 1/(4|\rho|)^2$ and $b = c_2/|\rho|^2$. For $p = \frac{u_0}{u_0+u_1} \in (0,1)$ and $q = 1-p$ we have

$$\frac{u_0}{p} = \frac{u_1}{q}. \tag{19}$$

Then we obtain

$$ap\left(\frac{|H|}{u_0}\right)^2+bp\left(\frac{t}{u_0}\right)^{\frac{2}{2-\alpha}}=p, \quad \text{and} \quad aq\left(\frac{|H_1|}{u_1}\right)^2+bq\left(\frac{t_1}{u_1}\right)^{\frac{2}{2-\alpha}}=q.$$

Summing these equations, using the fact that functions $x\to x^2$ and $t\to t^{2/(2-\alpha)}$ are convex, and applying (19), we get $u_2\le u_0/p=u_1/q$ and $u_2\le u_0+u_1$. This shows that u_0 is a norm. □

Proposition 5.2. *u_0 extends to a norm on $\mathfrak{a}\times\mathbb{R}$. $f(u_0)$ is a norm on $\mathfrak{a}\times\mathbb{R}^+$ and extends to a norm on $\mathfrak{a}\times\mathbb{R}$.*

Proof. Since u_0 is homogeneous, it is determined by its unit sphere $S=\{(H,t):\, u_0(H,t)=1\}$. Since $c_2^{(2-\alpha)/2}=\frac{\alpha}{2}$, S can be described by

$$t=\frac{2}{\alpha}\left(|\rho|^2-\frac{|H|^2}{4}\right)^{\frac{2-\alpha}{2}}.$$

One may deduce from this formula the convexity of the unit ball $B_0\subset\mathbb{R}^2$ of the 1-homogeneous function

$$U_0: U_0(y,t)=u_0(|H|,|t|)$$

with any $H\in\mathfrak{a}$ such that $|y|=|H|$, $y\in\mathbb{R}$. Thus U_0 is a norm on $\mathbb{R}^2$. The convexity of B_0 implies the convexity of

$$B=\big\{(H,t)\in\mathfrak{a}\times\mathbb{R}:(|H|,t)\in B_0\big\}=\big\{(H,t)\in\mathfrak{a}\times\mathbb{R}:u_0(H,|t|)=1\big\},$$

which is the unit ball of u_0 extended naturally to $\mathfrak{a}\times\mathbb{R}$. This gives the first statement.

For the second one, we solve $f(u_0)=1$ using (11) and the equality

$$\left(\frac{t}{u_0}\right)^{\frac{2}{2-\alpha}}=\frac{1}{c_2}\left(|\rho|^2-\frac{1}{4}\left(\frac{|H|}{u_0}\right)^2\right). \tag{20}$$

We obtain a quadratic equation for u_0,

$$\frac{2}{\alpha}\,|\rho|^2u_0^2-u_0+\frac{\alpha-1}{2\alpha}\,|H|^2=0, \tag{21}$$

and it follows that

$$u_0=\varphi(H)=\frac{\alpha}{4|\rho|^2}\left(1+P(H)\right), \qquad |H|\in\left(0,\frac{1}{|\rho|}\right), \tag{22}$$

where

$$P(H)=\sqrt{1-\frac{4(\alpha-1)}{\alpha^2}|\rho|^2|H|^2}.$$

Indeed, let $\varphi_m(H)$ be the alternative formal solution to (21) (this is possible at least for $\alpha>1$). Then positivity of the right-hand side of (20) implies that $\frac{|H|}{2\varphi_m(H)}\le|\rho|$ and, consequently, $P(H)\le 1-\frac{2|H||\rho|}{\alpha}$. This is possible only for $|H|\le\frac{\alpha}{2|\rho|}$. Solving this we obtain $|H|\ge\frac{1}{|\rho|}$. The contradiction shows that (21) admits at most one solution. An analogous argument for φ instead of φ_m establishes the range for H. Note that for $|H| \in (0, 1/|\rho|)$ the determinant of (21) is positive for every $\alpha \in (0,2)$. Thus, we arrive at (22). Putting $u_0 = \varphi(H)$ into (20) we get

$$t = \psi(H) = \frac{2}{\alpha}\,\varphi(H)\left(|\rho|^2 - \frac{|H|^2}{4\varphi(H)^2}\right)^{1-\frac{\alpha}{2}}.$$

We have $f(u_0) = 1$ iff $u_0 = \varphi(H)$ and $t = \psi(H)$. To finish the proof it is enough to show that the graph of $y \mapsto \psi(y)$ is concave on $\mathbb{R}^+$ (i.e., $\psi'' \le 0$, where we understand $y = |H|$ and ψ as a function of $|H|$ on $\mathbb{R}^+$) and the symmetry with respect to the t-axis does not affect the concavity (e.g., ψ is decreasing). We obtain

$$\psi''(y) = -\left(\frac{\alpha}{2}\right)^{\frac{\alpha}{2}} \frac{2r^{2-\alpha}(1+P(y))^\alpha}{s(y)^{1+\frac{\alpha}{2}}P(y)}$$

with the obvious meaning of $P(y)$ and

$$s(y) = \alpha - 2|\rho|^2|y|^2 + \sqrt{\alpha^2 - 4(\alpha-1)|\rho|^2|y|^2} > 0,$$

since for $y < 1/|\rho|$ we have

$$s(y) \ge \alpha - 2 + \sqrt{\alpha^2 - 4(\alpha - 1)} = \alpha - 2 + |\alpha - 2| = 0.$$

These formulas can be quickly checked using Maple or Mathematica. Thus $\psi''(y) \le 0$. It is easy to verify that $\psi'(0) = 0$, so $\psi'(y) \le 0$ as required. □

Corollary 5.3. *When $t + |x| \geqslant 1$,*

$$p_t(x) \asymp \begin{cases} \phi_0(x)\, t^{\frac{1}{2-\alpha}}\, (|x|+t)^{-r(\alpha)+\frac{1}{2}}\, e^{\sqrt{|x|^2+t^2}\, q_\alpha} & \text{if } u_0 \leqslant t^{2/\alpha}, \\ \phi_0(x)\, t\, (|x|+t)^{-s(\alpha)+\frac{1}{2}}\, e^{\sqrt{|x|^2+t^2}\, q_\alpha} & \text{if } u_0 \geqslant t^{2/\alpha}, \end{cases}$$

where the function q_α satisfies

$$q_\alpha(k \exp Hk', t) = q_\alpha\left(\frac{(H,t)}{\|(H,t)\|_2}\right), \quad k, k' \in K,$$

and is a continuous, bounded and bounded away from zero function on the Euclidean unit sphere in $\mathfrak{a} \times \mathbb{R}$.

Proof. Set $q_\alpha(\exp H, t) = f(u_0(\exp H, t))/\sqrt{|H|^2 + t^2}$. By Proposition 5.3 the function $(H,t) \to q_\alpha(\exp H, t)$ is homogeneous of degree 0. □

Remark. When $\alpha = 1$ the function q_α is constant. The bounds from the Corollary 5.3 are an extension of the bounds obtained in [**AJ**] for $\alpha = 1$.

Lemma 4.2 gives enough information to replace u_0 in an estimate when it is a multiplicative factor. To deal with the exponent $e^{-f(u_0)}$ in the estimates of $p_t(x)$ more delicate (additive) properties are required. Namely,

$$\exp(f(x)) \asymp \exp(g(x)) \quad \Longleftrightarrow \quad |f-g| \text{ is bounded.}$$

This motivates the following definition: we write

$$f(x) \overset{e}{\asymp} g(x)$$

if there exists $M \in \mathbb{R}$ such that $|f(x) - g(x)| \leq M$ for all x.

Proposition 5.4. *If $t \leqslant |x|^{\alpha/2}$ then*

$$\exp(-f(u_0)) \asymp \exp(-|\rho||x|). \tag{23}$$

In particular, this holds for fixed $t > 0$, as $x \to \infty$ (that is, for all sufficiently large $x > x_0(t)$). Moreover, if $t = |x|^s$ with $s > \alpha/2$ then (23) does not hold.

Proof. By our assumption, $t^{\frac{2}{2-\alpha}} \leq |x|^{\frac{\alpha}{2-\alpha}}$ and

$$t^{\frac{2}{2-\alpha}} u_0^{-\frac{\alpha}{2-\alpha}} \leq \left(\frac{|x|}{u_0}\right)^{\frac{\alpha}{2-\alpha}}. \tag{24}$$

The right-hand side of this inequality is bounded since $u_0 \asymp |x| + t$. Thus, Equation (14) multiplied by u_0 gives

$$f(u_0) = \frac{|x|^2}{4u_0} + |\rho|^2 u_0 + c_1 t^{\frac{2}{2-\alpha}} u_0^{-\frac{\alpha}{2-\alpha}} \overset{e}{\asymp} 2|\rho|^2 u_0. \tag{25}$$

Using (13) and (24) we deduce that $u_0 \overset{e}{\asymp} \frac{|x|}{2|\rho|}$, which together with (25) gives the first assertion.

By the definition of $f(u_0)$ for $t = |x|^s$ and by (13) we obtain

$$f(u_0) - |\rho||x| = (c_1 + c_2)\left(\frac{|x|}{u_0}\right)^{\frac{\alpha}{2-\alpha}} |x|^{\frac{2s-\alpha}{2-\alpha}} - |x|\left(|\rho| - \frac{|x|}{2u_0}\right). \tag{26}$$

By (14)

$$\left(|\rho| - \frac{|x|}{2u_0}\right)\left(|\rho| + \frac{|x|}{2u_0}\right) = c_2\left(\frac{t}{u_0}\right)^{\frac{2}{2-\alpha}} \tag{27}$$

so, transforming (26) with $t = |x|^s$ we get

$$\begin{aligned} &f(u_0) - |\rho||x| \\ &\quad = |x|^{\frac{2s-\alpha}{2-\alpha}}\left((c_1 + c_2)\left(\frac{|x|}{u_0}\right)^{\frac{\alpha}{2-\alpha}} - c_2\left(\frac{|x|}{u_0}\right)^{\frac{2}{2-\alpha}}\left(|\rho| + \frac{|x|}{2u_0}\right)^{-1}\right). \end{aligned} \tag{28}$$

Note that for $s \leq \alpha/2$ this is bounded (for $s < \alpha/2$ it even tends to 0), since $u_0 \asymp |x| + t$. (This gives once again the first assertion.) From (14) with $|x| = t^{1/s}$ we deduce that $\frac{t}{u_0} \to 0$. Observe that it is enough to show the second assertion for $s \in (\alpha/2, 1)$.

Now, from (14) it follows that

$$\frac{|x|}{2u_0} \to |\rho|. \tag{29}$$

Together with (28) it implies that for $s > \alpha/2$ we have $f(u_0) - |\rho||x| \to \infty$, which completes the proof. □

Proposition 5.5. *If* $|x| \leqslant \sqrt{t}$ *then* $\exp(-f(u_0)) \asymp \exp(-|\rho|^\alpha t)$. *This is true, in particular, when* x *is fixed and* $t \to \infty$ *(that is, for all sufficiently large* $t > t_0(x)$*). If* $|x| = t^s$ *with* $s > \frac{1}{2}$*, it is no longer true.*

Proof. Multiplying (14) by u_0 and using

$$\frac{|x|^2}{u_0} \leq \frac{t}{u_0}, \tag{30}$$

which is bounded since $u_0 \asymp |x| + t$, we get

$$f(u_0) \overset{e}{\asymp} |\rho|^2 u_0 + \frac{2-\alpha}{\alpha}|\rho|^2 u_0 = \frac{2}{\alpha}|\rho|^2 u_0. \tag{31}$$

We claim that

$$\frac{2}{\alpha}|\rho|^2 u_0 \overset{e}{\asymp} |\rho|^\alpha t. \tag{32}$$

We will use the following simple consequence of the Mean Value Theorem: for $0 < a < b$ and any γ we have

$$b^\gamma - a^\gamma = \gamma K^{\gamma-1}(b-a), \tag{33}$$

for some $K \in (a, b)$. From (14) multiplied by $u_0^{2/(2-\alpha)}$ we have

$$|\rho|^2 u_0^{\frac{2}{2-\alpha}} - c_2 t^{\frac{2}{2-\alpha}} = \frac{|x|^2}{4} u_0^{-2+\frac{2}{2-\alpha}}.$$

From this and (33) applied to $a = c_2 t^{\frac{2}{2-\alpha}}$, $b = |\rho|^2 u_0^{\frac{2}{2-\alpha}}$ and $\gamma = \frac{2-\alpha}{2}$ we get

$$|\rho|^{2-\alpha} u_0 - c_2^{\frac{2-\alpha}{2}} t = \frac{2-\alpha}{2} K^{-\frac{\alpha}{2}} \frac{|x|^2}{4} u_0^{-2+\frac{2}{2-\alpha}},$$

with some $K \in (a, b)$. Consequently, as $K > a$,

$$|\rho|^{2-\alpha} u_0 - c_2^{\frac{2-\alpha}{2}} t \leq c \left(\frac{u_0}{t}\right)^{\frac{\alpha}{2-\alpha}} \frac{|x|^2}{u_0},$$

which is bounded by virtue of (30) and the fact that $|x|^2 \leq t$ yields $u_0 \asymp |x| + t \asymp t$. It follows that

$$|\rho|^{2-\alpha} u_0 \overset{e}{\asymp} c_2^{\frac{2-\alpha}{2}} t. \tag{34}$$

To get our claim, multiply (34) by $\frac{2|\rho|^\alpha}{\alpha}$ and use the fact that $\frac{2}{\alpha} c_2^{\frac{2-\alpha}{2}} = 1$. The first assertion now follows from (31) and (32).

In order to prove the second assertion, we show in a way similar to the proof of the second assertion of the Proposition 5.4 (but using the Mean Value Theorem as above, for $\gamma = \alpha/2$) that the formula

$$f(u_0) - |\rho|^\alpha t = \frac{1}{2} t^{2s-1} \left(\frac{t}{u_0} - \frac{\alpha}{4} \left(\frac{t}{u_0} \right)^2 K^{\frac{\alpha-2}{2}} \right) \tag{35}$$

holds when $|x| = t^s$, with

$$c_2 \left(\frac{t}{u_0} \right)^{\frac{2}{2-\alpha}} < K < |\rho|^2. \tag{36}$$

The case $s > 1$ being evident by (14), we see that for $s < 1$, again using (14), we have $\frac{|x|}{u_0} \to 0$ and

$$\frac{t}{u_0} \to c_2^{\frac{\alpha-2}{2}} |\rho|^{2-\alpha}$$

when $|x| = t^s$.

Taking limits in (36) we get $K \to |\rho|^2$, so $K^{\frac{\alpha-2}{2}} \to |\rho|^{\alpha-2}$. Thus the expression between the brackets in (35) does not tend to 0. It follows that for $s > 1/2$ the difference $f(u_0) - |\rho|^\alpha t$ is unbounded when $t \to \infty$ and $|x| = t^s$. □

Corollary 5.6. *For any constants $K, M > 0$ we have*

$$p_t(x) \asymp \begin{cases} \phi_0(x) t^{\frac{1}{2-\alpha}} (|x| + t)^{-r(\alpha)+\frac{1}{2}} e^{-|\rho|^\alpha t} & \text{if } t + |x| \geqslant 1 \text{ and } |x| \leqslant K t^{1/2}, \\ \phi_0(x) t (|x| + t)^{-s(\alpha)+\frac{1}{2}} e^{-|\rho||x|} & \text{if } t + |x| \geqslant 1 \text{ and } |x| \geqslant M t^{2/\alpha}. \end{cases}$$

On the other hand, in a region $K t^{1/2} < |x| < M t^{2/\alpha}$ one cannot give a simpler estimate of $e^{-f(u_0)}$ and, consequently, a simpler estimate of $p_t(x)$ than the one given in the Theorem 4.3.

Proof. Only the last assertion needs to be justified. Suppose that $g(|x|, t)$ is a 1-homogeneous function such that $e^{-f(u_0)} \asymp e^{-g}$ on

$$R = \{(|x|, t) : K t^{1/2} < |x| < M t^{2/\alpha}\}.$$

It implies that $f(u_0) \overset{e}{\asymp} g$.

Observe that the region R contains half-lines of the form $|x| = at$, where $t > t_0(a)$, $a > 0$. The function $f(u_0) - g$ is homogeneous of degree 1 and bounded on R, so $f(u_0) = g$ on R.

Suppose that the function $f(u_0)$ is explicitly determined. Writing (11) in the form

$$f(u_0) = \frac{|x|^2}{4u_0} + |\rho|^2 u_0 + c_1 u_0 \left(\frac{t}{u_0} \right)^{\frac{2}{2-\alpha}}$$

and replacing $(\frac{t}{u_0})^{\frac{2}{2-\alpha}}$ by

$$c_2^{-1}\left(|\rho|^2 - \frac{1}{4}\left(\frac{|x|}{u_0}\right)^2\right)$$

we get according to (14) a quadratic equation with respect to u_0 that could be solved explicitly. Except for some special values of $\alpha = 1, \frac{3}{2}, \ldots,$ this is not possible. □

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