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The first disentanglement of a multigerm $f : (\mathbb{C}^{2n}, S) \to (\mathbb{C}^{3n}, 0)$ is shown to be homotopically equivalent to a wedge of $(n+1)$-spheres and 2-spheres. For corank-1 monogerms in the same dimensions, the second disentanglement is shown to be homotopically equivalent to a wedge of n-spheres and circles.

Good real perturbations of such maps are investigated and it is shown for multigerms in the case $n = 1$ with a good real perturbation that the real and complex disentanglements are homotopically equivalent.

1. Introduction

For a finitely $\mathscr{A}$-determined map-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, where S is a finite set, $n < p$, and (n, p) are in Mather's nice dimensions, there exists a disentanglement: the image of a stable perturbation of f. The images of any two stabilisations are homeomorphic; see [Marar 1993]. The definition of a disentanglement is analogous to that of the Milnor fibre in the $n \geq p$ and finitely $\mathscr{K}$-determined case. For monogerms, or when $p \leq n + 1$, it has been shown that the disentanglement of f is homotopically equivalent to a wedge of spheres. This is again analogous to Milnor fibre theory, but in contrast, the spheres may be of different dimensions in the case $p > n + 1$. See [Houston 1997] and [Mond 1991b] for details.

In this paper we study the situation for maps $f : (\mathbb{C}^{2n}, S) \to (\mathbb{C}^{3n}, 0)$. This is a natural generalisation of the case of germs of maps from surfaces to 3-space, which has been studied by Mond [1987; 1991a; 1991b; 1995] and by Marar and Mond [1996]. The usual generalisation of this work has been to map germs from $(\mathbb{C}^n, S)$ to $(\mathbb{C}^{n+1}, 0)$ since then the image of the map is still a hypersurface — a useful fact that is certainly not true in the case under consideration here, as the spaces are not, in general, even local complete intersections. Many of the results in this paper are direct generalisations of the work of Mond cited above, and, as well as being interesting in their own right, they also provide insight into the topology of disentanglements of multigerms, their real perturbations, and into higher disentanglements of monogerms in general.

The $\mathbb{C}^{2n}$ to $\mathbb{C}^{3n}$ case is tractable because the multiple point spaces (the closure of k-tuples of points that have the same image) of the stabilisation are particularly

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simple. The stable perturbation has a nonsingular source, each component of which is contractible, because this is true for all disentanglements defined as below, but in the situation considered, in addition, the double point space of the map has dimension n and the triple point space has dimension zero. One of the main tools in the theory is an image computing spectral sequence introduced in [Goryunov and Mond 1993], the E^1 terms of which depend on the alternating homology of the multiple point spaces. In our situation this means that $E^1_{p,q}$ is trivial for $p \notin \{0, 1, 2\}$, and furthermore, $E^1_{2,q} = 0$ for $q \neq 0$ as the triple point space is zero-dimensional. Thus, using the spectral sequence is a practical proposition.

Section 2 introduces definitions and notations. The first main result in Section 3 is that the disentanglement in the multigerm case is homotopically equivalent to a wedge of $(n+1)$-spheres and 2-spheres. This corresponds to what happens in general for monogerms [Houston 1997, Proposition 4.24], and thus provides evidence for multigerm disentanglements being homotopically equivalent to a wedge of spheres. (The multigerm problem is still open because there is a certain class of multigerms for which the methods employed in the monogerm case do not work.)

In a disentanglement we can look at the closure of the set of points with two or more preimages; this is called the second disentanglement. One can generalise to the k-th disentanglement: the closure of the set of points with k or more preimages. For f a corank-1 monogerm and $p = n+1$ it was shown in [Houston 2002a] that the k-th disentanglement is homotopically equivalent to a wedge of $(n-k+1)$-spheres; this followed [Goryunov and Mond 1993], where it was shown that such a disentanglement has reduced rational cohomology only in dimension $n-k+1$. In the situation of maps from $2n$-space to $3n$-space we again have to restrict to corank-1 monogerms to treat the second disentanglement. This is also homotopically equivalent to a wedge of spheres, this time of dimensions n and 1. The topology of the higher disentanglements ($k > 1$) is difficult to calculate in general and this is the first occasion on which we have a nontrivial result for $p > n+1$.

We do not formally treat the third disentanglement since this is just the set of triple points.

If one restricts the disentanglement map to the real part of the source, one gets a real perturbation. For maps $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ Mond [1996] defines a real perturbation to be good if the rank of the n-th integer homology group of the image of the real perturbation is equal to the rank of the n-th integer homology group of the disentanglement. In Section 4 this notion is generalised and it is shown in the case of $f : (\mathbb{C}^{2n}, S) \to (\mathbb{C}^{3n}, 0)$ that if the inclusion of the real disentanglement into the complex one produces an isomorphism on integer homology groups, then the spaces are homotopically equivalent. This is applied in the $n = 1$ case to show that if f has a good real perturbation, the real and complex disentanglements are homotopically equivalent. For $n = 1$, this improves a result in [Marar and Mond 1996] stating that the existence of a good real perturbation produces an isomorphism on the rational cohomology groups, and it is also proves a case of a conjecture from [Cooper and Mond 1998].

In [Marar and Mond 1996] it is shown that each member in the H_k series of singularities given by $f(x, y) = (x, y^3, xy + y^{3k-1})$ has a good real perturbation. Here, we generalise this series to the $2n$-space to $3n$-space situation and show that not only do the maps have good real perturbations but the real and complex r-th disentanglements for $r = 1, 2, 3$ are in fact homotopically equivalent as well.

2. Notation and definitions

Multiple point spaces. Let $f : X \to Y$ be a proper continuous map.

Definition 2.1. The *k-th multiple point space*, $D^k(f)$, of the map $f : X \to Y$ is the set of points

$$D^k(f) := \text{closure}\left\{(x_1, \dots, x_k) \in X^k \;\middle|\; f(x_1) = \dots = f(x_k),\ x_i \neq x_j \quad \text{for } i \neq j\right\}.$$

The group of permutations on k objects, denoted S_k, acts on $D^k(f)$ in the natural way. For each k there exists a map $\varepsilon_k : D^k(f) \to D^{k-1}(f)$ given by restriction of the natural projection map $(x_1, \dots, x_k) \mapsto (x_1, \dots, x_{k-1})$. We also have a map $\varepsilon^k : D^k(f) \to Y$ given by $\varepsilon^k(x_1, \dots, x_k) = f(x_1)$.

Define $D^2_1(f)$ to be the image of ε_2 in $D^1(f)$. The double point space of ε_2 is homeomorphic, by an S_3-equivariant map, to the triple point space of f. Furthermore, the k-th multiple point spaces, $k \geq 2$, for the map $D^2_1(f)$ to Y are the same as for f. This is part of a general phenomenon; see [Goryunov and Mond 1993].

Definition 2.2. The set of points $(x_1, \dots, x_k) \in D^k(f)$ such that $x_i = x_j$ for some $i \neq j$ is called the diagonal of $D^k(f)$.

If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a corank-1 multigerm, the multiple point spaces of f can be defined using adapted Vandermonde determinants [Marar and Mond 1989]. For finitely $\mathscr{A}$-determined maps this definition coincides with the one given above as long as $\dim D^k(f) > 0$ or f is stable. When $\dim D^k(f)$ is expected to be 0 then $D^k(f)$ can be empty while the Vandermonde definition gives a space lying in the diagonal. It will be clear from context which definition is being used.

The Vandermonde definition leads to the following: If the map f is finitely $\mathscr{A}$-determined, the multiple point spaces are isolated complete intersection singularities (of dimension $nk - p(k - 1)$) for all $nk - p(k - 1) \geq 0$, and vice versa. The multiple point spaces for the disentanglement map (see Section 2) are Milnor fibres of these singularities. See [Marar and Mond 1989] for details.

We finish with a definition for multiple point spaces in the image of a map.

Definition 2.3. The *k-th image multiple point space of f*, denoted $M_k(f)$, is

$$M_k(f) := \text{closure}\left\{y \in Y \;\middle|\; \#f^{-1}(y) \geq k\right\}.$$

This will be used in defining the k-th disentanglement.

Alternating homology. Let sign : $S_k \to \{1, -1\}$ be the usual sign representation of the symmetric group.

Definition 2.4. Let M be a module upon which S_k acts. Then

$$M^{\mathrm{alt}} := \{ m \in M \mid h(m) = \mathrm{sign}(h) m \text{ for all } h \in H \}$$

is the *alternating module of M with respect to S_k.*

Definition 2.5. The group S_k *acts cellularly* on a cellular complex if (i) S_k takes cells to cells and (ii) if a point of an open cell is fixed by an element of S_k, then the whole cell is fixed by that element.

Definition 2.6. Suppose Y is a space upon which S_k acts cellularly and let $C_*(Y)$ denote the cellular chain complex of Y. The *alternating homology of Y with respect to S_k and the coefficient group G* is

$$H_i^{\mathrm{alt}}(Y; G) := H_i\big((C_*(Y))^{\mathrm{alt}} \otimes G\big).$$

Define an operator $\mathrm{Alt}_{\mathbb{Z}} : M \to M$ by

$$\mathrm{Alt}_{\mathbb{Z}} = \sum_{h \in S_k} \mathrm{sign}(h)\, h.$$

Once can take the homology generated by $\mathrm{Alt}_{\mathbb{Z}}\, C_*(Y)$ — we call it the *alternated homology of Y.*

Theorem 2.7. *Suppose Y is an S_k-invariant subset of X^k and S_k acts on X^k through permutation of copies of X. Then*

$$H_i^{\mathrm{alt}}(Y; G) \cong H_i\big((\mathrm{Alt}_{\mathbb{Z}}\, C_*(Y)) \otimes G\big) \quad \textit{for all } i.$$

For a proof of this nontrivial theorem see [Houston 2000]. We can use the operator $\mathrm{Alt}_{\mathbb{Z}}$ to study alternating homology; this will be particularly useful for the zeroth alternating homology groups.

Definition 2.8. Two complexes K and L on which S_k acts are *S_k-homotopically equivalent* if there exist two equivariant maps $\alpha : K \to L$ and $\beta : L \to K$, such that the compositions $\alpha \circ \beta$ and $\beta \circ \alpha$ are homotopic to the identities in the class of S_k-equivariant maps.

Complexes that are S_k-homotopically equivalent have the same alternating homology.

We have a similar condition relating, not to homotopy, but to homology.

Definition 2.9. Suppose G is a group. Then two topological spaces X and Y are *G-homology equivalent* if there exists a continuous map $f : X \to Y$ such that $f_* : H_i(X; G) \to H_i(Y; G)$ is an isomorphism for all i. Such a map is called a *G-homology equivalence.*

Image computing spectral sequence. Now we state a general version of the image computing spectral sequence introduced in [Goryunov and Mond 1993] and further developed in [Goryunov 1995; Houston 2000].

Theorem 2.10. *Suppose that $f : X \to Y$ is a proper and finite analytic map between compact subanalytic spaces and that $\widetilde{X}$ is a subanalytic subset of X. Then, for any coefficient group G, there exists a spectral sequence*

$$E^1_{p,q}(f, f|\widetilde{X}) = H^{\mathrm{alt}}_q\big(D^{p+1}(f), D^{p+1}(f|\widetilde{X}); G\big) \implies H_*\big(f(X), f(\widetilde{X}); G\big).$$

The differential is induced from the map $\varepsilon_k : D^k(f) \to D^{k-1}(f)$.

Generally in this paper we take $\widetilde{X} = \varnothing$.

To produce this theorem one uses a *geometric realisation of a semisimplicial resolution of the image* (which, for simplicity as in [Goryunov 1995], we call the *semisimplicial resolution*). This is a space for which a point of the image is represented by a $(k-1)$-simplex if the point has k preimages. This construction will be used to describe the homotopy type of the disentanglements.

The resolution can be constructed by two methods. For the general method see [Houston 2000]; we only describe the case in which the image is embeddable in $\mathbb{R}^N$ for some N, as this will be true for all our images.

Let m be the maximal number of preimages of a point of f. Consider an embedding of X into some $\mathbb{R}^N$ so that any m distinct points of X do not lie in an $(m-2)$-dimensional affine plane. Let $x_1, \dots, x_k \in X$ be the preimages of a point $y \in Y$ and consider in $y \times \mathbb{R}^N$ the closed $(k-1)$-dimensional simplex with vertices $(y, x_1), \dots, (y, x_k)$. The union W of all such simplices for all points of Y is the semisimplicial resolution of Y.

The spectral sequence associated to the filtration of Y given by defining Y_k to be the union of simplices of dimension less than k, with $Y_1 = X$, leads to the spectral sequence of the theorem.

In the cases discussed in this paper the image of a map will be homotopically equivalent to its semisimplicial resolution.

Disentanglements. Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, with $n < p$, is a finitely $\mathscr{A}$-determined complex analytic map-germ. Then there exist

(i) a 1-parameter unfolding of f, $F : (\mathbb{C}^n \times \mathbb{C}, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}, 0 \times 0)$, such that $F(x, 0) = f(x)$,

(ii) a closed ball B_ϵ in $\mathbb{C}^p$ centred at 0 and of radius ϵ,

(iii) and a closed disc D_δ in $\mathbb{C}$ centred at 0 and of radius δ,

such that for a proper representative of F,

(a) the map $f_t := F\big|F^{-1}(B_\epsilon \times \{t\}) : F^{-1}(B_\epsilon \times \{t\}) \to B_\epsilon \times \{t\}$ is topologically stable for all $t \in D_\delta - \{0\}$ (and is stable if (n, p) is in the nice dimensions);

(b) the image of f_0 is Whitney stratified so that every $(2p-1)$-sphere $S_{\epsilon'} \subset B_\epsilon$ of radius ϵ' centred at 0 is stratified transverse to it for all $0 < \epsilon' \leq \epsilon$;
(c) the map f_t is topologically right-left equivalent to $f_{t'}$ for all $t, t' \in D_\delta - \{0\}$.

Details can be found in [Marar 1993] and [Goryunov and Mond 1993].

Definition 2.11. The ball B_ϵ is called a *Milnor ball* and ϵ is called a *Milnor radius*. The disc D_δ is called a *Milnor disc*.

The next definition is important in the rest of the paper.

Definition 2.12. The set $M_k(f_t)$ for $t \neq 0$ is called the *k-th disentanglement of f* and is denoted $\mathrm{Dis}_k(f)$.

We shall mainly be interested in $\mathrm{Dis}_1(f) = M_1(f_t)$, the *first disentanglement* (or more usually just the disentanglement), and $\mathrm{Dis}_2(f_t)$, the *second disentanglement*. We shall call f_t, for $t \neq 0$, the *disentanglement map*. The image computing spectral sequence of Theorem 2.10 exists for this map, and since $(2n, 3n)$ are in the nice dimensions the map is stable, not just topologically stable.

The disentanglement of f is essentially unique in that for any other unfolding for which a disentanglement can be constructed the two disentanglements are homeomorphic. Again see [Marar 1993; Goryunov and Mond 1993] for details.

3. Topology of the first disentanglement

Homology of the first disentanglement. For maps $f : (\mathbb{C}^{2n}, S) \to (\mathbb{C}^{3n}, 0)$ we will investigate the image computing spectral sequence for the disentanglement map defined above and show that the first disentanglement has free abelian homology in dimensions $n+1$ and 2.

Theorem 3.1. *Suppose that* $f : (\mathbb{C}^{2n}, S) \to (\mathbb{C}^{3n}, 0)$, *for* $n \geq 1$, *is a finitely* $\mathscr{A}$-*determined multigerm and that* f_t *is the disentanglement map. Let* $T(f_t)$ *be the number of triple points in the image of* f_t *and let* m *be the number of elements of* S. *Then the image computing spectral sequence* $E_{p,q}$ *for* f_t *degenerates at* E^2. *The* $E^1_{p,q}$ *terms have the following form*:

q			
$H_n^{\mathrm{alt}}(D^k(f_t))$	0	$\mathbb{Z}^{\mu_1}$	0
$H_{n-1}^{\mathrm{alt}}(D^k(f_t))$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$H_1^{\mathrm{alt}}(D^k(f_t))$	0	0	0
$H_0^{\mathrm{alt}}(D^k(f_t))$	$\mathbb{Z}^m$	$\mathbb{Z}^{\binom{m}{2}}$	$\mathbb{Z}^{T(f_t)}$
	$D^1(f_t)$	$D^2(f_t)$	$D^3(f_t)$

p

for some μ_1. *The* $E^2_{p,q}$ *groups have the form*

q ↑			
	0	$\mathbb{Z}^{\mu_1}$	0
	0	0	0
	$\vdots$	$\vdots$	$\vdots$
	0	0	0
	$\mathbb{Z}$	0	$\mathbb{Z}^{\mu_2}$
	$D^1(f_t)$	$D^2(f_t)$	$D^3(f_t)$
		p →	

where $\mu_2 = T(f_t) - \frac{1}{2}(m-2)(m-1)$. *All other terms are zero.*

Proof. Since f is finitely $\mathscr{A}$-determined, there exists a 1-parameter unfolding

$$F : (\mathbb{C}^{2n} \times \mathbb{C}, S \times \{0\}) \to (\mathbb{C}^{3n} \times \mathbb{C}, 0 \times 0)$$

such that $F(x, 0) = f(x)$ and $f_t(x) = F(x, t)$ gives the disentanglement map (which is stable). Through a choice of a good representative we get a map $F : U \to V$ such that the image of F is contractible and we get a fibration of the form of [Houston 1997, Propositions 3.20 and 3.21], i.e., maps $g_k : D^k(F) \to D_\delta$, with D_δ a sufficiently small Milnor disc, with fibres $D^k(f_t)$ over $D_\delta - \{0\}$.

Since f_t is stable, $\dim_{\mathbb{C}} D^k(f_t) = 2nk - 3n(k-1) = n(3-k)$, provided this is nonnegative. By [Houston 1997, Corollary 3.2] the group $H^{\text{alt}}_i(D^k(f_t); \mathbb{Z})$ is free abelian for $i = n(3-k)$, where $k \le 3$.

Theorems 3.13 and 3.30 of [Houston 1997] give $H^{\text{alt}}_i(D^k(F), D^k(f_t); \mathbb{Z}) = 0$ for $i \le n(3-k) = \dim_{\mathbb{C}} D^k(f_t)$. By [Houston 1997, Section 2.3 and Proposition 3.19], the set $D^k(F)$ contracts equivariantly onto $D^k(F) \cap (\varepsilon^k)^{-1}(0)$ for all k, and so the alternating homology of these multiple point spaces is the same as for the map $g : \{m \text{ points}\} \to \{1 \text{ point}\}$. Thus, by [Houston 1997, Example 2.5], we find

$$H^{\text{alt}}_0(D^1(F); \mathbb{Z}) = \mathbb{Z}^m,$$
$$H^{\text{alt}}_0(D^2(F); \mathbb{Z}) = \mathbb{Z}^{\binom{m}{2}},$$
$$H^{\text{alt}}_0(D^3(F); \mathbb{Z}) = \mathbb{Z}^{\binom{m}{3}}.$$

Note that $H^{\text{alt}}_i(D^k(f); \mathbb{Z}) = 0$ for $i \ne 0, n(3-k)$. From these facts we can calculate that $E^1_{p,q}$ for f_t has the stated form.

For E^2 we need only compare the $E^1_{*,0}$ complexes for F and f_t in the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & H^{\text{alt}}_0(D^1(f_t); \mathbb{Z}) & \longleftarrow & H^{\text{alt}}_0(D^2(f_t); \mathbb{Z}) & \longleftarrow & H^{\text{alt}}_0(D^3(f_t); \mathbb{Z}) & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & H^{\text{alt}}_0(D^1(F); \mathbb{Z}) & \longleftarrow & H^{\text{alt}}_0(D^2(F); \mathbb{Z}) & \longleftarrow & H^{\text{alt}}_0(D^3(F); \mathbb{Z}) & &
\end{array}$$

The bottom row is exact except at the first position, where it has homology equal to $\mathbb{Z}$ because the image of F is contractible and $E^1_{p,q}(F)=0$ for $q>0$. The first and second vertical arrows are isomorphisms and the third is surjective, so

$$E^2_{0,0}(f_t)=E^2_{0,0}(F)=\mathbb{Z}, \quad E^2_{1,0}(f_t)=E^2_{1,0}(F)=0,$$

and $E^2_{2,0}(f_t)$ is free abelian. Thus $E^2(f_t)$ has the form of the theorem statement.

That $\mu_2=T(f_t)-\frac{1}{2}(m-2)(m-1)$ follows from a simple calculation involving Euler characteristics applied to the complex

$$0 \leftarrow \mathbb{Z}^m \leftarrow \mathbb{Z}^{\binom{m}{2}} \leftarrow H_0^{\mathrm{alt}}(D^3(f_t);\mathbb{Z}) \cong \mathbb{Z}^{T(f_t)} \leftarrow 0,$$

which has homology $\mathbb{Z}$, 0, $\mathbb{Z}^{\mu_2}$. □

Corollary 3.2. *Suppose that $f:(\mathbb{C}^{2n},S)\to(\mathbb{C}^{3n},0)$, $n\geq 1$, is a finitely $\mathscr{A}$-determined multigerm. The reduced integer homology of the first disentanglement of f is free abelian in dimensions $n+1$ and 2 and zero in all other dimensions.*

Proof. The spectral sequence for f_t calculates the homology of $\mathrm{Dis}_1(f)$. Hence, from the theorem, $H_0(\mathrm{Dis}_1(f);\mathbb{Z})\cong E^2_{0,0}\cong\mathbb{Z}$. If $n=1$, then $H_2(\mathrm{Dis}_1(f);\mathbb{Z})\cong\mathbb{Z}^{\mu_1+\mu_2}$, else $H_2(\mathrm{Dis}_1(f);\mathbb{Z})\cong E^2_{2,0}\cong\mathbb{Z}^{\mu_2}$ and $H_{n+1}(\mathrm{Dis}_1(f);\mathbb{Z})\cong E^2_{1,n}\cong\mathbb{Z}^{\mu_2}$. □

Homotopy type of the first disentanglement. In this section we show that the first disentanglement of a finitely $\mathscr{A}$-determined multigerm $f:(\mathbb{C}^{2n},S)\to(\mathbb{C}^{3n},0)$ is homotopically equivalent to a wedge of spheres of dimension $n+1$ and 2. For monogerms this was proved in [Houston 1997, Theorem 4.24]. The case $n=1$ was effectively proved earlier in [Mond 1991b]; the statement there is made for monogerms but the proof works equally well for the multigerm case.

Lemma 3.3. *Suppose $g:U\to Y$ is a finite and proper surjective simplicial map of compact spaces* [May 1967], *with U contractible, $D^3(g)$ a finite set of points or curves with each component containing a point in the diagonal, and $D^4(f)=\varnothing$. Suppose that for the image computing spectral sequence we have*

$$E^1_{0,0}\cong\mathbb{Z}, \quad E^1_{1,n}\cong\mathbb{Z}^{\mu_1}, \quad E^1_{2,0}\cong\mathbb{Z}^{\mu_2}$$

for $n\geq 1$ and that all other terms at E^1 are zero. Then Y is homotopically equivalent to a wedge of μ_1 $(n+1)$-spheres and μ_2 2-spheres.

Proof. The proof uses some theorems from [Houston 1997] that were stated for complex analytic maps, but can be adapted to work in the current context. Also, the spectral sequence of such a map exists by [Houston 2000].

From the semisimplicial resolution of the image we have a filtration

$$Y_0=\varnothing, \quad U=Y_1\subset Y_2\subset Y_3,$$

such that Y_3, the semisimplicial resolution of Y, is homotopically equivalent to Y; see Section 2. As $H_0^{\mathrm{alt}}(D^2(g);\mathbb{Z})=0$ we get from [Houston 1997, Corollary 4.19] that Y is simply connected.

If $n=1$, then Y is a simply connected complex with free abelian integer homology in dimension 2 (by Corollary 3.2) and hence is homotopically equivalent to a wedge of 2-spheres. Thus assume $n>1$.

From the construction of the image computing spectral sequence (see [Goryunov 1995] or [Houston 2000]), we know $H_{i-k+1}^{\mathrm{alt}}(D^k(g);\mathbb{Z})\cong H_i(Y_k,Y_{k-1};\mathbb{Z})$. From $H_0^{\mathrm{alt}}(D^2(g);\mathbb{Z})=0$ we get $\pi(Y_2,Y_1)=0$ by [Houston 1997, Proposition 4.22]. Since $E^1_{1,i}=H_i^{\mathrm{alt}}(D^2(g);\mathbb{Z})\cong H_{i+1}(Y_2,Y_1;\mathbb{Z})$ we have $H_{n+1}(Y_2,Y_1;\mathbb{Z})\cong\mathbb{Z}^{\mu_1}$ and all other relative groups are trivial. Thus by [Houston 1997, Lemma 4.23] and the fact that $Y_1=U$ is contractible we deduce that $Y_2\simeq Y_1\vee(\bigvee_{\mu_1}S^{n+1})\simeq\bigvee_{\mu_1}S^{n+1}$.

Similarly $H_2(Y_3,Y_2)\cong\mathbb{Z}^{\mu_2}$ and all other groups are trivial, Y_2 is n-connected (so 1-connected since $n>1$), and $\pi_1(Y_3,Y_2)=0$ since we are adding 2-cells to Y_2 to get Y_3. Applying the same lemma again we find that $Y\simeq Y_3\simeq(\bigvee_{\mu_1}S^{n+1})\vee(\bigvee_{\mu_2}S^2)$. □

We now state one of the main theorems of this paper. The proof uses a novel trick that cannot immediately be applied to the general multigerm case but may still be useful in other contexts.

Theorem 3.4. *Suppose that $f:(\mathbb{C}^{2n},S)\to(\mathbb{C}^{3n},0)$, for $n\geq 1$, is a finitely $\mathscr{A}$-determined complex analytic multigerm. Then the first disentanglement*, $\mathrm{Dis}_1(f)$, *is homotopically equivalent to a wedge of spheres of dimension $n+1$ and* 2*. The number of* 2*-spheres arising from the triple point space of the disentanglement is equal to μ_2 in Theorem 3.1.* (*For $n>1$ the* 2*-spheres can only arise from the triple point set.*)

Proof. In the following proof let f denote the disentanglement map f_t, let Y be the image of f and let $U=U_1\sqcup U_2\sqcup\cdots\sqcup U_m$ be the source of f, where each U_i is path-connected. Effectively by induction we will construct a map $\tilde{f}$ for which the natural morphism $E^1_{p,q}(f)\to E^1_{p,q}(\tilde{f})$ is an isomorphism for all $q>0$, $E^2_{p,q}(f)\to E^2_{p,q}(\tilde{f})$ is an isomorphism for all p and q, and the image of $\tilde{f}$ has the same homotopy type as Y. At each step the main difference is that the complex $E^1_{p,0}$ is simpler than for the preceding map. We continue until we produce a map that satisfies the assumptions of Lemma 3.3.

Suppose that $(x_1,x_2,x_3)\in D^3(f)$ is a triple point such that $\varepsilon_{3*}(\mathrm{Alt}_{\mathbb{Z}}(x_1,x_2,x_3))$ is nonzero in $H_0^{\mathrm{alt}}(D^2(f);\mathbb{Z})$ and that each x_i is in a distinct component of U.

Then create $\tilde{f}:\widetilde{U}\to\widetilde{Y}$, where $\widetilde{U}$ is equal to U but with the three points x_1, x_2 and x_3 in U connected by intervals to a new point x. The set $\widetilde{Y}$ is Y with an interval attached to the point $y=f(x_1)=f(x_2)=f(x_3)$. Then let $\tilde{f}$ be the same as f but with the intervals x_i to x all mapped in the obvious way to the interval beginning at y.

Since f is the restriction of $\tilde{f}$ to a subset of $\widetilde{U}$ we have a morphism of spectral sequences $E^1_{p,q}(f) \to E^1_{p,q}(\tilde{f})$. The multiple point spaces $D^1(\tilde{f})$, $D^2(\tilde{f})$ and $D^3(\tilde{f})$ are the same as those of f but with previously unconnected components attached by intervals to x, (x,x) and (x,x,x) respectively. Thus $E^1_{p,q}(f) \to E^1_{p,q}(\tilde{f})$ is an isomorphism for all $q > 0$.

Since $\widetilde{Y}$ is Y with an interval attached by only one end, the two spaces are homotopically equivalent and the inclusion $Y \hookrightarrow \widetilde{Y}$ is a $\mathbb{Z}$-homology equivalence. This then implies, since both sequences collapse at E^2, that $E^2_{p,q}(f) \to E^2_{p,q}(\tilde{f})$ is an isomorphism for all p and q.

By doing this process repeatedly for triple points in distinct components of the (new) source we can produce a surjective map $f' : U' \to Y'$ such that $E^1_{p,q}(f) \to E^1_{p,q}(f')$ is an isomorphism for all $q > 0$, $E^2_{p,q}(f) \to E^2_{p,q}(f')$ is an isomorphism for all p and q and Y' is homotopically equivalent to Y.

We now wish to create a map that has $E^1_{0,0} = \mathbb{Z}$ and $E^1_{1,0} = 0$. Suppose that (x_1, x_2) is a point such that $\mathrm{Alt}_{\mathbb{Z}}(x_1, x_2)$ is nonzero, and thus is a generator of $H_0^{\mathrm{alt}}(D^2(f'); \mathbb{Z})$.

In a similar way to the case of triple points one can construct a map $\tilde{f}$ by joining the points x_1 and x_2 to a new point x. Again this map has the same properties as f' at E^2 and the image has the same homotopy type as Y'. Carry out this process for all generators of $H_0^{\mathrm{alt}}(D^2(\tilde{f}); \mathbb{Z})$. This produces a map $\tilde{f} : \widetilde{U} \to \widetilde{Y}$ satisfying

$$E^1_{p,q}(\tilde{f}) = \begin{cases} \mathbb{Z} & \text{for } (p,q) = (0,0), \\ \text{free abelian} & \text{for } (p,q) = (1,n) \text{ and } (2,0), \\ 0 & \text{otherwise.} \end{cases}$$

The set $\widetilde{U}$ is contractible as each original component of U was a contractible ball and $\widetilde{U}$ has all these components attached to a point.

The map has the form of Lemma 3.3 and so $Y \simeq \widetilde{Y}$ is homotopically equivalent to a wedge of spheres of dimensions $n+1$ and 2. □

4. Topology of the second disentanglement

In this section we investigate the topology of $\mathrm{Dis}_2(f)$, the second disentanglement, in the case that f is a corank-1 monogerm. When $n = 1$ the rational cohomology has been studied in [Houston 2002a] (admittedly, this is a rather simple situation, as the second disentanglement is a curve for $\mathbb{C}^2$ to $\mathbb{C}^3$).

The second disentanglement will be shown to be homotopically equivalent to the quotient space $D^2(f_t)/S_2$ union a finite number of hexagons and so we first look at the topology of $D^2(f_t)/S_2$.

Theorem 4.1. *Suppose $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^{3n}, 0)$, $n \geq 1$, is a finitely $\mathscr{A}$-determined map-germ and that f_t denotes the disentanglement map. Then*

(i) $$H_i(D^2(f_t)/S_2;\mathbb{Z}) \cong \begin{cases} H_i(D^2(f_t);\mathbb{Z}) & \textit{for } i \neq n, \\ H_n(D^2(f_t);\mathbb{Z})/\alpha_* H_n^{\mathrm{alt}}(D^2(f_t);\mathbb{Z}) & \textit{for } i = n, \end{cases}$$

where $\alpha_* : H_n^{\mathrm{alt}}(D^2(f_t);\mathbb{Z}) \to H_n(D^2(f_t);\mathbb{Z})$ *is the natural map induced from the identity map* $\mathrm{id} : D^2(f_t) \to D^2(f_t)$.

(ii) *If* $D^2(f_t)$ *is connected, the natural map* $\pi_1(D^2(f_t)) \to \pi_1(D^2(f_t)/S_2)$ *is surjective.*

Proof. The natural map $g : D^2(f_t) \to D^2(f_t)/S_2$ has no triple points, so the projection $\varepsilon_2 : D^2(g) \to D^1(g) = D^2(f_t)$ is injective. As it is also proper and surjective, it is a homeomorphism onto $D^1(g)$. It is also equivariant. Thus the natural map $H_i^{\mathrm{alt}}(D^2(g);\mathbb{Z}) \to H_i^{\mathrm{alt}}(D^2(f_t);\mathbb{Z})$ is an isomorphism for all i.

Now consider the image computing spectral sequence for g; we have

$$E^1_{0,i} \cong H_i(D^2(f_t);\mathbb{Z})$$

and, by [Houston 1997, Proposition 4.6],

$$E^1_{1,n} \cong H_n^{\mathrm{alt}}(D^2(f_t);\mathbb{Z}) = \mathbb{Z}^{\mu}$$

All other terms are zero. The only possible nontrivial differential is $d : E^1_{1,n} \to E^1_{0,n}$. As this arises from the homeomorphism $\varepsilon_2 : D^2(g) \to D^1(g)$ we have

$$E^2_{0,n} \cong H_n(D^2(f_t);\mathbb{Z})/\varepsilon_{2*} H_n^{\mathrm{alt}}(D^2(f_t);\mathbb{Z}).$$

The sequence collapses at E^2 since only the terms $E^2_{0,i}$ are nontrivial. We then find that

$$H_i(D^2(f_t)/S_2;\mathbb{Z}) \cong H_i(g(D^2(f_t));\mathbb{Z}) \cong E^2_{0,i}.$$

Thus we have proved (i).

Since $D^2(f_t) = D^2(g)$ is connected and $H_0^{\mathrm{alt}}(D^2(g);\mathbb{Z}) = 0$, the natural map $\pi_1(D^2(f_t)) = \pi_1(D^1(g)) \to \pi_1(g(D^2(f_t))) = \pi_1(D^2(f_t)/S_2)$ is surjective, by [Houston 1997, Corollary 4.19]. □

Corollary 4.2. *Suppose that* f *has corank* 1. *Then* $D^2(f_t)/S_2$ *is homotopically equivalent to a wedge of* n*-spheres, the number of which is equal to the rank of the symmetric part of* $H_n(D^2(f_t);\mathbb{Z})$.

Proof. As f has corank 1, $D^2(f_t)$ is the Milnor fibre of the n-dimensional isolated complete intersection singularity $D^2(f)$ (see [Marar and Mond 1989]), and so it is homotopically equivalent to a wedge of μ n-spheres. Thus by part (i) of the theorem we have

$$H_i(D^2(f_t)/S_2;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{\mu}/\varepsilon_{2*} H_0^{\mathrm{alt}}(D^2(f_t);\mathbb{Z}), & \text{for } i = n, \\ \mathbb{Z}, & \text{for } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

As $H_0^{\mathrm{alt}}(D^2(f_t);\mathbb{Z})$ is the alternating part of $H_n(D^2(f_t);\mathbb{Z})$ by [Goryunov 1995, Theorem 2.1.2], we conclude that $H_n(D^2(f_t)/S_2;\mathbb{Z})$ must be isomorphic to the symmetric part of $H_n(D^2(f_t);\mathbb{Z})$.

If $n = 1$, then $D^2(f_t)/S_2$ is homotopically equivalent to a compact curve, and since $D^2(f_t)$ is connected this implies the homotopy result in this case.

So now assume that $n > 1$. Then $D^2(f_t)$ is simply connected and hence by part (ii) of the theorem $D^2(f_t)/S_2$ is simply connected. But a simply connected n-dimensional CW-complex with nonreduced integer homology free abelian in dimension n is homotopically equivalent to a wedge of n-spheres. □

Remark 4.3. It would be interesting to find a map (with corank greater than one) for which $D^2(f_t)$ is not homotopically equivalent to a wedge of spheres. This could possibly be used to show that its second disentanglement is not homotopically equivalent to a wedge of spheres.

We now prove another of the main theorems of this paper. This one gives us nontrivial examples of the behaviour of the second disentanglement when the target dimension of the germ is greater than the source dimension plus 1.

Theorem 4.4. *Suppose that* $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^{3n}, 0)$, $n \geq 1$, *is a finitely* $\mathcal{A}$-*determined corank*-1 *map-germ. Then* $\mathrm{Dis}_2(f)$ *is homotopically equivalent to the wedge of* $\frac{1}{2}\big(\mu(D^2(f)) - \mu(D^2(f)|H)\big)$ *n-spheres and* $\frac{1}{3}(\mu(D^3(f)) + 1) = 2T(f_t)$ *circles. Here* H *is the diagonal in* $D^2(f)$ *and* μ *denotes Milnor number.*

Proof. As stated in Section 2 the images of maps are homotopically equivalent to their semisimplicial resolutions.

Outside the triple points the map $g : D^2_1(f_t) \to M_2(f_t) = \mathrm{Dis}_2(f)$ is the same as the map $D^2(f_t) \to D^2(f_t)/S_2$, because all the map g is doing is identifying (x_1, x_2) with (x_2, x_1) for all $(x_1, x_2) \in D^2(f_t)$.

Now, $D^2_1(f_t)$ is homotopically equivalent to $D^2(f_t)$ with intervals connecting (x_1, x_2) to (x_1, x_3), (x_2, x_1) to (x_2, x_3) and (x_3, x_1) to (x_3, x_2) for every triple point (x_1, x_2, x_3). This is because $\varepsilon_2 : D^2(f_t) \to D^2_1(f_t)$ is surjective and its double point space is homeomorphic to $D^3(f_t)$. Thus, the semisimplicial resolution of $D^2_1(f_t)$ is just $D^2(f_t)$ with points arising from triples connected with an interval as just explained. One can now connect every point (a, b) to (b, a) in this semisimplicial resolution of $D^2_1(f_t)$. For each triple point one can attach a hexagon along the edges

$$(x_1, x_2) \to (x_2, x_1) \to (x_2, x_3) \to (x_3, x_2) \to (x_3, x_1) \to (x_1, x_3) \to (x_1, x_2).$$

Call this new space W; it is homeomorphic to the semisimplicial resolution of $D^2(f_t)/S_2$ union some hexagons, one for each triple. If one contracts the edges (x_1, x_2) to (x_1, x_3), (x_2, x_1) to (x_2, x_3) and (x_3, x_1) to (x_3, x_2) in each hexagon, one gets the semisimplicial resolution of $M_2(f_t) = \mathrm{Dis}_2(f)$. In summary,

$$\text{resolution of } D^2(f_t)/S_2 \cup \{\text{hexagons}\} \simeq W \simeq \text{resolution of } \mathrm{Dis}_2(f) \simeq \mathrm{Dis}_2(f).$$

Through contracting the intervals connecting (x_1, x_2) to (x_2, x_1) for all $(x_1, x_2) \in D^2(f_t)$ in the semisimplicial resolution of $D^2(f_t)/S_2$ we produce $D^2(f_t)/S_2$. Then any hexagon attached to the resolution of $D^2(f_t)/S_2$ will have the edges (x_1, x_2) to (x_2, x_1), etc., contracted. Thus we will have a space homotopically equivalent to $D^2(f)/S_2$ with triangles attached at the vertices $(x_1, x_2) = (x_2, x_1)$, $(x_1, x_3) = (x_3, x_1)$ and $(x_3, x_2) = (x_2, x_3)$ for every triple point (x_1, x_2, x_3).

By Corollary 4.2 $D^2(f_t)/S_2$ is homotopically equivalent to a wedge of n-spheres, their number, μ say, being the rank of the symmetric part of $H_n(D^2(f_t); \mathbb{Z})$.

If $n > 1$, then $D^2(f_t)/S_2$ is simply connected and so adding the edges of a triangle gives a wedge of three circles. Adding the interior of the triangle causes one of the edges to be contracted onto the other two edges. Hence the wedge is now of two circles. This can be done for every triple point, so that we get $D^2(f_t)/S_2$ wedged with $2T(f_t)$ circles.

If $n = 1$, then $D^2(f_t)/S_2$ is homotopically equivalent to a wedge of circles. Adding $3T(f_t)$ 1-cells (the edges of triangles) and $T(f_t)$ 2-cells (the triangles) will alter the Euler characteristic by $2T(f_t)$. Contracting one edge of the triangle onto the other two will give a wedge of spheres.

Hence $\mathrm{Dis}_2(f)$ is homotopically equivalent to μ n-spheres and $2T(f_t)$ circles. That μ equals $\frac{1}{2}\big(\mu(D^2(f)) - \mu(D^2(f)|H)\big)$ follows from the fact that the rank of the alternating part of $H_n(D^2(f_t); \mathbb{Z})$ is equal to $\frac{1}{2}\big(\mu(D^2(f)) + \mu(D^2(f)|H)\big)$ (see [Houston and Kirk 1999] for example) and that the sum of the ranks of the alternating and symmetric parts is $\mu(D^2(f))$. The number of triple points is $\frac{1}{6}(\#D^3(f_t))$, which is equal to $\frac{1}{6}(\mu(D^3(f)) + 1)$. □

Remark 4.5. The image computing spectral sequence for the restriction

$$f_t \big| D^2_1(f_t) : D^2_1(f_t) \to \mathrm{Dis}_2(f)$$

is interesting because it exhibits a nontrivial second differential. For simplicity consider the case $n > 1$. The differential $d_2 : E^2_{2,0} \cong \mathbb{Z}^{T(f_t)} \to E^2_{0,1} \cong \mathbb{Z}^{3T(f_t)}$ must be nonzero, or else all of $E^2_{0,1}$ survives to infinity and thus $\mathrm{Dis}_2(f)$ has its first homology group isomorphic to $\mathbb{Z}^{3T(f_t)}$ and not $\mathbb{Z}^{2T(f_t)}$. A similar situation arises in the case of maps where $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$; see [Houston 2002a].

It would be interesting to get a proof of this fact about differentials from some other principle rather than working backwards as we have just done since this would be useful in the study of higher-order disentanglements of maps $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $n < p$.

5. Good real perturbations

General theorems. In this section we assume that f is the complexification of a finitely $\mathscr{A}$-determined real multigerm $f' : (\mathbb{R}^{2n}, S) \to (\mathbb{R}^{3n}, 0)$, where $n \geq 1$.

We will denote the disentanglement map by $f_{\mathbb{C}}$. The restriction of this map to the real part of the source will be denoted $f_{\mathbb{R}}$ and will be called a real perturbation. To ease notation, we will use $\mathrm{Dis}_k(f_{\mathbb{C}})$ for the k-th complex disentanglement of f, and $\mathrm{Dis}_k(f_{\mathbb{R}})$ for a k-th real disentanglement. We wish to compare the topology of real and complex disentanglements.

In [Marar and Mond 1996] and [Mond 1996], when $n = 1$, a real perturbation is called *good* if the vanishing homology of the complex map can be seen in the vanishing homology of the real map; that is, if $\operatorname{rank} H_2(\mathrm{Dis}_1(f_{\mathbb{R}}); \mathbb{Z}) = \operatorname{rank} H_2(\mathrm{Dis}_1(f_{\mathbb{C}}); \mathbb{Z})$. For general n the image of the complex map has homology in dimensions $n+1$ and 2 and so the following would seem a reasonable generalisation.

Definition 5.1. A real perturbation $f_{\mathbb{R}}$ is called *good* if

(i) $\operatorname{rank} H_{n+1}(\mathrm{Dis}_1(f_{\mathbb{R}}); \mathbb{Z}) = \operatorname{rank} H_{n+1}(\mathrm{Dis}_1(f_{\mathbb{C}}); \mathbb{Z})$, and

(ii) $\operatorname{rank} H_2(\mathrm{Dis}_1(f_{\mathbb{R}}); \mathbb{Z}) = \operatorname{rank} H_2(\mathrm{Dis}_1(f_{\mathbb{C}}); \mathbb{Z})$.

Of course, when $n = 1$ the two conditions in the definition are the same.

One of the aims of an investigation of this type is to show that if $f_{\mathbb{R}}$ is a good real perturbation, the real and complex disentanglements are homotopically equivalent. This is done for the case $n = 1$ in Theorem 5.5, but in general we use the weaker assumption that there is a $\mathbb{Z}$-homology equivalence between the real and complex disentanglements; see Theorem 5.4.

Lemma 5.2. *Suppose $D^2(f_{\mathbb{R}}) \cap (V_i \times V_j) \neq \varnothing$ for all i and j with $i \neq j$, where $V_1, \dots, V_{|S|}$ are the $|S|$ source components of $f_{\mathbb{R}}$. Then the natural homomorphism $H_0^{\mathrm{alt}}(D^2(f_{\mathbb{R}}); \mathbb{Z}) \to H_0^{\mathrm{alt}}(D^2(f_{\mathbb{C}}); \mathbb{Z})$ is surjective. Furthermore, generators of the two groups can be chosen so that generators of $H_0^{\mathrm{alt}}(D^2(f_{\mathbb{R}}); \mathbb{Z})$ map either to zero or to generators of $H_0^{\mathrm{alt}}(D^2(f_{\mathbb{C}}); \mathbb{Z})$.*

Proof. By [Houston 1997, Lemma 2.6] the orbit (under the action of S_2) of a path-connected component of an S_2-invariant subset X in the double point space of a map will either naturally correspond to a copy of $\mathbb{Z}$ or $\mathbb{Z}_2$ in $H_0^{\mathrm{alt}}(X; \mathbb{Z})$ or will not contribute anything to it. Thus we can choose natural generators for groups such as $H_0^{\mathrm{alt}}(D^2(f_{\mathbb{R}}); \mathbb{Z})$ and $H_0^{\mathrm{alt}}(D^2(f_{\mathbb{C}}); \mathbb{Z})$.

The map $f_{\mathbb{C}}$ is in effect a map from $U_1 \sqcup \cdots \sqcup U_m$ to $\mathbb{C}^{3n}$, where m is the number of elements of S and each U_i is path-connected, which we assume to contain V_i. The group $H_0^{\mathrm{alt}}(D^2(f_{\mathbb{C}}); \mathbb{Z})$ is isomorphic to the direct sum of the groups $H_0^{\mathrm{alt}}\big(D^2(f_{\mathbb{C}}) \cap ((U_i \times U_j) \cup (U_j \times U_i)); \mathbb{Z}\big)$ for $1 \le i \le j \le |S|$. A similar statement is true for $H_0^{\mathrm{alt}}(D^2(f_{\mathbb{R}}); \mathbb{Z})$.

From the proof of Theorem 3.1 we know that $H_0^{\mathrm{alt}}(D^2(f_{\mathbb{C}}); \mathbb{Z}) \cong \mathbb{Z}^{m(m-1)/2}$. Using the calculation of [Houston 1997, Example 2.5], we can also show that this group is a direct sum of all

$$H_0^{\mathrm{alt}}\big(D^2(f_{\mathbb{C}}) \cap ((U_i \times U_j) \cup (U_j \times U_i)); \mathbb{Z}\big) \cong \mathbb{Z}$$

for each distinct U_i and U_j with $i < j$. (The homology group is isomorphic to $\mathbb{Z}$ because, using the notation of Theorem 3.1, $H_0^{\mathrm{alt}}(D^2(f_{\mathbb{C}}); \mathbb{Z})$ is isomorphic to $H_0(D^2(F); \mathbb{Z})$ via the map induced by the natural inclusion of $D^2(f_{\mathbb{C}})$ into $D^2(F)$.

The homology of the latter space is a direct sum of groups arising from connected components of $D^2(F)$ that do not intersect the diagonal, and each summand is a copy of $\mathbb{Z}$.)

This direct-sum decomposition implies $D^2(f_{\mathbb{C}}) \cap (U_i \times U_j)$ is path-connected. The groups $H_0^{\text{alt}}\big(D^2(f_{\mathbb{C}}) \cap (U_i \times U_i); \mathbb{Z}\big)$ are all trivial, so every component of $D^2(f_{\mathbb{C}}) \cap (U_i \times U_i)$ must intersect the diagonal.

Since $D^2(f_{\mathbb{R}}) \cap (V_i \times V_j) \neq \varnothing$ for $i \neq j$, we have

$$H_0^{\text{alt}}(D^2(f_{\mathbb{R}}) \cap ((V_i \times V_j) \cup (V_j \times V_i)); \mathbb{Z}) \cong \mathbb{Z}^{\mu},$$

for some $\mu \geq 1$, where each copy of $\mathbb{Z}$ corresponds to some path-connected component of $D^2(f_{\mathbb{R}}) \cap (V_i \times V_j)$. The natural homomorphism to the alternating homology of the corresponding complex space is a surjection and so the natural homomorphism in the statement of the lemma is also a surjection.

Suppose that $(x_1, x_2) \in D^2(f_{\mathbb{R}})$ is contained in some path-connected component X such that $\mathrm{Alt}_{\mathbb{Z}}(x_1, x_2) \neq 0$ is a generator of $H_0^{\text{alt}}(D^2(f_{\mathbb{R}}); \mathbb{Z})$.

If x_1 and x_2 are in the same U_i then $H_0^{\text{alt}}\big(D^2(f_{\mathbb{R}}) \cap \mathrm{Orbit}(X); \mathbb{Z}\big)$ is $\mathbb{Z}_2$ or $\mathbb{Z}$, depending on whether $\mathrm{Orbit}(X)$ is connected or disconnected; see [Houston 1997, Lemma 2.6]. In both these situations this generator maps to zero, because $H_0^{\text{alt}}\big(D^2(f_{\mathbb{C}}) \cap (U_i \times U_i); \mathbb{Z}\big)$ is zero.

If x_1 and x_2 are in distinct U_i and U_j then $(x_1, x_2) \in D^2(f_{\mathbb{C}}) \cap (U_i \times U_j)$ is connected to all points of $D^2(f_{\mathbb{C}}) \cap (U_i \times U_j)$ and hence $\mathrm{Alt}_{\mathbb{Z}}(x_1, x_2)$ is a generator of $H_0^{\text{alt}}\big(D^2(f_{\mathbb{C}}) \cap ((U_i \times U_j) \cup (U_j \times U_i)); \mathbb{Z}\big)$. □

The following proposition is based on [Marar and Mond 1996, Lemma 2.4], which assumed that $n = 1$ and that $f_{\mathbb{R}}$ was a good real perturbation. The isomorphism assumption in the proposition holds for good real perturbations in the $n = 1$ case, by methods similar to those used in [Marar and Mond 1996, Theorem 2.3]; a natural question would be, does it also hold for good real perturbations with $n > 1$?

Proposition 5.3. *Suppose that f is finitely $\mathscr{A}$-determined and that the natural map $H_n^{\text{alt}}(D^2(f_{\mathbb{R}}); \mathbb{Z}) \to H_n^{\text{alt}}(D^2(f_{\mathbb{C}}); \mathbb{Z})$ is an isomorphism. Then*

$$D^2(f_{\mathbb{R}}) \cap (V_i \times V_j) \neq \varnothing \quad \textit{for } i \neq j.$$

Proof. We apply an argument similar to the proof of [Marar and Mond 1996, Lemma 2.4]. Suppose that $D^2(f_{\mathbb{R}}) \cap (V_i \times V_j) = \varnothing$ for some $i \neq j$. Then the images of V_i and V_j under $f_{\mathbb{R}}$ are disjoint in the perturbation and hence the corresponding spaces in the image of f restricted to the real parts of its source are not transverse. This nontransverse contact implies that $D^2(f_{\mathbb{C}}) \cap (U_i \times U_j)$ is the Milnor fibre of an isolated complete intersection singularity and thus has a nontrivial middle homology group. But, obviously,

$$H_n^{\text{alt}}\big(D^2(f_{\mathbb{C}}) \cap ((U_i \times U_j) \times (U_j \times U_i)); \mathbb{Z}\big) \cong H_n\big(D^2(f_{\mathbb{C}}) \cap (U_i \times U_j); \mathbb{Z}\big) \neq 0,$$

and so the isomorphism assumed in the statement restricts to give

$$H_n^{\mathrm{alt}}\big(D^2(f_{\mathbb{R}}) \cap ((V_i \times V_j) \times (V_j \times V_i)); \mathbb{Z}\big) \neq 0,$$

which contradicts $D^2(f_{\mathbb{R}}) \cap (V_i \times V_j) = \varnothing$. □

The next theorem allows us to convert $\mathbb{Z}$-homology equivalences into homotopy equivalences.

Theorem 5.4. *Suppose that $f : (\mathbb{C}^{2n}, S) \to (\mathbb{C}^{3n}, 0)$ is a finitely $\mathscr{A}$-determined multigerm. If the natural inclusion $\mathrm{Dis}_1(f_{\mathbb{R}}) \hookrightarrow \mathrm{Dis}_1(f_{\mathbb{C}})$ is a $\mathbb{Z}$-homology equivalence, then it is a homotopy equivalence and all the triple points of $f_{\mathbb{C}}$ are real.*

Proof. Consider the pair of maps $f_{\mathbb{R}}$ and $f_{\mathbb{C}}$. We have an image computing spectral sequence,

$$E^1_{p,q}(f_{\mathbb{C}}, f_{\mathbb{R}}) = H_q^{\mathrm{alt}}\big(D^{p+1}(f_{\mathbb{C}}), D^{p+1}(f_{\mathbb{R}}); \mathbb{Z}\big) \implies H_*\big(\mathrm{Dis}_1(f_{\mathbb{C}}), \mathrm{Dis}_1(f_{\mathbb{R}}); \mathbb{Z}\big).$$

The only possible nonzero groups in E^1 are $E^1_{1,q}$ for $0 \le q \le n = \dim_{\mathbb{C}} D^2(f_{\mathbb{C}})$, and $E^1_{2,0} = \mathbb{Z}^{\mu}$, (since $D^3(f_{\mathbb{C}})$ and $D^3(f_{\mathbb{R}})$ are finite), where μ is the number of complex triple points that are not also real.

The only possible nontrivial differential for E^* is $E^1_{2,0} \to E^1_{1,0}$, so $E^{\infty}_{p,q} = E^1_{p,q}$ for $q > 0$, or $q = 0$ and $p \neq 1, 2$. Also, $E^{\infty}_{p,q} = E^2_{p,q}$ for $q = 0$.

Because $\mathrm{Dis}_1(f_{\mathbb{R}}) \hookrightarrow \mathrm{Dis}_1(f_{\mathbb{C}})$ is a $\mathbb{Z}$-homology equivalence, $E^{\infty}_{p,q} = 0$ for all p and q. So, in particular, $E^1_{1,0} \cong E^1_{2,0}$, and $E^1_{1,n} = 0$. This latter means that

$$H_n^{\mathrm{alt}}(D^2(f_{\mathbb{R}}); \mathbb{Z}) \to H_n^{\mathrm{alt}}(D^2(f_{\mathbb{C}}); \mathbb{Z})$$

is an isomorphism. By Proposition 5.3 this means that $D^2(f_{\mathbb{R}}) \cap (V_i \times V_j) \neq \varnothing$ for all $i \neq j$, which in turn, by Lemma Lemma 5.2, implies that $H_0^{\mathrm{alt}}(D^2(f_{\mathbb{R}}); \mathbb{Z}) \to H_0^{\mathrm{alt}}(D^2(f_{\mathbb{C}}); \mathbb{Z})$ is surjective, that is,

$$0 = H_0^{\mathrm{alt}}\big(D^2(f_{\mathbb{C}}), D^2(f_{\mathbb{R}}); \mathbb{Z}\big) = E^1_{1,0} \cong E^1_{2,0} \cong \mathbb{Z}^{\mu}.$$

Hence all entries of $E^1_{p,q}(f_{\mathbb{C}}, f_{\mathbb{R}})$ are zero, and all complex triple points are real.

This implies that the spectral sequence morphism $E^1_{p,q}(f_{\mathbb{R}}) \to E^1_{p,q}(f_{\mathbb{C}})$ is an isomorphism. In particular the complex $E^1_{*,0}(f_{\mathbb{R}})$ has the same form as $E^1_{*,0}(f_{\mathbb{C}})$ and it is this form, via [Houston 1997, Proposition 4.21], that implies that the image of $f_{\mathbb{C}}$ is simply connected; i.e., the image of $f_{\mathbb{R}}$ is simply connected as well. (The β condition in that proposition is that the map should be complex analytic but the proof works just as well for real analytic maps.)

Thus $\mathrm{Dis}_1(f_{\mathbb{R}}) \to \mathrm{Dis}_1(f_{\mathbb{C}})$ is a $\mathbb{Z}$-homology equivalence between two simply connected CW-complexes and therefore is a homotopy equivalence by a theorem of Whitehead [1978, pp. 182 and 220]. □

The next theorem proves [Cooper and Mond 1998, Conjecture 4.1] in the $n = 1$ (surfaces to 3-space) situation.

Theorem 5.5. *Suppose* $f : (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ *is a finitely* $\mathscr{A}$*-determined multigerm such that* $\operatorname{rank} H_2(\mathrm{Dis}_1(f_{\mathbb{R}}); \mathbb{Z}) = \operatorname{rank} H_2(\mathrm{Dis}_1(f_{\mathbb{C}}); \mathbb{Z})$. *Then the inclusion of* $\mathrm{Dis}_1(f_{\mathbb{R}})$ *into* $\mathrm{Dis}_1(f_{\mathbb{C}})$ *is a homotopy equivalence.*

Proof. First, by [Mond 1996, Theorem 1.1], the natural map $H_2(\mathrm{Dis}_1(f_{\mathbb{R}}); \mathbb{Z}) \to H_2(\mathrm{Dis}_1(f_{\mathbb{C}}); \mathbb{Z})$ is an isomorphism. Second, a consequence of [Marar and Mond 1996, Theorem 2.3] is that $H^i(\mathrm{Dis}_1(f_{\mathbb{R}}); \mathbb{Q}) \to H^i(\mathrm{Dis}_1(f_{\mathbb{C}}); \mathbb{Q})$ is an isomorphism for all i. Using these facts and the homotopy equivalence of $\mathrm{Dis}_1(f_{\mathbb{C}})$ to a wedge of 2-spheres we can deduce that the only possible obstruction to an isomorphism on integer homology is if $H_1(\mathrm{Dis}_1(f_{\mathbb{R}}); \mathbb{Z})$ has nontrivial torsion. There are two ways that torsion could appear in the spectral sequence for a stable map with no quadruple points and having a set of simply connected surfaces as its source: through an appearance in the E^1 page or the E^2 page of the spectral sequence. We show that neither can occur in our situation.

If $D^2(f_{\mathbb{R}})$ contains a circle C upon which the natural S_2 action is the antipodal action, $H_0^{\mathrm{alt}}(D^2(f_{\mathbb{R}}); \mathbb{Z})$ contains the subgroup $H_0^{\mathrm{alt}}(C; \mathbb{Z}) \cong \mathbb{Z}_2$, by [Houston 1997, Lemma 2.6], and vice versa. If this torsion group survives in $E^2(f_{\mathbb{R}})$, it appears as a summand in $H_1(\mathrm{Dis}_1(f_{\mathbb{R}}); \mathbb{Z})$.

Thus, suppose such a circle C exists in $D^2(f_{\mathbb{R}})$. Then C projects to a curve C' in the source of $f_{\mathbb{R}}$, the nodes of which correspond to triple points of $f_{\mathbb{R}}$. By performing homotopies on $f_{\mathbb{R}}$ we can remove any other points arising from the double point space that lie within the compact region enclosed by C'. That is, we push them to outside the enclosed region.

There will exist a node such that on one side of it the curve forms a loop with no other nodes on it. That is, the loop will be equivalent to the interval $[0, 1]$ mapped so that the points $\left\{\frac{1}{4}\right\}$ and $\left\{\frac{3}{4}\right\}$ form the node. Call this space α (after its shape). To one side of the plane containing the node the surface will look like the product of an interval I with α. By a smooth homotopy one can contract the loop $\left\{\frac{1}{2}\right\} \times \alpha$; this will modify $I \times \alpha$ to a space with two cross caps. The plane containing the node can then be pushed through the nearest cross cap thus removing the node. See Figure 1.

This can be done for all nodes, so that the projection of C for the new map gives a circle C' in the source with no nodes and no points of the double point space touching it or contained within its interior.

Once this is done the disc in the source with C' as boundary is mapped into $\mathbb{R}^3$ with C' mapping two-to-one. The image of this disc is in effect a copy of $\mathbb{RP}^2$ embedded in $\mathbb{R}^3$. Such a situation is impossible to achieve and hence there are no loops like C in $D^2(f_{\mathbb{R}})$.

Since $H_1(\mathrm{Dis}_1(f_{\mathbb{R}}); \mathbb{Z}) \cong E^2_{1,0}(f_{\mathbb{R}})$ the next possible way to get torsion is if the group

$$E^2_{1,0} \cong \operatorname{Ker}(d : E^1_{1,0} \to E^1_{0,0}) / \operatorname{Im}(d : E^1_{2,0} \to E^1_{1,0})$$

contains some.

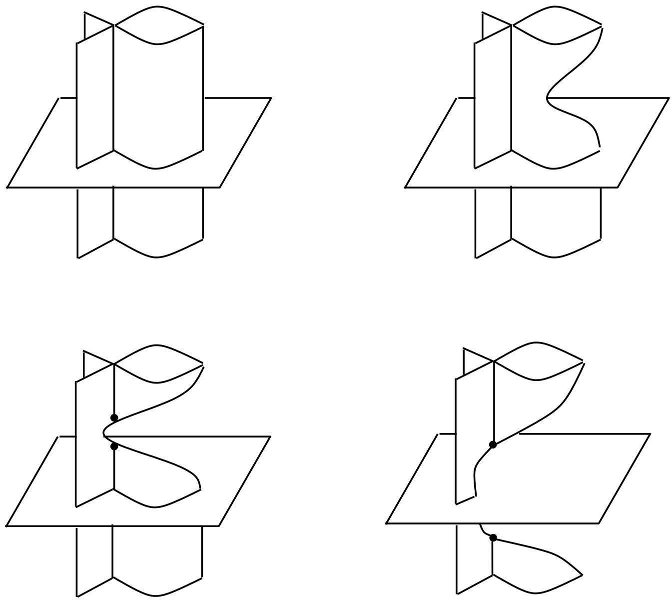

Figure 1. Removing nodes on C'.

The differential $d : E^1_{2,0} \to E^1_{1,0}$ is induced from ε_{3*}; see Theorem 2.10. Suppose that (a, b, c) is a triple point then it is easy to calculate that

$$\varepsilon_{3*}(\mathrm{Alt}_{\mathbb{Z}}(a, b, c)) = \mathrm{Alt}_{\mathbb{Z}}(a, b) + \mathrm{Alt}_{\mathbb{Z}}(b, c) + \mathrm{Alt}_{\mathbb{Z}}(c, a).$$

So, if for example (a, b), (b, c) and (c, a) are all contained in the same component of an ordinary loop in $D^2(f_{\mathbb{R}})$, then $\varepsilon_{3*}(\mathrm{Alt}_{\mathbb{Z}}(a, b, c)) = 3\,\mathrm{Alt}_{\mathbb{Z}}(a, b)$, which could lead to torsion.

The double point space $D^2(f_{\mathbb{R}})$ contains components homeomorphic to the following forms: intervals with a point in the diagonal, circles that intersect the diagonal in two points and pairs of loops such that the pair is invariant under S_2.

The first two situations do not contribute anything to $H^{\mathrm{alt}}_0(D^2(f_{\mathbb{R}}); \mathbb{Z})$ as for these any point (x_1, x_2) is homologous to one in the diagonal, (x, x) say, and so $\mathrm{Alt}_{\mathbb{Z}}(x_1, x_2) \equiv \mathrm{Alt}_{\mathbb{Z}}(x, x) = 0$.

Thus, only pairs of loops could cause difficulties. Let C be such a loop and (x_1, x_2) be a point of C. The generator $\mathrm{Alt}_{\mathbb{Z}}(x_1, x_2)$ of $H^{\mathrm{alt}}_0(D^2(f_{\mathbb{R}}) \cap \mathrm{Orbit}(C)) \cong \mathbb{Z}$ is also a generator of some $\mathbb{Z}$ in $H^{\mathrm{alt}}_0(D^2(f_{\mathbb{C}}); \mathbb{Z})$; see Lemma 5.2.

We have the natural commutative diagram

$$\begin{array}{ccccc} E^1_{0,1}(f_{\mathbb{R}}) \cong & \mathbb{Z}^{\mu} & \longleftarrow & \mathbb{Z}^T \cong & E^1_{2,0}(f_{\mathbb{R}}) \\ & \downarrow & & \downarrow & \\ E^1_{0,1}(f_{\mathbb{C}}) \cong & \mathbb{Z}^{m(m-1)/2} & \longleftarrow & \mathbb{Z}^T \cong & E^1_{2,0}(f_{\mathbb{C}}). \end{array}$$

The first vertical arrow is surjective and takes generators of $H_0^{\mathrm{alt}}(D^2(f_{\mathbb{R}});\mathbb{Z})$ to generators of $H_0^{\mathrm{alt}}(D^2(f_{\mathbb{C}});\mathbb{Z})$, as just described. The second is an isomorphism, since by [Marar and Mond 1996, Corollary 2.5] all triple points are real. Thus, if (a,b,c) is a triple point in $D^3(f_{\mathbb{R}})$ so that $\varepsilon_{3*}\operatorname{Alt}_{\mathbb{Z}}(a,b,c)\equiv r\operatorname{Alt}_{\mathbb{Z}}(x_1,x_2)$ with $r>1$ in the top row, a similar situation takes place in the bottom row leading to torsion in $E^2_{1,0}(f_{\mathbb{C}})\cong H_1(\operatorname{Dis}_1(f_{\mathbb{C}});\mathbb{Z})$. This is a contradiction.

Thus $H_1(\operatorname{Dis}_1(f_{\mathbb{R}});\mathbb{Z})=0$ and so there is an isomorphism $H_i(\operatorname{Dis}_1(f_{\mathbb{R}});\mathbb{Z})\to H_i(\operatorname{Dis}_1(f_{\mathbb{C}});\mathbb{Z})$ for all i. By Theorem 5.4 this leads to a homotopy equivalence. □

Compare the following theorem with [Houston 2002b, Theorem 3.6], where the homotopy conclusion is true when the image of the map is a hypersurface.

Theorem 5.6. *Suppose that $f:(\mathbb{C}^{2n},0)\to(\mathbb{C}^{3n},0)$, $n\geq 1$, is a corank-1 finitely $\mathscr{A}$-determined map-germ. If the natural map $D^k(f_{\mathbb{R}})\to D^k(f_{\mathbb{C}})$ is an S_k-homotopy equivalence for all $k\geq 2$, then the inclusion $\operatorname{Dis}_r(f_{\mathbb{C}})\to\operatorname{Dis}_r(f_{\mathbb{C}})$ is a homotopy equivalence for all $r\geq 1$.*

Proof. The natural maps $H_i^{\mathrm{alt}}(D^k(f_{\mathbb{R}});\mathbb{Z})\to H_i^{\mathrm{alt}}(D^k(f_{\mathbb{C}});\mathbb{Z})$ are isomorphisms for all $k\geq 1$ and thus the spectral sequence morphism $E^1_{p,q}(f_{\mathbb{R}})\to E^1_{p,q}(f_{\mathbb{C}})$ is an isomorphism for all p and q. This gives a $\mathbb{Z}$-homology equivalence between $\operatorname{Dis}_1(f_{\mathbb{R}})$ and $\operatorname{Dis}_1(f_{\mathbb{C}})$ and so by Theorem 5.4 the map is a homotopy equivalence.

The second disentanglements are the images of

$$f_{\mathbb{R}}\big|D^2_1(f_{\mathbb{R}}):D^2_1(f_{\mathbb{R}})\to M_2(f_{\mathbb{R}})\quad\text{and}\quad f_{\mathbb{C}}\big|D^2_1(f_{\mathbb{C}}):D^2_1(f_{\mathbb{C}})\to M_2(f_{\mathbb{C}}).$$

The maps $H_i(D^2_1(f_{\mathbb{R}});\mathbb{Z})\to H_i(D^2_1(f_{\mathbb{C}});\mathbb{Z})$ are isomorphisms as the spectral sequences for $g_{\mathbb{R}}:D^2(f_{\mathbb{R}})\to D^2_1(f_{\mathbb{R}})$ and $g_{\mathbb{C}}:D^2(f_{\mathbb{C}})\to D^2_1(f_{\mathbb{C}})$ are isomorphic (their multiple points spaces are S_k-homotopy equivalent). These then provide the isomorphisms between the spectral sequences for $f_{\mathbb{R}}\big|D^2_1(f_{\mathbb{R}})$ and $f_{\mathbb{C}}\big|D^2_1(f_{\mathbb{C}})$. Thus $M_2(f_{\mathbb{R}})=\operatorname{Dis}_2(f_{\mathbb{R}})$ and $M_2(f_{\mathbb{C}})=\operatorname{Dis}_2(f_{\mathbb{C}})$ are $\mathbb{Z}$-homology equivalent.

One can prove similarly that the spectral sequences for calculating $D^2(f_{\mathbb{R}})/S_2$ and $D^2(f_{\mathbb{C}})/S_2$ are isomorphic. If $n=1$, both $D^2(f_{\mathbb{R}})/S_2$ and $D^2(f_{\mathbb{C}})/S_2$ are homotopically equivalent to connected 1-dimensional CW-complexes, and since they are $\mathbb{Z}$-homology equivalent they must be homotopy equivalent. If $n>1$, then $D^2(f_{\mathbb{R}})\simeq D^2(f_{\mathbb{C}})\simeq\bigvee S^n$ (because $D^2(f_{\mathbb{C}})$ is the Milnor fibre of an isolated complete intersection singularity), and is thus simply connected; since $H_0^{\mathrm{alt}}(D^2(f_{\mathbb{R}});\mathbb{Z})$ vanishes we deduce that $D^2(f_{\mathbb{R}})/S_2$ is simply connected. This is essentially done the same way as the proof of Theorem 4.1. Thus $D^2(f_{\mathbb{R}})/S_2$ and $D^2(f_{\mathbb{C}})/S_2$ are simply connected and $\mathbb{Z}$-homology equivalent CW-complexes, and therefore homotopically equivalent by Whitehead's theorem.

Now, just as $\operatorname{Dis}_2(f_{\mathbb{C}})$ is homotopically equivalent to $D^2(f_{\mathbb{C}})/S_2$ union some hexagons, so is $\operatorname{Dis}_2(f_{\mathbb{R}})$ also homotopically equivalent to $D^2(f_{\mathbb{R}})/S_2$ union some hexagons. See the proof of Theorem 4.4. In fact, $\operatorname{Dis}_2(f_{\mathbb{C}})$ is homotopically equivalent to $D^2(f_{\mathbb{C}})/S_2$ wedged with $2T(f_{\mathbb{C}})$ circles; each hexagon contracts to give these loops. Similarly for $\mathrm{Dis}_2(f_{\mathbb{R}})$. The homotopy equivalence from $D^2(f_{\mathbb{R}})/S_2$ to $D^2(f_{\mathbb{C}})/S_2$ can then be extended to one from $\mathrm{Dis}_2(f_{\mathbb{R}})$ to $\mathrm{Dis}_2(f_{\mathbb{C}})$.

That $\mathrm{Dis}_3(f_{\mathbb{R}})$ and $\mathrm{Dis}_3(f_{\mathbb{C}})$ are homotopically equivalent follows from $D^3(f_{\mathbb{R}})$ and $D^3(f_{\mathbb{C}})$ being S_3-homotopically equivalent and from $M_3(f_{\mathbb{C}}) = D^3(f_{\mathbb{C}})/S_3$, and so on. □

A series with excellent real perturbations.

Definition 5.7. As in [Houston 2002b] we call a real perturbation *excellent* if the natural map

$$H_i(\mathrm{Dis}_r(f_{\mathbb{R}}); \mathbb{Z}) \to H_i(\mathrm{Dis}_r(f_{\mathbb{C}}); \mathbb{Z})$$

is an isomorphism for all i and r.

In [Marar and Mond 1996] it is shown that every member of the H_k series given by $f(x, y) = (x, y^3, xy + y^{3k-1})$, $k \geq 1$, has a good real perturbation.

This series can be generalised to a series for every value of n. We define H_k^n to be the map $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^{3n}, 0)$ given by

$$(x_1, \dots, x_{2n-1}, y) \mapsto \Big(x_1, \dots, x_{2n-1},\ y^3 + y\sum_{j=1}^{n-1} x_{n+j-1}^2,\ x_{2n-1}y + y^{3k-1},$$
$$x_1 y + x_n y^2, \dots, x_i y + x_{i+n-1}y^2, \dots, x_{n-1}y + x_{2n-2}y^2\Big).$$

By taking $n = 1$ we get the H_k series. We will show that the general series has more than a good real perturbation, in fact the corresponding r-th disentanglements are homotopy equivalent for all r.

It can be shown that the $\mathcal{A}_e$-codimension of H_k^1 is equal to k. This means that the $\mathcal{A}_e$-codimension is equal to the number of spheres in the first disentanglement; see [Marar and Mond 1996] or Proposition 5.9.

It is natural to compare $\mathcal{A}_e$-codimension and the number of spheres, since in the case of isolated singularities of complete intersections it is well known that the number of spheres in the Milnor fibre is greater than or equal to the $\mathcal{K}_e$-codimension of the map, with equality if the map is quasihomogeneous. Correspondingly, the number of spheres in the disentanglement for a map from $(\mathbb{C}^m, 0)$ to $(\mathbb{C}^q, 0)$ with $m \geq q$ and in the nice dimensions, is greater than or equal to its $\mathcal{A}_e$-codimension, with equality if the map is quasihomogeneous; see [Damon and Mond 1991]. For maps with $q = m + 1$, in the nice dimensions, the comparison is conjectured to be true; see [Mond 1991b], where the $m = 2$ case is proved.

Now, for $q > m + 1$, the comparison does not necessarily hold; the number of spheres in the disentanglement may be less than the $\mathcal{A}_e$-codimension, even for quasihomogeneous maps. See [Houston 1997, Example 4.26]. For $n > 1$, the series H_k^n also exhibits this inequality. For example, calculations with the computer package Transversal [Kirk 2000], show that the $\mathcal{A}_e$-codimension of H_2^2 is 3, yet, by Proposition 5.9, the number of spheres is 2. This is made more intriguing by the fact that the $\mathscr{A}_e$-codimension of H_1^2 is 1, and the number of spheres is also 1. That is, the result holds for H_1^2, but not for the next map in the series. It would, of course, be interesting to find some underlying reason for this.

Theorem 5.8. *The map* $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^{3n}, 0)$ *is finitely* $\mathscr{A}$*-determined.*

Proof. Since the map has corank 1, if we show that $D^2(f)$ and $D^3(f)$, as defined by the adapted Vandermonde method of [Marar and Mond 1989], are isolated complete intersection singularities and $D^4(f)$ is empty, it follows by Theorem 2.14 of the same reference that f is finitely determined.

If we let $f(x, y) = (x, f_1(x, y), \dots, f_{n+1}(x, y))$, then $D^2(f)$ is generated in $\mathbb{C}^{2n-1} \times \mathbb{C}^2$ by

$$\frac{f_j(x, y_1) - f_j(x, y_2)}{y_1 - y_2}, \quad \text{for } j = 1, \dots, n+1.$$

Thus it is generated by $y_1^2 + y_1y_2 + y_2^2 + \sum_{j=1}^{n-1} x_{n+j-1}^2$, $x_{2n-1} + O(3k-2)$, (i.e., terms of degree of $3k-2$ and above), and $x_i + x_{i+n-1}(y_1 + y_2)$ for $1 \le i \le n-1$, and it is equivalent to a quadratic isolated hypersurface singularity.

The set $D^3(f)$ is generated by

$$\frac{\begin{vmatrix} 1 & y_1 & f_j(x, y_1) \\ 1 & y_2 & f_j(x, y_2) \\ 1 & y_3 & f_j(x, y_3) \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & y_1^2 \\ 1 & y_2 & y_2^2 \\ 1 & y_3 & y_3^2 \end{vmatrix}} \quad \text{and} \quad \frac{\begin{vmatrix} 1 & f_j(x, y_1) & y_1^2 \\ 1 & f_j(x, y_2) & y_2^2 \\ 1 & f_j(x, y_3) & y_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & y_1^2 \\ 1 & y_2 & y_2^2 \\ 1 & y_3 & y_3^2 \end{vmatrix}}$$

for $j = 1, \dots, n+1$. These give the same generators as in the $n = 1$ case plus all x_i for $i = 1, \dots, 2n-2$. Thus the $D^3(f)$ singularity is equivalent to the singularity arising in the $n = 1$ case and hence is an isolated complete intersection singularity.

The set $D^4(f)$ is empty because the y^3 term is present in f_1. □

Proposition 5.9. *Let* f *be as above.*

(i) $\mathrm{Dis}_1(f)$ *is homotopically equivalent to the wedge of an* $(n+1)$*-sphere and* $k-1$ 2*-spheres.*
(ii) $\mathrm{Dis}_2(f)$ *is homotopically equivalent to a wedge of* $2(k-1)$ *circles.*

Proof. For part (i) we know from Theorem 3.4 that $\mathrm{Dis}_1(f)$ is homotopically equivalent to a wedge of $(n+1)$-spheres and 2-spheres. The number of $(n+1)$-spheres is rank $H_n^{\mathrm{alt}}(D^2(f_t); \mathbb{Z})$, and as $D^2(f)$ is given by an S_2-invariant quadratic singularity this rank must be 1. By Theorem 3.1 the number of 2-spheres equals $T(f_t)$, the number of triple points in the image of f_t. By the proof of Theorem 5.8 $D^3(f)$ is equivalent to the multiple point space in the case $n = 1$, and by [Marar and Mond 1996] the degree of this is $6(k-1)$, so the number of triple points of f_t is equal to $k-1$, as each triple point gives rise to an S_3-orbit of 6 points.

To prove part (ii) we use Theorem 4.4 to show that $\mathrm{Dis}_2(f)$ is homotopically equivalent to a wedge of n-spheres and $2T(f_t)$ circles. The number of n-spheres is the rank of the symmetric part of $H_n(D^2(f_t);\mathbb{Z})$ and hence is zero. □

Proposition 5.10. *The perturbation of f given by*

$$\Bigl(x_1,\dots,x_{2n-1},\ y^3+y\sum_{j=1}^{n-1}x_{n+j-1}^2-ty,\ x_{2n-1}y+y^2\prod_{i=1}^{k-1}(y^3-ty-c_i),$$
$$x_1y+x_ny^2,\dots,x_iy+x_{i+n-1}y^2,\dots,x_{n-1}y+x_{2n-2}y^2\Bigr),$$

where $t>0$ and c_i are real numbers with $|c_i|<\frac{2}{3}t^{3/2}$ and $c_i\neq c_j$ for $i\neq j$, is a good real perturbation. The natural map $\mathrm{Dis}_r(f_{\mathbb{R}})\hookrightarrow\mathrm{Dis}_r(f_{\mathbb{C}})$ is a homotopy equivalence for all $r\geq 1$.

Proof. The first statement obviously follows from the second.

To prove the second statement we note that by [Marar and Mond 1996, Section 3] in the case $n=1$ the map $D^3(f_{\mathbb{R}})\to D^3(f_{\mathbb{C}})$ is a homeomorphism respecting the S_3 action. Since $D^3(f_{\mathbb{C}})$ is $\mathcal{K}$-equivalent by an S_3-equivariant diffeomorphism to the triple point space singularity in the $n=1$ case we have an S_3-homotopy equivalence between the real and complex multiple points spaces of the disentanglement map in the general case. The map $D^2(f_{\mathbb{R}})\to D^2(f_{\mathbb{C}})$ is an S_2-equivariant homotopy equivalence; this follows from the description of $D^2(f)$. Thus the proposition is proved by Theorem 5.6. □

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