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CALABI–YAU THREEFOLDS WITH PICARD NUMBER $\rho(X) = 2$ AND THEIR KÄHLER CONE II

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We prove the rationality of the Kähler cone and the positivity of the Chern class $c_2(X)$ if X is a Calabi–Yau threefold with Picard number $\rho(X) = 2$ and has an embedding into a $\mathbb{P}^n$-bundle over $\mathbb{P}^m$ in the cases $(n, m) = (1, 3)$ and $(3, 1)$. The case $(n, m) = (2, 2)$ has been done in the first part of this paper. Moreover, if $(n, m) = (3, 1)$, we describe the 'other' contraction different from the projection.

1. Introduction

In this paper, a Calabi–Yau threefold is a compact complex Kähler manifold of dimension three with $K_X = \mathcal{O}_X$ and $H^1(\mathcal{O}_X) = 0$.

Wilson stated in 1994 [**Wi94b**] a conjecture about the rationality of the Kähler cone of a Calabi–Yau threefold. It says that the Kähler cone of a Calabi–Yau threefold X is rational and finitely generated in $N^1(X)$ if the Chern class $c_2(X)$ is positive, i.e., $D.c_2(X) > 0$ for every nef divisor D.

In [**Kü01b**] we dealt with the case $\rho(X) = 2$. We proved some general results about the Kähler cone and then concentrated on the case that X is embedded in a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$. For this class of Calabi–Yau manifolds we confirmed Wilson's conjecture.

We now finish this track by considering X embedded in either a $\mathbb{P}^1$-bundle over $\mathbb{P}^3$ or a $\mathbb{P}^3$-bundle over $\mathbb{P}^1$. The first case offers some interesting perspectives. The Calabi–Yau manifolds turn out to be generic double covers of $\mathbb{P}^3$ ramified over an octic. We compute the number of fibres of the bundle projection in X and describe the Kähler cone. The ramifying octics are discussed in greater detail in [**Kü02**].

Since $C = \mathbb{P}^1$ if $X \longrightarrow C$ is a fibration onto a normal curve C and X a Calabi–Yau manifold, it is also natural to turn our attention to those X that can be embedded in a $\mathbb{P}^3$-bundle over $\mathbb{P}^1$.

We will denote $K(X)$ for the Kähler cone of X. The main results are:

Theorem 1.1. *Let $X \subset \mathbb{P}(\mathcal{E})$ be a Calabi–Yau 3-fold with $\rho(X) = 2$, with $\mathcal{E}$ either a rank-2-bundle over $\mathbb{P}^3$ such that $h^0(-K_{\mathbb{P}(\mathcal{E})}) > 1$ or an arbitrary rank-4-bundle over $\mathbb{P}^1$. Then $\partial K(X)$ is rational and semiample. Furthermore all $D \in \overline{K(X)}$ satisfy $D.c_2(X) > 0$.*

In the case that $\mathcal{E}$ is a rank-4-bundle over $\mathbb{P}^1$ we can give a complete classification of the occurring contractions.

Theorem 1.2. *Let $X \subset \mathbb{P}(\mathcal{E}) := Z$ be a Calabi–Yau 3-fold with $\rho(X) = 2$, where $\mathcal{E} \longrightarrow \mathbb{P}^1$ is of rank 4. Fix $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \mathcal{O}(a_3)$, with $0 \leq a_1 \leq a_2 \leq a_3$, and let $\psi : X \longrightarrow X'$ be the second contraction. If E denotes the exceptional locus of ψ and $\mathcal{F} := \mathcal{O} \bigoplus_{i|a_i=0} \mathcal{O}(a_i)$ the maximal trivial subbundle of $\mathcal{E}$, then:*

(i) *If $c_1(\mathcal{E}) = 3$, then $\operatorname{rk}\mathcal{F} \leq 2$ and $E = \mathbb{P}(\mathcal{F}) \cong \mathbb{P}^1 \times \mathbb{P}^{\operatorname{rk}\mathcal{F}-1}$.*

(ii) *If $c_1(\mathcal{E}) = 2$, then $\operatorname{rk}\mathcal{F} \in \{2,3\}$ and $E = X \cap \mathbb{P}(\mathcal{F}) = \mathbb{P}^1 \times Y$, with $\dim Y = \operatorname{rk}\mathcal{F} - 2$.*
  - (a) *If $\operatorname{rk}\mathcal{F} = 2$, then Y consists of four points.*
  - (b) *If $\operatorname{rk}\mathcal{F} = 3$, then Y is a smooth plane quartic.*

(iii) *If $c_1(\mathcal{E}) = 1$, then Z is the blow-up of $\mathbb{P}^4$ in a linearly embedded $\mathbb{P}^2$; if $X \in |-K_Z|$ is general, then $E = \bigcup_{i=1}^{16} C_i$, with $C_i \cong \mathbb{P}^1$; furthermore, X' is a quintic in $\mathbb{P}^4$ with 16 double points on a linearly embedded $\mathbb{P}^2$.*

(iv) *If $c_1(\mathcal{E}) = 0$, then $Z = \mathbb{P}^1 \times \mathbb{P}^3$ and $E = \bigcup_{i=1}^{64} C_i$, with $C_i \cong \mathbb{P}^1$.*

For the proofs of these results we proceed as in [**Kü01b**] and prove first a generalization of a lemma of Kollár [**Bo89**]:

Theorem 1.3. *Let $X \subset \mathbb{P}(\mathcal{E})$ be a Calabi–Yau threefold with $\rho(X) = 2$, with $\mathcal{E}$ either a rank-2-bundle over $\mathbb{P}^3$ or a rank-4-bundle over $\mathbb{P}^1$. Then $K(X) = K(Z)|X$.*

In contrast to the generalized Kollár Lemma in [**Kü01b**], the situation here is simpler, since there are no exceptions in the theorem above. However, in the case that $\mathcal{E}$ is a rank-2-bundle over $\mathbb{P}^3$ it has to be investigated whether π^*h is not ample. It turns out that this is true if $\rho(X) > 1$. So the proofs of the rationality result and the positivity of $c_2(X)$ become much shorter than those in [**Kü01b**]. In the case of $\mathcal{E}$ a rank-4-bundle over $\mathbb{P}^1$, the rationality of the Kähler cone can even be proved without using the generalized Kollár Lemma.

This article grew out of the author's doctoral thesis at the University of Bayreuth.

2. Notation

In this section we summarize the most important notations of this paper. X will always denote a Calabi–Yau threefold, and Z always a fourfold.

$N^1(X)$: the $\mathbb{R}$-vector space of numerical classes of $\operatorname{Div}(X) \otimes \mathbb{R}$.

$K(X)$: the Kähler cone of X, i.e., the ample cone.

$W(X)$: the hypersurface $\{D^3 = 0\} \subset N^1(X)$.

$\mathcal{O}_X(1)$: the restriction $\mathcal{O}_Z(1)|X$, where $\mathcal{O}_Z(1)$ is the tautological bundle associated to $Z = \mathbb{P}(\mathcal{E})$.

K_Z: the canonical divisor of Z.

$c_i(\mathcal{E})$: the i-th Chern class of a bundle $\mathcal{E}$.

$c_i(M)$: the i-th Chern class of the tangent bundle of the complex manifold M.

3. Some general statements

For a more detailed description of properties of the Kähler cone of a Calabi–Yau threefold with $\rho(X) = 2$, see [**Kü01a**, **Kü01b**]. Here we mention only results necessary in this article.

The first fundamental theorem we want to cite in a specialized form is proved by Wilson in [**Wi94a**].

Theorem 3.1 (Wilson)**.** *Let X be a Calabi–Yau threefold and $\rho(X) = 2$. If $D \in \partial K(X)$ and $D^3 > 0$, then there is some $r \in \mathbb{R}$ with $rD \in \mathrm{Pic}\,(X)$.*

Hence it is natural to consider the cubic hypersurface

$$W(X) := \{D \in N^1(X) \mid D^3 = 0\}.$$

A useful statement is:

Lemma 3.2. *Let X be a Calabi–Yau threefold with $\rho(X) = 2$. If $W(X)$ contains a double line, then $W(X)$ is rational.*

Proof. $W(X)$ is in an appropriate affine neighbourhood of $\mathbb{P}(N^1(X)) \cong \mathbb{P}^1$ given by some cubic polynomial $w \in \mathbb{Z}[x]$. Let $Dw \in \mathbb{Q}[x]$ be the formal derivative of w. Then, if $w = (x-a)^2(x-b)$ and $a \neq b$,

$$(x-a) = \gcd(w, Dw) \in \mathbb{Q}[x].$$

If $a = b$, then

$$(x-a)^2 = \gcd(w, Dw) \in \mathbb{Q}[x].$$

In both cases, $a, b \in \mathbb{Q}$ follows. □

Putting both results together, we get in particular:

Corollary 3.3. *If X is a Calabi–Yau manifold with $\rho(X) = 2$ and $\phi : X \longrightarrow \mathbb{P}^1$ a fibration, then $\partial K(X)$ is rational.*

Finally, we need a lemma of Kollár, proved in [**Bo89**].

Lemma 3.4 (Kollár)**.** *If Z is a Fano 4-fold and $X \in |-K_Z|$ a Calabi–Yau manifold, then $i^* : N^1(Z) \longrightarrow N^1(X)$ is an isomorphism and $K(X) = K(Z)|X$.*

4. Calabi–Yau threefolds in $\mathbb{P}^1$-bundles over $\mathbb{P}^3$

We are interested in Calabi–Yau threefolds X of the form $X \subset \mathbb{P}(\mathcal{E}) =: Z$, $X \in |-K_{\mathbb{P}(\mathcal{E})}|$, with $\mathcal{E}$ a vector bundle of rank 2 over $\mathbb{P}^3$. Let $p : \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}^3$ be the bundle projection and $\pi : X \longrightarrow \mathbb{P}^3$ the restriction of p to X. The hyperplane class in $\mathbb{P}^2$ shall be denoted by h, the fibre of p by F. The expression $\gamma(\mathcal{E}) := (c_1^2(\mathcal{E}) - 4c_2(\mathcal{E})).h$ is invariant under $\mathcal{E} \mapsto \mathcal{E} \otimes L$, where L is a line bundle over $\mathbb{P}^2$. The line bundle $\mathcal{O}_Z(1)|X$ will be called $\mathcal{O}_X(1)$.

The following sequences are basic for our proofs and results:

$$0 \longrightarrow T_{Z|\mathbb{P}^3} \longrightarrow T_Z \longrightarrow p^*T_{\mathbb{P}^3} \longrightarrow 0, \tag{1}$$

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow p^*(\mathcal{E}^\vee) \otimes \mathcal{O}_Z(1) \longrightarrow T_{Z|\mathbb{P}^3} \longrightarrow 0, \tag{2}$$

$$0 \longrightarrow T_X \longrightarrow T_Z|X \longrightarrow N_{X|Z} \longrightarrow 0, \tag{3}$$

$$\mathcal{O}_Z(1)^2 - p^*c_1(\mathcal{E}).\mathcal{O}_Z(1) + p^*c_2(\mathcal{E}) = 0. \tag{4}$$

By the Künneth formula we get $b_1(Z)=0$, $b_2(Z)=2$, $b_3(Z)=0$, $b_4(Z)=2$.

The intersection theory on Z is computed inductively by $\mathcal{O}_Z(1).p^*h^3 = 1$ and Equation (4):

Lemma 4.1. *Let $\mathcal{E} \longrightarrow \mathbb{P}^3$ be a rank-2-bundle and $Z := \mathbb{P}(\mathcal{E})$. Then:*

(i) $\mathcal{O}_Z(1).p^*h^3 = 1$.
(ii) $\mathcal{O}_Z(1)^2.p^*h^2 = c_1(\mathcal{E}).h^2$.
(iii) $\mathcal{O}_Z(1)^3.p^*h = c_1^2(\mathcal{E}).h - c_2(\mathcal{E}).h$.
(iv) $\mathcal{O}_Z(1)^4 = c_1^3(\mathcal{E}) - 2c_1(\mathcal{E})c_2(\mathcal{E})$.

4.1. Intersection product and Picard number. By standard computations we get:

Lemma 4.2. *Let $\mathcal{E} \longrightarrow \mathbb{P}^3$ be a rank-2-bundle and $X \subset \mathbb{P}(\mathcal{E})$ a Calabi–Yau threefold. Then:*

(i) $c_3(X) = -8\gamma - 168$.
(ii) $\pi^*h.c_2(X) = 44$.
(iii) $\mathcal{O}_Z(1)|X.c_2(X) = 4\gamma + 22c_1(\mathcal{E}) + 24$.
(iv) $-K_Z|X.c_2(X) = 8\gamma + 224$.
(v) $\pi^*h^3 = 2$.
(vi) $\mathcal{O}_Z(1)|X.\pi^*h^2 = c_1(\mathcal{E}).h^2 + 4$.
(vii) $\mathcal{O}_Z(1)^2|X.\pi^*h = \frac{1}{2}\gamma + \frac{1}{2}c_1^2(\mathcal{E}).h + 4c_1(\mathcal{E}).h^2$.
(viii) $\mathcal{O}_Z(1)^3|X = \gamma + \frac{3}{4}\gamma c_1(\mathcal{E}).h^2 + 3c_1^2(\mathcal{E}).h + \frac{1}{4}c_1^3(\mathcal{E})$.

Compare the following result about the Picard number to the corresponding theorem in [**Kü01b**].

Theorem 4.3. *Let $X \subset \mathbb{P}(\mathcal{E})$ be a Calabi–Yau manifold, with $\mathcal{E} \longrightarrow \mathbb{P}^3$ a rank-2-bundle. If $\mathcal{E}$ is stable and $H^1(-K_Z) = H^2(-K_Z) = 0$ (for example, if $-K_Z$ is big and nef), then*

$$\rho(X) = 2 + h^2(\mathcal{E}^\vee \otimes \mathcal{E}).$$

Proof. We look at the sequences

$$0 \longrightarrow N^{\vee}_{X|Z} \longrightarrow \Omega_Z|X \longrightarrow \Omega_X \longrightarrow 0 \tag{5}$$

and

$$0 \longrightarrow \Omega_Z \otimes K_Z \longrightarrow \Omega_Z \longrightarrow \Omega_Z|X \longrightarrow 0. \tag{6}$$

First, we want to show $H^i(T_Z) = H^i(\mathcal{E}^{\vee} \otimes \mathcal{E})$ for $i > 1$. For this purpose we compute $R^i p_*(p^*(\mathcal{E}^{\vee}) \otimes \mathcal{O}_Z(1)) = \mathcal{E}^{\vee} \otimes R^i p_* \mathcal{O}_Z(1) = 0$ for $i > 0$. Hence the Leray spectral sequence implies

$$H^i(p^*(\mathcal{E}^{\vee}) \otimes \mathcal{O}_Z(1)) = H^i(\mathcal{E}^{\vee} \otimes \mathcal{E}).$$

Sequence (2) shows that $H^i(p^*(\mathcal{E}^{\vee}) \otimes \mathcal{O}_Z(1)) = H^i(T_{Z|\mathbb{P}^3})$ for $i > 0$, since $R^i p_* \mathcal{O}_Z = 0$ for $i > 0$ and therefore $H^i(\mathcal{O}_Z) = H^i(\mathcal{O}_{\mathbb{P}^3}) = 0$ for $i > 0$. To apply sequence (1), we compute $R^i p_* p^* T_{\mathbb{P}^3} = T_{\mathbb{P}^3} \otimes R^i p_* \mathcal{O}_Z = 0$ for $i > 0$. Therefore we see again by the Leray spectral sequence that

$$H^i(p^* T_{\mathbb{P}^3}) = H^i(T_{\mathbb{P}^3}) = 0$$

for $i > 0$. This implies by sequence (1) that

$$H^i(T_{Z|\mathbb{P}^3}) = H^i(T_Z)$$

for $i > 1$. Hence

$$H^i(T_Z) = H^i(\mathcal{E}^{\vee} \otimes \mathcal{E})$$

for $i > 1$.

We now know

$$H^i(\Omega_Z \otimes K_Z) = H^{4-i}(T_Z)^{\vee} = H^{4-i}(\mathcal{E}^{\vee} \otimes \mathcal{E})^{\vee}$$

for $i < 3$. In particular,

$$H^1(\Omega_Z \otimes K_Z) = H^3(\mathcal{E}^{\vee} \otimes \mathcal{E})^{\vee} = H^0(\mathcal{E}^{\vee} \otimes \mathcal{E} \otimes \mathcal{O}(-4)) = 0,$$

the last equality holding since $\mathcal{E}$ is stable and therefore simple. Since $b_3(Z)$ vanishes, so does $H^{2,1}(Z)$. Since $N^{\vee}_{X|Z} = K_Z|X$ the cohomology sequences of (6) and (5) contain

$$0 \longrightarrow H^1(\Omega_Z) \longrightarrow H^1(\Omega_Z|X) \longrightarrow H^2(\mathcal{E}^{\vee} \otimes \mathcal{E})^{\vee} \longrightarrow 0 \tag{7}$$

and

$$0 \longrightarrow H^1(K_Z|X) \longrightarrow H^1(\Omega_Z|X) \longrightarrow H^1(\Omega_X) \longrightarrow H^2(K_Z|X), \tag{8}$$

respectively.

By the assumption $H^1(-K_Z) = H^2(-K_Z) = 0$ we conclude from

$$0 \longrightarrow 2K_Z \longrightarrow K_Z \longrightarrow K_Z|X \longrightarrow 0$$

and from

$$H^1(K_Z) = H^3(\mathcal{O}_Z)^\vee = 0,$$
$$H^2(K_Z) = H^2(\mathcal{O}_Z)^\vee = 0,$$
$$H^2(2K_Z) = H^2(-K_Z)^\vee = 0,$$
$$H^3(2K_Z) = H^1(-K_Z)^\vee = 0$$

that

$$H^1(K_Z|X) = H^2(K_Z|X) = 0.$$

Therefore by (8) and (7)

$$\rho(X) = h^1(\Omega_Z|X) = \rho(Z) + h^2(\mathcal{E}^\vee \otimes \mathcal{E}).$$ □

Example 4.4 (A Calabi–Yau manifold with $\rho(X) = 1$)**.** Let $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(4)$. We prove as in Theorem 4.3 that

$$H^i(\Omega_Z \otimes K_Z) = H^{4-i}(T_Z)^\vee = H^{4-i}(\mathcal{E}^\vee \otimes \mathcal{E})^\vee.$$

But now

$$H^3(\mathcal{E}^\vee \otimes \mathcal{E})^\vee = H^0(\mathcal{E}^\vee \otimes \mathcal{E} \otimes \mathcal{O}(-4)) = H^0(\mathcal{O}(-8) \oplus 2\mathcal{O}(-4) \oplus \mathcal{O}) = \mathbb{C}.$$

The cohomology sequence of (5) starts

$$0 \longrightarrow H^0(N^\vee_{X|Z}) \longrightarrow H^0(\Omega_Z|X) \longrightarrow H^0(\Omega_X).$$

Since $H^0(\Omega_X) = 0$ and $H^0(N^\vee_{X|Z}) = 0$, we see that

$$H^0(\Omega_Z|X) = 0.$$

This we use in the cohomology sequence of (6) and get the sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow H^1(\Omega_Z) \longrightarrow H^1(\Omega_Z|X) \longrightarrow H^2(\mathcal{E}^\vee \otimes \mathcal{E})^\vee \longrightarrow 0.$$

We know $H^2(\mathcal{E}^\vee \otimes \mathcal{E})^\vee = 0$. Since $-K_Z = \mathcal{O}_Z(2)$ is big and nef, $H^i(N^\vee_{X|Z}) = H^i(K_Z|X) = 0$ for $i = 1, 2$. By using the cohomology sequence of (5) we get

$$H^1(\Omega_X) = H^1(\Omega_Z|X).$$

This finally implies

$$\rho(X) = 1.$$

As a last subject in this section, we are interested in some bounds for γ. This yields a total lower bound for $c_3(X)$ for the Calabi–Yau threefolds considered here, and, what is more important within this framework, it allows us to compute the number of full fibres of p contained in X. That will be done in the next section.

Lemma 4.5. *Let $\mathcal{E} \longrightarrow \mathbb{P}^3$ be a rank-2-bundle. Denote its generic splitting type by (a, b). Then*

$$\gamma(\mathcal{E}) \le \gamma(\mathcal{O}(a) \oplus \mathcal{O}(b)) = (a-b)^2.$$

Moreover, equality holds if and only if $\mathcal{E} = \mathcal{O}(a) \oplus \mathcal{O}(b)$.

Proof. Set $\mathcal{E}' := \mathcal{O}(a) \oplus \mathcal{O}(b)$. By tensoring with $\mathcal{O}(m)$ for $m \gg 0$, we may assume that $H^i(\mathcal{E}) = 0$ for $i > 0$. For a general hyperplane we $H \subset \mathbb{P}^3$ we look at the sequence

$$0 \longrightarrow \mathcal{E}(-1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}|H \longrightarrow 0.$$

We see that

$$h^0(\mathcal{E}) \le h^0(\mathcal{E}(-1)) + h^0(\mathcal{E}|H).$$

The same argument shows inductively that

$$h^0(\mathcal{E}) \le h^0(\mathcal{E}(-k)) + \sum_{i=0}^{k-1} h^0(\mathcal{E}(-i)|H)$$

for all k, and hence

$$h^0(\mathcal{E}) \le \sum_{i \ge 0} h^0(\mathcal{E}(-i)|H).$$

We choose a general line $L \subset H$ and conclude, by replacing $\mathbb{P}^3$ by H, that

$$h^0(\mathcal{E}(-i)|H) \le \sum_{j \ge 0} h^0(\mathcal{E}(-i-j)|L).$$

Therefore

$$\begin{aligned} h^0(\mathcal{E}) &\le \sum_{0 \le i,j} h^0(\mathcal{E}(-i-j)|L) \\ &= \sum_{0 \le j \le a, i+j \le a} a+1-i-j + \sum_{0 \le j \le b, i+j \le b} b+1-i-j. \\ &= \sum_{j=0}^{a} \binom{a+2-j}{2} + \sum_{j=0}^{b} \binom{b+2-j}{2} \\ &= \sum_{j=2}^{a+2} \binom{j}{2} + \sum_{j=2}^{b+2} \binom{j}{2} = \binom{a+3}{3} + \binom{b+3}{3} = h^0(\mathcal{E}'). \end{aligned}$$

By assumption we have

$$\chi(\mathcal{E}) = h^0(\mathcal{E}) \le h^0(\mathcal{E}') = \chi(\mathcal{E}').$$

Using Riemann–Roch, this inequality transforms to

$$\begin{aligned} &\tfrac{1}{8}\gamma(c_1(\mathcal{E}).h^2 + 4) + \tfrac{1}{24}c_1^3(\mathcal{E}) + \tfrac{1}{2}c_1^2(\mathcal{E}).h + \tfrac{11}{6}c_1(\mathcal{E}).h^2 + 2 \\ &\qquad \le \tfrac{1}{8}\gamma(c_1(\mathcal{E}').h^2 + 4) + \tfrac{1}{24}c_1^3(\mathcal{E}') + \tfrac{1}{2}c_1^2(\mathcal{E}').h + \tfrac{11}{6}c_1(\mathcal{E}').h^2 + 2. \end{aligned}$$

Since by assumption $c_1(\mathcal{E}) = c_1(\mathcal{E}')$ and $c_1(\mathcal{E}).h^2 > 0$, this implies

$$\gamma(\mathcal{E}) \leq \gamma(\mathcal{E}').$$

Equality holds if and only if all connecting homomorphisms

$$H^0(\mathcal{E}(-i)|L) \longrightarrow H^1(\mathcal{E}(-i-1)|H)$$

are the zero map, what means that

$$h^1(\mathcal{E}(-i-1)|H) \leq h^1(\mathcal{E}(-i)|H)$$

for all $i \geq 0$. This implies that $H^1(\mathcal{E}(n)|H) = 0$ for all $n \in \mathbb{Z}$, by the choice of m. We conclude, using Horrocks' splitting criterion, that

$$\mathcal{E}|H = \mathcal{O}_H(a) \oplus \mathcal{O}_H(b).$$

By [**OSS88**, p. 42] we know that $\mathcal{E}$ splits if and only if $\mathcal{E}|H$ splits for some hyperplane H. Hence

$$\mathcal{E} = \mathcal{O}(a) \oplus \mathcal{O}(b). \qquad \square$$

Theorem 4.6. *Let $X \subset \mathbb{P}(\mathcal{E})$ be a Calabi–Yau manifold, with $\mathcal{E} \longrightarrow \mathbb{P}^3$ a rank-2-bundle. For the generic splitting type (a, b) of $\mathcal{E}$ (where $a \leq b$) we have $b - a \leq 4$ and hence*

$$\gamma(\mathcal{E}) \leq 16,$$

or equivalently,

$$c_3(X) \geq -296.$$

Proof. Let $X = \{s = 0\}$, where $s \in H^0(-K_{\mathbb{P}(\mathcal{E})})$, $L \subset \mathbb{P}^3$ is a general line and

$$\mathcal{E}|L = \mathcal{O}(a) \oplus \mathcal{O}(b),$$

with $a \leq b$. Then s induces a section

$$t = p_* s \in H^0(S^2\mathcal{E} \otimes \det \mathcal{E}^{-1} \otimes \mathcal{O}(4)).$$

We have

$$S^2\mathcal{E} \otimes \det \mathcal{E}^{-1} \otimes \mathcal{O}(4)|L \cong \mathcal{O}(a - b + 4) \oplus \mathcal{O}(4) \oplus \mathcal{O}(b - a + 4)$$

and by the general choice of L the intersection $X \cap p^*L$ can be assumed to be smooth. Let $d_{00} := a - b + 4$, $d_{01} := 4$, $d_{11} := b - a + 4$. We denote by $[x_0 : x_1]$ the coordinates of the fibres of p in a trivializing neighbourhood $p^{-1}(U)$, with $U \subset \mathbb{P}^3$. In this neighbourhood we can express $s|p^*L$ as

$$s|p^*L = \sum s_{ij} x_i x_j,$$

with $s_{ij} \in H^0(\mathcal{O}(d_{ij}))|U$. If $b - a > 4$, then $H^0(\mathcal{O}(d_{00})) = 0$ and hence $s_{00} = 0$. Therefore

$$s|p^*L = x_1(s_{01}x_0 + s_{11}x_1)$$

is reducible and so $X \cap p^*L$ is singular along

$$S := \{x_1 = 0\} \cap \{s_{01}x_0 + s_{11}x_1 = 0\} = \{x_1 = 0\} \cap \{s_{01} = 0\}.$$

Since $s_{01} \in H^0(\mathcal{O}(4))$, we conclude $S \neq \emptyset$. Hence this case does not occur and therefore $b - a \leq 4$.

By applying Lemma 4.5 we conclude finally that

$$\gamma(\mathcal{E}) \leq \gamma(\mathcal{O} \oplus \mathcal{O}(b-a)) \leq \gamma(\mathcal{O} \oplus \mathcal{O}(4)) = 16. \qquad \square$$

4.2. The discriminant map. To be able to compute $K(X)$ we need information about the morphism $\pi : X \longrightarrow \mathbb{P}^3$, which is the natural projection.

Construction 4.7. Let $X = \{s = 0\}$, with $s \in H^0(-K_{\mathbb{P}(\mathcal{E})})$ and

$$V := \{p \in \mathbb{P}^3 \mid \pi \text{ is locally in } p \text{ not an étale covering}\}.$$

We define the discriminant

$$\Delta_{(\mathcal{E},\mathcal{F})} : S^2\mathcal{E} \otimes \mathcal{F} \longrightarrow (\det(\mathcal{E}) \otimes \mathcal{F})^{\otimes 2}$$

by

$$\Delta_{(\mathcal{E},\mathcal{F})}\left(\sum_{1 \leq i < j \leq 2} c_{ij} s_i s_j \otimes f \right) := (c_{12}^2 - 4c_{11}c_{22})(s_1 \wedge s_2 \otimes f)^{\otimes 2},$$

where s_1, s_2 form an $\mathcal{O}(U)$-basis of $\mathcal{E}(U)$, $\mathcal{F} \longrightarrow \mathbb{P}^3$ is a line bundle and $f \in \mathcal{F}(U)$ is a generator of $\mathcal{F}(U)$ for a small open set $U \subset \mathbb{P}^3$. An easy computation shows that this definition is independent of the chosen bases.

Now we specify

$$\mathcal{F} = \det \mathcal{E}^\vee \otimes \mathcal{O}(4).$$

Then the discriminant is a map

$$\Delta_{\mathcal{E}} : p_*(-K_{\mathbb{P}(\mathcal{E})}) \longrightarrow \mathcal{O}(8),$$

with

$$\{\Delta_{\mathcal{E}}(p_* s) = 0\} = V$$

set-theoretically: in local coordinates

$$s = \sum s_{ij} x_i x_j,$$

where $[x_0 : x_1]$ denotes the coordinates of the fibre, V is the locus where the zeroes of $\sum s_{ij}(z) x_i x_j = 0$ are not two distinct points. By definition this is the discriminant locus of the quadratic equation in x_0, x_1, given by

$$s_{01}^2 - 4s_{00}s_{11} = 0.$$

This coincides with the discriminant locus of $p_* s$.

Since on a trivializing neighbourhood $U \subset \mathbb{P}^3$ the map is given by

$$\Delta_{\mathcal{E}}(t)|U = t_{12}^2 - 4t_{11}t_{22},$$

if $t \in H^0(p_*(-K_{\mathbb{P}(\mathcal{E})}))$ and $t|U = (t_{11}, t_{12}, t_{22})$, we see that, in particular, $H^0(\Delta_{\mathcal{E}}) : H^0(-K_{\mathbb{P}(\mathcal{E})}) \longrightarrow H^0(\mathcal{O}(8))$ is a holomorphic map.

Moreover,

$$H^0(\Delta_{\mathcal{E}})(rt) = r^2 H^0(\Delta_{\mathcal{E}})(t)$$

for $r \in \mathbb{C}, t \in H^0(-K_{\mathbb{P}(\mathcal{E})})$. Hence we can projectivize, but cannot exclude that $H^0(\Delta_{\mathcal{E}})(s') = 0$ for some $s' \neq 0$. Therefore we get a rational map

$$\delta_{\mathcal{E}} : \mathbb{P}(H^0(-K_{\mathbb{P}(\mathcal{E})})) \cdots\to \mathbb{P}(H^0(\mathcal{O}(8))) \cong \mathbb{P}^{164}.$$

Let for the moment $V' := \{z \in \mathbb{P}^3 \mid H^0(\Delta_{\mathcal{E}})(s)(z) = 0\}$ in the sense of ideals. If we set

$$P := \{z \in \mathbb{P}^3 \mid \dim \pi^{-1}(z) = 1\},$$

we see that

$$P = \left\{z \in \mathbb{P}^3 \mid \sum s_{ij}x_ix_j = 0 \text{ for all } [x_0 : x_1]\right\}$$

and hence

$$P = \{z \in \mathbb{P}^3 \mid s_{00}(z) = s_{01}(z) = s_{11}(z) = 0\} \subset \operatorname{Sing}(V').$$

Moreover, this shows that

$$P = \{z \in \mathbb{P}^3 \mid \pi^{-1}(z) \cong \mathbb{P}^1\}.$$

Now let $z \in \operatorname{Sing}(V')$. If $s_{00}(z) = s_{01}(z) = s_{11}(z) = 0$, then $z \in P$. So let us assume $s_{00}(z) \neq 0$ or $s_{01}(z) \neq 0$. Define

$$x := [s_{01}(z) : -2s_{00}(z)] \in p^{-1}(z).$$

Since $z \in V$, we get that $\Delta_{\mathcal{E}}(s)(z) = s_{01}(z)^2 - 4s_{00}(z)s_{11}(z) = 0$. Therefore

$$\begin{aligned} s(x) &= s_{00}(z)s_{01}(z)^2 - 2s_{00}(z)s_{01}(z)^2 + 4s_{11}(z)s_{00}(z)^2 \\ &= -s_{00}(z)\Delta_{\mathcal{E}}(s)(z) = 0, \end{aligned}$$

hence $x \in X$.

We want to show that $x \in X$ is singular. For this we have to compute in the point x

$$\frac{\partial s}{\partial x_0} = 2s_{00}x_0 + s_{01}x_1 = 0, \tag{9}$$

$$\frac{\partial s}{\partial x_1} = s_{01}x_0 + 2s_{11}x_1 = 0, \tag{10}$$

$$\frac{\partial s}{\partial z_i} = \frac{\partial s_{00}}{\partial z_i}x_0^2 + \frac{\partial s_{01}}{\partial z_i}x_0x_1 + \frac{\partial s_{11}}{\partial z_i}x_1^2 = 0, \tag{11}$$

and since $z \in \operatorname{Sing}(V')$ we know moreover that at the point z we have

$$s_{01}^2 - 4s_{00}s_{11} = 0 \tag{12}$$

$$2s_{01}\frac{\partial s_{01}}{\partial z_i} - 4s_{11}\frac{\partial s_{00}}{\partial z_i} - 4s_{00}\frac{\partial s_{11}}{\partial z_i} = 0. \tag{13}$$

Using the expression for x in (9), (10) and (11) we compute

$$\begin{aligned}
\frac{\partial s}{\partial x_0} &= 2s_{00}s_{01} - 2s_{00}s_{01} = 0, \\
\frac{\partial s}{\partial x_1} &= s_{01}^2 - 4s_{00}s_{11} = 0, \\
\frac{\partial s}{\partial z_i} &= \frac{\partial s_{00}}{\partial z_i}s_{01}^2 - 2\frac{\partial s_{01}}{\partial z_i}s_{00}s_{01} + 4\frac{\partial s_{11}}{\partial z_i}s_{00}^2 \\
&= 4\frac{\partial s_{00}}{\partial z_i}s_{00}s_{11} - 2\frac{\partial s_{01}}{\partial z_i}s_{00}s_{01} + 4\frac{\partial s_{11}}{\partial z_i}s_{00}^2 \\
&= -s_{00}\Big(2s_{01}\frac{\partial s_{01}}{\partial z_i} - 4s_{11}\frac{\partial s_{00}}{\partial z_i} - 4s_{00}\frac{\partial s_{11}}{\partial z_i}\Big) = 0,
\end{aligned}$$

where the last equation used (12) as well as (13).

Thus we have proved that $x \in X$ is singular. But we assumed X to be smooth. Hence it has been proved that $P = \operatorname{Sing}(V')$. In particular, V' is reduced and therefore $V' = V$ in the sense of ideals.

Now we know that $V = \delta_{\mathcal{E}}(X) \in |\mathcal{O}(8)|$ and

$$P = \operatorname{Sing}(V).$$

If we assume $\rho(X) = 2$, we will see in the next section that in this case $P \neq \emptyset$. The image of $\delta_{\mathcal{E}}$ then is a subvariety of the singular octics in $\mathbb{P}^3$; in particular $\delta_{\mathcal{E}}$ is not surjective.

At this point we should mention the work of Clemens, Cynk and Szemberg [**Cl83**, **CS99**, **Cy99**]. The first article mentioned describes the construction of Calabi–Yau threefolds as resolutions of double covers of $\mathbb{P}^3$ ramified over a given (singular) octic. The latter two papers are concerned with the Euler number of such Calabi–Yau threefolds. The track followed in the present paper reverses the direction, since we are given first the Calabi–Yau threefold and then construct the octic. Hence our method can be used to construct octic hypersurfaces with many nodes. This is done in [**Kü02**].

To prove now the finiteness of P we will fall back upon the bounds for γ proved in the last section.

Lemma 4.8. *Let $X \subset \mathbb{P}(\mathcal{E}) =: Z$ be a Calabi–Yau manifold, with $\mathcal{E} \longrightarrow \mathbb{P}^3$ a rank-2-bundle. Let $P := \{p \in \mathbb{P}^3 \mid \pi^{-1}(p) \cong \mathbb{P}^1\}$. Then* $\dim P \leq 0$.

Proof. Assume first that $\dim P = 2$. Then $\dim \pi^*P = 3$ and, since $\pi^*P \neq X$, we conclude that X is reducible, hence not smooth.

If $\dim P = 1$, then $D := \pi^*P \in \operatorname{Pic}(X)$ is an effective divisor satisfying

$$D.\pi^*h^2 = 0.$$

If we write

$$\mu D = \mathcal{O}_X(2) + k\pi^*h,$$

with $\mu \in \mathbb{R}^+$, we compute

$$\begin{aligned} 0 &= D.\pi^* h^2 \\ &= \big(\mathcal{O}_Z(2) + kp^*h\big).p^*h^2.\big(\mathcal{O}_Z(2) + (4 - c_1(\mathcal{E}).h^2)p^*h\big) \\ &= 8 + 2k + 2c_1(\mathcal{E}).h^2. \end{aligned}$$

Hence

$$\mu D = -K_Z - 8p^*h|X.$$

But we know additionally that

$$\begin{aligned} 0 < \deg P &= D.\pi^* h.\mathcal{O}_X(1) \\ &= \frac{1}{\mu}(-K_Z - 8p^*h).\pi^*h.\mathcal{O}_Z(1).(-K_Z) = \frac{1}{\mu}(\gamma - 16) \le 0, \end{aligned}$$

what is a contradiction. □

This we now use to prove a result about the number of full fibres in X:

Theorem 4.9. *Let $X \subset \mathbb{P}(\mathcal{E}) =: Z$ be a Calabi–Yau manifold, with $\mathcal{E} \longrightarrow \mathbb{P}^3$ a rank-2-bundle. X contains exactly $64 - 4\gamma$ fibres of p.*

Proof. Let $s \in H^0(-K_Z)$ be such that $X = \{s = 0\}$. This section s induces a section $t \in H^0(p_*(-K_Z))$. Let

$$P := \{t = 0\}.$$

Since $\dim P \le 0$ by Lemma 4.8, we know that

$$[P] = c_3(p_*(-K_Z)).$$

To compute $c_3(p_*(-K_Z))$ we make the usual ansatz

$$c(\mathcal{E}) = (1 + at)(1 + bt).$$

Then we can express

$$c(S^2\mathcal{E} \otimes \mathcal{O}(r)) = (1 + (2a + r)t)(1 + (a + b + r)t)(1 + (2b + r)t).$$

If we do the multiplications and replace again by Chern classes of $\mathcal{E}$, we get

$$c_3(S^2\mathcal{E} \otimes \mathcal{O}(r)) = 4c_2(\mathcal{E})c_1(\mathcal{E}) + 2r(c_1^2(\mathcal{E}).h + 2c_2(\mathcal{E}).h) + 3r^2c_1(\mathcal{E}) + r^3.$$

Setting $r = 4 - c_1(\mathcal{E}).h^2$ finally leads to

$$c_3(p_*(-K_Z)) = 64 - 4\gamma.$$ □

4.3. The generalized Kollár Lemma. By Theorem 4.9 the only case in which $\pi^* h$ can be ample is $\gamma = 16$. But Lemma 4.5 and Lemma 4.6 state that in this case $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(4)$. We computed in Example 4.4 that then $\rho(X) = 1$. Hence we have proved:

Corollary 4.10. *Let $X \subset \mathbb{P}(\mathcal{E})$ be a Calabi–Yau manifold, with $\mathcal{E} \longrightarrow \mathbb{P}^3$ a rank-2-bundle. Assume that $\rho(X) = 2$. Then $\pi^* h \in \partial K(X)$.*

So we turn our attention to the 'other' side of the ample cone. The following arguments are similar to the corresponding section of [**Kü01b**].

Lemma 4.11. *Let $X \subset \mathbb{P}(\mathcal{E}) =: Z$ be a Calabi–Yau manifold with $\rho(X) = 2$ and with $\mathcal{E} \longrightarrow \mathbb{P}^3$ a rank-2-bundle. If $-K_Z$ is not nef, then $K(X) = K(Z)|X$.*

Proof. Take $D \in \mathrm{Pic}(Z) \otimes \mathbb{Q}$ such that $D|X$ is nef. Without loss of generality we may assume $D = \mathcal{O}_Z(2) + kp^* h$, with $k \in \mathbb{Q}$. Set $E := D + K_Z$. Then $E = lp^* h$, with $l \in \mathbb{Q}$; hence E or $-E$ is nef. If $-E$ is nef, so is $-K_Z$, and we get a contradiction. Therefore E is nef. Now let $C \subset Z$ be a curve. If $C \subset X$, then by assumption $D.C \geq 0$. But if $C \not\subset X$, then $-K_Z.C \geq 0$ since $X \in |-K_Z|$ and hence

$$D.C = (E - K_Z).C > 0.$$

This shows that D is nef. □

The generalization of Kollár's lemma is possible in an unrestricted way:

Theorem 4.12. *Let $X \subset \mathbb{P}(\mathcal{E}) =: Z$ be a Calabi–Yau manifold with $\rho(X) = 2$, with $\mathcal{E} \longrightarrow \mathbb{P}^3$ a rank-2-bundle. Then*

$$K(X) = K(Z)|X.$$

We divide the proof into several steps:

Lemma 4.13. *Let Z be a fourfold such that $-K_Z$ is big and nef but not ample. If $\Phi_{|-mK_Z|} : Z \longrightarrow Z'$ contracts only a finite number of curves, then these are smooth and rational.*

Proof. Let C be an irreducible curve contracted by Φ. Since $-mK_Z = \Phi^* \mathcal{O}_{Z'}(1)$ we also have $-mK_{Z'} = \mathcal{O}_{Z'}(1)$, and Z' has only canonical singularities; in particular, they are rational. This tells us that $R^1 \Phi_* \mathcal{O}_Z = 0$, and since

$$0 \longrightarrow \mathcal{I}_C \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

we conclude that $R^1 \Phi_* \mathcal{O}_C = 0$. By the Leray spectral sequence, $H^1(\mathcal{O}_C) = 0$. Hence C is smooth and rational. □

Lemma 4.14. *Let Z be a fourfold such that $-K_Z$ is big and nef but not ample. Then the exceptional locus of $\Phi_{|-mK_Z|} : Z \longrightarrow Z'$ contains a two-dimensional component.*

Proof. Assume Φ contracts only a finite number of curves. By Lemma 4.13 these are smooth and rational. Let C be such a curve. By the adjunction formula, $K_Z.C = 0$ implies that $c_1(N_{C|Z}) = c_1(K_C) = -2$. Now we compute

$$\chi(N_{C|Z}) = 3(1 - g(C)) + c_1(N_{C|Z}) = 1 > 0.$$

Therefore C deforms in Z, hence there is a surface contracted, contradicting the assumption. □

Lemma 4.15. *Let $X \subset \mathbb{P}(\mathcal{E}) =: Z$ be a Calabi–Yau manifold with $\rho(X) = 2$, with $\mathcal{E} \longrightarrow \mathbb{P}^3$ a rank-2-bundle. Then*

$$H^4(Z, \mathbb{Z}) = \left\langle \mathcal{O}_Z(1).p^*h, (p^*h)^2 \right\rangle.$$

Proof. By the Künneth formula $b_4(Z) = 2$.

In order to show that $v_1 := (p^*h)^2$ and $v_2 := \mathcal{O}_Z(1).p^*h$ form a $\mathbb{Z}$-basis for $H^4(Z, \mathbb{Z})$, it suffices to show that the matrix $A = (a_{ij}) = (v_i.v_j)$ is invertible over $\mathbb{Z}$. But by Lemma 4.1

$$A = \begin{pmatrix} 0 & 1 \\ 1 & c_1(\mathcal{E}).h^2 \end{pmatrix},$$

which proves the lemma. □

Lemma 4.16. *Let $X \subset \mathbb{P}(\mathcal{E}) =: Z$ be a Calabi–Yau manifold with $\rho(X) = 2$, with $\mathcal{E} \longrightarrow \mathbb{P}^3$ a rank-2-bundle. Assume that $-K_Z$ is big and nef but not ample. Then $\Phi_{|-mK_Z|}|X : X \longrightarrow X'$ is not an isomorphism.*

Proof. Let $E \longrightarrow V$ be the exceptional locus of Φ. We obtain

$$kX = \phi^*H,$$

for a $k \in \mathbb{Z}$. Since H is ample, H intersects every positive-dimensional component of V. This implies that $\phi|X$ can be an isomorphism only if $\dim V = 0$.

So assume $\dim V = 0$. By Lemma 4.14 there is a surface $G \subset Z$ that gets contracted to a point by Φ. In particular, $-K_Z.G \equiv 0$. Let $a, b \in \mathbb{Z}$ be such that

$$[G] = a\mathcal{O}_Z(1).p^*h + b(p^*h)^2.$$

Then $-K_Z.G.p^*h = 0$ and $-K_Z.G.\mathcal{O}_Z(1) = 0$ are equivalent to

$$\begin{aligned} (c_1(\mathcal{E}).h^2 + 4)a + 2b &= 0 \\ (c_1^2(\mathcal{E}).h - 2c_2(\mathcal{E}).h + 4c_1(\mathcal{E}).h^2)a + (c_1(\mathcal{E}).h^2 + 4)b &= 0, \end{aligned}$$

which has a nontrivial solution only if

$$\det \begin{pmatrix} c_1(\mathcal{E}).h^2 + 4 & 2 \\ c_1^2(\mathcal{E}).h - 2c_2(\mathcal{E}).h + 4c_1(\mathcal{E}).h^2 & c_1(\mathcal{E}).h^2 + 4 \end{pmatrix} = 0.$$

This means exactly $\gamma = 16$. As argued above, this amounts to $\rho(X) = 1$, contradicting our assumption. □

Proof of Theorem 4.12. We assume $K(X) \neq K(Z)|X$. The Kollár lemma says that then $-K_Z$ is not ample. But by Corollary 4.10 the pullbacks p^*h and π^*h are nef and not ample. So, if $K(X) \neq K(Z)|X$, by Lemma 4.11 the divisor $-K_Z|X$ has to be ample and $-K_Z$ has to be big and nef, but not ample. From

$$0 \longrightarrow -(m-1)K_Z \longrightarrow -mK_Z \longrightarrow -mK_Z|X \longrightarrow 0$$

and $H^1(-(m-1)K_Z) = 0$ for $m > 0$ we see that

$$H^0(-mK_Z) \longrightarrow H^0(-mK_Z|X)$$

is surjective, hence $\phi_{|-mK_Z|X|} = \phi_{|-mK_Z|}|X$. Since $-K_Z|X$ is ample, this means that $\phi_{|-mK_Z|}|X$ is an isomorphism for $m \gg 0$. This contradicts Lemma 4.16. □

4.4. Rationality of $K(X)$ and positivity of $c_2(X)$.

Corollary 4.17. *Let $X \subset \mathbb{P}(\mathcal{E}) =: Z$ be a Calabi–Yau manifold with $\rho(X) = 2$, with $\mathcal{E} \longrightarrow \mathbb{P}^3$ a rank-2-bundle. Then*

$$D.c_2(X) > 0 \quad \textit{for all } D \in \overline{K(X)}.$$

Proof. If $-K_Z$ is not ample, neither is $-K_Z|X$, by Theorem 4.12. Hence there is some nonnegative $k \in \mathbb{Q}$ such that

$$D := -K_Z|X + k\pi^*h \in \partial K(X).$$

By Lemma 4.2,

$$D.c_2(X) = -K_Z|X.c_2(X) + k\pi^*h.c_2(X) \geq 56 + 44k > 0.$$

By Theorem 4.12, $\pi^*h \in \partial K(X)$. The claim follows since $\pi^*h.c_2(X) = 44$.

If $-K_Z$ is ample, the claim of the theorem has been proven by Oguiso and Peternell in [**OP98**]. □

The rationality of the Kähler cone also follows easily under mild restrictions:

Theorem 4.18. *Let $X \subset \mathbb{P}(\mathcal{E})$ be a Calabi–Yau manifold with $\rho(X) = 2$, with $\mathcal{E} \longrightarrow \mathbb{P}^3$ a rank-2-bundle satisfying $h^0(-K_{\mathbb{P}(\mathcal{E})}) > 1$. Then $\partial K(X)$ is rational.*

Proof. Set $Z := \mathbb{P}(\mathcal{E})$ as before. In view of Theorem 4.12 it suffices to show that $\partial K(Z)$ is rational.

We consider three cases: $-K_Z$ is ample, $-K_Z$ is nef but not ample, and $-K_Z$ is not nef.

Let $-K_Z$ be ample. Then the cone theorem states the rationality of $\partial K(Z)$.

Now let $-K_Z$ be nef, but not ample. Then $-K_Z \in \partial K(X)$ and hence $\partial K(X)$ is rational.

Finally, let $-K_Z$ be not nef. By Theorem 4.12,

$$K(X) = K(Z)|X.$$

Since $h^0(-K_Z) > 1$ holds, $-K_Z|X$ is effective. By assumption also $-K_Z|X$ is not nef. Now we want to use the log-rationality theorem ([**KMM87**, Thm. 4-1-1]), which states that

$$\sup\{r \in \mathbb{R} \mid H + r(K_X + \Delta) \in K(X)\} \in \mathbb{Q}$$

if $H \in \operatorname{Pic}(X)$ is ample, $K_X + \Delta$ not nef and Δ an effective $\mathbb{Q}$-divisor such that (X, Δ) has only weak log-terminal singularities. The latter property can be reached by choosing $\varepsilon\Delta$ instead of Δ for $0 < \varepsilon \ll 1$. Note that $K_X = 0$.

Because $-K_Z|X$ is not nef, we apply the log-rationality theorem for an arbitrary ample $H \in \operatorname{Pic}(X)$ and $\Delta := \varepsilon(-K_Z|X)$ for $0 < \varepsilon \ll 1$. In this way we get that $\partial K(X)$ is rational. □

A result of Wilson [**Wi94b**, Facts A,B,C] says that the rationality of $\partial K(X)$ and the positivity of $c_2(X)$ imply that $\partial K(X)$ is semiample. Hence:

Corollary 4.19. *Let $X \subset \mathbb{P}(\mathcal{E})$ be a Calabi–Yau manifold with $\rho(X) = 2$, with $\mathcal{E} \longrightarrow \mathbb{P}^3$ a rank-2-bundle. If $\partial K(X)$ is rational, there is a second contraction $X \longrightarrow X'$ (apart from the first Stein factor of π).*

5. Calabi–Yau threefolds in $\mathbb{P}^3$-bundles over $\mathbb{P}^1$

According to Corollary 3.3, if X is a Calabi–Yau threefold in a $\mathbb{P}^3$-bundle over $\mathbb{P}^1$ with $\rho(X) = 2$, the Kähler cone is rational. With similar arguments as in the previous section, it can also be proved that $K(X) = K(Z)|X$.

We adopt the convention that $\mathcal{E} \longrightarrow \mathbb{P}^1$ is normalized in such a way that

$$\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \mathcal{O}(a_3)$$

with $0 \leq a_1 \leq a_2 \leq a_3$. As before, $X \subset \mathbb{P}(\mathcal{E}) =: Z$ is the Calabi–Yau manifold under consideration.

Theorem 5.1. *Let $X \subset \mathbb{P}(\mathcal{E}) =: Z$ be a Calabi–Yau threefold with $\mathcal{E} \longrightarrow \mathbb{P}^1$ a rank-4-bundle. Then $K(X) = K(Z)|X$.*

Proof. We consider $\mathcal{E}$ normalized as above. In particular, $\mathcal{O}_Z(1) \in \partial K(Z)$. The canonical bundle of Z is

$$-K_Z = \mathcal{O}_Z(4) + (2 - c_1(\mathcal{E}))p^*h$$

and $X \in |-K_Z|$.

If $c_1(\mathcal{E}) > 2$, then

$$-K_Z.\mathbb{P}(\mathcal{O}) < 0,$$

hence $\mathbb{P}(\mathcal{O}) \subset X$. Therefore $\mathcal{O}_X(1)$ is also nef and not ample.

If $c_1(\mathcal{E}) < 2$, then $-K_Z$ is ample and we use Lemma 3.4 (Kollár).

If $c_1(\mathcal{E}) = 2$, then $a_1 = 0$, so $\mathbb{P}(\mathcal{O})$ deforms in Z. Since $-K_Z.\mathbb{P}(\mathcal{O}) = 0$, either there is a curve of the form $\mathbb{P}(\mathcal{O})$ lying in X or the surface $G := \mathbb{P}(\mathcal{O} \oplus \mathcal{O})$ has the property

$$X \cap G = \emptyset.$$

In the first case $K(X) = K(Z)|X$ is proved.

In the second case we compute as in the previous section

$$\langle \mathcal{O}_Z(1).p^*h, \mathcal{O}_Z(1)^2 \rangle_{\mathbb{Z}} = H^4(Z, \mathbb{Z}). \tag{14}$$

Again, we show this by setting $v_1 := \mathcal{O}_Z(1).p^*h$, $v_2 := \mathcal{O}_Z(1)^2$ and computing the matrix $A = (v_i v_j)_{ij}$:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & c_1(\mathcal{E}) \end{pmatrix}.$$

Obviously, $A \in \mathrm{Gl}(2, \mathbb{Z})$ and therefore (14) is proved.

Now we can write $[G] = kv_1 + lv_2$. With this we get $0 = -K_Z.G.p^*h = 4l$ and hence

$$0 = -K_Z.G.\mathcal{O}_Z(1) = 4k + 4lc_1(\mathcal{E}) + (2 - c_1(\mathcal{E}))l = 4k;$$

therefore $[G] = 0$, contradicting the projectivity of Z. □

The intersection theory on X looks as follows:

(i) $c_3(X) = -168$.
(ii) $\mathcal{O}_Z(1)|X.c_2(X) = 6c_1(\mathcal{E}) + 44, \pi^*h.c_2(X) = 24$, hence $-K_Z|X.c_2(X) = 224$.
(iii) $(-K_Z|X)^3 = 512, (-K_Z|X)^2.\pi^*h = 64$.

With this, we already are able to prove the positivity of $c_2(X)$:

Theorem 5.2. *Let $X \subset \mathbb{P}(\mathcal{E}) =: Z$ be a Calabi–Yau threefold with $\mathcal{E} \longrightarrow \mathbb{P}^1$ a rank-4-bundle normalized as above. Then*

$$D.c_2(X) > 0$$

for all $D \in \partial K(X)$.

Proof. By the chosen normalization, $\mathcal{O}_Z(1)$ is nef and not ample, hence Theorem 5.1 shows that $\mathcal{O}_X(1)$ is also nef and not ample. Since $c_1(\mathcal{E}) \geq 0$, the formulas above imply $\mathcal{O}_X(1).c_2(X) > 0$ and $\pi^*h.c_2(X) > 0$. Hence the claim is proved. □

Again we refer to

$$0 \longrightarrow T_{Z|\mathbb{P}^1} \longrightarrow T_Z \longrightarrow p^*T_{\mathbb{P}^1} \longrightarrow 0, \tag{15}$$

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow p^*(\mathcal{E}^\vee) \otimes \mathcal{O}_Z(1) \longrightarrow T_{Z|\mathbb{P}^1} \longrightarrow 0, \tag{16}$$

$$0 \longrightarrow T_X \longrightarrow T_Z|X \longrightarrow N_{X|Z} \longrightarrow 0. \tag{17}$$

For the later description we compute the Picard number.

Theorem 5.3. *Let $X \subset \mathbb{P}(\mathcal{E}) =: Z$ be a Calabi–Yau threefold with $\mathcal{E} \longrightarrow \mathbb{P}^1$ a rank-4-bundle normalized as above. Then*

$$\rho(X) = 2 + h^1(-K_Z).$$

In particular,

$$\rho(X) = 2 \iff c_1(\mathcal{E}) \leq 3.$$

Proof. We look at the two sequences

$$0 \longrightarrow N^\vee_{X|Z} \longrightarrow \Omega_Z|X \longrightarrow \Omega_X \longrightarrow 0 \tag{18}$$

and

$$0 \longrightarrow \Omega_Z \otimes K_Z \longrightarrow \Omega_Z \longrightarrow \Omega_Z|X \longrightarrow 0. \tag{19}$$

Our first aim is to show that $H^i(T_Z) = H^i(\mathcal{E}^\vee \otimes \mathcal{E})$ for $i > 1$. For this purpose we calculate $R^i p_*(p^*(\mathcal{E}^\vee) \otimes \mathcal{O}_Z(1)) = \mathcal{E}^\vee \otimes R^i p_* \mathcal{O}_Z(1) = 0$ for $i > 0$. Therefore the Leray spectral sequence yields

$$H^i(p^*(\mathcal{E}^\vee) \otimes \mathcal{O}_Z(1)) = H^i(\mathcal{E}^\vee \otimes \mathcal{E}).$$

Sequence (16) shows that $H^i(p^*(\mathcal{E}^\vee) \otimes \mathcal{O}_Z(1)) = H^i(T_{Z|\mathbb{P}^1})$ for $i > 0$, since by $R^i p_* \mathcal{O}_Z = 0$ for $i > 0$ we get $H^i(\mathcal{O}_Z) = H^i(\mathcal{O}_{\mathbb{P}^1}) = 0$ for $i > 0$. For applying (15), we verify by the projection formula that $R^i p_* p^* T_{\mathbb{P}^1} = 0$ for $i > 0$. Hence again the Leray spectral sequence implies

$$H^i(p^* T_{\mathbb{P}^1}) = H^i(T_{\mathbb{P}^1}) = 0$$

for $i > 0$. This implies, now with (15), that

$$H^i(T_{Z|\mathbb{P}^1}) = H^i(T_Z)$$

for $i > 1$. Therefore

$$H^i(T_Z) = H^i(\mathcal{E}^\vee \otimes \mathcal{E})$$

for $i > 1$.

So we see that

$$H^i(\Omega_Z \otimes K_Z) = H^{4-i}(T_Z)^\vee = H^{4-i}(\mathcal{E}^\vee \otimes \mathcal{E})^\vee$$

for $i < 3$. In particular,

$$H^1(\Omega_Z \otimes K_Z) = H^3(\mathcal{E}^\vee \otimes \mathcal{E})^\vee = 0.$$

Since $H^{2,1}(Z) = 0$, $H^2(\mathcal{E}^\vee \otimes \mathcal{E}) = 0$ and $N^\vee_{X|Z} = K_Z|X$, the cohomology sequences of (18) and (19) contain

$$0 \longrightarrow H^1(\Omega_Z) \longrightarrow H^1(\Omega_Z|X) \longrightarrow 0 \tag{20}$$

and

$$0 \longrightarrow H^1(K_Z|X) \longrightarrow H^1(\Omega_Z|X) \longrightarrow H^1(\Omega_X) \longrightarrow H^2(K_Z|X), \tag{21}$$

respectively.

The sequence

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow -K_Z \longrightarrow -K_Z|X \longrightarrow 0$$

and Serre duality imply

$$h^i(-K_Z) = h^i(-K_Z|X) = h^{3-i}(K_Z|X)$$

for $i > 0$. Since $R^i p_*(-K_Z) = 0$ for $i > 0$ we compute

$$h^2(-K_Z) = h^2(p_*(-K_Z)) = 0.$$

Considering this together with the sequences (20) and (21) we get

$$\rho(X) = 2 + h^2(K_Z|X) = 2 + h^1(-K_Z).$$

Since $R^1 p_*(-K_Z) = 0$, we compute further

$$h^1(-K_Z) = h^1(S^4\mathcal{E} \otimes \mathcal{O}(2 - c_1(\mathcal{E}))).$$

Hence the condition $\rho(X) = 2$ is equivalent to

$$2 - c_1(\mathcal{E}) \geq -1,$$

what proves the claim. □

In the chosen normalization of $\mathcal{E}$ the divisor $\mathcal{O}_Z(1)$ is nef but not ample. By $K(X) = K(Z)|X$ also $\mathcal{O}_X(1)$ is nef and not ample. The computation

$$\mathcal{O}_X(1)^3 = \mathcal{O}_Z(1)^3.(-K_Z) = 3c_1(\mathcal{E}) + 2 > 0$$

and the basepoint-free theorem show that there is a birational contraction

$$\psi : X \longrightarrow X'.$$

This contraction is described as follows:

Theorem 5.4. *Let $X \subset \mathbb{P}(\mathcal{E}) := Z$ be a Calabi–Yau manifold with $\rho(X) = 2$, with $\mathcal{E} \longrightarrow \mathbb{P}^1$ a rank-4-bundle normalized as above. Let $\psi : X \longrightarrow X'$ be the birational contraction and E its exceptional locus. Denote by $\mathcal{F} := \mathcal{O} \bigoplus_{i|a_i=0} \mathcal{O}(a_i)$ the maximal trivial subbundle of $\mathcal{E}$.*

(i) *If $c_1(\mathcal{E}) = 3$, then $\operatorname{rk}\mathcal{F} \leq 2$ and $E = \mathbb{P}(\mathcal{F}) \cong \mathbb{P}^1 \times \mathbb{P}^{\operatorname{rk}\mathcal{F}-1}$.*

(ii) *If $c_1(\mathcal{E}) = 2$, then $\operatorname{rk}\mathcal{F} \in \{2, 3\}$ and $E = X \cap \mathbb{P}(\mathcal{F}) = \mathbb{P}^1 \times Y$, with $\dim Y = \operatorname{rk}\mathcal{F} - 2$.*
   (a) *If $\operatorname{rk}\mathcal{F} = 2$, then Y consists of four points.*
   (b) *If $\operatorname{rk}\mathcal{F} = 3$, then Y is a smooth plane quartic curve.*

(iii) *If $c_1(\mathcal{E}) = 1$, then Z is the blowup of $\mathbb{P}^4$ in a linearly embedded $\mathbb{P}^2$; if $X \in |-K_Z|$ is chosen generally, then $E = \bigcup_{i=1}^{16} C_i$, with $C_i \cong \mathbb{P}^1$; in this case, X' is a quintic in $\mathbb{P}^4$ with 16 double points on a linearly embedded $\mathbb{P}^2$.*

(iv) *If $c_1(\mathcal{E}) = 0$, then $Z = \mathbb{P}^1 \times \mathbb{P}^3$ and $E = \bigcup_{i=1}^{64} C_i$, with $C_i \cong \mathbb{P}^1$.*

Proof. $\mathcal{E}$ is normalized in such a way that $\mathcal{O}_Z(1) \in \partial K(Z)$. First we show that all curves C satisfying $\mathcal{O}_Z(1).C = 0$ have the form $\mathbb{P}(\mathcal{O})$ for some quotient $\mathcal{E} \longrightarrow \mathcal{O} \longrightarrow 0$: If $\mathcal{O}_Z(1).C = 0$, then $s|C$ is constant for all $s \in H^0(\mathcal{O}_Z(1))$. Writing

$$s = \sum s_i x_i,$$

where $[x_0 : \cdots : x_3]$ are fibre coordinates and $s_i \in H^0(\mathcal{O}(a_i))$, the constancy of $s|C$ for all $s \in H^0(\mathcal{O}_Z(1))$ implies that

$$x_i = 0, \text{ if } a_i > 0 \text{ and } x_i = c_i, \text{ if } a_i = 0$$

for $c_i \in \mathbb{C}$. This proves the claim.

(i) In case $c_1(\mathcal{E}) = 3$ we compute for $C = \mathbb{P}(\mathcal{O})$

$$-K_Z.C = (\mathcal{O}_Z(4) + (2 - c_1(\mathcal{E}))p^*h).C = 2 - c_1(\mathcal{E}) < 0,$$

hence $C \subset X$. Since every curve C satisfying $\mathcal{O}_Z(1).C = 0$ is of the form $\mathbb{P}(\mathcal{O})$, this implies

$$E = \mathbb{P}(\mathcal{F}).$$

The divisor E is also the exceptional locus of $\widetilde{\psi} := \Phi_{|\mathcal{O}_Z(m)|} : Z \longrightarrow Z'$ and hence by Lemma 4.14 we know that $\dim E \le 2$. Therefore also $\operatorname{rk} \mathcal{F} \le 2$.

(ii) If $c_1(\mathcal{E}) = 2$, then for $C = \mathbb{P}(\mathcal{O})$

$$-K_Z.C = 0,$$

hence $C \subset X$ or $C \cap X = \emptyset$. Thus,

$$E = X \cap \mathbb{P}(\mathcal{F}) = \mathbb{P}^1 \times Y.$$

Since $-K_Z = \mathcal{O}_Z(4)$, there is an ample hypersurface $V \subset Z'$ such that

$$kX = \widetilde{\psi}^* V,$$

with $\widetilde{\psi} : Z \longrightarrow Z'$ as above the birational map given by the linear system $|\mathcal{O}_Z(m)|$. In particular,

$$\dim E = \dim \mathbb{P}(\mathcal{F}) - 1 = \operatorname{rk} \mathcal{F} - 1,$$

and hence

$$\dim Y = \operatorname{rk} \mathcal{F} - 2.$$

If $\operatorname{rk} \mathcal{F} = 2$, the number of contracted curves is computed as

$$Y.F = \mathcal{O}_Z(4).\mathbb{P}(\mathcal{F}).F = 4,$$

since every contracted curve is of the form $\mathbb{P}(\mathcal{O})$.

If $\operatorname{rk} \mathcal{F} = 3$, i.e., $\mathcal{E} = 3\mathcal{O} \oplus \mathcal{O}(2)$, then by [**Gr97**] Y is a smooth curve, since $\rho(X) = 2$ implies that every contraction is primitive. For a general fibre F of p the intersection $E \cap F$ is smooth ([**Ha77**, Corollary 10.9.1]) and

$$\psi|_{E \cap F} : E \cap F \longrightarrow Y$$

is an isomorphism, since $F.\mathbb{P}(\mathcal{O}) = 1$. Furthermore $E \cap F \subset \mathbb{P}(\mathcal{F}) \cap F \cong \mathbb{P}^2$ and we compute

$$\deg E \cap F = E.F.\mathcal{O}_Z(1) = -K_Z.\mathbb{P}(\mathcal{F}).F.\mathcal{O}_Z(1) = 4,$$

because $-K_Z = \mathcal{O}_Z(4)$.

(iii) If $c_1(\mathcal{E}) = 1$, only $\mathcal{E} = 3\mathcal{O} \oplus \mathcal{O}(1)$ is possible and therefore Z is the blowup of $\mathbb{P}^4$ in a linearly embedded $\mathbb{P}^2$. Denote the blowup by $\phi : Z \longrightarrow \mathbb{P}^4$. As above the exceptional curves have the form $C = \mathbb{P}(\mathcal{O})$. Let E' be the exceptional divisor of the blowup. Then

$$-K_Z = \phi^*\mathcal{O}_{\mathbb{P}^4}(5) - E' = \mathcal{O}_Z(5) - E';$$

on the other hand

$$-K_Z = \mathcal{O}_Z(4) + p^*h,$$

hence

$$E' = \mathcal{O}_Z(1) - p^*h.$$

Set $Y := X \cap E'$. The sequence

$$0 \longrightarrow -K_Z - E' \longrightarrow -K_Z \longrightarrow -K_Z|E' \longrightarrow 0$$

and the equality $H^1(-K_Z - E') = H^1(\mathcal{O}_Z(3) + 2p^*h) = 0$ (with Kodaira vanishing) imply that

$$H^0(-K_Z) \longrightarrow H^0(-K_Z|E')$$

is surjective. Since $-K_Z|E'$ is globally generated, we see that by a general choice of X also Y can be assumed to be smooth.

Since $\mathcal{O}_Z(1).\mathbb{P}(\mathcal{O}) = 0$ and $p^*h.\mathbb{P}(\mathcal{O}) = 1$ we conclude that $-K_Z.\mathbb{P}(\mathcal{O}) = 1$ and therefore

$$\phi|Y : Y \longrightarrow \mathbb{P}^2$$

is generically one-to-one. If $C = \mathbb{P}(\mathcal{O}) \subset Y$, the sequence

$$0 \longrightarrow T_{\mathbb{P}(\mathcal{O})} \longrightarrow T_Y|\mathbb{P}(\mathcal{O}) \longrightarrow N_{\mathbb{P}(\mathcal{O})|Y} \longrightarrow 0$$

and the equalities $T_{\mathbb{P}(\mathcal{O})} = \mathcal{O}(2)$ and

$$c_1(T_Y|\mathbb{P}(\mathcal{O})) = c_1(N^\vee_{Y|X}|\mathbb{P}(\mathcal{O})) = -E'.\mathbb{P}(\mathcal{O}) = (-\mathcal{O}_Z(1) + p^*h).\mathbb{P}(\mathcal{O}) = 1$$

imply that

$$N_{\mathbb{P}(\mathcal{O})|Y} = \mathcal{O}(-1).$$

This shows that $\phi|Y$ is the blowup of $\mathbb{P}^2$ in k points. To compute k, we apply the canonical bundle formula,

$$K_Y^2 = \left(\phi|Y^*(\mathcal{O}_{\mathbb{P}^2}(3)) - \sum_{i=1}^k E_i\right)^2 = 9 - k,$$

where E_i denotes the exceptional curves. We know that

$$E = \bigcup_{i=1}^{k} E_i.$$

At the same time,

$$\begin{aligned} K_Y^2 = N_{Y|X}^2 &= (-K_Z).E^3 \\ &= (\mathcal{O}_Z(4) + p^*h).(\mathcal{O}_Z(1) - p^*h)^3 \\ &= 4c_1(\mathcal{E}) - 12 + 1 = -7. \end{aligned}$$

Hence we arrive at $k = 16$.

Since $-K_Z = \phi^*\mathcal{O}_{\mathbb{P}^4}(5) - E'$ the characterization of $\phi|Y$ as a blowup shows that X' is a quintic in $\mathbb{P}^4$ with 16 double points on a linearly embedded $\mathbb{P}^2 \subset \mathbb{P}^4$.

(iv) In the last case $c_1(\mathcal{E}) = 0$, obviously $\mathcal{E} = 4\mathcal{O}$ and hence $Z = \mathbb{P}^1 \times \mathbb{P}^3$. Therefore we may use Theorem 4.9 to conclude that E consists of 64 curves $C \cong \mathbb{P}^1$. □

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