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We use circle packing techniques to construct approximate solutions to the generalized Beltrami equations with simply and multiply connected regions in the plane. We show convergence of the approximate solutions. This gives a constructive proof for the existence of quasiconformal mappings with a given pair of complex dilations.

1. Introduction

Circle packings — configurations of circles with specified patterns of tangency and having disjoint interiors — came to prominence with analysts in 1985 when Thurston conjectured that maps between such configurations would approximate conformal maps. Shortly, Burt Rodin and Dennis Sullivan [**7**] proved the convergence of Thurston's scheme, and much research on circle packings followed.

In [**1**], Z.-X. He used circle packing methods to solve the Beltrami equation

$$\partial_{\bar{z}} f = \lambda \partial_z f.$$

In this paper we shall construct homeomorphic solutions $w = f(z)$ of the generalized Beltrami equation

$$\partial_{\bar{z}} f = \lambda \partial_z f + \mu \overline{\partial_z f}, \quad z \in \Omega. \tag{1}$$

We suppose that Ω is a simply or multiply connected Jordan domain in the complex plane $\mathbb{C}$ and that $\mathbb{U}$ is either, as the corresponding case may be, the unit disk with center at the origin or the unit circular domain obtained from the unit disk by the deletion of a closed disjoint union of finitely many closed disks and points. Further we assume that $\lambda, \mu : \Omega \to \mathbb{C}$ are measurable functions with

$$\big\| |\lambda| + |\mu| \big\|_\infty = \operatorname{esssup}_{z\in\Omega}\big(|\lambda| + |\mu|\big) < 1. \tag{2}$$

The homeomorphic solution in L_2 to (1) and (2) is called a quasiconformal mapping with complex dilations (λ, μ).

This paper can be viewed as a development of [**1**] in terms of both equation and region. Our method of constructing functions is different from that of [**1**], and avoids an essential difficulty that one is faced with in discussing the Riemann's existence theorem of Equation (1): namely, the composition of a quasiconformal map with complex dilations (λ, μ) and a conformal map is a quasiconformal map, but its complex dilations are different from (λ, μ) (see [5], for example).

2. Construction of Approximate Solutions

We shall construct approximate solutions of Equation (1) in this section.

Let δ be a small positive number. We use the grid

$$\mathrm{RL}\,(\delta) = \left\{x + iy : x = p\delta,\, y = \tfrac{\sqrt{3}}{2}q\delta,\, (p, q) \in \mathbb{Z} \times \mathbb{Z}\right\}$$

to decompose $\mathbb{C}$ into a number of rectangles with side lengths δ and $\sqrt{3}\delta/2$, and choose the side lengths of the rectangles in such a way that one is just able to use the regular hexagonal grid of mesh δ/n $(n \in \mathbb{N})$ to triangulate it. Suppose $\overline{\Omega}_\delta$ is the minimal Jordan domain containing all rectangles that intersect Ω. Then $\overline{\Omega}_\delta$ is a closed polyhedral Jordan domain, which is the union of squares of side length δ and $\sqrt{3}\delta/2$. It is obvious that $\overline{\Omega} \subset \overline{\Omega}_\delta$ and $\overline{\Omega}_\delta$ converges to $\overline{\Omega}$ when $\delta \to 0$ in the sense that any compact subset of $\hat{\mathbb{C}} \setminus \overline{\Omega}$ is contained in $\hat{\mathbb{C}} \setminus \overline{\Omega}_\delta$ for δ small enough.

For any rectangle R in $\overline{\Omega}_\delta$, set

$$\lambda(R) = \begin{cases} \frac{1}{A(R)} \iint_R \lambda(z)\, dx\, dy, & R \subset \Omega, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mu(R) = \begin{cases} \frac{1}{A(R)} \iint_R \mu(z)\, dx\, dy, & R \subset \Omega, \\ 0, & \text{otherwise}, \end{cases}$$

where $A(R)$ denotes the area of R. Define piecewise constant functions $\lambda_\delta, \mu_\delta : \Omega \to \mathbb{C}$ by

$$\lambda_\delta(z) = \lambda(R), \quad \mu_\delta(z) = \mu(R), \quad \text{a.e. } z \in R \cap \Omega. \tag{3}$$

Then

$$\lim_{\delta \to 0} \lambda_\delta(z) = \lambda(z), \quad \lim_{\delta \to 0} \mu_\delta(z) = \mu(z), \quad \text{a.e. } z \in \Omega. \tag{4}$$

Suppose that R is an interior rectangle in $\overline{\Omega}_\delta$ with $R \cap \partial\overline{\Omega}_\delta = \emptyset$. Denote by z_1, z_2, z_3 and z_4 its lower left, lower right, upper right, and upper left vertices, respectively. Let $\Psi_R : \mathbb{C} \to \mathbb{C}$ be the map defined by

$$\Psi_R(z) = \frac{\left|1 + \lambda(R) + \mu(R)\right|}{1 - \left|\lambda(R) + \mu(R)\right|^2}\big((z - z_1) + (\lambda(R) + \mu(R))(\bar{z} - \bar{z}_1)\big).$$

It is easy to verify that $\partial_{\bar{z}}\Psi_R(z) = \lambda(R)\partial_z\Psi_R(z) + \mu(R)\overline{\partial_z\Psi_R(z)}$. Since $|\lambda(R) + \mu(R)| \le \||\lambda| + |\mu|\|_\infty < 1$, the function $\Psi_R(z)$ is an affine conformal homeomorphism with complex dilations $(\lambda(R), \mu(R))$ and maps the square R

onto the parallelogram R' with vertices $z_1' = \Psi_R(z_1) = 0$, $z_2' = \Psi_R(z_2) = a$, $z_3' = \Psi_R(z_3) = a + b$, $z_4' = \Psi_R(z_4) = b$, where

$$a = \frac{\left|1 + \lambda(R) + \mu(R)\right|}{1 - \left|\lambda(R) + \mu(R)\right|^2}\,\delta\big(1 + \lambda(R) + \mu(R)\big), \tag{5}$$

$$b = \frac{\left|1 + \lambda(R) + \mu(R)\right|}{2\big(1 - \left|\lambda(R) + \mu(R)\right|^2\big)}\,\sqrt{3}\,\delta\big(1 - (\lambda(R) + \mu(R))\big)\,i. \tag{6}$$

Write $n = 2[1/(2\delta)]$, $\omega = e^{i\pi/3}$ and $\gamma = \delta/n$. Let $\mathrm{HL}(\gamma)$ denote the regular hexagonal grid with mesh γ, whose vertices form the lattice $V(\gamma) = \{\gamma p + \gamma q\omega : p, q \in \mathbb{Z}\}$. The neighbors of $\alpha \in V(\gamma)$ are the points $\alpha + \gamma\omega^k$, $0 \le k \le 5$. Suppose $T(\gamma)$ is the complex obtained by using $\mathrm{HL}(\gamma)$ to triangulate the plane $\mathbb{C}$. Then the 1-skeleton of $T(\gamma)$ is the nerve of some regular hexagonal circle packing P_γ of the complex plane $\mathbb{C}$ formed by closed disks of radii $\gamma/2$.

By (5), (6) and the fact that $\left|\lambda(R) + \mu(R)\right| \le \||\lambda| + |\mu|\|_\infty < 1$, it is easy to deduce that the angle $z_4'z_1'z_2'$ of the parallelogram R' is bounded from below, the distance between $z_1'z_2'$ and $z_4'z_3'$ is $\sqrt{3}\delta/2$ and the lengths of sides of R' are between $c\delta$ and $K\delta$, where $0 < c \le 1$ and $K \ge 1$ are constants depending only on $\||\lambda| + |\mu|\|_\infty$. So one can suppose that $H(R')$ is the collection of all triangles of $T^{(r)}(\gamma)$ lying in R' and having distance at least $\gamma/2$ from $\partial R'$, where $T^{(r)}(\gamma) = e^{i\theta}T(\gamma)$, for $\theta = \arg\big(1 + \lambda(R) + \mu(R)\big)$. Thus $H(R')$ is a simplicial complex and $|H(R')|$ is some polyhedral Jordan domain in R'.

Since triangles in $H(R')$ have length δ/n one gets, as in [2], a triangulation $T(R')$ on R' that agrees with the triangulation $H(R')$ on $|H(R')|$ and whose restriction on each side of R' is the triangulation consisting of n 1-simplexes of equal length, the length lying between $c\delta/n$ and $K\delta/n$.

Set $(T(R), t) = \big(T(R'), (\Psi_R)^{-1}\big)$. Then $(T(R), t)$ is a K-quasiconformal affine triangulation on the rectangle R, and its restriction on each side of R is the affine triangulation with n 1-simplexes of equal length. So these affine triangulations can be glued together to form a K-quasiconformal affine triangulation for

$$\widetilde{\Omega}_\delta = \bigcup_{R \cap \partial\overline{\Omega} = \phi} R.$$

In addition, by intersecting the region

$$\overline{\Omega}_\delta \setminus \operatorname{int}\widetilde{\Omega}_\delta = \bigcup_{R \cap \partial\overline{\Omega}_\delta \neq \phi} R$$

with the regular hexagonal triangulation $\mathrm{HL}(\gamma)$ of the plane, one gets an affine triangulation $(\widetilde{T}_\delta, I)$ for $\overline{\Omega}_\delta \setminus \operatorname{int}\widetilde{\Omega}_\delta$, where I denotes the identity map.

Pasting together the affine triangulations above for $\widetilde{\Omega}_\delta$ and $\overline{\Omega}_\delta \setminus \operatorname{int} \widetilde{\Omega}_\delta$, we obtain an affine triangulation (T_δ, t_δ) for $\overline{\Omega}_\delta$.

Denote by $L_m(R)$ the subcomplex of $H(R) = \Psi_R^{-1}(H(R'))$ consisting of faces of triangles with the property that $G(T_\delta, v, m) \subset H(R)$ for each vertex v of Δ, where $G(T_\delta, v, m)$ is m generations about v in the complex T_δ. From the preceding discussion, it is not difficult to obtain:

Lemma 1.

(a) *For* $\widetilde{\Omega}_\delta = \bigcup_{R \cap \partial\overline{\Omega} = \phi} R$, *one has* $\lim_{\delta \to 0} A(\Omega \setminus \widetilde{\Omega}_\delta) = 0$.

(b) *Let* m *be a nonnegative integer with* $m \le cn$, *for any square* R *in* $\widetilde{\Omega}_\delta$. *Then*

$$A\big(|H(R) \setminus L_m(R)|\big) \le cA(R)(m+1)/n,$$

where A *denotes area.*

In addition, for the hexagonal circle packing, we have:

Lemma 2. *Let* H_n *be the* n*-generation of a circle packing* P *about some circle* c_0 *of* P *such that* H_n *is combinatorially equivalent to* H'_n, *where* H'_n *is the* n*-generation of some regular hexagonal circle packing about one of its circles, say* c'_0. *For* n *large enough, there is a quasiconformal mapping* Φ *from the complex plane* $\mathbb{C}$ *to* $\mathbb{C}$ *that maps the subpacking* H'_m *of* H'_n *(with* $m \sim n/2$*) to the corresponding subpacking* H_m *of* H_n *and whose restriction* $\Phi|_{D'_0}$ *to the closure* D'_0 *of the interior of* c'_0 *is Lipschitz and can be expressed by* $\Phi|_{D'_0} = \big(r/r'(z - a'_0) + a_0\big) + h(z)/n$, *where* a_0, a'_0, r, r' *are the centers and radii of* c_0, c'_0, *and* $|h(z)|$ *is bounded above by a positive constant* M.

Proof. By the transformation $\zeta = m(z) = r/r'(z - a'_0) + a_0$, H'_n becomes H''_n. It is clear that H''_n is combinatorially equivalent to H'_n and $c''_0 = c_0$, where $c''_0 = m(c'_0)$. Since H'_n is combinatorially equivalent to H_n, so is H''_n. We now construct the quasiconformal mapping Φ. Fix any three mutually tangent circles c''_1, c''_2, c''_3 in H''_{n-1} and the corresponding circles c_1, c_2, c_3 in H_{n-1}. Let φ be the orientation-preserving Möbius transformation sending $c''_j \cap c''_k$ to $c_j \cap c_k$, where (j,k) is any pair in$\{(1,2),(1,3),(2,3)\}$. Obviously, φ maps the interstice bounded by c''_1, c''_2 and c''_3 to that bounded by c_1, c_2 and c_3. These conformal mappings may be glued together to form a conformal mapping from the union of interstice bounded by circles of H''_{n-1} to the union of interstice bounded by circles of H_{n-1}. The mapping maps each circle of H''_{n-2} to the corresponding circle of H_{n-2}. Next, we extend the mapping radially on each disk bounded by circles of H''_{n-2} and the result is a K-quasiconformal mapping φ_1 from the union of interstices and disks bounded by circles of H''_{n-2} to the corresponding union bounded by circles of H_{n-2}. Finally, by a property of quasiconformal mappings, φ_1 can be extended to a K_1-quasiconformal mapping $\Phi_1 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $\Phi_1(\infty) = \infty$ such that Φ_1

equals φ_1 on the union of interstice and the disks bounded by circles of H''_m, where $m = [(n-3)/2]$. From [2], we obtain $\Phi_1|_{D''_0}(\zeta) = \zeta + h_1(\zeta)/n$, where $|h_1(\zeta)| \le M$ and M is a positive constant. Let $\Phi(z) = \Phi_1\big(m(z)\big)$; then $\Phi|_{D'_0} = \big(r/r'(z - a'_0) + a_0\big) + h(z)/n$, where $h(z) = h_1\big(m(z)\big)$. From the construction of Φ, it is easy to get that Φ is Lipschitz on D'_0. So Φ is the desired mapping. □

According to the circle packing theorem [8], given an abstract triangulation T of a topological sphere, there exists a circle packing P on the Riemann sphere $\mathbb{S}^2$ having the combinatorics of T. P is unique up to Möbius transformations of $\mathbb{S}^2$. We further obtain:

Lemma 3. *Let T be a simply or multiply connected complex in the plane. There exists a circle packing P in the unit disk or some unit circle domain, as the case may be, whose tangency graph is isotopic to the 1-skeleton of T. Moreover, P is unique up to Möbius transformation.*

Proof. If T is simply connected, the conclusion of the lemma holds from [7]. If T is a multiply connected complex, by adding an *ideal* vertex for each hole and connecting each *ideal* vertex and boundary vertices of T neighboring it with disjoint Jordan arcs, we get the augmented complex T^*, which is simply connected. By the result in the simply connected case, there exists a circle packing P^* in the unit disk whose tangency graph is isomorphic to the 1-skeleton of T^*. By removing the circles corresponding to the ideal vertices in P^*, we obtain a circle packing P in the unit circle domain with the combinatorics of T and P is unique up to Möbius transformation. □

For the affine triangulation (T_δ, t_δ) of $\overline{\Omega}_\delta$, by Lemma 3, there is a circle packing P_δ in the closed region $\overline{\mathbb{U}}$, which has the combinatorics of T_δ and is unique up to Möbius transformation of the unit disk, where $\mathbb{U}$ is the unit disk or the unit circle domain.

We construct approximate solutions of Equation (1) as follows:

(a) For any $|L_m(R')| \subset |T_\delta|$, with $L_m(R') = \Psi_R(L_m(R))$, let ζ_0 be any vertex of $L_m(R')$. There is an m-generation of some regular hexagonal circle packing about one of its circles, say, c_0 of center ζ_0 and radius $\delta/(2n)$, which is combinatorially equivalent to some m-generation of the circle packing P_δ about one of its circle, say c'_0, in $\overline{\mathbb{U}}$. We define $g_\delta : D_0 \to D'_0$ by $g_\delta = \Phi|_{D_0}$, where Φ is as in Lemma 2 and D_0, D'_0 are regions bounded by c_0 and c'_0. By pasting these mappings, we obtain a quasiconformal mapping from the union of disks bounded by the circles whose centers are the vertices of $L_m(R')$ and whose radii are all equal to $\delta/(2n)$ to the corresponding union of bounded circles of P_δ in $\overline{\mathbb{U}}$. We then extend linearly the mapping to the interstices bounded by these circles and the result is a K-quasiconformal mapping G_δ from the union of interstices and disks bounded by circles whose centers are the vertices of $L_m(R')$ and whose radii are all equal to $\delta/(2n)$ to the corresponding union of bounded by circles of P_δ in $\overline{\mathbb{U}}$. Let $\widetilde{G_\delta} = G_\delta|_{|L_m(R')|}$. Then $\widetilde{G_\delta}$ is a K-quasiconformal mapping from $|L_m(R')|$ to the corresponding portion of $|\mathrm{carr}(P_\delta)|$, where $\mathrm{carr}(P_\delta)$ denotes the concrete geometric complex in $\overline{\mathbb{U}}$ formed by connecting the centers of tangent circles of P_δ with line segments.

(b) For $|T_\delta| \setminus \left(\bigcup_{R\cap\partial\overline{\Omega}=\phi} |L_m(R)|\right)$, let $\hat{G}_\delta$ be the map sending any vertex of $T_\delta \setminus \left(\bigcup_{R_\delta\cap\partial\overline{\Omega}=\phi} L_m(R_\delta)\right)$ to the center of the corresponding disk and then extend it linearly on each simplex of $T_\delta \setminus \left(\bigcup_{R_\delta\cap\partial\overline{\Omega}=\phi} L_m(R_\delta)\right)$. Then $\hat{G}_\delta$ is a K-quasiconformal mapping from $|T_\delta| \setminus \left(\bigcup_{R\cap\partial\overline{\Omega}=\phi} |L_m(R_\delta)|\right)$ to the corresponding portion of $|\mathrm{carr}(P_\delta)|$.

(c) By pasting $\hat{G}_\delta$ and the $\widetilde{G_\delta}$, we get a K-quasiconformal mapping H_δ from $|T_\delta|$ to $|\mathrm{carr}(P_\delta)| \subset \overline{\mathbb{U}}$. Let $F_\delta = H_\delta \circ t_\delta^{-1}$, then F_δ is a homeomorphic mapping from $\overline{\Omega}_\delta$ to $|\mathrm{carr}(P_\delta)| \subset \overline{\mathbb{U}}$.

We next prove that f_δ converges to some solution of the generalized Beltrami equation (1) as δ approaches 0.

3. Convergence of approximate solutions

In this section, we show the existence of homeomorphic solutions for the generalized Beltrami equation (1).

Theorem 1. *Let λ, μ be two measurable functions with $\||\lambda| + |\mu|\|_\infty < 1$. For any positive sequence δ_k with $\delta_k \to 0$, let $f_{\delta_k} : \Omega \to \mathbb{U}$ be the sequence of map constructed for each δ_k through the procedure in Section 2. Then f_{δ_k} has a subsequence converging uniformly on every compact subset of Ω to some quasiconformal homeomorphism $f : \Omega \to \mathbb{U}$ that solves the generalized Beltrami equation*

$$\partial_{\bar{z}} f(z) = \lambda \partial_z f(z) + \mu \overline{\partial_z f(z)}, \quad a.e.\ z \in \Omega.$$

Proof. First, $t_\delta^{-1} : \overline{\Omega}_\delta \to |T_\delta|$ is an affine map, and from its construction we see that $F_\delta = H_\delta \circ t_\delta^{-1} : \overline{\Omega}_\delta \to |\mathrm{carr}(P_\delta)|$ is a K-quasiconformal homeomorphism. Set $\widetilde{f}_{\delta_k} = F_{\delta_k} = H_{\delta_k} \circ t_{\delta_k}^{-1}$, where $\delta_k \to 0$. Then $f_{\delta_k} = \widetilde{f}_{\delta_k}|_\Omega : \Omega \to \mathbb{U}$ is K-quasiconformal, too. Based on the convergence theorem for quasiconformal maps (see [4] or [5]), there is a subsequence of f_{δ_k}, still denoted by f_{δ_k}, that converges uniformly on every compact subset of Ω to f, where f is some quasiconformal homeomorphism from Ω to $\mathbb{U}$.

Next we show that f has complex dilations (λ, μ). Suppose that m_k is a sequence of positive integer such that $m_k \to \infty$ and $m_k/n_k \to 0$. For any $z \in \sigma \subset |L_{m_k}(R)|$, by the definition of f_{δ_k}, we have $f_{\delta_k}(z) = H_{\delta_k} \circ t_{\delta_k}^{-1}(z) = H_{\delta_k} \circ \Psi_R(z)$. There are two situations for $\Psi_R(z)$:

(i) $\Psi_R(z)$ lies in some disk D_0 of center ζ_0 and radius $r_0 = \delta_k/(2n_k)$. Then

$$f_{\delta_k}(z) = \big(r_0'/r_0(\Psi_R(z) - \zeta_0) + w_0\big) + h(\Psi_R(z))/m_k,$$

where w_0 and r_0' are the center and radius of the corresponding circle in P_{δ_k}. Taking partial derivatives of f_{δ_k}, we get

$$\partial_{\bar z} f_{\delta_k}(z) = r_0'/r_0 \partial_{\bar z}\Psi_R(z) + \partial_{\bar z} h(\Psi_R(z))/m_k,$$
$$\partial_z f_{\delta_k}(z) = r_0'/r_0 \partial_z\Psi_R(z) + \partial_z h(\Psi_R(z))/m_k.$$

Since $\partial_{\bar z}\Psi_R(z) = \lambda_{\delta_k}\partial_z\Psi_R(z) + \mu_{\delta_k}\overline{\partial_z\Psi_R(z)}$, we have

$$\begin{aligned}(7)\qquad \partial_{\bar z} f_{\delta_k}(z) = \lambda_{\delta_k}\partial_z f_{\delta_k}(z) + \mu_{\delta_k}\overline{\partial_z f_{\delta_k}(z)}\\ + \big(\partial_{\bar z} h(\Psi_R(z))/m_k + 2\,\mathrm{Re}(\partial_z h(\Psi_R(z)))/m_k\big);\end{aligned}$$

by Lemma 2 and the definition of $\Psi_R(z)$, we conclude that $\partial_{\bar z} h(\Psi_R(z))$ and $\partial_z h(\Psi_R(z))$ are bounded on $L_{m_k}(R)$. Combining with (4) and letting $k \to \infty$, because r_0'/r_0 converges to the modulus of the derivative of some Riemann mapping function [**6**], we obtain from (7)

$$\partial_{\bar z} f(z) = \lambda\partial_z f(z) + \mu\overline{\partial_z f(z)}.$$

(ii) $\Psi_R(z)$ lies in the interstice bounded by three circles c_0, c_1 and c_2. Then $\Psi_R(z) = (1-t)\zeta' + t\zeta''$, where $0 \le t \le 1$, $\zeta' \in c_0$, and $\zeta'' \in c_l$, for $l = 1$ or $l = 2$. Write ζ_j and r_j $(j = 0, 1, 2)$ for the center and radius of c_j. It follows from the definition of $f_{\delta_k}(z)$ that

$$\begin{aligned}(8)\qquad f_{\delta_k}(z) &= (1-t)\big((r_0'/r_0(\zeta' - \zeta_0) + w_0) + h(\zeta')/m_k\big)\\ &\quad + t\big((r_l'/r_l(\zeta' - \zeta_l) + w_l) + h(\zeta'')/m_k\big)\\ &= r_0'/r_0\Psi_R(z) + (r_l' - r_0')/r_l q\big(\Psi_R(z)\big) + \widetilde{q}\big(\Psi_R(z)\big) + C,\end{aligned}$$

where

$$q\big(\Psi_R(z)\big) = t\zeta'' = \Psi_R(z) - (1-t)\zeta', \widetilde{q}\big(\Psi_R(z)\big) = (1-t)h(\zeta') + th(\zeta''),$$
$$C = (1-t)(w_0 - r_0'/r_0\zeta_0) + t(w_j - r_l'/r_l\zeta_l)$$

and w_j and r_j' $(j = 0, 1, 2)$ are the center and radius of the corresponding circle. By [**6**] and [**2**], we have $(r_l' - r_0')/r_l = r_0'/r_l O(1/m_k)$, and note that $\partial_{\bar z} q\big(\Psi_R(z)\big)$, $\partial_z q\big(\Psi_R(z)\big)$, $\partial_{\bar z}\widetilde{q}\big(\Psi_R(z)\big)$ and $\partial_z\widetilde{q}\big(\Psi_R(z)\big)$ are bounded on $|L_{m_k}(R)|$. As in (i), it follows from (8) that

$$\partial_{\bar z} f(z) = \lambda\partial_z f(z) + \mu\overline{\partial_z f(z)}, \quad k \to \infty.$$

Thus, for any $R \subset \widetilde{\Omega}_{\delta_k}$, we obtain from (i) and (ii) that

$$\partial_{\bar z} f(z) = \lambda\partial_z f(z) + \mu\overline{\partial_z f(z)}, \quad z \in |L_{m_k}(R)|,\ k \to \infty.$$

On the other hand, for any $z \in \Omega \setminus L_k$, where $L_k = \bigcup_{R\cap\partial\overline{\Omega}_{\delta_k}=\phi} |L_{m_k}(R)|$, we get easily from Lemma 1 that

$$\lim_{k\to\infty} A(\Omega \setminus L_k) = 0.$$

Therefore, as $k \to \infty$, we conclude that

$$\partial_{\bar{z}} f(z) = \lambda \partial_z f(z) + \mu \overline{\partial_z f(z)}, \quad \text{a.e. } z \in \Omega,$$

that is, the complex dilations of the quasiconformal mapping f are (λ, μ). □

Theorem 2. *The maps f_δ constructed in Section 2, as $\delta \to 0$, converge uniformly on each compact subset of Ω to a solution $f : \Omega \to \mathbb{U}$ of (1).*

Proof. By their construction, the f_δ must converge to a homeomorphic map $\Omega \to \mathbb{U}$ as $\delta \to 0$. Using Theorem 1, we can get that the f_δ, as $\delta \to 0$, converge uniformly on every compact subset of Ω to some solution $f : \Omega \to \mathbb{U}$ of (1). □

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