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We introduce the Chevalley cohomology for the graded Lie algebra of polyvector fields on $\mathbb{R}^d$. This cohomology occurs naturally in the problem of construction and classification of formalities on the space $\mathbb{R}^d$. Considering only graph formalities, that is, formalities defined with the help of graphs as in the original construction of Kontsevich, we define (as the first and third authors did earlier for the Hochschild cohomology) the Chevalley cohomology directly on spaces of graphs. More precisely, observing first a noteworthy property for Kontsevich's explicit formality on $\mathbb{R}^d$, we restrict ourselves to graph formalities with that property. With this restriction, we obtain some simple expressions for the Chevalley coboundary operator; in particular, we can write this cohomology directly on the space of purely aerial, nonoriented graphs. We also give examples and applications.

1. Introduction

In this article, we study formalities on the space $\mathbb{R}^d$, which are defined as follows. Let $T_{\text{poly}}(\mathbb{R}^d)[1]$ be the space of polyvector fields on $\mathbb{R}^d$ graded by $|\alpha| = \text{degree}(\alpha) = k-2$ if α is a k-vector field (the [1] stands for this choice of translation on degrees). Similarly, $D_{\text{poly}}(\mathbb{R}^d)[1])$ will denote the polydifferential operators on $\mathbb{R}^d$ graded by $|D| = m-2$ if D is an m-differential operator. We view both spaces as formal graded manifolds; see [Kontsevich 1997; 2003]. A *formality* is a formal nonlinear mapping $\mathcal{F}$ between $T_{\text{poly}}(\mathbb{R}^d)[1]$ and $D_{\text{poly}}(\mathbb{R}^d)[1]$, intertwining their natural vector fields Q and Q'.

The monomial functions $\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_n$ on $T_{\text{poly}}(\mathbb{R}^d)$ are elements of the space $S^n(T_{\text{poly}}(\mathbb{R}^d)[1])$ of symmetric n-polyvector fields on $T_{\text{poly}}(\mathbb{R}^d)[1]$ (this means that $\alpha_2 \cdot \alpha_1 = (-1)^{|\alpha_1||\alpha_2|}\alpha_1 \cdot \alpha_2$). The manifold $T_{\text{poly}}(\mathbb{R}^d)[1]$ is equipped with the formal bilinear vector field $Q = Q_2$, defined with the help of the Schouten bracket $[\ ,\ ]_S$:

$$Q_2(\alpha_1 \cdot \alpha_2) = (-1)^{(|\alpha_1|-1)|\alpha_2|}[\alpha_1, \alpha_2]_S.$$

Similarly, $D_{\text{poly}}(\mathbb{R}^d)[1]$ is equipped with the formal vector field

$$Q' = Q'_1 + Q'_2,$$

defined by

$$Q'_1(D_1) = -d_H D_1, \quad Q'_2(D_1 \cdot D_2) = (-1)^{(|D_1|-1)|D_2|}[D_1, D_2]_G.$$

Here $[\ ,\ ]_G$ is the Gerstenhaber bracket and d_H denotes the usual Hochschild coboundary operator: if D is an m-differential operator,

$$d_H D(f_1, \dots, f_{m+1}) \\ = f_1 D(f_2, \dots, f_{m+1}) - D(f_1 f_2, \dots, f_{m+1}) + \dots + (-1)^m D(f_1, \dots, f_m) f_{m+1}$$

A formality $\mathcal{F}$ is then given by a sequence of mappings

$$\mathcal{F}_n : S^n\big(T_{\text{poly}}(\mathbb{R}^d)[1]\big) \to D_{\text{poly}}(\mathbb{R}^d)[1],$$

homogeneous of degree 0 and satisfying the *formality equation*

$$d_H(\mathcal{F}_n)(\alpha_1 \cdot \ldots \cdot \alpha_n) = \frac{1}{2} \sum_{\substack{I \sqcup J = \{1,\dots,n\} \\ |I| \neq 0,\ |J| \neq 0}} \varepsilon_\alpha(I, J) Q'_2\big(\mathcal{F}_{|I|}(\alpha_I) \cdot \mathcal{F}_{|J|}(\alpha_J)\big) \\ - \frac{1}{2} \sum_{k \neq \ell} \varepsilon_\alpha(k\ell, 1 \dots \widehat{k\ell} \dots n) \mathcal{F}_{n-1}\big(Q_2(\alpha_k \cdot \alpha_\ell) \cdot \alpha_1 \cdot \ldots \cdot \widehat{\alpha_k \alpha_\ell} \cdot \ldots \cdot \alpha_n\big).$$

Here, if $I = \{i_1 < \dots < i_\ell\}$, the notation α_I means $\alpha_{i_1} \cdot \ldots \cdot \alpha_{i_\ell}$.

We shall impose moreover the condition that $\mathcal{F}_1$ is the canonical mapping $\mathcal{F}_1^{(0)}$ from $T_{\text{poly}}(\mathbb{R}^d)$ to $D_{\text{poly}}(\mathbb{R}^d)$ defined by

$$\mathcal{F}_1^{(0)}(\xi_1 \wedge \ldots \wedge \xi_n)(f_1, \dots, f_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} \xi_{\sigma(i)}(f_i),$$

for any vector field ξ_k and any function f_i.

Now choose a coordinate system (x_t) on $\mathbb{R}^d$. M. Kontsevich [2003] has built explicitly a formality $\mathcal{U}$ for $\mathbb{R}^d$, using families of graphs drawn on configuration spaces. A graph Γ has aerial and terrestrial vertices. The aerial vertices are labeled $p_1, \dots, p_n$ and are elements of the Poincaré half-plane

$$\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

The terrestrial vertices $q_1 < \dots < q_m$ are on the real line. The edges of Γ are arrows starting from an aerial vertex and ending in a terrestrial or aerial vertex; there are no arrows of the form $\overrightarrow{p_i p_i}$ and no multiple arrows. If we fix a total ordering O on the edges of Γ, we get an oriented graph (Γ, O). We say that O is compatible if, for all i, the arrows starting from p_i precede those starting from p_{i+1}. We denote by $GO_{n,m}$ the set of oriented graphs (Γ, O) with n labeled aerial vertices and m labeled terrestrial vertices, and such that O is compatible.

Consider such an oriented graph $(\Gamma, O) \in GO_{n,m}$. Suppose there are k_i edges starting from the vertex p_i $(1 \leq i \leq n)$. Kontsevich [2003] defines a natural operator $B_{(\Gamma,O)}$ assigning an m-differential operator $B_{(\Gamma,O)}(\alpha_1 \otimes \cdots \otimes \alpha_n)$ to an n-uple $(\alpha_1, \ldots, \alpha_n)$ of polyvector fields α_i. This operator vanishes unless, for each i, α_i belongs to $T_{\text{poly}}^{k_i-1}(\mathbb{R}^d)$ (α_i is a k_i-polyvector field). We first consider all the multi-indexes $(t_1, \ldots, t_{|k|})$ with $|k| = \sum k_i$ and $1 \leq t_r \leq d$ for all $1 \leq r \leq |k|$. We denote by $\text{end}(a)$ the set of edges ending at the vertex a; if these edges are $e_{i_1}, \ldots, e_{i_\ell}$, we let $\partial_{\text{end}(a)}$ be the operator

$$\partial_{\text{end}(a)} = \frac{\partial^l}{\partial x_{t_{i_1}} \ldots \partial x_{t_{i_\ell}}}.$$

Then, we denote by $\text{strt}(p_i)$ the ordered set $e^i_{j_1} < \cdots < e^i_{j_{k_i}}$ of edges starting from p_i and, if α_i is a k_i-vector field, by $\alpha_i^{\text{strt}(p_i)}$ the following component of α_i:

$$\alpha_i^{\text{strt}(p_i)} = \alpha_i^{t_{j_1} \ldots t_{j_{k_i}}}.$$

Finally, if α_i is a k_i-vector field for each i, we set

$$B_{(\Gamma,O)}(\alpha_1 \otimes \cdots \otimes \alpha_n)(f_1, \ldots, f_m) = \sum_{1 \leq t_1, \ldots, t_{|k|} \leq d} \prod_{i=1}^{n} \partial_{\text{end}(p_i)} \alpha_i^{\text{strt}(p_i)} \prod_{j=1}^{m} \partial_{\text{end}(q_j)} f_j.$$

$B_{(\Gamma,O)}$ will be called the graph operator associated with (Γ, O).

The explicit formality $\mathcal{U}$ of Kontsevich can now be written as a sum $\mathcal{U} = \sum_n \mathcal{U}_n$ with

$$\mathcal{U}_n = \sum_{m \geq 0} \sum_{(\Gamma,O) \in GO_{n,m}} w_{(\Gamma,O)} B_{(\Gamma,O)},$$

where the coefficient $w_{(\Gamma,O)}$, the *weight* of (Γ, O), is an integral on a compactified configuration space. To be precise, for $2n + m - 2 \geq 0$, let $\text{Conf}(n, m)$ be the space of $(n+m)$-tuples consisting of n distinct points p_i in $\mathcal{H}$ and m distinct points q_j on the real line $\partial\mathcal{H}$. Consider on $\text{Conf}(n, m)$ the action of the group G of transformations $z \mapsto az + b$ ($a > 0$ and b real), and form the quotient space

$$C_{n,m} = \text{Conf}(n, m)/G.$$

Kontsevich associates with each oriented graph (Γ, O) the form

$$\omega_{(\Gamma,O)} = \frac{1}{k!} \bigwedge_{i=1}^{n} \left(d\Phi_{e^i_1} \wedge \cdots \wedge d\Phi_{e^i_{k_i}}\right)$$

on $C_{n,m}$, where $\{e^i_1 < e^i_2 < \cdots < e^i_{k_i}\}$ denotes the ordered set $\operatorname{strt}(p_i)$ formed by the k_i edges starting from p_i, $k! := k_1! \dots k_n!$ and, if $e^i_\ell = \overrightarrow{p_i a}$,

$$\Phi_{e^i_\ell} = \Phi_{\overrightarrow{p_i a}} = \frac{1}{2\pi} \operatorname{Arg} \frac{a - p_i}{a - \bar{p}_i}.$$

The weight $w_{(\Gamma,O)}$ is then defined as the value of the integral $\omega_{(\Gamma,O)}$ on the connected component $C^+_{n,m}$ of $C_{n,m}$ for which $q_1 < \cdots < q_m$.

In this work, we are looking for graph formalities, that is, formalities on the space $\mathbb{R}^d$ of the form $\mathcal{F} = \sum_n \mathcal{F}_n$, where the $\mathcal{F}_n$ are homogeneous mappings (of degree 0) of the form

$$\mathcal{F}_n = \sum_{m\geq 0} \sum_{(\Gamma,O)\in GO_{n,m}} c_{(\Gamma,O)} B_{(\Gamma,O)},$$

with real coefficients $c_{(\Gamma,O)}$. We shall use the notation $\mathcal{F}_n = B_{\gamma_n}$, where γ_n is the linear combination

$$\gamma_n = \sum_{m\geq 0} \sum_{(\Gamma,O)\in GO_{n,m}} c_{(\Gamma,O)} (\Gamma, O).$$

Now assume we have found $\mathcal{F}_1, \dots, \mathcal{F}_{n-1}$ (with $\mathcal{F}_1 = \mathcal{F}^{(0)}_1 = \mathcal{U}_1$) such that the formality equation holds up to order $n-1$. The next term $\mathcal{F}_n$, if it exists, must be a solution of an equation

$$d_H \circ \mathcal{F}_n = E_n,$$

that is,

$$d_H(\mathcal{F}_n(\alpha_1 \cdot \dots \cdot \alpha_n)) = E_n(\alpha_1 \cdot \dots \cdot \alpha_n) = E_n(\alpha_{\{1,\dots,n\}}),$$

where $E_n(\alpha_{\{1,\dots,n\}})$ is a Hochschild cocycle. The Hochschild cohomology is localized in $T_{\text{poly}}(\mathbb{R}^d)[1]$; more precisely, the total skewsymmetrization $\mathfrak{a} \circ E_n(\alpha_{\{1,\dots,n\}})$ of $E_n(\alpha_{\{1,\dots,n\}})$ is a polydifferential operator of order $1, \dots, 1$, that is, the image under $\mathcal{F}^{(0)}_1$ of a polyvector field. Moreover, there exists an operator A_n such that

$$E_n(\alpha_{\{1,\dots,n\}}) = (\mathfrak{a} \circ E_n + d_H \circ A_n)(\alpha_{\{1,\dots,n\}}).$$

Now put

$$\varphi_n = \mathcal{F}^{-1}_1 \circ \mathfrak{a} \circ E_n,$$

that is,

$$\varphi_n(\alpha_{\{1,\dots,n\}}) = \mathcal{F}^{-1}_1\big(\mathfrak{a}(E_n(\alpha_{\{1,\dots,n\}}))\big);$$

then $\varphi_n : S^n\big(T_{\text{poly}}(\mathbb{R}^d)[1]\big) \to T_{\text{poly}}(\mathbb{R}^d)[1]$ is homogeneous of degree $|\varphi_n| = 1$.

In Section 2, we define the Chevalley coboundary operator ∂ on $T_{\text{poly}}(\mathbb{R}^d)$. We show that the mapping φ_n described above is a Chevalley cocycle, and, if it is a coboundary ($\varphi_n = \partial\phi_{n-1}$), we can add to $\mathcal{F}_{n-1}$ a Hochschild coboundary so that

$\mathfrak{a}(E_n)$ vanishes and thus find a $\mathcal{F}_n$ for which the formality equation holds up to order n.

In Section 3, we establish a noteworthy property for the Kontsevich weights. For any graph Γ (with k_i edges starting from p_i), denote by Δ the purely aerial graph obtained by removing the legs $\overrightarrow{p_i q_j}$ and the feet q_j of Γ, and by ℓ_i the number of aerial edges starting from p_i. We prove that

$$\mathfrak{a}\Bigg(\sum_{(\Gamma,O)\in GO^{(1)}_{n,m}} w_{(\Gamma,O)} B_{(\Gamma,O)}\Bigg) = \sum_{(\Delta,O_\Delta)\in GO^{(0)}_n} w_{(\Delta,O_\Delta)} \frac{1}{m!} \sum_{\substack{(\Gamma,O)\in GO^{(1)}_{n,m}\\ (\Gamma,O)\supset(\Delta,O_\Delta)}} \frac{\ell!}{k!} \varepsilon(\Gamma) B_{(\Gamma,O)}.$$

Here $GO^{(1)}_{n,m}$ denotes the subspace of $GO_{n,m}$ formed by the oriented graphs having exactly one leg for each foot, $GO^{(0)}_n$ is the set of purely aerial oriented graphs (Δ, O_Δ) with n aerial vertices and O_Δ compatible and $\varepsilon(\Gamma)$ is an explicit sign depending only on Γ.

This property suggests that we study what we call K-graph formalities. A *K-graph formality* up to order n is a graph formality $\mathcal{F}$ at order $n-1$ such that $\varphi_n = \mathcal{F}_1^{-1} \circ \mathfrak{a} \circ E_n$ has the form

$$\varphi_n = \sum_{(\Delta,O_\Delta)\in GO^{(0)}_n} c_{(\Delta,O_\Delta)} C_{(\Delta,O_\Delta)}$$

with real coefficients $c_{(\Delta,O_\Delta)}$ and where

$$C_{(\Delta,O_\Delta)} = \sum_{m\geq 0} \frac{1}{m!} \sum_{\substack{(\Gamma,O)\in GO^{(1)}_{n,m}\\ (\Gamma,O)\supset(\Delta,O_\Delta)}} \frac{\ell!}{k!} \varepsilon(\Gamma) B_{(\Gamma,O)}.$$

In Section 4 we give some simple expressions of our Chevalley coboundary operator. Then we restrict ourselves to K-graph formalities and study the Chevalley cohomology related to the question of building such formalities.

In Section 5 we show that the coboundary operator ∂ can be written directly on the aerial part of the graphs.

We devote Section 6 to explicit computations and applications. In particular, we prove the triviality of the cohomology for small values of n and give the restriction of the cohomology for linear formalities.

2. Chevalley cohomology and formalities

We start by defining a graded Chevalley cohomology in a general algebraic setting — that is, for cochains $C : S^n(\mathfrak{g}[1]) \to \mathfrak{M}[1]$, where $\mathfrak{g}$ is a graded Lie algebra and $\mathfrak{M}$ a graded $\mathfrak{g}$-module. In fact two Chevalley coboundary operators are naturally associated with the formality equation for $\mathbb{R}^d$. The first, ∂', is obtained by endowing $D_{\mathrm{poly}}(\mathbb{R}^d)$ with a $T_{\mathrm{poly}}(\mathbb{R}^d)$-graded module structure; cochains are mappings $C : S^n\big(T_{\mathrm{poly}}(\mathbb{R}^d)[1]\big) \to D_{\mathrm{poly}}(\mathbb{R}^d)[1]$. The other one, ∂, is obtained by considering $T_{\mathrm{poly}}(\mathbb{R}^d)$ as a graded module over itself; cochains are mappings $C : S^n\big(T_{\mathrm{poly}}(\mathbb{R}^d)[1]\big) \to T_{\mathrm{poly}}(\mathbb{R}^d)[1]$. Using both ∂ and ∂', we show that the obstructions to formalities can be interpreted as cocycles for ∂.

2.1. *Chevalley cohomology.* Let $(\mathfrak{g}, [\ ,\ ])$ be a graded Lie algebra and $\mathfrak{M}$ a graded module over $\mathfrak{g}$. For reasons of homogeneity, we prefer to work with $\mathfrak{g}[1]$ and $\mathfrak{M}[1]$. Thus, we replace $[\ ,\ ]$ and the action of $\mathfrak{g}$ on $\mathfrak{M}$ respectively by $[\ ,\ ]'$ and $[\ ,\ ]_{\mathfrak{M}}$, defined for homogeneous α, β in $\mathfrak{g}[1]$ of degrees $|\alpha|$, $|\beta|$ and for m in $\mathfrak{M}[1]$ of degree $|m|$ by

$$[\alpha, \beta]' = (-1)^{(|\alpha|+1)|\beta|}[\alpha, \beta],$$
$$[\alpha, m]_{\mathfrak{M}} = (-1)^{(|\alpha|+1)|m|}\alpha.m.$$

The space $C^n(\mathfrak{g}, \mathfrak{M})$ of n-cochains consists of mappings C from $S^n(\mathfrak{g}[1])$ to $\mathfrak{M}[1]$. The Chevalley coboundary ∂C of an n-cochain C, homogeneous of degree $|C|$, is the $(n+1)$-cochain defined by

$$\begin{aligned}\partial C(\alpha_1 \cdot \ldots \cdot \alpha_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{|C||\alpha_i|} \varepsilon_\alpha(i, 1 \ldots \hat{\imath} \ldots n+1)\big[\alpha_i, C(\alpha_1 \cdot \ldots \cdot \hat{\alpha_i} \cdot \ldots \cdot \alpha_{n+1})\big]_{\mathfrak{M}} \\ &\quad - \frac{1}{2}\sum_{i \neq j} \varepsilon_\alpha(ij, 1 \ldots \widehat{ij} \ldots, n+1)(-1)^{|C|} C\big([\alpha_i, \alpha_j]' \cdot \alpha_1 \ldots \widehat{\alpha_i \alpha_j} \cdot \ldots \cdot \alpha_{n+1}\big).\end{aligned}$$

Here the α_i are homogeneous elements of $\mathfrak{g}$, $|\alpha_i|$ denotes the degree of α_i in $\mathfrak{g}[1]$ and for any permutation σ of $\{1, \ldots, n\}$, $\varepsilon_\alpha(\sigma)$ is the sign of σ in the graded sense. We shall denote by $C^n_{[q]}(\mathfrak{g}, \mathfrak{M})$ the subspace of $C^n(\mathfrak{g}, \mathfrak{M})$ formed by the n-cochains of degree q and by $H^n_{[q]}(\mathfrak{g}, \mathfrak{M})$ the corresponding cohomology group. Note that ∂ sends $C^n_{[q]}(\mathfrak{g}, \mathfrak{M})$ into $C^{n+1}_{[q+1]}(\mathfrak{g}, \mathfrak{M})$.

Extending usual techniques to the graded case (See [Gammella 2001] for an explicit computation), it is possible to prove:

Lemma 2.1. *The operator ∂ is a cohomology operator, that is,* $\partial^2 = \partial \circ \partial = 0$.

We now return to the graded Lie algebras

$$\big(T_{\mathrm{poly}}(\mathbb{R}^d), [\ ,\ ]_S\big) \quad \text{and} \quad \big(D_{\mathrm{poly}}(\mathbb{R}^d), [\ ,\ ]_G\big),$$

where $[\ ,\ ]_S$ is the Schouten bracket and $[\ ,\ ]_G$ the Gerstenhaber bracket. Let us first make our conventions for these spaces and brackets precise.

Let α be a k-vector field and $\{e_i\}$ the canonical basis of $\mathbb{R}^d$. We put

$$\alpha = \sum_{j_1,\dots,j_k} \alpha^{j_1\cdots j_k} e_{j_1} \otimes \cdots \otimes e_{j_k} = \sum_{j_1<j_2<\cdots<j_k} \alpha^{j_1\cdots j_k} e_{j_1} \wedge \cdots \wedge e_{j_k}$$
$$= \frac{1}{k!} \sum_{j_1\cdots j_k} \alpha^{j_1\cdots j_k} e_{j_1} \wedge \cdots \wedge e_{j_k}.$$

For any k_1-vector field α_1 and k_2-vector field α_2 (the degree of α_i is $k_i - 1$ in $T_{\mathrm{poly}}(\mathbb{R}^d)$), we define first a polyvector field $\alpha_1 \bullet \alpha_2$ with components

$$\alpha_1 \bullet \alpha_2^{i_1\cdots_{k_1+k_2-1}} = \frac{1}{k_1!\,k_2!} \sum_{\sigma\in S_{k_1+k_2-1}} \Bigg(\varepsilon(\sigma) \sum_{\ell=1}^{k_1} (-1)^{\ell-1} \times \sum_{s=1}^{d} \alpha_1^{i_{\sigma(1)}\dots i_{\sigma(\ell-1)} s i_{\sigma(\ell)}\dots i_{\sigma(k_1-1)}} \partial_s \alpha_2^{i_{\sigma(k_1)}\dots i_{\sigma(k_1+k_2-1)}} \Bigg).$$

Now, $[\alpha_1, \alpha_2]_S$ can be written as

$$[\alpha_1, \alpha_2]_S = (-1)^{k_2(k_1-1)} \alpha_1 \bullet \alpha_2 - (-1)^{k_2-1} \alpha_2 \bullet \alpha_1.$$

(This choice for the Schouten bracket is denoted $[\ ,\ ]'_S$ in [Arnal et al. 2002] and [Manchon and Torossian 2003].)

On the other hand, for any m_1-differential operator D_1 and any m_2-differential operator D_2 (the degree of D_i is $m_i - 1$ in $D_{\mathrm{poly}}(\mathbb{R}^d)$), we may write $[D_1, D_2]_G$ in the form

$$[D_1, D_2]_G = D_1 \circ D_2 - (-1)^{(m_1-1)(m_2-1)} D_2 \circ D_1,$$

where

$$D_1 \circ D_2(f_1, \dots, f_{m_1+m_2-1}) = \sum_{j=1}^{m_1} (-1)^{(m_2-1)(j-1)} D_1\big(f_1, \dots, f_{j-1}, D_2(f_j, \dots, f_{j+m_2-1}), f_{j+m_2}, \dots, f_{m_1+m_2-1}\big).$$

Recall the canonical mapping $\mathscr{F}_1^{(0)}$ from $T_{\mathrm{poly}}(\mathbb{R}^d)$ into $D_{\mathrm{poly}}(\mathbb{R}^d)$: each k-vector field α can be viewed as a k-differential operator $\mathscr{F}_1^{(0)}(\alpha)$ of order $1, \dots, 1$:

$$\big(\mathscr{F}_1^{(0)}(\alpha)\big)(f_1, \dots, f_k) = \langle \alpha, df_1 \wedge \cdots \wedge df_k \rangle = \frac{1}{k!} \alpha^{i_1\cdots_k} \partial_{i_1} f_1 \dots \partial_{i_k} f_k.$$

Now consider the action of $T_{\mathrm{poly}}(\mathbb{R}^d)$ given by

$$\alpha.D = \mathfrak{a} \circ \big[\mathscr{F}_1^{(0)}(\alpha), D\big]_G \qquad \text{for } \alpha \in T_{\mathrm{poly}}(\mathbb{R}^d) \text{ and } D \in D_{\mathrm{poly}}(\mathbb{R}^d),$$

where $\mathfrak{a}$ denotes the usual skewsymmetrization of differential operators and $[\ ,\ ]_G$ is the Gerstenhaber bracket. This action defines a $T_{\text{poly}}(\mathbb{R}^d)$-graded module structure on $D_{\text{poly}}(\mathbb{R}^d)$. Indeed, one can prove:

Proposition 2.2. *The following equalities hold for any D_1, D_2, D in $D_{\text{poly}}(\mathbb{R}^d)$, any k_1-vector field α_1 and k_2-vector field α_2 in $T_{\text{poly}}(\mathbb{R}^d)$:*

(i) $\mathfrak{a}\circ[D_1, D_2]_G = \mathfrak{a}\circ[D_1, \mathfrak{a}\circ D_2]_G$;

(ii) $\mathcal{F}_1^{(0)}([\alpha_1,\alpha_2]_S) = \mathfrak{a}\circ[\mathcal{F}_1^{(0)}(\alpha_1), \mathcal{F}_1^{(0)}(\alpha_2)]_G$;

(iii) $$\mathfrak{a}\circ\big[\mathcal{F}_1^{(0)}([\alpha_1,\alpha_2]_S), D\big]_G = \mathfrak{a}\circ\big[\mathcal{F}_1^{(0)}(\alpha_1), \mathfrak{a}\circ[\mathcal{F}_1^{(0)}(\alpha_2), D]_G\big]_G - (-1)^{(k_1-1)(k_2-1)}\mathfrak{a}\circ\big[\mathcal{F}_1^{(0)}(\alpha_2), \mathfrak{a}\circ[\mathcal{F}_1^{(0)}(\alpha_1), D]_G\big]_G.$$

From (iii) *it follows that*

$$[\alpha_1,\alpha_2]_S.D = \alpha_1.(\alpha_2.D) - (-1)^{(k_1-1)(k_2-1)}\alpha_2.(\alpha_1.D),$$

and thus $D_{\text{poly}}(\mathbb{R}^d)$ is a $T_{\text{poly}}(\mathbb{R}^d)$-module.

Now endow $D_{\text{poly}}(\mathbb{R}^d)$ with the $T_{\text{poly}}(\mathbb{R}^d)$-graded structure described above. If $C : \bigwedge^n\big(T_{\text{poly}}(\mathbb{R}^d)\big) = S^n\big(T_{\text{poly}}(\mathbb{R}^d)[1]\big) \to D_{\text{poly}}(\mathbb{R}^d)[1]$ is a homogeneous mapping of degree $|C|$, we can define its Chevalley coboundary $\partial' C$. The latter can be written using the vector fields Q and Q', associated respectively with $T_{\text{poly}}(\mathbb{R}^d)$ and $D_{\text{poly}}(\mathbb{R}^d)$:

$$\begin{aligned}\partial' C(\alpha_1 \cdot \ldots \cdot \alpha_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{|C||\alpha_i|}\varepsilon_\alpha(i, 1\ldots\hat{\imath}\ldots n+1)\mathfrak{a}\circ Q_2'\big(\mathcal{F}_1^{(0)}(\alpha_i)\cdot C(\alpha_1\cdot\ldots\hat{\alpha}_i\cdot\ldots\cdot\alpha_{n+1})\big)\\ &\quad - \frac{1}{2}\sum_{i\neq j}\varepsilon_\alpha(ij, 1\ldots\widehat{ij}\ldots n+1)(-1)^{|C|}C\big(Q_2(\alpha_i\cdot\alpha_j)\cdot\alpha_1\ldots\widehat{\alpha_i\alpha_j}\cdot\ldots\cdot\alpha_{n+1}\big).\end{aligned}$$

To simplify the writing, we will sometimes write α_i instead of $\mathcal{F}_1^{(0)}(\alpha_i)$.

At the same time, considering $T_{\text{poly}}(\mathbb{R}^d)$ as a graded module over itself, one can define the Chevalley cohomology for $T_{\text{poly}}(\mathbb{R}^d)$. If $C : S^n\big(T_{\text{poly}}(\mathbb{R}^d)[1]\big) \to T_{\text{poly}}(\mathbb{R}^d)[1]$ is an n-cochain, homogeneous of degree $|C|$, its coboundary ∂C is

$$\begin{aligned}\partial C(\alpha_1 \cdot \ldots \cdot \alpha_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{|C||\alpha_i|}\varepsilon_\alpha(i, 1\ldots\hat{\imath}\ldots n+1) Q_2\big(\alpha_i\cdot C(\alpha_1\cdot\ldots\hat{\alpha}_i\cdot\ldots\cdot\alpha_{n+1})\big)\\ &\quad - \frac{1}{2}\sum_{i\neq j}\varepsilon_\alpha(ij, 1\ldots\widehat{ij}\ldots n+1)(-1)^{|C|}C\big(Q_2(\alpha_i\cdot\alpha_j)\cdot\alpha_1\ldots\widehat{\alpha_i\alpha_j}\cdot\ldots\cdot\alpha_{n+1}\big).\end{aligned}$$

Remark. For any $\varphi : S^n\big(T_{\text{poly}}(\mathbb{R}^d)[1]\big) \to T_{\text{poly}}(\mathbb{R}^d)[1]$, we have

$$\partial'(\mathcal{F}_1^{(0)} \circ \varphi) = \mathcal{F}_1^{(0)} \circ \partial\varphi.$$

2.2. *Obstruction to formalities.* The two Chevalley coboundary operators ∂ and ∂' enable us to reformulate the formality equation. Indeed, suppose we want to construct a formality $\mathcal{F}$ from $T_{\text{poly}}(\mathbb{R}^d)$ to $D_{\text{poly}}(\mathbb{R}^d)$. We thus need to solve recursively the formality equation (see [Kontsevich 1997; Arnal et al. 2002] for notations)

$$d_H(\mathcal{F}_n)(\alpha_1 \cdot \ldots \cdot \alpha_n) = \frac{1}{2} \sum_{\substack{I \sqcup J = \{1,\ldots,n\} \\ |I|\geq 1, |J|\geq 1}} \varepsilon_\alpha(I, J) Q_2'\big(\mathcal{F}_{|I|}(\alpha_I) \cdot \mathcal{F}_{|J|}(\alpha_J)\big)$$
$$- \frac{1}{2} \sum_{k\neq \ell} \varepsilon_\alpha(k\ell, 1 \ldots \widehat{k\ell} \ldots n) \mathcal{F}_{n-1}\big(Q_2(\alpha_k \cdot \alpha_\ell) \cdot \alpha_1 \cdot \ldots \cdot \widehat{\alpha_k \alpha_\ell} \partial \cdot \ldots \cdot \alpha_n\big),$$

where d_H is the Hochschild coboundary operator.

Now impose the condition that the first component $\mathcal{F}_1$ be $\mathcal{F}_1^{(0)}$. Assume there are mappings $\mathcal{F}_2, \ldots, \mathcal{F}_{n-1}$, homogeneous of degree 0, and satisfying the formality equation up to order $n-1$. Denote by E_n the right-hand side of the equation at the order n. Then E_n is a Hochschild cocycle: $d_H E_n = 0$ (see [Arnal and Masmoudi 2002] for instance). Thus

$$E_n = \mathfrak{a} \circ E_n + d_H C,$$

where $\mathfrak{a} \circ E_n$ is a differential operator of order $1, \ldots, 1$ and E_n is a coboundary if and only if $\mathfrak{a} \circ E_n = 0$. But

$$\mathfrak{a} \circ E_n(\alpha_1 \cdot \ldots \cdot \alpha_n) = \partial' \mathfrak{a} \mathcal{F}_{n-1}(\alpha_1 \cdot \ldots \cdot \alpha_n) + \mathfrak{a} R_n(\alpha_1 \cdot \ldots \cdot \alpha_n),$$

where

$$R_n(\alpha_1 \cdot \ldots \cdot \alpha_n) = \frac{1}{2} \sum_{I \sqcup J = \{1,\ldots,n\}, |I|\geq 2, |J| \geq 2} \varepsilon_\alpha(I, J) Q_2'\Big(\mathcal{F}_{|I|}(\alpha_I) \cdot \mathcal{F}_{|J|}(\alpha_J)\Big).$$

It follows directly from this expression that R_n and $\mathfrak{a} \circ R_n$ both have degree 1: $|R_n| = |\mathfrak{a} \circ R_n| = 1$. Moreover,

Theorem 2.3. *The skewsymmetrization $\mathfrak{a} \circ E_n$ of E_n can be identified through the inverse mapping of $\mathcal{F}_1$ with a ∂-cocycle. If this cocycle is exact, we can find $\mathcal{F}'_{n-1}$ and $\mathcal{F}'_n$, homogeneous of degree 0, such that $\mathcal{F}_2, \ldots, \mathcal{F}_{n-2}, \mathcal{F}'_{n-1}, \mathcal{F}'_n$ satisfy the formality equation up to order n.*

Proof. The proof proceeds in three steps.

Step 1. First we check that $\mathfrak{a}\circ R_n$ is a cocycle for ∂':

$$\begin{aligned}
&\partial'\mathfrak{a}R_n(\alpha_1\ldots\ldots\alpha_{n+1})\\
&\quad=\sum_{i=1}^{n+1}(-1)^{|\alpha_i|}\varepsilon_\alpha(i,1\ldots\hat{\imath}\ldots n+1)\mathfrak{a}Q_2'\big(\alpha_i\,.\,\mathfrak{a}R_n(\alpha_1\ldots\ldots\widehat{\alpha_i}\ldots\ldots\alpha_{n+1})\big)\\
&\qquad+\frac{1}{2}\sum_{i\neq j}\varepsilon_\alpha(ij,1\ldots\widehat{\imath\jmath}\ldots n+1)\mathfrak{a}R_n\big(Q_2(\alpha_i\,.\,\alpha_j)\,.\,\alpha_1\ldots\ldots\widehat{\alpha_i\alpha_j}\ldots\ldots\alpha_{n+1}\big)\\
&\quad=\frac{1}{2}\sum_{i=1}^{n+1}\Bigg((-1)^{|\alpha_i|}\varepsilon_\alpha(i,1\ldots\hat{\imath}\ldots n+1)\\
&\qquad\qquad\times\sum_{\substack{I\sqcup J=\{1\ldots\hat{\imath}\ldots n+1\}\\|I|\geq 2,|J|\geq 2}}\varepsilon_{\alpha'}(I,J)\mathfrak{a}Q_2'\big(\alpha_i\,.\,\mathfrak{a}Q_2'(\mathcal{F}_{|I|}(\alpha_I)\,.\,\mathcal{F}_{|J|}(\alpha_J))\big)\Bigg)\\
&\qquad+\frac{1}{4}\sum_{i\neq j}\Bigg(\varepsilon_\alpha(ij,1\ldots\widehat{\imath\jmath}\ldots n+1)\\
&\qquad\qquad\times\sum_{\substack{I\sqcup J=\{0,1\ldots\widehat{\imath\jmath}\ldots n+1\}\\|I|\geq 2,|J|\geq 2}}\varepsilon_{\alpha''}(I,J)\mathfrak{a}Q_2'\big(\mathcal{F}_{|I|}(\alpha_I)\,.\,\mathcal{F}_{|J|}(\alpha_J)\big)\Bigg)\\
&\quad=\tfrac{1}{2}(\mathrm{I})+\tfrac{1}{4}(\mathrm{II}),
\end{aligned}$$

where we have set $\alpha_0:=Q_2(\alpha_i\,.\,\alpha_j)$, $\varepsilon_{\alpha'}:=\varepsilon_{\alpha\setminus\{\alpha_i\}}$ and $\varepsilon_{\alpha''}:=\varepsilon_{(\alpha\cup\{\alpha_0\})\setminus\{\alpha_i,\alpha_j\}}$.

The term (I) above equals

$$\sum_{i=1}^{n+1}\sum_{\substack{I\sqcup J=\{1\ldots\hat{\imath}\ldots n+1\}\\|I|\geq 2,|J|\geq 2}}(-1)^{(|\alpha_I|+|\alpha_J|)|\alpha_i|}\varepsilon_\alpha(i,I,J)\mathfrak{a}Q_2'\big(\mathfrak{a}Q_2'(\mathcal{F}_{|I|}(\alpha_I)\,.\,\mathcal{F}_{|J|}(\alpha_J))\,.\,\alpha_i\big).$$

By Proposition 2.2, $\mathfrak{a}Q_2'$ satisfies the graded Jacobi identity; thus (I) equals

$$\begin{aligned}
&-\sum_{i=1}^{n+1}\sum_{\substack{I\sqcup J=\{1\ldots\hat{\imath}\ldots n+1\}\\|I|\geq 2,|J|\geq 2}}(-1)^{|\alpha_J|(|\alpha_I|+|\alpha_i|)}\varepsilon_\alpha(i,I,J)\mathfrak{a}Q_2'\big(\mathfrak{a}Q_2'(\mathcal{F}_{|J|}(\alpha_J)\,.\,\alpha_i)\,.\,\mathcal{F}_{|I|}(\alpha_I)\big)\\
&\qquad-\sum_{\substack{I\sqcup J=\{1\ldots\hat{\imath}\ldots n+1\}\\|I|\geq 2,|J|\geq 2}}\varepsilon_\alpha(i,I,J)\mathfrak{a}Q_2'\big(\mathfrak{a}Q_2'(\alpha_i\,.\,\mathcal{F}_{|I|}(\alpha_I))\,.\,\mathcal{F}_{|J|}(\alpha_J)\big)\\
&\quad=-2\sum_{\substack{I\sqcup J=\{1\ldots\hat{\imath}\ldots n+1\}\\|I|\geq 2,|J|\geq 2}}\varepsilon_\alpha(i,I,J)\mathfrak{a}Q_2'\big(\mathfrak{a}Q_2'(\alpha_i\,.\,\mathcal{F}_{|I|}(\alpha_I))\,.\,\mathcal{F}_{|J|}(\alpha_J)\big).
\end{aligned}$$

Similarly, the second term, (II), is equal to

$$\sum_{i\neq j}\varepsilon_\alpha(ij,1\ldots\widehat{ij}\ldots n+1)\sum_{\substack{I\sqcup J=\{0,1\ldots\widehat{ij}\ldots n+1\}\\|I|\geq 2,|J|\geq 2}}\varepsilon_{\alpha''}(I,J)\mathfrak{a}Q_2'\big(\mathcal{F}_{|I|}(\alpha_I)\centerdot\mathcal{F}_{|J|}(\alpha_J)\big)$$

$$=\sum_{i\neq j}\sum_{\substack{I=I_1\sqcup\{0\}\\I_1\sqcup J=\{1\ldots\widehat{ij}\ldots n+1\}}}\varepsilon_\alpha(ij,I_1,J)\mathfrak{a}Q_2'\big(\mathcal{F}_{|I|}(Q_2(\alpha_i\centerdot\alpha_j)\centerdot\alpha_{I_1})\centerdot\mathcal{F}_{|J|}(\alpha_J)\big)$$

$$+\sum_{i\neq j}\Bigg(\sum_{\substack{J=J_1\sqcup\{0\}\\I\sqcup J_1=\{1\ldots\widehat{ij}\ldots n+1\}}}\varepsilon_\alpha(ij,1\ldots\widehat{ij}\ldots n+1)\varepsilon_{\alpha''}(I,\{0\},J_1)\times\mathfrak{a}Q_2'\big(\mathcal{F}_{|I|}(\alpha_I)\centerdot\mathcal{F}_{|J|}(Q_2(\alpha_i\centerdot\alpha_j)\centerdot\alpha_{J_1})\big)\Bigg)$$

$$=\sum_{i\neq j}\Bigg(\sum_{\substack{I=I_1\sqcup\{0\}\\I_1\sqcup J=\{1\ldots\widehat{ij}\ldots n+1\}}}\varepsilon_\alpha(ij,I_1,J)\mathfrak{a}Q_2'\big(\mathcal{F}_{|I|}(Q_2(\alpha_i\centerdot\alpha_j)\centerdot\alpha_{I_1})\centerdot\mathcal{F}_{|J|}(\alpha_J)\big)$$

$$+\sum_{\substack{J=J_1\sqcup\{0\}\\I\sqcup J_1=\{1\ldots\widehat{ij}\ldots n+1\}}}\varepsilon_\alpha(ij,I,J_1)(-1)^{(|\alpha_i|+|\alpha_j|+1)|\alpha_J|}\times\mathfrak{a}Q_2'\big(\mathcal{F}_{|I|}(\alpha_I)\centerdot\mathcal{F}_{|J|}(Q_2(\alpha_i\centerdot\alpha_j)\centerdot\alpha_{J_1})\big)\Bigg)$$

$$=2\sum_{i\neq j}\sum_{\substack{I=I_1\sqcup\{0\}\\I_1\sqcup J=\{1\ldots\widehat{ij}\ldots n+1\}}}\varepsilon_\alpha(ij,I_1,J)\mathfrak{a}Q_2'\big(\mathcal{F}_{|I|}(Q_2(\alpha_i\centerdot\alpha_j)\centerdot\alpha_{I_1})\centerdot\mathcal{F}_{|J|}(\alpha_J)\big).$$

Putting (I) and (II) together, we get

$$\partial'(\mathfrak{a}R_n)(\alpha_1\centerdot\ldots\centerdot\alpha_{n+1})=\tfrac{1}{2}(\mathrm{I})+\tfrac{1}{4}(\mathrm{II})$$

$$=\sum_{\substack{I'\sqcup J=\{1\ldots n+1\}\\|J|\geq 2,|I'|\geq 3}}\varepsilon_\alpha(I',J)\Bigg(\sum_{\substack{i\in I'\\(I'=I\sqcup\{i\})}}\varepsilon_{\alpha_{\{i\}\sqcup I}}(i,I)\mathfrak{a}Q_2'\big(\mathfrak{a}Q_2'(\alpha_i\centerdot\mathcal{F}_{|I|}(\alpha_I))\centerdot\mathcal{F}_{|J|}(\alpha_J)\big)$$

$$+\frac{1}{2}\sum_{\substack{i\neq j\in I',I'=I_1\sqcup\{ij\}\\I=I_1\sqcup\{0\}}}\varepsilon_{\alpha_{\{ij\}\cup I_1}}(ij,I_1)\mathfrak{a}Q_2'\big(\mathcal{F}_{|I|}(Q_2(\alpha_i\centerdot\alpha_j)\centerdot\alpha_{I_1})\centerdot\mathcal{F}_{|J|}(\alpha_J)\big)\Bigg).$$

Now, Proposition 2.2 and the definition of ∂' yield
(∗)
$$\partial'(\mathfrak{a}R_n)(\alpha_1\centerdot\ldots\centerdot\alpha_{n+1})=-\sum_{\substack{I'\sqcup J=\{1\ldots n+1\}\\|J|\geq 2,\ |I'|\geq 3}}\varepsilon_\alpha(I,J)\mathfrak{a}Q_2'\big(\partial'\mathfrak{a}\mathcal{F}_{|I'|-1}(\alpha_{I'})\centerdot\mathcal{F}_{|J|}(\alpha_J)\big).$$

On the other hand, since the formality equation holds up to order $n-1$, we have

$$\partial'\mathfrak{a}\mathcal{F}_{p-1}+\mathfrak{a}R_p=\mathfrak{a}(E_p)=\mathfrak{a}(d_H(\mathcal{F}_p))=0\quad\text{for } p\leq n-1.$$

But $|I'| \leq n-1$ for all I' in the expression $(*)$; thus

$$-\partial' \mathfrak{a}\mathcal{F}_{|I'|-1}(\alpha_{I'}) = \mathfrak{a}R_{|I'|}(\alpha_{I'}) = \frac{1}{2} \sum_{\substack{S\sqcup T=I' \\ |S|\geq 2, |T|\geq 2}} \varepsilon_{\alpha_{S\sqcup T}}(S,T)\mathfrak{a}Q'_2(\mathcal{F}_{|S|}(\alpha_S) \cdot \mathcal{F}_{|T|}(\alpha_T)).$$

Finally, $(*)$ becomes

$$\begin{aligned}
\partial'(\mathfrak{a}R_n)(\alpha_1 \cdot \ldots \cdot \alpha_{n+1})
&= \frac{1}{2} \sum_{\substack{S\sqcup T\sqcup J=\{1\ldots n+1\} \\ |S|\geq 2, |T|\geq 2, |J|\geq 2}} \varepsilon_\alpha(S\cup T, J)\varepsilon_{\alpha_{S\sqcup T}}(S,T) \\
&\qquad\qquad \times \mathfrak{a}Q'_2\big(\mathfrak{a}Q'_2(\mathcal{F}_{|S|}(\alpha_S) \cdot \mathcal{F}_{|T|}(\alpha_T)) \cdot \mathcal{F}_{|J|}(\alpha_J)\big) \\
&= \frac{1}{2} \sum_{\substack{S\sqcup T\sqcup J=\{1\ldots n+1\} \\ |S|\geq 2, |T|\geq 2, |J|\geq 2}} \varepsilon_\alpha(S,T,J)\mathfrak{a}Q'_2\big(\mathfrak{a}Q'_2(\mathcal{F}_{|S|}(\alpha_S) \cdot \mathcal{F}_{|T|}(\alpha_T)) \cdot \mathcal{F}_{|J|}(\alpha_J)\big).
\end{aligned}$$

Thanks to the Jacobi identity, the quantity on the last line vanishes. Hence $\partial'(\mathfrak{a}R_n)$ and $\partial'(\mathfrak{a}E_n)$ both vanish.

Step 2. Put

$$\varphi_n = \mathcal{F}_1^{-1} \circ \mathfrak{a} \circ E_n.$$

Since

$$\partial'(\mathfrak{a} \circ E_n) = \partial'\mathcal{F}_1(\varphi_n) = \mathcal{F}_1(\partial\varphi_n) = 0,$$

φ_n is a cocycle for ∂.

Step 3. Assume that $\varphi_n = \partial\phi_{n-1}$, where $\phi_{n-1} : S^{n-1}\big(T_{\text{poly}}(\mathbb{R}^d)[1]\big) \to T_{\text{poly}}(\mathbb{R}^d)[1]$. Of course, $d_H\mathcal{F}_1(\phi_{n-1}) = 0$. Therefore, the mappings $\mathcal{F}'_2 = \mathcal{F}_2, \ldots, \mathcal{F}'_{n-2} = \mathcal{F}_{n-2}$, $\mathcal{F}'_{n-1} = \mathcal{F}_{n-1} - \mathcal{F}_1 \circ \phi_{n-1}$ satisfy the formality equation up to order $n-1$. Moreover, the Hochschild cocycle E'_n corresponding to these $\mathcal{F}'_p$ satisfies

$$\mathfrak{a} \circ E'_n = \mathfrak{a} \circ E_n - \partial'(\mathcal{F}_1 \circ \phi_{n-1}) = \mathfrak{a} \circ E_n - \mathcal{F}_1(\partial\phi_{n-1}) = 0.$$

We are now able to find $\mathcal{F}'_n$ such that $E'_n = d_H\mathcal{F}'_n$. This ends the proof. □

3. Skewsymmetrization

The aim of this section is to prove a noteworthy property of Kontsevich's weights and the definition of K-graph formalities.

3.1. ***Skewsymmetrization and 1-graphs.*** Consider an m-differential operator D on $\mathbb{R}^d$, vanishing on constants. We can decompose D as

$$D = D^{(1)} + D^{(>1)},$$

where $D^{(1)}$ has order 1 in each of its arguments and $D^{(>1)}$ has order greater than 1 in at least one of its arguments. The skewsymmetrization $\mathfrak{a}(D)$ of D, defined by

$$\mathfrak{a}(D)(f_1, \dots, f_m) = \frac{1}{m!} \sum_{\sigma \in S_m} \varepsilon(\sigma) D\big(f_{\sigma^{-1}(1)}, \dots, f_{\sigma^{-1}(m)}\big),$$

satisfies $\mathfrak{a}(D) = \mathfrak{a}(D^{(1)}) + \mathfrak{a}(D^{(>1)})$, and therefore

$$\mathfrak{a}(D)^{(1)} = \mathfrak{a}(D^{(1)}).$$

We assume D is defined with the help of graphs:

$$D_{(\alpha_1,\dots,\alpha_n)} = \sum_{\Gamma} c_\Gamma B_\Gamma(\alpha_1 \otimes \cdots \otimes \alpha_n),$$

where the c_Γ are real. To compute $a(D)^{(1)}$, we need only consider

$$D^{(1)}_{(\alpha_1,\dots,\alpha_n)} = \sum_{\Gamma \in G^{(1)}} c_\Gamma B_\Gamma(\alpha_1 \otimes \cdots \otimes \alpha_n),$$

where $G^{(1)}$ denotes the family of 1-graphs, that is, graphs having exactly one leg for each foot.

However, as in [Kontsevich 2003], to define B_Γ we need to choose a total ordering O on the set $E(\Gamma)$ of edges of Γ. To be precise, we first choose a labeling on the aerial vertices of Γ, say $p_1, \dots, p_n$. Then we put away the arrows starting from p_1, from p_2, and so on, and finally from p_n. We get a total ordering of $E(\Gamma)$ compatible with the ordering $p_1 < p_2 < \cdots < p_n$ in the sense that the arrows starting from p_i precede those starting from p_{i+1}.

From now on, we denote by $GO_{n,m}$ the set of oriented graphs (Γ, O) with n labeled aerial vertices, m labeled terrestrial vertices and compatible ordering O, and by $GO^{(1)}_{n,m}$ the subset of $GO_{n,m}$ formed by the oriented 1-graphs. Our earlier notation $\sum c_\Gamma B_\Gamma$ actually means

$$\sum_{\Gamma} c_\Gamma B_\Gamma = \sum_{(\Gamma,O)\in GO_{n,m}} c_{(\Gamma,O)} B_{(\Gamma,O)} \quad \text{and} \quad \sum_{\Gamma\in G^{(1)}} c_\Gamma B_\Gamma = \sum_{(\Gamma,O)\in GO^{(1)}_{n,m}} c_{(\Gamma,O)} B_{(\Gamma,O)}.$$

3.2. *A noteworthy property of Kontsevich weights.*

Kontsevich weights. Let (Γ, O) be an oriented graph in $GO^{(1)}_{n,m}$ with aerial vertices $p_1 < \cdots < p_n$. We denote by k_i the number of edges starting from p_i, by U_i the ordered set of legs starting from p_i, and by V_i the ordered set of aerial edges starting from the same point. Let ℓ_i be the number of elements in V_i, U_i has $m_i = k_i - \ell_i$ elements. By the definition of $GO^{(1)}_{n,m}$, the number of legs is exactly the number of terrestrial vertices; that is, $m = \sum_{i=1}^n m_i$.

Given (Γ, O), it will be helpful to consider the permutation s_O defined by

$$s_O : E(\Gamma) \mapsto V_1 \cup \cdots \cup V_n \cup U_1 \cup \cdots \cup U_n.$$

After this permutation we get a new (and no longer compatible) ordering O' on $E(\Gamma)$ such that all the legs are put at the end, and we can define a permutation τ_O of the legs of (Γ, O') by putting first the leg ending at q_1, then the leg ending at q_2, and so on, with the the leg ending at q_m last. We extend the permutation τ_O to $V_1 \cup \dots V_n \cup U_1 \cup \cdots \cup U_n$ by setting $\tau_O(v) = v$ for all v in $\bigcup V_i$. Finally, note Δ the aerial graph obtained from Γ by cutting the legs and the feet and by O_Δ the (compatible) ordering on Δ induced by O.

Let $GO_n^{(0)}$ be the set of oriented, purely aerial graphs (Δ, O_Δ) with n vertices.

With these notations, the Kontsevich weight associated with (Γ, O) can be written as

$$w_{(\Gamma,O)} = \frac{1}{k!}\varepsilon(s_O)\varepsilon(\tau_O)\int_{C^+_{n,0}} \bigwedge_{r=1}^{|\ell|} d\Phi_{e^\Delta_r} \int_{\substack{q_1<\cdots<q_m \text{ oriented}\\ \text{by } dq_1\wedge\cdots\wedge dq_m}} \bigwedge_{j=1}^{m} d\Phi_{\overrightarrow{p_{i_j}q_j}},$$

where $k! = k_1!\dots k_n!$, $|\ell| := \sum \ell_i$, $V_1 \cup \cdots \cup V_n := \{e^\Delta_1 < \cdots < e^\Delta_{|\ell|}\}$ and i_j stands for the unique index i such that the leg arriving on q_j is exactly $\overrightarrow{p_i q_j}$.

The Kontsevich weight of (Δ, O_Δ) is just

$$w_{(\Delta,O_\Delta)} = \frac{1}{\ell!}\int_{C^+_{n,0}} \bigwedge_{r=1}^{|\ell|} d\Phi_{e^\Delta_r},$$

$(\ell! = \ell_1!\dots\ell_n!)$. Thus

$$w_{(\Gamma,O)} = w_{(\Delta,O_\Delta)}\frac{\ell!}{k!}\varepsilon(s_O)\varepsilon(\tau_O)\int_{\substack{q_1<\cdots<q_m \text{ oriented}\\ \text{by } dq_1\wedge\cdots\wedge dq_m}} \bigwedge_{j=1}^{m} d\Phi_{\overrightarrow{p_{i_j}q_j}}.$$

The S_m action on $GO^{(1)}_{n,m}$. Let σ be an element in the permutation group S_m. With any graph (Γ, O) in $GO^{(1)}_{n,m}$, we associate a new graph $(\sigma(\Gamma), \sigma(O))$. We keep for $\sigma(\Gamma)$ the vertices of Γ. But, if $E(\Gamma) = \{e_1 < \cdots < e_{|k|}\}$, we put $E(\sigma(\Gamma)) = \{e'_1 < \cdots < e'_{|k|}\}$, where $e'_r := e_r$ if e_r is an aerial edge and $e'_r := \overrightarrow{p_i q_{\sigma(j)}}$ if $e_r = \overrightarrow{p_i q_j}$ is a leg (see Figure 1). In this way we get a free action of S_m on $GO^{(1)}_{n,m}$.

Lemma 3.1. *For all σ in S_m and all (Γ, O) in $GO^{(1)}_{n,m}$,*

$$B_{(\sigma(\Gamma),\sigma(O))}(\alpha)(f_1, \dots, f_m) = B_{(\Gamma,O)}(\alpha)(f_{\sigma(1)}, \dots, f_{\sigma(m)}) \quad f_i \in C^\infty(\mathbb{R}^d).$$

Proof. Let r_j be the label of the leg arriving on q_j in (Γ, O). In $(\sigma(\Gamma), \sigma(O))$, this leg has the same label r_j, but it ends at $q_{\sigma(j)}$. The aerial edges are kept unchanged. The result follows easily. ☐

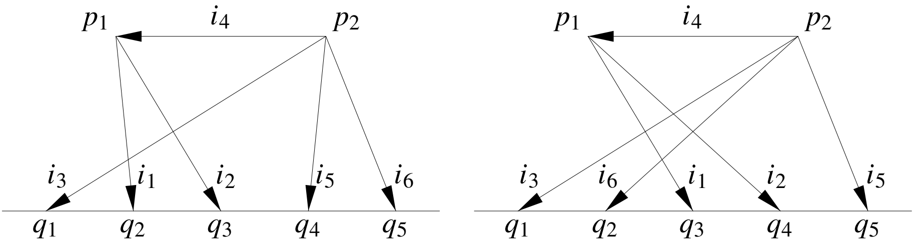


Figure 1. Left: (Γ, O). Right: $(\sigma(\Gamma), \sigma(O))$ with $\sigma = (2345)$.

Lemma 3.2. *Let σ be in S_m and (Γ, O) in $GO^{(1)}_{n,m}$. Then*

$$\varepsilon(s_{\sigma(O)}) = \varepsilon(s_O) \quad \textit{and} \quad \varepsilon(\tau_{\sigma(O)}) = \varepsilon(\sigma)\varepsilon(\tau_O).$$

Proof. When building $(\sigma(\Gamma), \sigma(O))$, we get a bijective mapping from $E(\Gamma)$ to $E(\sigma(\Gamma))$, say $\tilde{\sigma}$. In fact, $s_{\sigma(O)} = \tilde{\sigma} \circ s_O \circ \tilde{\sigma}^{-1}$. Thus $\varepsilon(s_{\sigma(O)}) = \varepsilon(s_O)$.

Now let $q_{a^i_1}, \ldots, q_{a^i_{m_i}}$ be the feet of the legs starting from p_i. By definition, τ_O is the permutation

$$\overrightarrow{p_1 q_{a^1_1}}, \overrightarrow{p_1 q_{a^1_2}}, \ldots, \overrightarrow{p_n q_{a^n_{m_n}}} \mapsto \overrightarrow{p_{i_1} q_1}, \ldots, \overrightarrow{p_{i_m} q_m}.$$

We may write

$$\tau_O^{-1} : (1, \ldots, m) \mapsto (a^1_1, \ldots, a^n_{m_n}).$$

By the definition of $(\sigma(\Gamma), \sigma(O))$,

$$\tau^{-1}_{\sigma(O)} : (1, \ldots, m) \mapsto \big(\sigma(a^1_1), \ldots, \sigma(a^n_{m_n})\big).$$

Thus $\tau^{-1}_{\sigma(O)} \circ \tau_O = \sigma$. The result follows. □

A noteworthy property.

Proposition 3.3. *We keep our notations. In particular, for any (Γ, O) in $GO^{(1)}_{n,m}$ and any (Δ, O_Δ) in $GO^{(0)}_n$, we denote by $w_{(\Gamma,O)}$ and $w_{(\Delta,O_\Delta)}$ the corresponding weights. Then*

$$\mathfrak{a}\Bigg(\sum_{(\Gamma,O)\in GO^{(1)}_{n,m}} w_{(\Gamma,O)} B_{(\Gamma,O)}(\alpha) \Bigg) = \sum_{(\Delta,O_\Delta)\in GO^{(0)}_n} w_{(\Delta,O_\Delta)} \sum_{m\geq 0} \frac{1}{m!} \sum_{\substack{(\Gamma,O)\supset(\Delta,O_\Delta)\\ (\Gamma,O)\in GO^{(1)}_{n,m}}} \frac{\ell!}{k!} \varepsilon(s_O)\varepsilon(\tau_O) B_{(\Gamma,O)}(\alpha).$$

Proof. Skewsymmetrizing and using Lemma 3.1, we get

$$\mathfrak{a}\Bigg(\sum_{(\Gamma,O)\in GO_{n,m}^{(1)}} w_{(\Gamma,O)}B_{(\Gamma,O)}(\alpha)\Bigg)(f_1\otimes\cdots\otimes f_m)$$

$$=\frac{1}{m!}\sum_{\sigma\in S_m}\varepsilon(\sigma)\sum_{(\Gamma,O)\in GO_{n,m}^{(1)}} w_{(\Gamma,O)}B_{(\Gamma,O)}(\alpha)(f_{\sigma^{-1}(1)}\otimes\cdots\otimes f_{\sigma^{-1}(m)})$$

$$=\frac{1}{m!}\sum_{\sigma\in S_m}\varepsilon(\sigma)\sum_{(\Gamma,O)\in GO_{n,m}^{(1)}} w_{(\Gamma,O)}B_{(\sigma^{-1}(\Gamma),\sigma^{-1}(O))}(\alpha)(f_1\otimes\cdots\otimes f_m)$$

$$=\frac{1}{m!}\sum_{\sigma\in S_m}\varepsilon(\sigma)\sum_{(\Gamma,O)\in GO_{n,m}^{(1)}} w_{(\sigma(\Gamma),\sigma(O))}B_{(\Gamma,O)}(\alpha)(f_1\otimes\cdots\otimes f_m).$$

By definition,

$$w_{(\sigma(\Gamma),\sigma(O))}=\varepsilon(s_{\sigma(O)})\varepsilon(\tau_{\sigma(O)})\frac{\ell!}{k!}\int_{C_{n,0}^+}\bigwedge_{r=1}^{|\ell|}d\Phi_{e_r^\Delta}\int_{\substack{q_1<\cdots<q_m\text{ oriented}\\ \text{by } dq_1\wedge\cdots\wedge dq_m}}\bigwedge_{j=1}^{m}d\Phi_{\overrightarrow{p_{i'_j}q_j}},$$

where i'_j stands for the unique index i' such that the leg arriving on q_j is exactly $\overrightarrow{p'_iq_j}$. Now $\bigwedge_{j=1}^m d\Phi_{\overrightarrow{p_{i'_j}q_j}}=\varepsilon(\sigma)\bigwedge_{j=1}^m d\Phi_{\overrightarrow{p_{i_j}q_{\sigma(j)}}}$; then, by Lemma 3.2,

$$w_{(\sigma(\Gamma),\sigma(O))}=\frac{\ell!}{k!}\varepsilon(s_O)\varepsilon(\tau_O)\int_{C_{n,0}^+}\bigwedge_{r=1}^{|\ell|}d\Phi_{e_r^\Delta}\int_{\substack{q_1<\cdots<q_m\text{ oriented}\\ \text{by } dq_1\wedge\cdots\wedge dq_m}}\bigwedge_{j=1}^{m}d\Phi_{\overrightarrow{p_{i_j}q_{\sigma(j)}}}.$$

With the new variables $q'_j=q_{\sigma(j)}$, we get

$$w_{(\sigma(\Gamma),\sigma(O))}=\frac{\ell!}{k!}w_{(\Delta,O_\Delta)}\varepsilon(s_O)\varepsilon(\tau_O)\int_{D^\sigma}\bigwedge_{j=1}^{m}d\Phi_{\overrightarrow{p_{i_j}q'_j}}$$

where D^σ is the domain $q'_{\sigma^{-1}(1)}<\cdots<q'_{\sigma^{-1}(m)}$ oriented by $dq'_1\wedge\cdots\wedge dq'_m$. Thus

$$\mathfrak{a}\Bigg(\sum_{(\Gamma,O)\in GO_{n,m}^{(1)}} w_{(\Gamma,O)}B_{(\Gamma,O)}(\alpha)\Bigg)$$

$$=\frac{1}{m!}\sum_{\sigma\in S_m}\sum_{(\Gamma,O)\in GO_{n,m}^{(1)}} w_{(\Delta,O_\Delta)}\frac{\ell!}{k!}\varepsilon(s_O)\varepsilon(\tau_O)\int_{D^\sigma}\bigwedge_{j=1}^{m}d\Phi_{\overrightarrow{p_{i_j}q'_j}}B_{(\Gamma,O)}(\alpha)$$

$$=\sum_{(\Delta,O_\Delta)\in GO_n^{(0)}} w_{(\Delta,O_\Delta)}\frac{1}{m!}\sum_{\substack{(\Gamma,O)\in GO_{n,m}^{(1)}\\ (\Gamma,O)\supset(\Delta,O_\Delta)}}\frac{\ell!}{k!}\varepsilon(s_O)\varepsilon(\tau_O)\Bigg(\sum_{\sigma\in S_m}\int_{D^\sigma}\bigwedge_{j=1}^{m}d\Phi_{\overrightarrow{p_{i_j}q'_j}}\Bigg)B_{(\Gamma,O)}(\alpha)$$

$$=\sum_{(\Delta,O_\Delta)\in GO_n^{(0)}} w_{(\Delta,O_\Delta)}\frac{1}{m!}\sum_{\substack{(\Gamma,O)\in GO_{n,m}^{(1)}\\ (\Gamma,O)\supset(\Delta,O_\Delta)}}\frac{\ell!}{k!}\varepsilon(s_O)\varepsilon(\tau_O)B_{(\Gamma,O)}(\alpha).\qquad\square$$

3.3. ***K*-graph formalities.** Consider the explicit Kontsevich formality $\mathcal{U}=\sum_n \mathcal{U}_n$ on $\mathbb{R}^d$. If (Γ, O) is an oriented graph with O not compatible, we write, as in [Arnal et al. 2002],

$$B_{(\Gamma,O)}=\varepsilon(\sigma_{(O,O_0)})B_{(\Gamma,O_0)},$$

where O_0 is any compatible orientation on Γ and $\sigma_{(O,O_0)}$ stands for the permutation of $E(\Gamma)$ obtained by changing (Γ, O) into (Γ, O_0). We also put

$$\omega'_{(\Gamma,O)}=\frac{k!}{|k|!}\omega_{(\Gamma,O)} \quad \text{and} \quad w'_{(\Gamma,O)}=\int_{C^+_{n,m}}\omega'_{(\Gamma,O)},$$

where $k!=k_1!\dots k_n!$ and $|k|=\sum k_i$ if k_i is the number of edges emanating from the vertex p_i of Γ, and $\omega_{(\Gamma,O)}=d\Phi_{e_1}\wedge\cdots\wedge d\Phi_{e_{|k|}}$ if $E(\Gamma)=\{e_1<\cdots<e_{|k|}\}$.

We denote by $GO'_{n,m}$ the set of oriented graphs (Γ', O'), with O' not necessarily compatible. Then

$$\mathcal{U}_n=\sum_{m\geq 0}\sum_{(\Gamma',O')\in GO'_{n,m}} w'_{(\Gamma',O')}B_{(\Gamma',O')}.$$

We write the formality equation for $\mathcal{U}$ as

$$F_n=E_n-d_H(\mathcal{U}_n)=0.$$

Rewriting the proof of the formality theorem by Kontsevich, one can see that F_n looks like a sum over the faces F of the boundary $\partial C^+_{n,m}$ of $C^+_{n,m}$ (see [Arnal et al. 2002] for details):

$$F_n=\sum_{m\geq 0}\sum_{F\subset\partial C^+_{n,m}}\sum_{(\Gamma',O')\in GO'_{n,m}} {w'}^F_{(\Gamma',O')}B_{(\Gamma',O')},$$

where ${w'}^F_{(\Gamma',O')}$ is the integral over F of the closed 2-form ${\omega'}_{(\Gamma',O')}$.

That $F_n=0$ then follows directly from the Stokes formula. In particular, we have $\mathfrak{a}(E_n)=0$.

Now, we saw that $\mathfrak{a}(E_n)=\mathfrak{a}({E_n}^{(1)})$. Thus, for a fixed face F of $\partial C^+_{n,m}$, the corresponding term in $\mathfrak{a}(E_n)$ is a sum over 1-graphs of the form

$$\mathfrak{a}\Bigg(\sum_{(\Gamma',O')\in {GO'}^{(1)}_{n,m}} {w'}^F_{(\Gamma',O')}B_{(\Gamma',O')}\Bigg).$$

Each term of this sum satisfies our relation

$$\mathfrak{a}\Bigg(\sum_{(\Gamma',O')\in {GO'}^{(1)}_{n,m}} {w'}^F_{(\Gamma',O')}B_{(\Gamma',O')}(\alpha)\Bigg) = \sum_{(\Delta,O_\Delta)} w^F_{(\Delta,O_\Delta)}\frac{1}{m!}\sum_{GO^{(1)}_{n,m}\ni(\Gamma,O)\supset(\Delta,O_\Delta)}\frac{\ell!}{k!}\varepsilon(s_O)\varepsilon(\tau_O)B_{(\Gamma,O)}(\alpha),$$

where $w^F_{(\Delta,O_\Delta)} = \int_F \omega_{(\Delta,O_\Delta)}$. Let's prove this:

A face is of either type 1 or type 2 (see [Kontsevich 2003] or [Arnal et al. 2002]). We consider only the faces such that $w'^F_{(\Gamma',O')}$ can be different from 0.

(i) If the face F has type 1: Two vertices p_i, p_j of Γ', related by exactly one edge, are collapsing and the face is $F = C_{\{p_i,p_j\}} \times C^+_{\{p,p_1,\dots,\widehat{p_i p_j},\dots,p_n\};\{q_1,\dots,q_m\}}$. We parametrize $C^+_{n,m}$ by

$$\rho = \frac{|p_j - p_i|}{\operatorname{Im} p_i}, \quad p'_j = \frac{p_j - p_i}{|p_j - p_i|} \quad p'_r = \frac{p_r - \operatorname{Re} p_i}{\operatorname{Im} p_i} \quad q'_s = \frac{q_s - \operatorname{Re} p_i}{\operatorname{Im} p_i}.$$

With the signs computed in [Arnal et al. 2002], we can write

$$w'^F_{(\Gamma',O')} = -\int_{C_2} d\Phi_{\overrightarrow{p_i p_j}} \int_{C^+_{n-1,m}} \omega'_{(\Gamma_2,O_2)},$$

where Γ_2 is the graph obtained from Γ' by gluing together p_i and p_j at the point p and suppressing the edge $\overrightarrow{p_i p_j}$. This weight $w'^F_{(\Gamma',O')}$ corresponds to a limit when ρ tends to zero. In fact, if we put

$$C^+_{n,m}(\varepsilon) = C^+_{n,m} \cap \{(p,q) : \rho = \varepsilon\},$$

we get

$$w'^F_{(\Gamma',O')} = \lim_{\varepsilon\to 0} \frac{k!}{|k|!} \int_{C^+_{n,m}(\varepsilon)} \omega_{(\Gamma',O')} := \lim_{\varepsilon\to 0} w'^F_{(\Gamma',O')}(\varepsilon).$$

This limit vanishes for graphs (Γ', O') whose vertices p_i and p_j are linked by two edges or no edges at all. We can thus also consider these graphs in our sum. Then

$$\mathfrak{a}\Bigg(\sum_{(\Gamma',O')\in GO'^{(1)}_{n,m}} w'^F_{(\Gamma',O')} B_{(\Gamma',O')}(\alpha)\Bigg) = \lim_{\varepsilon\to 0} \mathfrak{a}\Bigg(\sum_{(\Gamma',O')\in GO'^{(1)}_{n,m}} w'^F_{(\Gamma',O')}(\varepsilon) B_{(\Gamma',O')}(\alpha)\Bigg).$$

Passing to compatible orderings, we obtain

$$\mathfrak{a}\Bigg(\sum_{(\Gamma',O')\in GO'^{(1)}_{n,m}} w'^F_{(\Gamma',O')} B_{(\Gamma',O')}(\alpha)\Bigg) = \lim_{\varepsilon\to 0} \mathfrak{a}\Bigg(\sum_{(\Gamma,O)\in GO^{(1)}_{n,m}} w^F_{(\Gamma,O)}(\varepsilon) B_{(\Gamma,O)}(\alpha)\Bigg).$$

By Proposition 3.3, we get, as announced,

$$\mathfrak{a}\Big(\sum_{(\Gamma',O')\in GO'^{(1)}_{n,m}} w'^{F}_{(\Gamma',O')}B_{(\Gamma',O')}(\alpha)\Big)$$
$$= \lim_{\varepsilon\to 0}\frac{1}{m!}\sum_{(\Delta,O_\Delta)\in GO^{(0)}_n} w^F_{(\Delta,O_\Delta)}(\varepsilon)\sum_{\substack{(\Gamma,O)\in GO^{(1)}_{n,m}\\(\Gamma,O)\supset(\Delta,O_\Delta)}}\frac{l!}{k!}\varepsilon(s_O)\varepsilon(\tau_O)B_{(\Gamma,O)}(\alpha)$$
$$= \frac{1}{m!}\sum_{(\Delta,O_\Delta)\in GO^{(0)}_n} w^F_{(\Delta,O_\Delta)}\sum_{\substack{(\Gamma,O)\in GO^{(1)}_{n,m}\\(\Gamma,O)\supset(\Delta,O_\Delta)}}\frac{\ell!}{k!}\varepsilon(s_O)\varepsilon(\tau_O)B_{(\Gamma,O)}(\alpha).$$

(ii) If F has type 2: Since our graphs (Γ', O') have exactly one leg for each foot, F is isomorphic to $C^+_{n_1,m_1}\times C^+_{n_2,m_2}$ with $n_2>0$ and $n_1>0$. This case corresponds to the subcase 1 of [Arnal et al. 2002]. Suppose that $p_{i_1},\dots,p_{i_{n_1}}$ and $q_{\ell+1},\dots,q_{\ell+m}$ are collapsing on $q\in\mathbb{R}$. Denote by p_j the first aerial vertex of Γ' that is not a p_{i_s}, and impose the condition $p_j=\sqrt{-1}$. The other parameters are then fixed and we get a parametrization of our configuration space $C^+_{n,m}$ by variables a_r, b_s, q_t (see the notation of [Arnal et al. 2002]). We put $a_{i_1}=q$, $b=\operatorname{Im} p_{i_1}$, and

$$p'_{i_k}=\frac{p_{i_k}-q}{b}\quad(2\le k\le n_1),\qquad q'_{\ell+r}=\frac{q_{\ell+r}-q}{b}\quad(1\le r\le m_1).$$

That is, $p_{i_k}=bp'_{i_k}+qb$ and $q_{\ell+r}=q'_{\ell+r}+qb$, and when b tends to zero, the p_{i_k} and the $q_{\ell+r}$ tend to q. We finally set

$$C^+_{n,m}(\varepsilon)=\{(p,q)\in C^+_{n,m} : b=\varepsilon\}.$$

We get

$$w'^{F}_{(\Gamma',O')}=\lim_{\varepsilon\to 0}\frac{k!}{|k|!}\int_{C^+_{n,m}(\varepsilon)}\omega_{(\Gamma',O')}=\lim_{\varepsilon\to 0}w'^{F}_{(\Gamma',O')}(\varepsilon).$$

If Γ' has a bad edge, the weight $w'^{F}_{(\Gamma',O')}$ vanishes. We can thus consider also these graphs in our sum. Now, a computation similar to that of (i) gives the result.

From now on, for any aerial oriented graph (Δ, O_Δ) in $GO^{(0)}_n$, denote by $C_{(\Delta,O_\Delta)}$ the operator $C_{(\Delta,O_\Delta)}: T^{\otimes n}_{\mathrm{poly}}(\mathbb{R}^d)\to D_{\mathrm{poly}}(\mathbb{R}^d)^{(1)}\simeq T_{\mathrm{poly}}(\mathbb{R}^d)$ defined by

$$C_{(\Delta,O_\Delta)}(\alpha_1\otimes\cdots\otimes\alpha_n)=\sum_{m\ge 0}\frac{1}{m!}\sum_{\substack{(\Gamma,O)\in GO^{(1)}_{n,m}\\(\Gamma,O)\supset(\Delta,O_\Delta)}}\frac{\ell!}{k!}\varepsilon(s_O)\varepsilon(\tau_O)B_{(\Gamma,O)}(\alpha_1\otimes\cdots\otimes\alpha_n),$$

where $\varepsilon(s_O)$ and $\varepsilon(\tau_O)$ have the same meaning as above.

Remark. The definition of $C_{(\Delta,O_\Delta)}$ can be extended naturally to the space $GO'^{(0)}_n$ of aerial graphs (Δ', O'_Δ) with O'_Δ not necessarily compatible just by putting

$$C_{(\Delta',O'_\Delta)} = \sum_{m\geq 0} \frac{1}{m!} \sum_{\substack{(\Gamma,O)\in GO^{(1)}_{n,m} \\ (\Gamma,O)\supset(\Delta,O_\Delta)}} \frac{\ell!}{k!} \varepsilon(s_{O'})\varepsilon(\tau_{O'}) B_{(\Gamma',O')}.$$

We will need to use this extension in Section 5.

Summing up:

Proposition 3.4. *Consider the explicit Kontsevich formality $\mathcal{U}$ on $\mathbb{R}^d$. The formality equation can be read as*

$$F_n = E_n - d_H \mathcal{U}_n = 0,$$

and the skewsymmetrization of E_n has the form

$$\mathfrak{a} \circ E_n = \sum_{m\geq 0} \sum_{F \text{ face of } \partial C^+_{n,m}} \sum_{(\Gamma',O')\in GO'^{(1)}_{n,m}} w'^F_{(\Gamma',O')} B_{(\Gamma',O')},$$

where $w'^F_{(\Gamma',O')} = \int_{F\in\partial C^+_{n,m}} \omega'_{(\Gamma',O')}$. Then, for each face F,

$$\mathfrak{a}\Bigg(\sum_{(\Gamma',O')\in GO'^{(1)}_{n,m}} w'^F_{(\Gamma',O')} B_{(\Gamma',O')}(\alpha) \Bigg) = \sum_{(\Delta,O_\Delta)\in GO^{(0)}_n} w^F_{(\Delta,O_\Delta)} C_{(\Delta,O_\Delta)}(\alpha).$$

This proposition suggests that we define:

Definition 3.5. A mapping φ from $T_{\text{poly}}(\mathbb{R}^d)^{\otimes n}$ to $D_{\text{poly}}(\mathbb{R}^d)^{(1)} \simeq T_{\text{poly}}(\mathbb{R}^d)$ is called a *K-graph mapping* if it can be written

$$\varphi = \sum_{(\Delta,O_\Delta)\in GO^{(0)}_n} c_{(\Delta,O_\Delta)} C_{(\Delta,O_\Delta)}$$

with real coefficients $c_{(\Delta,O_\Delta)}$. Such a mapping is homogeneous of degree s if $c_{(\Delta,O_\Delta)} = 0$ for all Δ such that $\# E(\Delta) + s \neq 2n-2$.

Definition 3.6. A *K-graph formality* $\mathcal{F}$ at order n is a graph formality up to order $n-1$ such that $\varphi_n = \mathcal{F}_1^{-1} \circ \mathfrak{a} \circ E_n$ is a K-graph mapping.

4. Symmetrization

4.1. ***Expressions for* ∂.** If B is an n-linear mapping $B : T_{\text{poly}}(\mathbb{R}^d)^{\otimes n} \to T_{\text{poly}}(\mathbb{R}^d)$, we define SB by setting

$$SB(\alpha_1 \otimes \cdots \otimes \alpha_n) = \frac{1}{n!} \sum_{\sigma\in S_n} \varepsilon_\alpha(\sigma) B(\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(n)}),$$

and say that B is symmetric if $SB = B$. Any symmetric mapping can be viewed as a map $\varphi : S^n(T_{\text{poly}}(\mathbb{R}^d)) \to T_{\text{poly}}(\mathbb{R}^d)$. With this symmetrization operator S, the expression of the Chevalley coboundary operator can be conveniently simplified:

Proposition 4.1. *Let $\varphi : S^n(T_{\text{poly}}(\mathbb{R}^d)[1]) \to T_{\text{poly}}(\mathbb{R}^d)[1]$ be an n-cochain for ∂, homogeneous of degree $|\varphi|$. Then we can write*

$$\partial\varphi = S(\tilde{\partial}\varphi),$$

where $\tilde{\partial}\varphi$ is given by

$$\begin{aligned}\tilde{\partial}\varphi(\alpha_1 \otimes \cdots \otimes \alpha_{n+1}) &= (n+1)\big(\varphi(\alpha_1 \otimes \cdots \otimes \alpha_n) \bullet \alpha_{n+1} \\ &\quad + (-1)^{|\varphi||\alpha_1|}\alpha_1 \bullet \varphi(\alpha_2 \otimes \cdots \otimes \alpha_{n+1}) + (-1)^{|\varphi|+1} n\varphi(\alpha_1 \bullet \alpha_2 \otimes \alpha_3 \otimes \cdots \otimes \alpha_{n+1})\big),\end{aligned}$$

or else by an expression imitating the Hochschild coboundary operator :

$$\begin{aligned}\tilde{\partial}\varphi(\alpha_1 \otimes \cdots \otimes \alpha_{n+1}) = (n+1)\Bigg(&\varphi(\alpha_1 \otimes \cdots \otimes \alpha_n) \bullet \alpha_{n+1} \\ &+ (-1)^{|\varphi|+1}\sum_{k=2}^{n+1}(-1)^{\sum_{s=1}^{k-2}|\alpha_s|}\varphi(\alpha_1 \otimes \cdots \otimes \alpha_{k-1} \bullet \alpha_k \otimes \cdots \otimes \alpha_{n+1}) \\ &+ (-1)^{|\varphi||\alpha_1|}\alpha_1 \bullet \varphi(\alpha_2 \otimes \cdots \otimes \alpha_{n+1})\Bigg).\end{aligned}$$

Proof. By the definition of ∂, we have

$$\partial\varphi(\alpha_1 \ldotp \ldots \ldotp \alpha_{n+1}) = (1) + (2) + (3),$$

with

$$\begin{aligned}(1) &= \sum_{i=1}^{n+1} \varepsilon_\alpha(1 \ldots \hat{\imath} \ldots n{+}1, i)\varphi(\alpha_1 \ldotp \ldots \ldotp \hat{\alpha}_i \ldotp \ldots \ldotp \alpha_{n+1}) \bullet \alpha_i, \\ (2) &= \sum_{i=1}^{n+1}(-1)^{|\varphi||\alpha_i|}\varepsilon_\alpha(i, 1 \ldots \hat{\imath} \ldots n{+}1)\alpha_i \bullet \varphi(\alpha_1 \ldotp \ldots \ldotp \hat{\alpha}_i \ldotp \ldots \ldotp \alpha_{n+1}), \\ (3) &= \sum_{i \neq j}(-1)^{|\varphi|+1}\varepsilon_\alpha(ij, 1 \ldots \widehat{ij} \ldots n{+}1)\varphi(\alpha_i \bullet \alpha_j \ldotp \alpha_1 \ldots \widehat{\alpha_i\alpha_j} \ldotp \ldots \ldotp \alpha_{n+1}).\end{aligned}$$

Now, put

$$\begin{aligned}\psi_1(\alpha_1 \otimes \cdots \otimes \alpha_{n+1}) &= (n{+}1)\varphi(\alpha_1 \otimes \cdots \otimes \alpha_n) \bullet \alpha_{n+1}, \\ \psi_2(\alpha_1 \otimes \cdots \otimes \alpha_{n+1}) &= (-1)^{|\varphi||\alpha_1|}(n{+}1)\alpha_1 \bullet \varphi(\alpha_2 \otimes \cdots \otimes \alpha_{n+1}), \\ \psi_3(\alpha_1 \otimes \cdots \otimes \alpha_{n+1}) &= (-1)^{|\varphi|+1}(n{+}1)n\varphi(\alpha_1 \bullet \alpha_2 \otimes \cdots \otimes \alpha_{n+1}), \\ \psi_3'(\alpha_1 \otimes \cdots \otimes \alpha_{n+1}) &= (-1)^{|\varphi|+1}\sum_{k=2}^{n+1}\big((-1)^{\sum_{s=1}^{k-2}|\alpha_s|} \\ &\qquad \times \varphi(\alpha_1 \otimes \cdots \otimes \alpha_{k-1} \bullet \alpha_k \otimes \cdots \otimes \alpha_{n+1})\big).\end{aligned}$$

First

$$
\begin{aligned}
S\psi_1(\alpha_1 \cdot \dots \cdot \alpha_{n+1}) &= \frac{(n+1)}{(n+1)!} \sum_{\sigma \in S_{n+1}} \varepsilon_\alpha(\sigma)\varphi(\alpha_{\sigma(1)} \cdot \dots \cdot \alpha_{\sigma(n)}) \bullet \alpha_{\sigma(n+1)} \\
&= \frac{(n+1)}{(n+1)!} \sum_{i=1}^{n+1} \sum_{\sigma:\sigma(n+1)=i} \varepsilon_\alpha(\sigma)\varphi(\alpha_{\sigma(1)} \cdot \dots \cdot \alpha_{\sigma(n)}) \bullet \alpha_i \\
&= \frac{(n+1)}{(n+1)!} \sum_{i=1}^{n+1} \sum_{\tau:\tau(i)=i} \varepsilon_\alpha(\tau \circ \sigma_i)\varphi(\alpha_{\tau(\sigma_i(1))} \cdot \dots \cdot \alpha_{\tau(\sigma_i(n))}) \bullet \alpha_i .
\end{aligned}
$$

Here σ_i is the permutation of S_{n+1} sending $(1, \dots, n+1)$ to $(1, \dots \hat{\imath} \dots, n+1, i)$. And, denoting by $\bar{\tau}$ the restriction of τ to $\{1, \dots \hat{\imath} \dots, n+1\}$, we easily get

$$
\begin{aligned}
&S\psi_1(\alpha_1 \cdot \dots \cdot \alpha_{n+1}) \\
&\quad = \frac{n+1}{(n+1)!} \sum_{i=1}^{n+1} \sum_{\bar{\tau} \in S_n} \varepsilon_{\alpha\setminus\{\alpha_i\}}(\bar{\tau})\varepsilon_\alpha(\sigma_i)\varphi(\alpha_{\bar{\tau}(1)} \cdot \dots \cdot \alpha_{\bar{\tau}(n+1)}) \bullet \alpha_i \\
&\quad = \frac{(n+1)}{(n+1)!} n! \sum_{i=1}^{n+1} \varepsilon_\alpha(1 \dots \hat{\imath} \dots n+1, i)\varphi(\alpha_1 \cdot \dots \hat{\alpha}_i \dots \cdot \alpha_{n+1}) \bullet \alpha_i = (1).
\end{aligned}
$$

With exactly the same argument, we obtain

$$
S\psi_2(\alpha_1 \cdot \dots \cdot \alpha_{n+1}) = (2).
$$

Now,

$$
\begin{aligned}
&S\psi_3(\alpha_1 \cdot \dots \cdot \alpha_{n+1}) \\
&\quad = \sum_{\sigma \in S_{n+1}} \frac{1}{(n+1)!}(-1)^{|\varphi|+1}\varepsilon_\alpha(\sigma)(n+1)n\varphi(\alpha_{\sigma(1)} \bullet \alpha_{\sigma(2)} \otimes \alpha_{\sigma(3)} \otimes \cdots \otimes \alpha_{\sigma(n+1)}) \\
&\quad = \sum_{i \neq j} \sum_{\sigma:\sigma(1)=i,\sigma(2)=j} \Big(\varepsilon_\alpha(\sigma)\frac{1}{(n+1)!}(-1)^{|\varphi|+1}(n+1)n \\
&\qquad\qquad \varphi(\alpha_i \bullet \alpha_j \otimes \alpha_1 \otimes \alpha_{\sigma(3)} \otimes \cdots \otimes \alpha_{\sigma(n+1)})\Big) \\
&\quad = \sum_{i \neq j} \sum_{\tau:\tau(i)=i,\tau(j)=j} \Big(\varepsilon_\alpha(\tau)\frac{1}{(n+1)!}\varepsilon_\alpha(\sigma_{ij})(-1)^{|\varphi|+1}(n+1)n \\
&\qquad\qquad \varphi(\alpha_i \bullet \alpha_j \otimes \alpha_{\tau(\sigma_{ij}(3))} \dots \cdots \otimes \alpha_{\tau(\sigma_{ij}(n+1))})\Big),
\end{aligned}
$$

where σ_{ij} is the permutation of S_{n+1} sending $(1, \dots, n+1)$ to $(ij, 1 \dots \widehat{ij} \dots n+1)$. Now, if $\bar{\tau}$ denotes the restriction of τ to $\{1, \dots \widehat{ij} \dots, n+1\}$, we get

$$
\begin{aligned}
&S\psi_3(\alpha_1 \cdot \dots \cdot \alpha_{n+1}) \\
&\quad = \sum_{i \neq j} \frac{(-1)^{|\varphi|+1}}{(n-1)!} \sum_{\bar{\tau}:\tau(i)=i,\tau(j)=j} \Big(\varepsilon_\alpha(\sigma_{ij})\varepsilon_\alpha(\bar{\tau}) \\
&\qquad\qquad \varphi(\alpha_i \bullet \alpha_j \otimes \alpha_{\bar{\tau}(1)} \otimes \cdots \widehat{\alpha_i\alpha_j} \cdots \otimes \alpha_{\bar{\tau}(n+1)})\Big)
\end{aligned}
$$

$$= \sum_{i \neq j} (-1)^{|\varphi|+1} \varepsilon_\alpha(ij, 1 \dots \widehat{ij} \dots n+1) \varphi(\alpha_i \bullet \alpha_j \bullet \dots \bullet \widehat{\alpha_i \alpha_j} \bullet \dots \bullet \alpha_{n+1}) = (3).$$

Finally,

$$\begin{aligned} &S\psi_3'(\alpha_1 \bullet \dots \bullet \alpha_{n+1}) \\ &= \frac{(n+1)}{(n+1)!} \sum_{\sigma \in S_{n+1}} (-1)^{|\varphi|+1} \sum_{k=2}^{n+1} (-1)^{\sum_{s=1}^{k-2} |\alpha_s|} \big(\varepsilon_\alpha(\sigma) \\ &\qquad \varphi(\alpha_{\sigma(1)} \otimes \dots \otimes \alpha_{\sigma(k-2)} \otimes \alpha_{\sigma(k-1)} \bullet \alpha_{\sigma(k)} \otimes \dots \otimes \alpha_{\sigma(n+1)})\big) \\ &= \frac{(-1)^{|\varphi|+1}}{n!} \sum_{k=2}^{n+1} \sum_{i \neq j} \sum_{\sigma : \sigma(k-1)=i,\, \sigma(k)=j} (-1)^{\sum_{s=1}^{k-2} |\alpha_s|} \big(\varepsilon_\alpha(\sigma) \\ &\qquad \varphi(\alpha_{\sigma(1)} \dots \otimes \alpha_i \bullet \alpha_j \otimes \dots \otimes \alpha_{\sigma(n+1)})\big). \end{aligned}$$

Let σ_{ij}^k be the permutation

$$\sigma_{ij}^k : (1 \dots n+1) \mapsto (1, \dots, k-2, i, j, k-1, k, \dots, n+1).$$

Then

$$\begin{aligned} &(-1)^{|\varphi|+1} S\psi_3'(\alpha_1 \bullet \dots \bullet \alpha_{n+1}) \\ &= \frac{1}{n!} \sum_{\substack{2 \le k \le n+1 \\ i \neq j}} \big((-1)^{\sum_{s=1}^{k-2} |\alpha_s|} \varepsilon_\alpha(\sigma_{ij}^k)(n-1)! \\ &\qquad \varphi(\alpha_1 \otimes \dots \otimes \alpha_{(k-2)} \otimes \alpha_i \bullet \alpha_j \otimes \dots \otimes \alpha_{n+1})\big) \\ &= \frac{1}{n} \sum_{\substack{2 \le k \le n+1 \\ i \neq j}} \big((-1)^{\sum_{s=1}^{k-2} |\alpha_s|} \varepsilon_\alpha(\sigma_{ij}) \varepsilon_\alpha(\rho_{ij}^k) \varphi(\alpha_i \bullet \alpha_j \otimes \alpha_1 \otimes \dots \otimes \alpha_{n+1})(-1)^{a_{ijk}}\big), \end{aligned}$$

with $a_{ijk} = (|\alpha_i| + |\alpha_j| + 1)\big(\sum_{s=1}^{k-2} |\alpha_s|\big)$. Here $\sigma_{ij} = (ij1 \dots \widehat{ij} \dots n+1)$ and ρ_{ij}^k is the permutation

$$\rho_{ij}^k : (ij1 \dots \widehat{ij} \dots n+1) \mapsto (1, \dots, k-2, i, j, k-1, k, \dots, n+1);$$

thus we have used the composition $\sigma_{ij}^k = \rho_{ij}^k \circ \sigma_{ij}$. Now, since

$$\varepsilon_\alpha(\rho_{ij}^k) = (-1)^{(|\alpha_i|+|\alpha_j|)(\sum_{s=1}^{k-2} |\alpha_s|)},$$

we get

$$S\psi_3'(\alpha_1 \bullet \dots \bullet \alpha_{n+1}) = \sum_{i \neq j} (-1)^{|\varphi|+1} \varepsilon_\alpha(\sigma_{ij}) \varphi(\alpha_i \bullet \alpha_j \otimes \alpha_1 \otimes \dots \otimes \alpha_{n+1}) = (3).$$

This ends the proof. □

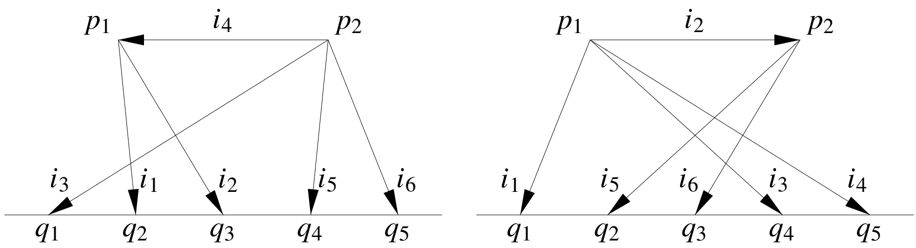


Figure 2. Left: (Γ, O). Right: $(\sigma(\Gamma), \sigma(O))$ with $\sigma = (12)$.

4.2. ***Symmetrization on graphs.*** We now want to describe the symmetrization directly on the space of graphs. Since we are mainly interested in K-graph formalities, we will restrict ourselves to linear combinations of graphs for which the associated operator is a K-graph mapping (see Section 3.3).

The S_n action on $GO_{n,m}$ and $GO_n^{(0)}$. There is a natural action of S_n on $GO_{n,m}$ and $GO_n^{(0)}$, which we now define. Let σ be a permutation in S_n. Let (Γ, O) be in $GO_{n,m}$; for the moment, denote by P_i the set $\mathrm{strt}(p_i)$, ordered by O. Let σ_Γ be the permutation of the ordered set $E(\Gamma)$ of edges of Γ sending $P_1 \cup \cdots \cup P_n$ to $P_{\sigma(1)} \cup \cdots \cup P_{\sigma(n)}$. We denote by $\varepsilon_\Gamma(\sigma_\Gamma)$ the sign of σ_Γ and by $\sigma(\Gamma, O) := (\sigma(\Gamma), \sigma(O))$ the graph with aerial vertices $p'_1 = p_{\sigma(1)}, \ldots, p'_n = p_{\sigma(n)}$ oriented by $\sigma_\Gamma(E(\Gamma))$ (see Figure 2). We apply the same definition to aerial graphs in $GO_n^{(O)}$. Clearly, σ sends $GO_{n,m}$ (and $GO_n^{(0)}$) onto itself.

This S_n action on $GO_{n,m}^{(1)}$ is entirely different from the action of S_m defined in Section 3. But there is an analog of Lemma 3.1:

Lemma 4.2. *For all σ in S_n, all (Γ, O) in $GO_{n,m}^{(1)}$ and all polyvector fields α_i,*

$$B_{(\sigma(\Gamma),\sigma(O))}(\alpha_1 \otimes \cdots \otimes \alpha_n) = B_{(\Gamma,O)}(\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(n)}).$$

Proof. With our notations,

$$B_{(\Gamma,O)}(\alpha_{\sigma(1)}\otimes\cdots\otimes\alpha_{\sigma(n)})(f_1, \ldots, f_m) = \sum_{1\le i_{t_1}\ldots i_{t_{|k|}}\le d} \prod_{i=1}^{n} \partial_{\mathrm{end}(p_i)}\alpha_{\sigma(i)}^{P_i} \prod_{j=1}^{m} \partial_{\mathrm{end}(q_j)} f_j.$$

Since the permutation σ_Γ does not affect the order inside each P_i, we have

$$\begin{aligned} B_{(\Gamma,O)}(\alpha_{\sigma(1)}\otimes\cdots\otimes\alpha_{\sigma(n)})(f_1, \ldots, f_m) &= \sum_{1\le i_{t_1}\ldots i_{t_{|k|}}\le d} \prod_{i=1}^{n} \partial_{\mathrm{end}(p_{\sigma(i)})}\alpha_{\sigma(i)}^{P_{\sigma(i)}} \prod_{j=1}^{m} \partial_{\mathrm{end}(q_j)} f_j \\ &= \sum_{1\le i_{t_1}\ldots i_{t_{|k|}}\le d} \prod_{i'=1}^{n} \partial_{\mathrm{end}(p'_i)}\alpha_{i'}{}^{P_{i'}} \prod_{j=1}^{m} \partial_{\mathrm{end}(q_j)} f_j \\ &= B_{\sigma(\Gamma,O)}(\alpha_1\otimes\cdots\otimes\alpha_n)(f_1, \ldots, f_m). \end{aligned}$$ □

Symmetrization for K-graph mappings.

Definition 4.3. Let

$$(\delta, O_\delta) = \sum_{(\Delta, O_\Delta)\in GO_n^{(0)}} c_{(\Delta, O_\Delta)}(\Delta, O_\Delta)$$

be a linear combination of aerial graphs with n vertices. We say that (δ, O_δ) is symmetric if

$$c_{(\sigma(\Delta),\sigma(O_\Delta))} = \varepsilon_\Delta(\sigma_\Delta)c_{(\Delta,O_\Delta)} \quad \text{for all } (\Delta, O_\Delta) \text{ and } \sigma \in S_n.$$

Proposition 4.4. *If* $(\delta, O_\delta) = \sum_{(\Delta,O_\Delta)\in GO_n^{(0)}} c_{(\Delta,O_\Delta)}(\Delta, O_\Delta)$ *is symmetric, so is the corresponding K-graph mapping*

$$C_{(\delta,O_\delta)} = \sum_{(\Delta,O_\Delta)\in GO_n^{(0)}} c_{(\Delta,O_\Delta)}C_{(\Delta,O_\Delta)}.$$

Proof. Let σ be in S_n and let $\alpha_1, \dots, \alpha_n$ be n polyvector fields on $\mathbb{R}^d$. By Lemma 4.2 and using the fact that δ is symmetric, we have

$$\begin{aligned}
&C_{(\delta,O_\delta)}(\alpha_{\sigma(1)}\otimes\cdots\otimes\alpha_{\sigma(n)}) \\
&\quad= \sum_{(\Delta,O_\Delta)} c_{(\Delta,O_\Delta)} \sum_{m\geq 0}\frac{1}{m!}\sum_{(\Gamma,O)\supset(\Delta,O_\Delta)} \frac{\ell!}{k!}\varepsilon(s_O)\varepsilon(\tau_O)B_{(\sigma(\Gamma),\sigma(O))}(\alpha_1\otimes\cdots\otimes\alpha_n) \\
&\quad= \sum_{\sigma^{-1}(\Delta,O_\Delta)} c_{(\sigma^{-1}(\Delta),\sigma^{-1}(O_\Delta))}\sum_{m\geq 0}\frac{1}{m!}\sum_{\sigma^{-1}(\Gamma,O)\supset\sigma^{-1}(\Delta,O_\Delta)} \\
&\qquad\qquad \frac{\ell!}{k!}\varepsilon(s_{\sigma^{-1}(O)})\varepsilon(\tau_{\sigma^{-1}(O)})B_{(\Gamma,O)}(\alpha_1\otimes\cdots\otimes\alpha_n) \\
&\quad= \sum_{(\Delta,O_\Delta)} \varepsilon_\Delta(\sigma_\Delta)c_{(\Delta,O_\Delta)}\sum_{m\geq 0}\frac{1}{m!}\sum_{(\Gamma,O)\supset(\Delta,O_\Delta)} \\
&\qquad\qquad \frac{\ell!}{k!}\varepsilon(s_{\sigma^{-1}(O)})\varepsilon(\tau_{\sigma^{-1}(O)})B_{(\Gamma,O)}(\alpha_1\otimes\cdots\otimes\alpha_n).
\end{aligned}$$

Extending σ_Δ to $E(\Gamma)$ in the obvious way, we can write

$$\tau_O\circ s_O = \sigma_\Delta\circ\tau_{\sigma^{-1}(O)}\circ s_{\sigma^{-1}(O)}\circ\sigma_\Gamma^{-1}.$$

Thus

$$\begin{aligned}
&C_\delta(\alpha_{\sigma(1)}\otimes\cdots\otimes\alpha_{\sigma(n)}) \\
&= \sum_{(\Delta,O_\Delta)} c_{(\Delta,O_\Delta)}\sum_{m\geq 0}\frac{1}{m!}\sum_{(\Gamma,O)\supset(\Delta,O_\Delta)} \varepsilon_\Gamma(\sigma_\Gamma)\frac{l!}{k!}\varepsilon(s_O)\varepsilon(\tau_O)B_{(\Gamma,O)}(\alpha_1\otimes\cdots\otimes\alpha_n).
\end{aligned}$$

Since each $\varepsilon_\Gamma(\sigma_\Gamma)$ clearly coincides with the sign $\varepsilon_\alpha(\sigma)$ of σ, we get

$$C_\delta(\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(n)}) = \varepsilon_\alpha(\sigma) C_\delta(\alpha_1 \otimes \cdots \otimes \alpha_n).$$

This proves the result. □

5. Chevalley cohomology for graphs

We will now prove that, on K-graph mappings, the Chevalley coboundary operator can be nicely reduced to an operator acting on purely aerial graphs.

5.1. ***Purely aerial and compatible oriented graphs.*** For any (Δ, O_Δ) in $GO_n^{(0)}$ with vertices $p_1 < \cdots < p_n$, we still write $\ell_i = \#\,\mathrm{strt}^\Delta(p_i)$. We also put $|\Delta| = \sum \ell_i = |\ell|$.

Fix two indexes $i \neq j$. We say that an aerial graph $(\Delta', O_{\Delta'})$ in $GO'^{(0)}_{n+1}$ (with $O_{\Delta'}$ not necessarily compatible) with vertices $p'_1 < \cdots < p'_{n+1}$ reduces to (Δ, O_Δ) in the indexes i, j if the two following assertions hold:

(i) The vertices p'_i and p'_j of Δ' are linked by only the edge $\overrightarrow{p'_i p'_j}$.

(ii) the new graph $(\Delta'_{ij}, O_{\Delta'_{ij}})$, obtained by gluing together the vertices p'_i, p'_j of Δ', by suppressing the edge $\overrightarrow{p'_i p'_j}$ and considering the induced ordering, coincides with (O, Δ).

We say that $(\Delta', O_{\Delta'})$ reduces properly to (Δ, O_Δ) in the indexes i, j if $(\Delta', O_{\Delta'})$ reduces to (Δ, O_Δ) in the same indexes and in addition

$$\inf\big(\#\,\mathrm{strt}^{\Delta'}(p'_i) + \#\,\mathrm{end}^{\Delta'}(p'_i),\ \#\,\mathrm{strt}^{\Delta'}(p'_j) + \#\,\mathrm{end}^{\Delta'}(p'_j)\big) > 1.$$

In the situations above we write

$$(\Delta', O_{\Delta'}) \to_{i,j} (\Delta, O_\Delta) \quad \text{and} \quad (\Delta', O_{\Delta'}) \to^{\mathrm{prop}}_{i,j} (\Delta, O_\Delta),$$

respectively. We use the same notation for graphs (Γ, O) in $GO^{(1)}_{n,m}$.

Definition 5.1. If (Δ, O_Δ) is an aerial oriented graph in $GO_n^{(0)}$, we define the coboundary $\partial(\Delta, O_\Delta)$ of (Δ, O_Δ) by

$$\partial(\Delta, O_\Delta) = (-1)^{|\Delta|+1} \sum_{i \neq j} \sum_{(\Delta', O_{\Delta'}) \to^{\mathrm{prop}}_{i,j} (\Delta, O_\Delta)} \varepsilon(\Delta', O_{\Delta'}, \Delta, O_\Delta)(\Delta', O_{\Delta'}).$$

Here $\varepsilon(\Delta', O_{\Delta'}, \Delta, O_\Delta)$ is the sign of the permutation of $E(\Delta')$, that consists in putting first the edge $\overrightarrow{p'_i p'_j}$, then the other edges starting from p'_i (with the ordering induced by $O_{\Delta'}$), then the edges starting from p'_j (also with the induced ordering), and finally all the remaining edges (with the ordering given by O_Δ).

We extend ∂ by linearity to all combinations $(\delta, O_\delta) = \sum_{(\Delta, O_\Delta)} c_{(\Delta, O_\Delta)}(\Delta, O_\Delta)$. Note that the restriction of ∂ to symmetric combinations of graphs is an operator of cohomology.

More precisely:

Proposition 5.2. *With the same notations as above and for any symmetric combination of graphs* (δ, O_δ), *we have*

$$\partial(C_{(\delta, O_\delta)}) = C_{\partial(\delta, O_\delta)}.$$

Proof. First, $C_{(\Delta, O_\Delta)}$ is a linear combination of m-differential operators $B_{(\Gamma,O)}(\alpha)$, for certain k_i-vector fields α_i:

$$m-2 = \left|B_{(\Gamma,O)}(\alpha_1 \otimes \cdots \otimes \alpha_n)\right| = \sum_{i=1}^{n} |\alpha_i| + |B_{(\Gamma,O)}| = \sum_{i=1}^{n} k_i - 2n + |B_{(\Gamma,O)}|,$$

where $|\ |$ stands for the degree in $T_{\text{poly}}(\mathbb{R}^d)[1]$ and $D_{\text{poly}}(\mathbb{R}^d)[1]$. Now, since the graphs (Γ, O) occurring in $C_{(\Delta, O_\Delta)}$ are 1-graphs, we have $k_i = \ell_i + m_i$ for each i and $m = \sum_{i=1}^{n} m_i$. Thus

$$|B_{(\Gamma,O)}| = \sum_{i=1}^{n} \ell_i = |\Delta| \bmod 2.$$

Now, by the definition of ∂ on operators,

$$\begin{aligned}
&\partial C_{(\Delta,O_\Delta)}(\alpha_1 \cdot \ldots \cdot \alpha_{n+1}) \\
&\quad = \sum_{j=1}^{n+1} \varepsilon_\alpha(1 \ldots \hat{j} \ldots n{+}1, j) C_{(\Delta,O_\Delta)}(\alpha_1 \cdot \ldots \cdot \hat{\alpha}_j \cdot \ldots \cdot \alpha_{n+1}) \bullet \alpha_j \\
&\qquad + \sum_{i=1}^{n+1} (-1)^{|\Delta||\alpha_i|} \varepsilon_\alpha(i, 1 \ldots \hat{\imath} \ldots n{+}1) \alpha_i \bullet C_{(\Delta,O_\Delta)}(\alpha_1 \cdot \ldots \cdot \hat{\alpha}_i \cdot \ldots \cdot \alpha_{n+1}) \\
&\qquad + \sum_{i \neq j} (-1)^{|\Delta|+1} \varepsilon_\alpha(ij, 1 \ldots \widehat{\imath\jmath} \ldots n{+}1) C_{(\Delta,O_\Delta)}(\alpha_i \bullet \alpha_j \cdot \alpha_1 \ldots \widehat{\alpha_i \alpha_j} \cdot \ldots \cdot \alpha_{n+1}) \\
&\quad = \text{(i)} + \text{(ii)} + \text{(iii)}.
\end{aligned}$$

We first consider the term (iii). We have

$$\begin{aligned}
&C_{(\Delta,O_\Delta)}(\alpha_i \bullet \alpha_j \cdot \ldots \cdot \widehat{\alpha_i \alpha_j} \cdot \ldots \cdot \alpha_{n+1}) \\
&\quad = \sum_{m \geq 0} \frac{1}{m!} \sum_{GO^{(1)}_{n,m} \ni (\Gamma,O) \supset (\Delta, O_\Delta)} \frac{\ell!}{k!} \varepsilon(s_O) \varepsilon(\tau_O) B_{(\Gamma,O)}(\alpha_i \bullet \alpha_j \cdot \ldots \cdot \widehat{\alpha_i \alpha_j} \cdot \ldots \cdot \alpha_{n+1}).
\end{aligned}$$

Now, we can write (see [Arnal et al. 2002] for details)

$$B_{(\Gamma,O)}(\alpha_i \bullet \alpha_j \centerdot \alpha_1 \centerdot \dots \centerdot \alpha_{n+1}) = \sum_{(\Gamma',O')\to_{i,j}(\Gamma,O)} (-1)^{\ell_{\Gamma'}-1} B_{(\Gamma',O')}(\alpha_1 \centerdot \dots \centerdot \alpha_{n+1}),$$

where $\ell_{\Gamma'}$ denotes the position of the edge $\overrightarrow{p'_i p'_j}$ in Γ', and the sign $(-1)^{\ell_{\Gamma'}-1}$ comes directly from the definition of $\bullet$.

Next consider a graph (Γ', O') that reduces to (Γ, O) in the indexes i, j. We permute the edges as follows: we put first the edge $\overrightarrow{p'_i p'_j}$, then the other edges starting from p'_i, then the edges starting from p'_j, and finally we put all the legs at the end in order of their feet. This gives a sign that can be written as

$$\varepsilon_\alpha(ij, 1 \dots \widehat{ij} \dots n{+}1)(-1)^{\ell_{\Gamma'}-1}\varepsilon(s_O)\varepsilon(\tau_O).$$

Starting from (Γ', O'), one can also place the legs at the end in order of their feet, preceded by the aerial edges starting from p'_i and those starting from p'_j, and then by the aerial edge $\overrightarrow{p'_i p'_j}$ at the first position. If we denote by Δ' the aerial part of Γ' and by $\ell_{\Delta'}$ the position of the edge $\overrightarrow{p'_i p'_j}$ in Δ', the resulting sign is

$$\varepsilon(s_{O'})\varepsilon(\tau_{O'})\varepsilon(\Delta', \Delta)(-1)^{\ell_{\Delta'}-1}.$$

These two permutations of the edges of Γ' obviously coincide; thus

$$\varepsilon_\alpha(ij, 1 \dots \widehat{ij} \dots n{+}1)(-1)^{\ell_{\Gamma'}-1}\varepsilon(s_O)\varepsilon(\tau_O) = \varepsilon(s_{O'})\varepsilon(\tau_{O'})\varepsilon(\Delta', \Delta)(-1)^{\ell_{\Delta'}-1}.$$

It follows that

$$\begin{aligned}
&C_{(\Delta,O_\Delta)}(\alpha_i \bullet \alpha_j \centerdot \dots \centerdot \widehat{\alpha_i\alpha_j} \centerdot \dots \centerdot \alpha_{n+1}) \\
&\quad = \sum_{m\ge 0} \frac{1}{m!} \sum_{\substack{(\Gamma,O)\in GO^{(1)}_{n,m}\\ (\Gamma,O)\supset(\Delta,O_\Delta)}} \varepsilon_\alpha(ij1\dots n{+}1)\Bigg(\sum_{(\Gamma',O')\underset{i,j}{\to}(\Gamma,O)} \frac{\ell!}{k!}\varepsilon(s_{O'})\varepsilon(\tau_{O'})(-1)^{\ell_{\Delta'}-1} \\
&\qquad\qquad \varepsilon(\Delta', \Delta) B_{(\Gamma',O')}(\alpha_1 \centerdot \dots \centerdot \alpha_{n+1})\Bigg) \\
&\quad = \varepsilon_\alpha(ij, 1\dots n{+}1) \sum_{m\ge 0}\frac{1}{m!}\Bigg(\sum_{(\Delta',O_{\Delta'})\to_{i,j}(\Delta,O_\Delta)} (-1)^{\ell_{\Delta'}-1}\varepsilon(\Delta',\Delta) \\
&\qquad\qquad \sum_{(\Gamma',O')\supset(\Delta',O_{\Delta'})} \frac{\ell!}{k!}\varepsilon(s'_O)\varepsilon(\tau'_O) B_{(\Gamma',O')}(\alpha_1 \centerdot \dots \centerdot \alpha_{n+1})\Bigg) \\
&\quad = \varepsilon_\alpha(ij, 1\dots n{+}1) \sum_{(\Delta',O_{\Delta'})\to_{i,j}(\Delta,O_\Delta)} (-1)^{\ell_{\Delta'}-1}\varepsilon(\Delta', \Delta) C_{(\Delta',O_{\Delta'})}(\alpha_1 \centerdot \dots \centerdot \alpha_{n+1}).
\end{aligned}$$

Finally,

$$(\text{iii}) = (-1)^{|\Delta|+1} \sum_{(\Delta', O_{\Delta'}) \to_{i,j} (\Delta, O_\Delta)} (-1)^{\ell_{\Delta'}-1} \varepsilon(\Delta', \Delta) C_{(\Delta', O_{\Delta'})}(\alpha_1 \bullet \dots \bullet \alpha_{n+1})$$
$$= (-1)^{|\Delta|+1} \sum_{(\Delta', O_{\Delta'}) \to_{i,j} (\Delta, O_\Delta)} \varepsilon(\Delta', O_{\Delta'}, \Delta, O_\Delta) C_{(\Delta', O_{\Delta'})}(\alpha_1 \bullet \dots \bullet \alpha_{n+1}).$$

Now let $(\delta, O_\delta) = \sum c_{(\Delta, O_\Delta)}(\Delta, O_\Delta)$ be a symmetric combination of graphs and put

$$C_{(\delta, O_\delta)} = (\text{i})_\delta + (\text{ii})_\delta + (\text{iii})_\delta.$$

We have to prove that $-\big((\text{i})_\delta + (\text{ii})_\delta\big)$ coincides with the nonproper terms of $(\text{iii})_\delta$, that is, with

$$\sum_{(\Delta, O_\Delta)} c_{(\Delta, O_\Delta)} (-1)^{|\Delta|+1} \sum_{i \neq j} \sum_{(\Delta', O_{\Delta'}) \to^{\text{nonprop}}_{i,j} (\Delta, O_\Delta)} (-1)^{\ell_{\Delta'}-1} \varepsilon(\Delta', O_{\Delta'})(\Delta', O_{\Delta'}).$$

Consider first the term

$$(\text{ii})_\delta = \sum_{(\Delta, O_\Delta)} c_{(\Delta, O_\Delta)} \sum_{i=1^n} (-1)^{|\Delta||\alpha_i|} \varepsilon_\alpha(i, 1 \dots n+1) \alpha_i \bullet C_{(\Delta, O_\Delta)}(\alpha_1 \dots \hat{\alpha}_i \dots \alpha_{n+1}).$$

We identify $C_{(\Delta, O_\Delta)}(\alpha)$ with a polyvector field, and put

$$C_{(\Delta, O_\Delta)}(\alpha_1 \dots \hat{\alpha}_i \dots \alpha_{n+1}) = \big(C_{(\Delta, O_\Delta)}(\alpha_1 \dots \hat{\alpha}_i \dots \alpha_{n+1})\big)^{r_1 \dots r_m} \partial_{r_1} \wedge \dots \wedge \partial_{r_m}.$$

Thus

$$\alpha_i \bullet C_{(\Delta, O_\Delta)}(\alpha_1 \dots \hat{\alpha}_i \dots \alpha_{n+1})$$
$$= \sum_{j \neq i} \sum_{l \leq k_i} (-1)^{l-1} \alpha_i^{i_1 \dots s \dots k_i - 1} \big(C_{(\Delta, O_\Delta)}(\alpha_1 \dots \partial_s(\alpha_j) \dots \hat{\alpha}_i \dots \alpha_{n+1})\big)^{r_1 \dots r_m}$$
$$\partial_{i_1} \wedge \dots \partial_{i_{k_1-1}} \wedge \partial_{r_1} \wedge \dots \wedge \partial_{r_m}.$$

Let σ be the permutation $(j 1 \dots \widehat{ij} \dots n+1)$ and $(\Delta^\sigma, O_{\Delta^\sigma})$ be the aerial graph obtained by relabeling the vertices of Δ in the ordering given by σ. Then

$$C_{(\Delta, O_\Delta)}(\alpha_1 \dots \partial_s(\alpha_j) \dots \hat{\alpha}_i \dots \alpha_{n+1}) = C_{(\Delta^\sigma, O_{\Delta^\sigma})}(\partial_s(\alpha_j) \alpha_1 \dots \widehat{\alpha_i \alpha_j} \dots \alpha_{n+1}).$$

But (δ, O_δ) is symmetric; thus

$$c_{(\Delta^\sigma, O_{\Delta^\sigma})} = c_{(\Delta, O_\Delta)} \varepsilon_\alpha(j, 1 \dots \widehat{ij} \dots n+1).$$

Hence,

$$(\text{ii})_\delta = \sum_{(\Delta, O_\Delta)} \sum_{i \neq j} (-1)^{|\Delta||\alpha_i|} \varepsilon_\alpha(ij 1 \dots n+1) c_{(\Delta, O_\Delta)} \sum_{\ell \leq k_i} (-1)^{\ell-1} \alpha_i^{i_1 \dots s \dots i_{k_i-1}}$$
$$\big(C_{(\Delta, O_\Delta)}(\partial_s \alpha_j \alpha_1 \dots \widehat{\alpha_i \alpha_j} \dots \alpha_{n+1})\big)^{r_1 \dots r_m} \partial_{i_1} \wedge \dots \wedge \partial_{i_{k_i-1}} \wedge \partial_{r_1} \wedge \dots \wedge \partial_{r_m}.$$

It is now easy to see that $-(\mathrm{ii})_\delta$ coincides with certain nonproper terms of $(\mathrm{iii})_\delta$ — more precisely, with those corresponding to the graphs Δ' with

$$(\Delta', O_{\Delta'}) \to_{i,j} (\Delta, O_\Delta) \quad \text{and} \quad \big(\#\,\mathrm{strt}^{\Delta'}(p'_i) + \#\,\mathrm{end}^{\Delta'}(p'_i)\big) = 1.$$

(In this case, $\ell_{\Delta'} = 1$.) In the same way, one can check that $-(\mathrm{i})_\delta$ coincides with the remaining nonproper terms of $(\mathrm{iii})_\delta$, that is, with the nonproper terms corresponding to the case

$$(\Delta', O_{\Delta'}) \to_{i,j} (\Delta, O_\Delta) \quad \text{and} \quad (\#\,\mathrm{strt}^{\Delta'}(p'_j) + \#\,\mathrm{end}^{\Delta'}(p'_j)) = 1.$$

The result follows. □

5.2. ***Purely aerial and nonoriented graphs.*** We say that a graph is nonoriented if there is an ordering only on the aerial vertices but no ordering on the edges of the graph. We are now interested in translating our cohomology on nonoriented graphs. Let Δ be an aerial nonoriented graph with n vertices $p_1 < \cdots < p_n$. We still write $\ell_i = \mathrm{strt}^\Delta(p_i)$ and $\ell! = \ell_1! \ldots \ell_n!$. We order the edges of Δ lexicographically:

$$\overrightarrow{ab} \le \overrightarrow{a'b'} \quad \text{if and only if} \quad (a = a' \text{ and } a < b') \text{ or } (a < a').$$

This yields a compatible ordering on Δ, called the standard ordering. We denote by $(\Delta, O_\Delta^{\mathrm{std}})$ the resulting oriented graph.

Now put

$$\Delta = \frac{1}{\ell!} \sum_{O_\Delta : (\Delta, O_\Delta) \in GO_n^{(0)}} \varepsilon(\sigma_{(O_\Delta^{std}, O_\Delta)})(\Delta, O_\Delta).$$

By the definition of ∂ on compatible oriented graphs, we have:

$$\partial\Delta = \frac{1}{\ell!}\Bigg(\sum_{O_\Delta : (\Delta, O_\Delta) \in GO_n^{(0)}} \varepsilon(\sigma_{(O_\Delta^{std}, O_\Delta)})(-1)^{|\Delta|+1} \sum_{i \ne j} \sum_{(\Delta', O_{\Delta'}) \to_{i,j}^{\mathrm{prop}} (\Delta, O_\Delta)} \varepsilon(\Delta', O_{\Delta'}, \Delta, O_\Delta)(\Delta', O_{\Delta'}) \Bigg).$$

Note that the sign

$$\tilde{\varepsilon}(\Delta, \Delta') := \varepsilon(O_\Delta^{\mathrm{std}}, O_\Delta)\,\varepsilon(\Delta', O_{\Delta'}, \Delta, O_\Delta)\,\varepsilon(O_{\Delta'}^{\mathrm{std}}, O_{\Delta'})$$

does not depend on O_Δ or O'_Δ. This yields a very simple expression for the coboundary $\partial\Delta$ of Δ:

$$\partial\Delta = \frac{1}{\ell!} \sum_{i \ne j} \sum_{\Delta' \supset \Delta} \tilde{\varepsilon}(\Delta', \Delta)\Delta'.$$

We extend ∂ to linear combination of graphs $\delta = \sum_\Delta c_\Delta \Delta$.

Now, if Δ is a nonoriented graph with vertices $p_1 < \cdots < p_n$ and if σ is a permutation in S_n, we denote by $\sigma(\Delta)$ the nonoriented graph with vertices $p_{\sigma(1)} < \cdots < p_{\sigma(n)}$. A linear combination $\delta = \sum_\Delta c_\Delta \Delta$ of nonoriented graphs with n labeled vertices is said to be symmetric if for any σ in S_n, we have $c_\Delta = c_{\sigma(\Delta)}$. Our operator ∂ restricted to symmetric δ is clearly a cohomology operator.

More precisely, for an aerial nonoriented graph Δ, let

$$C_\Delta = \frac{1}{\ell!} \sum_{O_\Delta:(\Delta,O_\Delta)\in GO_n^{(0)}} \varepsilon(\sigma_{(O_\Delta^{\mathrm{std}}),O_\Delta)})C_{(\Delta,O_\Delta)}.$$

Extend this definition by linearity to all linear combinations. Then, by computations similar to those we did before for oriented graphs, we can prove:

Proposition 5.3. *For any symmetric combination $\delta = \sum_\Delta c_\Delta C_\Delta$ of graphs with n labeled vertices, we have*

$$\partial(C_\delta) = C_{\partial(\delta)}.$$

5.3. ***Examples.*** Let Δ_1 be the graph with only one vertex p_1. Let α_1 be a k_1-vector field. Then

$$C_{\Delta_1}(\alpha_1) = \frac{1}{(k_1!)^2} \sum_{GO_{n,m}^{(1)}\ni(\Gamma,O)\supset\Delta_1} \varepsilon(s_O)\varepsilon(\tau_O)B_{(\Gamma,O)}(\alpha_1).$$

There is only one graph occurring in this sum, namely the graph Γ with one aerial vertex p_1, k_1 terrestrial vertices $q_1, \ldots, q_{k_1}$ and k_1 edges $\overrightarrow{p_1q_1}, \ldots, \overrightarrow{p_1q_{k_1}}$. For any σ in S_{k_1}, denote by (Γ, O^σ) the graph Γ endowed with the ordering given by $\overrightarrow{p_1q_{\sigma(1)}} \ldots \overrightarrow{p_1q_{\sigma(k_1)}}$. Clearly,

$$C_{\Delta_1}(\alpha_1) = \frac{1}{(k_1!)^2} \sum_{\sigma\in S_{k_1}} \varepsilon(\sigma)B_{(\Gamma,O^\sigma)}(\alpha_1) = \mathcal{F}_1^{(0)}(\alpha_1) \simeq \alpha_1,$$

and C_{Δ_1} just corresponds to the identity mapping.

Now let Δ_2 be the aerial graph with two vertices $p_1 < p_2$ and one edge $\overrightarrow{p_1p_2}$. Let α_1 be a k_1-vector field and α_2 a k_2-vector field. Then

$$C_{\Delta_2}(\alpha_1 \otimes \alpha_2) = \frac{1}{(k_1+k_2-1)!} \sum_{\substack{(\Gamma,O)\supset\Delta_2 \\ (\Gamma,O)\in GO_{n,m}^{(1)}}} \frac{1}{k_1!\,k_2!}\varepsilon(s_O)\varepsilon(\tau_O)B_{(\Gamma,O)}(\alpha_1 \otimes \alpha_2).$$

There are exactly $(k_1+k_2-1)!/\big((k_1-1)!\,k_2!\big)$ graphs Γ containing Δ_2 and having exactly (k_1-1) legs starting from p_1 and k_2 legs starting from p_2. For each of them, we choose a compatible ordering. There are $k_1!\,k_2!$ possibilities to do it. Thus, there are exactly $k_1(k_1+k_2-1)!$ compatible oriented graphs (Γ, O) occurring in C_{Δ_2}. For each of these graphs, $\varepsilon(s_O)$ corresponds to the permutation of S_{k_1} that consists in putting the aerial edge of (Γ, O) at the first position and $\varepsilon(\tau_O)$ corresponds to the permutation of $S_{k_1+k_2-1}$ that consists in putting the legs in the order of the feet. There is thus $k_1(k_1+k_2-1)!$ terms in C_{Δ_2}, each of which looks like

$$\frac{1}{(k_1+k_2-1)!\,k_1!\,k_2!}\varepsilon(s_O)\varepsilon(\tau_O)B_{(\Gamma,O)} = \frac{1}{(k_1+k_2-1)!\,k_1!\,k_2!}(-1)^{\ell-1}\varepsilon(\sigma)$$
$$\alpha_1^{i_{\sigma(1)}\dots i_{\sigma(\ell-1)}s\,i_{\sigma(\ell)}\dots i_{\sigma(k_1-1)}}\partial_s(\alpha_2^{i_{\sigma(k_1)}\dots i_{\sigma(k_1+k_2-1)}})\partial_{i_{\sigma(1)}}\otimes\cdots\otimes\partial_{i_{s(k_1+k_2-1)}}.$$

Thus

$$C_{\Delta_2}(\alpha_1\otimes\alpha_2) = \mathscr{F}_1^{(0)}(\alpha_1\bullet\alpha_2) \simeq \alpha_1\bullet\alpha_2.$$

Now consider the aerial graph Δ_2^- with two vertices $p_1 < p_2$ and one edge $\overrightarrow{p_2p_1}$. In the same way as above, one can see that

$$C_{\Delta_2^-}(\alpha_1\otimes\alpha_2) = (-1)^{k_1k_2}\alpha_2\bullet\alpha_1.$$

In other words, $C_{\Delta_2+\Delta_2^-}$ coincides with Q_2.

The identity map Id and Q_2 are thus easy examples of K-graph mappings, and the fact that Q_2 is the Chevalley coboundary of Id can be checked directly on the graphs. Indeed, we have with our notations:

$$\partial\Delta_1 = \tilde{\varepsilon}(\Delta_2,\Delta_1)\Delta_2 + \tilde{\varepsilon}({\Delta_2}^-,\Delta_1){\Delta_2}^- = \Delta_2+\Delta_2^-.$$

Hence,

$$Q_2 = C_{\Delta_2+\Delta_2^-} = C_{\partial\Delta_1} = \partial C_{\Delta_1} = \partial\,\mathrm{Id}\,.$$

6. Triviality of the cohomology for small *n*

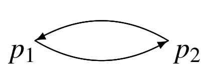
Our first example proves that the first cohomology group H^1 is trivial, since, for $n=1$, there is only one purely aerial graph, namely Δ_1.

Now suppose $n=2$. There is one graph Δ with two vertices and with degree 0 $|\Delta|=0$, the nonconnected symmetric graph denoted $\Delta_1\times\Delta_1$ without any edges. Its coboundary does not vanish; in the obvious notation, we have

$$\partial(\Delta_1\times\Delta_1) = S\big((\Delta_2^+ + \Delta_2^-)\times\Delta_1 + \Delta_1\times(\Delta_2^2+\Delta_2^-)\big) \neq 0.$$

In degree 1 ($|\Delta|=1$), there is only one symmetrized graph, $\Delta_2^+ + \Delta_2^-$. Our second example shows that this graph is a coboundary.

Finally, there is no graph with degree larger than 1; indeed, the number of edges for a graph with 2 vertices is at most 2, but there is only one graph Δ with $|\Delta|=2$, the graph $\Delta_{2,2}$ given by



But the symmetrization of this graph is $\Delta_{2,2} - \Delta_{2,2} = 0$. Thus the second cohomology group H^2 vanishes.

It is possible to prove with elementary arguments that $H^3 = 0$ too. For that, we consider the different cases, $|\Delta| = 0, \dots, 6$, then we define the order of a graph in the following way:

We define the order o_i of a vertex p_i as the pair (ℓ_i, r_i) of number ℓ_i of edges starting from p_i and the number r_i of edges ending at p_i, we shall say that $o = (\ell, r)$ is smaller than $o' = (\ell', r')$ and note $o < o'$ if and only if $\ell + r < \ell' + r'$ or $\ell + r = \ell' + r'$ and $\ell < \ell'$.

We define then the order $o(\Delta)$ of a graph Δ as $o(\Delta) = (o_1, \dots, o_n)$ if Δ has n vertices. The order $o(\delta)$ of a linear combination $\delta = \sum c_\Delta \Delta$ of graphs is the maximum of $o(\Delta)$ for $c_\Delta \neq 0$ for the lexicographic ordering. We define the symbol of δ by

$$\operatorname{symb} \delta = \sum_{o(\Delta)=o(\delta)} c_\Delta C_\Delta.$$

Case 1: $|\Delta| = 0$. There is only one graph, disconnected and symmetric: the graph $\Delta_1 \times \Delta_1 \times \Delta_1$. It is not a cocycle since

$$\partial(\Delta_1 \times \Delta_1 \times \Delta_1) = S\big((\Delta_2^+ + \Delta_2^-) \times \Delta_1 \times \Delta_1\big) \neq 0.$$

Case 2: $|\Delta| = 1$. There is, up to the ordering of vertices, only one symmetrized, disconnected graph: $\delta = S(\Delta_2^+ \times \Delta_1)$. This graph is a coboundary:

$$\partial(\Delta_1 \times \Delta_1) = \tfrac{1}{3} S\big((\Delta_2^+ + \Delta_2^-) \times \Delta_1\big) = \tfrac{2}{3}\delta.$$

Case 3: $|\Delta| = 2$. There is, up to the ordering of vertices, a disconnected graph $\Delta_{2,2} \times \Delta_1$ and three connected graphs, listed below. (We choose the ordering of vertices that maximizes the order, and for a given order maximizes, for the lexicographic ordering, the set $E(\Delta)$ of edges of graphs Δ.)

$$\begin{aligned} &\Delta_{3,2,1} \quad \text{with} \quad E(\Delta_{3,2,1}) = \{\overrightarrow{p_1 p_2}, \overrightarrow{p_1 p_3}\},\\ &\Delta_{3,2,2} \quad \text{with} \quad E(\Delta_{3,2,2}) = \{\overrightarrow{p_2 p_1}, \overrightarrow{p_1 p_3}\},\\ &\Delta_{3,2,3} \quad \text{with} \quad E(\Delta_{3,2,3}) = \{\overrightarrow{p_2 p_1}, \overrightarrow{p_3 p_1}\}.\end{aligned}$$

After symmetrization, we get $S(\Delta_{2,2} \times \Delta_1) = 0$, $S(\Delta_{3,2,1})) = S(\Delta_{3,2,3}) = 0$ and

$$\operatorname{symb} S(\Delta_{3,2,2}) = \tfrac{1}{6}\Delta_{3,2,2}, \quad o(S(\Delta_{3,2,2})) = \big((1,1), (1,0), (0,1)\big).$$

When we compute $\partial(S(D))$, we have to consider the blow-up of each vertex of each graph in $S(\Delta)$. If the vertex p has order $o = (\ell, r)$, we get a few graphs with two vertices p' and p" at the place of p; these vertices have order $o' = (\ell', r')$,

$o" = (\ell", r")$, with conditions

$$\ell' + r' \geq 2, \quad \ell'' + r'' \geq 2, \quad \ell' + \ell'' = \ell + 1, \quad r' + r'' = r + 1.$$

Then we look for $o(\partial\Delta)$. If $r > 0$, the maximal possible order among those $(o', o")$ is $\big((\ell+1, r-1), (0, 2)\big)$; if $r = 0$, it is $\big((\ell, r), (1, 1)\big) = \big((\ell, 0), (1, 1)\big)$.

Thus $o(\partial(S(D_{3,2,2}))) \leq \big((2, 0), (0, 2), (1, 0), (0, 1)\big)$; more precisely,

$$\text{symb } \partial(S(\Delta_{3,2,2})) = \tfrac{1}{6}\Delta', \quad E(\Delta') = \{\overrightarrow{p_1 p_2}, \overrightarrow{p_1 p_4}, \overrightarrow{p_3 p_2}\},$$

and, since there is only one graph in the symbol,

$$o(\partial(S(\Delta_{3,2,2}))) = \big((2, 0), (0, 2), (1, 0), (0, 1)\big).$$

No vector in this case is a cocycle; ∂ is an one-to-one mapping.

Case 4: $|\Delta| = 3$. From now on, all our graphs are connected. Repeating the argument of the preceding case, we get the following results:

They are, up to a permutation of vertices, four graphs:

$$\begin{aligned}
&\Delta_{3,3,1} \quad \text{with} \quad E(\Delta_{3,3,1}) = \{\overrightarrow{p_1 p_2}, \overrightarrow{p_1 p_3}, \overrightarrow{p_2 p_1}\},\\
&\Delta_{3,3,2} \quad \text{with} \quad E(\Delta_{3,3,2}) = \{\overrightarrow{p_1 p_2}, \overrightarrow{p_2 p_1}, \overrightarrow{p_3 p_1}\},\\
&\Delta_{3,3,3} \quad \text{with} \quad E(\Delta_{3,3,3}) = \{\overrightarrow{p_1 p_2}, \overrightarrow{p_1 p_3}, \overrightarrow{p_2 p_3}\},\\
&\Delta_{3,3,4} \quad \text{with} \quad E(\Delta_{3,3,4}) = \{\overrightarrow{p_1 p_2}, \overrightarrow{p_2 p_3}, \overrightarrow{p_3 p_1}\}.
\end{aligned}$$

Their symmetrizations do not vanish:

$$\begin{aligned}
o(S(\Delta_{3,3,1})) &= \big((2, 1), (1, 1), (0, 1)\big),\\
o(\partial(S(\Delta_{3,3,1}))) &= \big((3, 0), (1, 1), (0, 2), (0, 1)\big),\\
o(S(\Delta_{3,3,2})) &= \big((1, 2), (1, 1), (1, 0)\big),\\
o(\partial(S(\Delta_{3,3,2}))) &= \big((2, 1), (1, 1), (0, 2), (0, 1)\big),\\
o(S(\Delta_{3,3,3})) &= \big((2, 0), (1, 1), (0, 2)\big),\\
o(\partial(S(\Delta_{3,3,3}))) &= \big((2, 0), (2, 0), (0, 2), (0, 2)\big),\\
o(S(\Delta_{3,3,4})) &= \big((1, 1), (1, 1), (1, 1)\big),\\
o(\partial(S(\Delta_{3,3,4}))) &= \big((2, 0), (1, 1), (1, 1), (0, 2)\big).
\end{aligned}$$

Then ∂ is still a one-to-one mapping on that space of graphs.

Case 5: $|\Delta| = 4$. They are, up to a permutation of vertices, four graphs:

$$\begin{aligned}
&\Delta_{3,4,1} \quad \text{with} \quad E(\Delta_{3,4,1}) = \{\overrightarrow{p_1p_2}, \overrightarrow{p_1p_3}, \overrightarrow{p_2p_1}, \overrightarrow{p_3p_1}\},\\
&\Delta_{3,4,2} \quad \text{with} \quad E(\Delta_{3,4,2}) = \{\overrightarrow{p_1p_2}, \overrightarrow{p_1p_3}, \overrightarrow{p_2p_1}, \overrightarrow{p_2p_3}\},\\
&\Delta_{3,4,3} \quad \text{with} \quad E(\Delta_{3,4,3}) = \{\overrightarrow{p_1p_2}, \overrightarrow{p_1p_3}, \overrightarrow{p_2p_1}, \overrightarrow{p_3p_2}\},\\
&\Delta_{3,4,4} \quad \text{with} \quad E(\Delta_{3,4,4}) = \{\overrightarrow{p_1p_2}, \overrightarrow{p_2p_1}, \overrightarrow{p_3p_1}, \overrightarrow{p_3p_2}\}.
\end{aligned}$$

Their symmetrizations do not vanish:

$$\begin{aligned}
o(S(\Delta_{3,4,1})) &= \big((2,2),(1,1),(1,1)\big),\\
o(\partial(S(\Delta_{3,4,1}))) &= \big((3,1),(1,1),(1,1),(0,2)\big),\\
o(S(\Delta_{3,4,2})) &= \big((2,1),(2,1),(0,2)\big),\\
o(\partial(S(\Delta_{3,4,2}))) &= \big((3,0),(2,1),(0,2),(0,2)\big),\\
o(S(\Delta_{3,4,3})) &= \big((2,1),(1,2),(1,1)\big),\\
o(\partial(S(\Delta_{3,4,3}))) &= \big((3,0),(1,2),(1,1),(0,2)\big),\\
o(S(\Delta_{3,4,4})) &= \big((1,2),(1,2),(2,0)\big),\\
o(\partial(S(\Delta_{3,4,4}))) &= \big((2,1),(1,2),(2,0),(0,2)\big).
\end{aligned}$$

Then ∂ is still a one-to-one mapping on that space of graphs.

Case 6: $|\Delta| = 5$. Up to a permutation of vertices, this space contains only one graph:

$$\Delta_{3,5,1} \quad \text{with} \quad E(\Delta_{3,5,1}) = \{\overrightarrow{p_1p_2}, \overrightarrow{p_1p_3}, \overrightarrow{p_2p_1}, \overrightarrow{p_2p_3}, \overrightarrow{p_3p_1}\}.$$

Its symmetrization does not vanish,

$$\begin{aligned}
o(S(\Delta_{3,5,1})) &= \big((2,2),(2,1),(1,2)\big),\\
o(\partial(S(\Delta_{3,6,1}))) &= \big((3,1),(2,1),(1,2),(0,2)\big).
\end{aligned}$$

Then ∂ is still a one-to-one mapping on that space of graphs.

Case 7: $|\Delta| = 6$. In this last case, there is only one graph:

$$\Delta_{3,6,1} \quad \text{with} \quad E(\Delta_{3,6,1}) = \{\overrightarrow{p_1p_2}, \overrightarrow{p_1p_3}, \overrightarrow{p_2p_1}, \overrightarrow{p_2p_3}, \overrightarrow{p_3p_1}, \overrightarrow{p_3p_2}\}.$$

But its symmetrization does vanish.

This proves:

Proposition 6.1. *The three first spaces H^1, H^2 and H^3 of the Chevalley cohomology for graphs vanish.*

7. Canonical cocycles for the linear case

We first recall the construction of the relevant cocycles for the cohomology of the Lie algebra of vector fields $\mathscr{X}(\mathbb{R}^d)$ associated to the Lie derivative of smooth functions; see for instance [De Wilde and Lecomte 1983] for an explicit presentation of this cohomology.

A basis of the Lie algebra $\bigwedge^{\text{inv}}\big(\mathfrak{gl}(d,\mathbb{R})\big)$ of multilinear, skewsymmetric, invariant forms on $\mathfrak{gl}(d,\mathbb{R})$ is given by

$$\zeta^{(j_1)}\wedge\cdots\wedge\zeta^{(j_q)}\quad\text{with } j_k \text{ odd and } j_1<j_2\cdots<j_q<2d,$$

where the $\zeta^{(j)}$ are the mappings

$$\zeta^{(j)}(A_1,\dots,A_j)=\mathfrak{a}\big(\operatorname{Tr}(A_1\dots A_j)\big).$$

Then, for each odd n, the linear form θ defined on $\bigwedge^n\mathscr{X}(\mathbb{R}^d)$ by

$$\theta(\xi_1,\xi_2,\dots,\xi_n)=\zeta^{(n)}\big(\operatorname{Jac}(\xi_1),\dots,\operatorname{Jac}(\xi_n)\big)$$

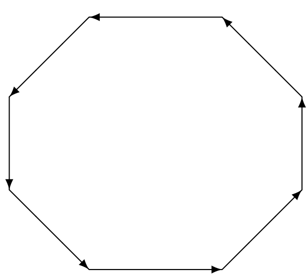
is a cocycle for the coboundary operator associated to the Lie derivative:

$$d\theta(\xi_0\cdot\ldots\cdot\xi_n)=\sum_{i=0}^{n}(-1)^i\mathscr{L}_{\xi_i}\theta(\xi_0\cdot\ldots\cdot\hat{\xi}_i\cdot\ldots\cdot\xi_n)\\ +\frac{1}{2}\sum_{i\neq j}(-1)^{i+j}\theta\big([\xi_i,\xi_j]\cdot\xi_0\dots\widehat{\xi_i\xi_j}\cdot\ldots\cdot\xi_n\big).$$

This cocycle is not a coboundary; see [De Wilde and Lecomte 1983].

Let Ψ be an n-cochain on $T_{\text{poly}}(\mathbb{R}^d)$ with values in the space $T_{\text{poly}}(\mathbb{R}^d)^{-1}$ (that is, in $C^\infty(\mathbb{R}^d)$), and let ψ be its restriction to $\mathscr{X}(\mathbb{R}^d)$. Then the restriction of $\partial\Psi$ to $\mathscr{X}(\mathbb{R}^d)$ is exactly $d\psi$.

For instance, we consider the "wheel without an axis", the graph Δ of this form:



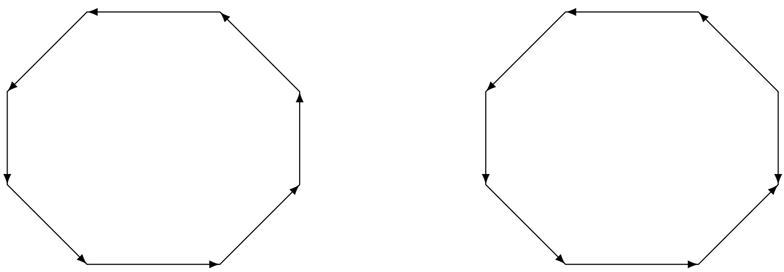
Denote by δ its symmetrization, which defines a cochain $\Psi = C_\delta$. By construction, on vector fields ξ_i, we get

$$\begin{aligned}\psi(\xi_1 \cdot \ldots \cdot \xi_n) = \Psi(\xi_1 \cdot \ldots \cdot \xi_n) = C_d(\xi_1 \cdot \ldots \cdot \xi_n)\\ = \frac{1}{n!}\sum_{\sigma\in S_n} \varepsilon(\sigma)\partial_{i_n}\xi^{i_1}_{\sigma(1)}\partial_{i_1}\xi^{i_2}_{\sigma(2)}\cdots\partial_{i_{n-1}}\xi^{i_n}_{\sigma(n)}\\ = \theta(\xi_1 \cdot \ldots \cdot \xi_n).\end{aligned}$$

Thus

$$\begin{aligned}C_{\partial\delta}(\xi_0 \cdot \ldots \cdot \xi_n) = \partial C_\delta(\xi_0 \cdot \ldots \cdot \xi_n) = \partial\Psi(\xi_0 \cdot \ldots \cdot \xi_n)\\ = d\theta(\xi_0 \cdot \ldots \cdot \xi_n) = 0.\end{aligned}$$

We now restrict ourselves to the space of linear polyvector fields. This is a subalgebra of $T_{\text{poly}}(\mathbb{R}^d)$ equipped with the Schouten bracket; thus we can restrict our coboundary operator to cochains defined on this subalgebra. We get a new operator ∂_{lin}. Our previous computation tells us that the graphs happening in $\partial\delta$ are of the following forms:



For linear polyvector fields, only the first case appears. Then $B_{\partial_{\text{lin}}(\delta)}(\alpha_0 \cdot \ldots \cdot \alpha_n)$ vanishes if one of the α_j is not a vector field. And

$$B_{\partial_{\text{lin}}\delta}(\xi_0 \cdot \ldots \cdot \xi_n) = C_{\partial\delta}(\xi_0 \cdot \ldots \cdot \xi_n) = 0.$$

Since the mapping $\gamma \mapsto B_\gamma$ is one-to-one, $\partial_{\text{lin}}\delta = 0$.

Now, suppose δ is a coboundary $d = \partial_{\text{lin}}\beta$. Then β has $n-1$ vertices and $n-1$ edges. At each vertex there ends exactly one edge. If there is a vertex p from which no edge emanates, denote by $\overrightarrow{p'p}$ the edge ending at p. Since the graphs in β can be deduced from the graphs $\partial_{\text{lin}}\beta$ only by proper reduction, there is no reduction at the vertex p, and in $\partial_{\text{lin}}\beta$ there remains a unique edge $\overrightarrow{p'p}$. But there is no such graph in δ, so we can eliminate in β all the graphs with a vertex without emanating edges (we consider only "nonhanded" graphs). Now from each vertex of a graph in β, there is exactly one edge starting. As previously, the restriction of

$\partial\beta$ to the vector fields coincides with $\partial_{\text{lin}}\beta$, and

$$\begin{aligned} dC_\beta(\xi_0 \centerdot \dots \centerdot \xi_n) &= \partial C_\beta(\xi_0 \centerdot \dots \centerdot \xi_n) = C_{\partial\beta}(\xi_0 \centerdot \dots \centerdot \xi_n) \\ &= C_{\partial_{\text{lin}} b}(\xi_0 \centerdot \dots \centerdot \xi_n) = C_\delta(\xi_0 \centerdot \dots \centerdot \xi_n) = \theta(\xi_0 \centerdot \dots \centerdot \xi_n). \end{aligned}$$

This is impossible.

Thus any "wheel without an axis" Δ having an odd number of vertices gives rise to a canonical true cocycle for ∂_{lin}.

Remark. Suppose we want to build a linear formality $\mathscr{F}$ from the space of linear polyvector fields to the space of multidifferential operators. As we saw in Section 2, the obstruction to such a construction is a mapping φ, of degree 1, with $n \geq 4$ arguments. Such a mapping corresponds to purely aerial graphs with n vertices and $2n-3$ edges; in the linear case, we should have $2n-3 \leq n$, which is impossible. Every linear formality at order n can be extended to a linear formality.

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