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We construct a Jacobian of dimension three whose theta divisor contains an elliptic curve. We work over an algebraically closed field of characteristic zero.

Let E be an elliptic curve and F a principally polarized abelian variety of dimension 3. Let $\mathcal{L}$ and $\mathcal{N}$ be their principal polarizations.

Lemma 1. *There exist E' and F' such that we have isogenies $\psi_E : E' \to E$ and $\psi_F : F' \to F$ of degree two.*

Proof. Let $F' = (F\check{}/\mathbf{Z}/2\mathbf{Z})\check{}$ and the same with E. □

Let $\mathcal{L}' = \psi_E^*\mathcal{L}$ and $\mathcal{N}' = \psi_F^*\mathcal{N}$. Then $\varphi_{\mathcal{L}'} : E' \to (E')\check{}$ and $\varphi_{\mathcal{N}'} : F' \to (F')\check{}$ have degree four. Let $H_{\mathcal{L}'}$ and $H_{\mathcal{N}'}$ be their kernels. Then we have theta groups

$$1 \to \mathbf{G}_m \to G_{\mathcal{L}'} \to H_{\mathcal{L}'} \to 0$$

and

$$1 \to \mathbf{G}_m \to G_{\mathcal{N}'} \to H_{\mathcal{N}'} \to 0.$$

By Mumford theory we have two torsion elements $\alpha_{\mathcal{L}}$ and $\beta_{\mathcal{L}}$ of $G_{\mathcal{L}'}$ such that $\alpha_{\mathcal{L}}\cdot\beta_{\mathcal{L}} = (-1)\beta_{\mathcal{L}}\cdot\alpha_{\mathcal{L}}$ and the same with $\mathcal{N}$. Here the images of $\alpha_{\mathcal{L}}$ and $\beta_{\mathcal{L}}$ generate $H_{\mathcal{L}'}$. Consider $\mathcal{M} = \pi_{E'}^*\mathcal{L}' \otimes \pi_{F'}^*\mathcal{N}'$. Then $H_{\mathcal{M}} = H_{\mathcal{L}'} \times H_{\mathcal{N}'}$.

Lemma 2. *We have an inclusion $K = (\mathbf{Z}/2\mathbf{Z})^2 \subset G_{\mathcal{M}}$.*

Proof. $\alpha_{\mathcal{L}} \otimes \alpha_{\mathcal{N}}$ and $\beta_{\mathcal{L}} \otimes \beta_{\mathcal{N}}$ generate the group. □

Let $X = E' \otimes F'/\mathrm{Im}\ K$ and let R be the quotient of $\mathcal{M}$ by $(\mathbf{Z}/2\mathbf{Z})^2$. Then R gives a principal polarization on X. Let γ be a nonzero section of X. Let θ be the zeroes of γ.

Lemma 3. *θ contains some translate of* $\mathrm{Im}\ E'$.

Proof. γ corresponds to a section of $\mathcal{M}$ that is invariant under K. Let τ and μ be nonzero sections of $\mathcal{L}'$ and $\mathcal{N}'$ invariant under $\alpha_{\mathcal{L}}$ and $\alpha_{\mathcal{N}}$. Let $\tau' = \beta_{\mathcal{L}}(\tau)$ and $\mu' = \beta_{\mathcal{N}}(\mu)$. Then τ' and μ' are anti-invariant under $\alpha_{\mathcal{L}}$ and $\alpha_{\mathcal{N}}$. Consider the section $\eta = \tau \otimes \mu + \tau' \otimes \mu' \neq 0$ of $\mathcal{M}$. Then η is invariant under $(\mathbf{Z}/2\mathbf{Z})^2$. Then the inverse image of θ is the zeroes $\eta \supset E \times (\mu = \mu' = 0)$, where the second set is nonempty as $\mathcal{N}^2$ is ample. □

Assume that F contains no elliptic curve.

Lemma 4. $(\mu = \mu' = 0)$ *is a finite set.*

Proof. Let D be the largest divisor in the intersection. Then D is invariant under the group P generated by the image of $\alpha_{\mathcal{N}}$ and $\beta_{\mathcal{N}}$. Then D comes from an effective divisor D' on F'/P where $\#P = 4$. So $\frac{(D^2)}{4} = 4\frac{(D')^2}{2}$ and $\frac{(D^2)}{2} \leq \frac{(\mu=0)^2}{2} = 2$. So by [**1**], D' comes from a divisor on a quotient of F'/D which is a point. So D' is empty. □

Lemma 5. (X, θ) *is a Jacobian.*

Proof. We need to see that θ is irreducible. If θ is reducible, we have $X = E \oplus R$ by [**1**], where θ is the sum of divisors depending on the factors. Thus $\theta \supset E \times x$ for a curve x. But this contradicts Lemma 4. □

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Department of Mathematics
Johns Hopkins University
Baltimore MD 21218