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Alexandrov spaces are a large class of metric spaces that includes Hilbert spaces, Riemannian manifolds and convex surfaces. In the framework of Alexandrov spaces, we examine the ambiguous locus of analysis and the cut locus of differential geometry, proving a general bisecting property, showing how small the ambiguous locus must be, and proving that typically the ambiguous locus and a fortiori the cut locus are dense.

Introduction

A metric space is called an *Alexandrov space* if it is an intrinsic metric space with curvature bounded below in the sense of Alexandrov (see definition on page 376). Such spaces were introduced by A.D. Alexandrov in 1955, along with spaces with curvature bounded above.

Hilbert spaces, Riemannian manifolds, convex surfaces, and convex subsets of these are examples of Alexandrov spaces.

This paper considers the ambiguous locus of analysis and the cut locus of differential geometry in the framework of Alexandrov spaces. We prove a general bisecting property, then show how small the ambiguous locus must be, and finally prove that typically the ambiguous locus and a fortiori the cut locus must be dense.

In fact, we discover that the apparently distant notions of cut locus and ambiguous locus share a common soul. Some more (metamathematical) searching brings to light another two independent developments of the same notion.

The ambiguous locus has been investigated by — among others — de Blasi and Myjak [3], [4], [5]; de Blasi, Kenderov and Myjak [6]; Myjak and Rudnicki [21]; Zhivkov [41]; de Blasi and Zamfirescu [7].

The cut locus, introduced by Poincaré in 1905 [25], received its name from Whitehead in 1936; it was studied by Myers in 1935–36 [19], [20], and subsequently by many other authors in the context of Riemannian geometry. Their contributions grew to what is today a large, well-established body of results.

More recently, Otsu and Shioya [22], Shiohama and Tanaka [27], Itoh [15], Zamfirescu [38] and others have investigated the cut locus in Alexandrov spaces. Shiohama and Tanaka considered in [27] the cut locus with respect to a compact set.

Independently, A. Rivière [26] studied the cut locus (under the name of “nervure”) with respect to closed subsets of a Euclidean space.

A fourth independent appearance of the cut locus (or, more exactly, of the ambiguous locus), with a more applied background, under the name of “medial axis”, appears in papers by Lee [17], Lee and Drysdale [18], Yap [29], Choi, Choi and Moon [10], and others.

Kunze [16] and Zamfirescu [39] have treated problems on the cut locus in the case of closed convex hypersurfaces without any differentiability assumptions.

By making use of the notion of porosity we shall see how small (σ -porous) the ambiguous locus must always be, and also how large (dense) it can sometimes be. These results relate to work in Banach spaces and in Riemannian manifolds.

Definitions. We work in a metric space (\mathcal{A}, ρ) .

A *segment between two distinct points* is a shortest path between them — a minimal segment in Otsu and Shioya’s terminology [22].

A *segment between a point x and a closed set K not containing x* is an arc between x and a point of K , not longer than any other such arc.

A *geodesic* is a curve which is locally a segment (for a precise definition see, e.g., [38]).

A *geodesic triangle* is a triangle with segments as sides.

Let S_k denote the 2-dimensional complete simply-connected Riemannian manifold of constant curvature $k < 0$ (a Lobachevskii plane).

If a, b, c belong to the metric space (\mathcal{A}, ρ) , let $\angle^* abc$ denote the angle of the geodesic triangle in S_k of side-lengths $\rho(a, b)$, $\rho(b, c)$, $\rho(c, a)$, opposite to the side of length $\rho(c, a)$.

Here, (\mathcal{A}, ρ) is called an *intrinsic metric space* if any two points admit a midpoint. An *Alexandrov space* is a complete intrinsic metric space (\mathcal{A}, ρ) such that every point of \mathcal{A} has a neighbourhood in which, for any four distinct points a, b, c, d , we have

$$\angle^* bac + \angle^* cad + \angle^* dab \leq 2\pi.$$

This angle condition says that (\mathcal{A}, ρ) has *curvature bounded below by k* in the sense of A.D. Alexandrov (note the dependence of \angle^* on k).

Berestovskii [2] proved that a complete intrinsic metric space (\mathcal{A}, ρ) is an Alexandrov space if and only if every point of \mathcal{A} has a neighbourhood in which any four points admit an isometric embedding in $S_{k'}$ for some $k' \geq k$.

Several other characterizations of Alexandrov spaces are given by Burago, Gromov and Perelman in [9]. We use some basic concepts and results developed in [9] and further investigated by Perelman [23], [24]. So, for example, in any Alexandrov space a geodesic starting at x has a direction at x (see the definition below) and the angle between two geodesics exists. Moreover, geodesics do not branch. The interested reader should consult the recent book of Burago, Burago and Ivanov [8].

Prerequisites

If x is a point in the Alexandrov space \mathcal{A} , the *space Σ_x of directions at x* is defined as the completion of the metric space Σ'_x consisting of classes of geodesics starting at x , all the geodesics of the same class overlapping, and the distance being the angle between representatives (see [9], p. 23). We shall identify the elements of Σ'_x with the directions of representatives. Every space of directions is known to be itself an Alexandrov space if \mathcal{A} has finite dimension [9].

Let \overline{xy} denote the direction of the segment xy at x .

We say that a set $A \in \mathcal{A}$ has a *direction τ* at its accumulation point x if, for $y \rightarrow x$ with $y \in A \setminus \{x\}$, \overline{xy} converges in Σ_x to τ .

If pa, pb are geodesics, let $\angle apb$ denote the angle between pa and pb , i.e., the distance from \overline{pa} to \overline{pb} .

We now recall three basic results from [9].

Lemma 1 ([9], p. 6). *If pa, pb, pc are geodesics in an Alexandrov space,*

$$\angle apb + \angle bpc + \angle cpa \leq 2\pi.$$

Here is a generalized form of Toponogov’s comparison theorem:

Lemma 2 ([9], p. 7). *For any geodesic triangle abc in an Alexandrov space,*

$$\angle^* abc \leq \angle abc, \quad \angle^* bca \leq \angle bca, \quad \angle^* cab \leq \angle cab.$$

Lemma 3 ([9], p. 6). *If in an Alexandrov space the segment $p_i q_i$ converges to pq and the segment $p_i r_i$ converges to pr , then*

$$\angle qpr \leq \liminf_{i \rightarrow \infty} \angle q_i p_i r_i.$$

Let \mathcal{A} be an Alexandrov space and $K \subset \mathcal{A}$ a closed set.

The *cut locus $C(K)$* of K is the set of all points $x \in \mathcal{A}$ such that there is a segment from x to K not extendable as a segment beyond x .

The *multijointed locus* of K is the set $M(K)$ of all points $x \in \mathcal{A}$ such that the distance from x to K is realized by at least two distinct segments from x to (not necessarily distinct) points in K . The points of $M(K)$ are said to be multijointed to K (see also [37]).

The *ambiguous locus* of K is the set $A(K)$ of all points $x \in \mathcal{A}$ such that the distance from x to K is realized by at least two segments from x to distinct points in K .

If K consists of a single point x , we write $C(x)$ and $M(x)$ instead of $C(\{x\})$ and $M(\{x\})$, respectively.

A point in \mathcal{A} is called an *endpoint* if it is not interior to any segment; let E denote the set of all endpoints of \mathcal{A} .

Clearly, $A(K) \subset M(K) \subset C(K)$ and $E \subset C(K)$.

The open ball with centre $x \in \mathcal{A}$ and radius α is denoted by $B(x, \alpha)$.

The boundary of a compact convex set with interior points in \mathbb{R}^n is called a *convex hypersurface*. The space of all convex hypersurfaces, equipped with the Pompeiu–Hausdorff metric, is a Baire space.

Most (or *typical*) means “all, except those in a set of first Baire category”.

Lemma 4 ([31]). *On most convex hypersurfaces, most points are endpoints.*

Equidistant sets

Both the ambiguous locus and the cut locus are known to enjoy a bisecting property with respect to certain pairs of segments.

Let a, b be (possibly coinciding) points in \mathcal{A} . The set $E(a, b)$ of all points joined by equally long but distinct segments with a and b will be called the *equidistant set* of $\{a, b\}$. Thus $E(a, b)$ coincides with $A(\{a, b\})$ if $a \neq b$, and with $M(\{a\})$ if $a = b$.

We shall prove a bisecting property of $E(a, b)$ which lies at the core of these phenomena.

Let $x \in E(a, b)$, and consider distinct segments $xa, xb \subset \mathcal{A}$ and $\alpha, \beta \subset \Sigma_x$ such that $\alpha = \overline{xa}$, $\beta = \overline{xb}$. Also, suppose $\alpha\beta$ is a segment in Σ_x .

A *bisector* of xa, xb is a nondegenerate arc $\Gamma \subset E(a, b)$ starting at x such that, for any $y \in \Gamma \setminus \{x\}$:

- 1) There is a segment xy with $\overline{xy} \in \alpha\beta$.
- 2) There are segments ya, yb such that $ya \rightarrow xa$ and $yb \rightarrow xb$ for $y \rightarrow x$.

This name is explained by the next theorem.

There may exist no bisector of xa, xb . This is the case, for example, if $a = b$ and a, x are antipodal on the standard sphere, because then $E(a, a)$ is a single point and cannot include any nondegenerate arc. But we may also have no bisector if $a \neq b$: consider a smooth convex (2-dimensional) surface with precisely two segments from x to a , say σ_1, σ_2 , and precisely one segment σ_3 from x to b such that, for $s_i \in \text{int}\sigma_i$,

$$\angle s_1 x s_2 + \angle s_2 x s_3 = \angle s_1 x s_3 < \pi.$$

Then σ_1 and σ_3 admit no bisector, because condition 2 is not satisfied. In this case, however, σ_1, σ_2 , as well as σ_2, σ_3 , would admit a bisector each.

Theorem 1. *Any bisector of xa, xb has a direction at x , namely the midpoint of $\alpha\beta$, where $\alpha = \overline{xa}$ and $\beta = \overline{xb}$.*

Proof. Let Γ be a bisector of xa, xb . If $y \in \Gamma \setminus \{x\}$ converges to x then $\angle^*xay \rightarrow 0$, which implies $\angle^*axy + \angle^*ayx \rightarrow \pi$. Analogously, $\angle^*bxy + \angle^*byx \rightarrow \pi$. Hence

$$\angle^*axy + \angle^*bxy + \angle^*ayx + \angle^*byx \rightarrow 2\pi.$$

By Lemma 2,

$$\liminf_{y \rightarrow x} (\angle axy + \angle bxy + \angle ayx + \angle byx) \geq 2\pi.$$

Since $\overline{xy} \in \alpha\beta$,

$$\angle axb = \angle axy + \angle bxy.$$

Hence

$$\liminf_{y \rightarrow x} (\angle axb + \angle ayx + \angle byx) \geq 2\pi.$$

Suppose

$$\limsup_{y \rightarrow x} (\angle axb + \angle ayx + \angle byx) > 2\pi.$$

By Lemma 3,

$$\liminf_{y \rightarrow x} \angle ayb \geq \angle axb.$$

Then

$$\limsup_{y \rightarrow x} (\angle ayb + \angle ayx + \angle byx) > 2\pi,$$

which contradicts the inequality

$$\angle ayb + \angle ayx + \angle byx \leq 2\pi,$$

which holds, by Lemma 1, for all y . Hence

$$\angle axb + \angle ayx + \angle byx \rightarrow 2\pi.$$

Using Lemma 2 again, we get $\angle axy - \angle^*axy \rightarrow 0$, $\angle bxy - \angle^*bxy \rightarrow 0$, $\angle ayx - \angle^*ayx \rightarrow 0$, $\angle byx - \angle^*byx \rightarrow 0$. But $\angle^*axy = \angle^*bxy$; therefore

$$\angle axy - \angle bxy \rightarrow 0$$

and

$$\lim_{y \rightarrow x} \angle axy = \lim_{y \rightarrow x} \angle bxy = \frac{1}{2} \angle axb.$$

Thus, the proof is finished.

Multijointed loci in Alexandrov spaces: the arbitrary case

A set A in a metric space (\mathcal{M}, ρ) is said to be *porous at* $x \in \mathcal{M}$ if there is a number $\alpha > 0$ such that every ball centered at x contains a ball $B(y, \alpha\rho(x, y))$ disjoint from A . It is called *porous* if it is porous at all its points, and *σ -porous* if it is a countable union of porous sets.

The main result of this section establishes the σ -porosity of the multijointed locus of any closed set in an arbitrary Alexandrov space \mathcal{A} . This extends results of Gruber [12], [13] and the author [35].

Otsu and Shioya [22] have shown that the cut locus $C(x)$ of a point x in an n -dimensional Alexandrov space \mathcal{A} has n -dimensional Hausdorff measure 0. However we cannot expect $C(x)$ to be σ -porous because, by Lemma 4, there are Alexandrov spaces in which $C(x)$ is residual. About the multijointed locus Otsu and Shioya proved the stronger assertion $\dim M(x) \leq n - 1$ [22].

Also, Shiohama and Tanaka described in detail, for $n = 2$, the structure of $C(K)$ and $M(K)$ for any compact set K [27]. According to them, $M(K)$ is a countable union of Jordan arcs.

Unfortunately, the Hausdorff dimension being at most $n - 1$ does not imply σ -porosity (see Zajíček [30]), nor are Jordan arcs necessarily σ -porous (see Foran [11]).

Lemma 5. *Let \mathcal{A} be an Alexandrov space and let $A \subset \mathcal{A}$ be such that, for some points x, x' in \mathcal{A} and some positive number ε , all points z in a neighbourhood of x that satisfy $\angle^* zxx' \leq \varepsilon$ do not belong to A . Then A is porous at x .*

Proof. Let N be the neighbourhood of x considered in the statement. It suffices to take $\alpha = \sin \varepsilon$ in the definition of porosity. Then, for any segment xx' and any point $y \in xx'$,

$$z \in B(y, \alpha\rho(x, y)) \cap N \implies \sin \angle^* zxy \leq \alpha \implies \angle^* zxx' \leq \varepsilon.$$

Hence $z \in N \setminus A$, and A is porous at x .

Theorem 2. *In an Alexandrov space, the multijointed locus of any closed set is σ -porous.*

Proof. Let \mathcal{A} be an Alexandrov space and let $K \subset \mathcal{A}$ be closed. For any $m \in \mathbb{N}$, let $M_m \subset M(K)$ be the set of all points u joined with K by (at least) two segments making an angle at x not less than m^{-1} . We have

$$M(K) = \bigcup_{n=1}^{\infty} M_n,$$

and we only have to prove that M_m is porous for each $m \in \mathbb{N}$.

Consider $u, v \in M_m$ and let a, b be interior points of the two segments from u to K under consideration. If simultaneously

$$\angle auv > \pi - (2m)^{-1} \quad \text{and} \quad \angle buv > \pi - (2m)^{-1},$$

then

$$\angle auv + \angle buv + \angle aub > 2\pi,$$

which contradicts Lemma 1. Thus, there exists a segment ux from u to K such that, in the geodesic triangle xuv ,

$$\angle xuv \leq \pi - (2m)^{-1}.$$

By Lemma 2, $\angle^*xuv \leq \pi - (2m)^{-1}$ too.

Now, keep v fixed and let $u \in M_m$ converge to v . Then $\angle^*uxv \rightarrow 0$. But this yields $\angle^*xvu > (3m)^{-1}$ for all $u \in M_m$ in some neighbourhood of v . Then, by Lemma 5, M_m is porous at v . Thus, M_m is porous, $M(K)$ is σ -porous, and the proof is finished.

For the special case of a point x on a convex surface, the σ -porosity of $M(x)$ was proved in [35] (see also Gruber [12], [13]).

Ambiguous and multijointed loci in Alexandrov spaces: the generic case

In this section we consider the ambiguous locus in typical cases. We investigate typical compact sets or take the ambient space to be typical. Let $\mathcal{K}(\mathcal{A})$ be the space of all compact sets in the Alexandrov space (\mathcal{A}, ρ) , endowed with its usual Pompeiu–Hausdorff metric h , based on ρ .

The next theorem generalizes Theorem 1 from [34] considerably. This result of [34] has already been repeatedly strengthened, for example in [3], [4], [5], [6], [41], [7].

Theorem 3. *For most compact sets in a separable Alexandrov space of dimension at least 2, the ambiguous locus is dense.*

Proof. Let $B(x_0, \varepsilon)$ be an open ball in the Alexandrov space \mathcal{A} . We prove that the compact sets $K \subset \mathcal{A}$ for which the ambiguous locus $A(K)$ does not meet $B(x_0, \varepsilon)$ form a nowhere dense set.

Indeed, in any open set $\mathcal{O} \subset \mathcal{K}(\mathcal{A})$, there exists a compact set K avoiding x_0 . Let $y_0 \in K$ be a point closest to x_0 and take y_1 different from y_0 on a segment σ from x_0 to y_0 , so that $K \cup \{y_1\} \in \mathcal{O}$. There is a whole ball $B(y_1, \eta)$ disjoint from K such that for any finite set $F \subset B(y_1, \eta)$, $K \cup F \in \mathcal{O}$.

Since $\dim \mathcal{A} \geq 2$, we can choose $y_2 \in B(y_1, \eta) \setminus \sigma$.

Consider the point $y_3 \in \sigma$ with $\rho(x_0, y_2) = \rho(x_0, y_3)$. Then $y_3 \in B(y_1, \eta)$ too. Let σ_2, σ_3 be segments from x_0 to y_2, y_3 respectively. Choose points $x_2 \in \sigma_2, x_3 \in \sigma_3$ such that $\rho(x, x_2) = \rho(x, x_3) < \varepsilon$. Clearly, $\rho(x_2, y_3) > \rho(x_2, y_2), \rho(x_3, y_2) > \rho(x_3, y_3)$.

Let

$$\nu < \min \{ \rho(x_2, y_3) - \rho(x_2, y_2), \rho(x_3, y_2) - \rho(x_3, y_3) \}.$$

If $h(K', K \cup \{y_2, y_3\}) < \nu/2$ in $\mathcal{K}(\mathcal{A})$, then K' meets both $B(y_1, \nu/2)$ and $B(y_2, \nu/2)$. Therefore, for $i = 2, 3$, the point of K' closest to x_i lies in $B(y_i, \nu/2)$. The function

$$f(x) = d(x, K' \cap B(y_2, \nu/2)) - d(x, K' \cap B(y_3, \nu/2)),$$

where $d(x, M)$ means the infimum of $\rho(x, y)$ for $y \in M$, is continuous, $f(x_2) < 0$ and $f(x_3) > 0$. Thus there is a point $x \in x_2x_0 \cup x_0x_3$ with $f(x) = 0$; that is, $x \in A(K')$.

Hence the set $\mathcal{K}_{m,n} \subset \mathcal{K}(\mathcal{A})$ of all compact sets K for which $A(K) \cap B(x_m, 1/n) = \emptyset$ is nowhere dense. Since \mathcal{A} is separable, $\{x_m\}_{m=1}^\infty$ can be chosen to be dense in \mathcal{A} , and then the set $\bigcup_{m,n=1}^\infty \mathcal{K}_{m,n}$ of all compact sets with nondense ambiguous locus is of first Baire category.

For an interesting analog of Theorem 3 concerning convex hypersurfaces instead of compact sets, see [40].

There are Alexandrov spaces in which the multijointed locus is dense for more compact sets than “just” those of a second Baire family. More precisely, in these spaces, the multijointed locus of any element of an open subset of $\mathcal{K}(\mathcal{A})$ is dense in its complement.

The following result strengthens in several directions Theorem 2 in [35] and C. Vilcu’s second assertion in the first theorem of [28].

Theorem 4. *Let the compact Alexandrov space \mathcal{A} be an n -dimensional topological manifold ($n \geq 2$, finite), and $A \subset \mathcal{A}$. Assume that the set of endpoints of \mathcal{A} is dense in A (with respect to its relative topology) and K is a closed subset of a union B of components of $\mathcal{A} \setminus A$. If B is not dense in \mathcal{A} then the multijointed locus $M(K)$ is dense in the interior of $\mathcal{A} \setminus B$.*

Proof. Suppose there is an open set $O \subset \mathcal{A} \setminus B$ homeomorphic to \mathbb{R}^n and disjoint from $M(K)$. Since every point $y \in \mathcal{A} \setminus M(K)$ is joined by precisely one segment σ_y to the closest point π_y of K , the mapping $y \mapsto \sigma_y$ is continuous in O . The mapping $y \mapsto \pi_y$ is continuous too.

If $0 \leq r \leq 1$ and $\phi(y, r)$ denotes the point of σ_y at distance $r\rho(y, \pi_y)$ from y , then ϕ is continuous in both variables.

Moreover the function ψ_r defined by $\psi_r(y) = \phi(y, r)$ is injective for $0 \leq r < 1$, since geodesics do not branch and therefore $y \neq y'$ implies $\sigma_y \subset \sigma_{y'}$ or $\sigma_{y'} \subset \sigma_y$ or $\sigma_y \cap \sigma_{y'} = \{\pi_y\} \cap \{\pi_{y'}\}$. The inverse of ψ_r , defined on $\psi_r(O)$, is obviously continuous too.

Let $y_0 \in O$. For some r distinct from 1, $\psi_r(y_0)$ belongs to A and is different from y_0 . We may consider O chosen such that $\psi_r(O)$ lies in a neighbourhood of y_0 homeomorphic to \mathbb{R}^n . Then $\psi_r(O)$ is open (see, e.g., [14, Theorem 6-54]). Hence $\psi_r(O) \cap A$ is nonempty and open in the relative topology of A . Now choose the endpoint z in this set. Then z must belong to a segment σ_y with $y \in O$, and is not an endpoint of σ_y . This contradiction completes the proof.

Corollary 1. *In a space \mathcal{A} as in Theorem 4, assume that the closed set K is strictly included in the open set O , and that the set of endpoints of \mathcal{A} is dense in $O \setminus K$. Then $M(K)$ is dense in $\mathcal{A} \setminus K$.*

Example. Consider a smooth torus T embedded in \mathbb{R}^3 and a point $p \in T$ of positive Gauss curvature. Near p and outside T take a convex hypersurface C of the type described in Lemma 4.

If C is small enough and close enough to p , each line segment s joining a point of C with a point of T and not meeting the interiors (bounded components of the complements) of C and T has an endpoint on T of positive Gauss curvature. The union of all line segments s has a boundary S . Then the (topological) closure of

$$C \cup S \cup T \setminus ((C \cap S) \cup (S \cap T))$$

is a torus T' , the endpoints of which lie densely in $C \setminus S$. By Corollary 1, for any compact set $K \subset C$ disjoint from S , $M(K)$ is dense in $T' \setminus K$.

Theorem 5. *On most convex hypersurfaces S , for any compact set $K \subset S$, the multijointed locus $M(K)$ is dense in $S \setminus K$.*

This follows from Corollary 1 and Lemma 4.

By Theorems 3 and 5, we can say that every compact set on most convex hypersurfaces and most compact sets on every convex hypersurface have multijointed loci dense in their complements.

In a Baire metric space, “nearly all” means “all, except those in a σ -porous set” [32].

Lemma 6. *On any convex hypersurface, nearly all compact sets are porous.*

This follows from Theorem 2 in [33] applied to \mathbb{R}^{n-1} , together with the observation that the convex hypersurface can be tiled into finitely many pieces admitting bi-Lipschitz bijections to pieces of \mathbb{R}^{n-1} , while porosity and σ -porosity are invariant under bi-Lipschitz transformations.

Lemma 6 and Theorem 5 immediately imply the following result:

Theorem 6. *On most convex hypersurfaces S , for nearly all compact sets the multijointed locus is dense on S .*

Suggestion. Show that Theorem 6 is also true for the ambiguous locus instead of the multijointed locus, although Theorem 5 is not.

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