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We compute the structure of the Lie algebras associated to two examples of branch groups, and show that one has finite width while the other, the Gupta–Sidki group, has unbounded width and Lie algebra of Gelfand–Kirillov dimension $\log 3/\log(1+\sqrt{2})$.

We then draw a general result relating the growth of a branch group, of its Lie algebra, of its graded group ring, and of a natural homogeneous space we call *parabolic space*, namely the quotient of the group by the stabilizer of an infinite ray. The growth of the group is bounded from below by the growth of its graded group ring, which connects to the growth of the Lie algebra by a product-sum formula, and the growth of the parabolic space is bounded from below by the growth of the Lie algebra.

Finally we use this information to explicitly describe the normal subgroups of $\mathfrak{G}$, the *Grigorchuk group*. All normal subgroups are characteristic, and the number b_n of normal subgroups of $\mathfrak{G}$ of index 2^n is odd and satisfies $\limsup b_n/n^{\log_2 3} = 5^{\log_2 3}$, $\liminf b_n/n^{\log_2 3} = \frac{2}{9}$.

1. Introduction

The first purpose of this paper is to describe explicitly the Lie algebra associated to the Gupta–Sidki group $\ddot{\Gamma}$ [Gupta and Sidki 1983], and show in this way that this group is not of finite width (Corollary 3.9). We shall describe in Theorem 3.8 the Lie algebra as a graph, somewhat similar to a Cayley graph, in a formalism close to that introduced in [Bartholdi and Grigorchuk 2000a].

We shall then consider another group, Γ, and show in Corollary 3.14 that although many similarities exist between $\ddot{\Gamma}$ and Γ, the Lie algebra of Γ does have finite width.

These results follow from a description of group elements as branch portraits, exhibiting the relation between the group and its Lie algebra. They lead to the

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notion of infinitely iterated wreath algebras, similar to wreath products of groups [Bartholdi ≥ 2005].

We shall show in Theorem 4.4 that, in the class of branch groups, the growth of the homogeneous space G/P (where P is a parabolic subgroup) is larger than the growth of the Lie algebra $\mathscr{L}(G)$. This result parallels a lower bound on the growth of G by that of its graded group ring $\overline{\Bbbk G}$ (Proposition 1.10).

Finally, we shall describe all the normal subgroups of the first Grigorchuk group, using the same formalism as that used to describe the lower central series. We confirm the description by Ceccherini et al. [2001] of the low-index normal subgroups of $\mathfrak{G}$. It turns out that all nontrivial normal subgroups are characteristic, and have finite index a power of 2. Call b_n the number of normal subgroups of index 2^n (Finite-index, not necessarily normal subgroups always have index a power of 2; this follows from G being a 2-torsion group.) Then there are 3^k+2 subgroups of index $2^{5\cdot 2^k+1}$ and $\frac{2}{9}3^k+1$ subgroups of index 2^{2^k+2}; these two values are extreme, in the sense that $b_n/n^{\log_2 3}$ has lower limit $5^{-\log_2 3}$ and upper limit $\frac{2}{9}$. Also, b_n is odd for all n (see Corollaries 5.4 and 5.5).

1.1. ***Philosophy.*** One can hardly exaggerate the importance of Lie algebras in the study of Lie groups. Lie subgroups correspond to subalgebras, normal subgroups correspond to ideals; simplicity, nilpotence and other properties match perfectly. This is due to the existence of mutually-inverse functions exp and log between a group and its algebra, and the Campbell–Hausdorff formula expressing the group operation in terms of the Lie bracket.

In the context of (discrete) p-groups and Lie algebras of characteristic p, the correspondence is not so perfect. First, in general, there is no exponential, and the best one can consider is the degree-1 truncations

$$\exp x = 1 + x + \mathbb{O}(x^2),$$
$$\log(1+x) = x + \mathbb{O}(x^2);$$

more terms would introduce denominators that in general are not invertible; and no reasonable definition of convergence can be imposed on $\mathbb{F}_p$. As a consequence, the group has to be subjected to a filtration to yield a Lie algebra. Then there is no perfect bijection between group and Lie-algebra objects.

However, the numerous results obtained in the area show that much can be gained from consideration of these imperfect algebras. To name a few, the theory of groups of finite width is closely related to the classification of finite p-groups (see [Leedham-Green 1994; Shalev and Zelmanov 1992]) and the theory of pro-p-groups is intimately Lie-algebraic; see [Shalev 1995a], [Shalev 1995b, §8] and [Klaas et al. 1997] with its bibliography. The solution to Burnside's problems by Efim Zelmanov relies also on Lie algebras. The results by Lev Kaloujnine on the

p-Sylow subgroups of $\mathfrak{S}_{p^n}$, even if in principle independent, can be restated in terms of Lie algebras in a very natural way (see Theorem 3.4).

In this paper, I argue that questions of growth, geometry and normal subgroup structure are illuminated by Lie-algebraic considerations.

1.2. ***Notation.*** We shall always write commutators as $[g,h]=g^{-1}h^{-1}gh$, conjugates as $g^h=h^{-1}gh$, and the adjoint operators $\mathsf{Ad}(g)=[g,-]$ and $\mathsf{ad}(x)=[x,-]$ on the group and Lie algebra respectively. $\mathfrak{S}_n$ is the symmetric group on n letters, and $\mathfrak{A}_n$ is the alternate subgroup of $\mathfrak{S}_n$. Polynomials and power series are all written over the formal variable $\hbar$, as is customary in the theory of quantum algebras. The Galois field with p elements is written $\mathbb{F}_p$. The cyclic group of order n is written C_n.

The lower central series of G is $\{\gamma_n(G)\}$, the lower p-central series is $\{P_n(G)\}$, the dimension series is $\{G_n\}$, the Lie dimension series is $\{L_n(G)\}$, and the derived series is $G^{(n)}$, and in particular $G'=[G,G]$ — the definitions shall be given below.

For $H\le G$, the subgroup of H generated by n-th powers of elements in H is written $\mho_n(H)$, and $H^{\times n}$ denotes the direct product of n copies of H, avoiding the ambiguous H^n. The normal closure of H in G is H^G.

Finally, $*$ stands for anything — something a speaker would abbreviate as “blah, blah, blah” in a talk. It is used to mean either that the value is irrelevant to the rest of the computation, or that it is the only unknown in an equation and therefore does not warrant a special name.

1.3. ***N-series.*** We first recall a classical construction of Magnus [1940], described for instance in [Lazard 1954] and [Huppert and Blackburn 1982, Chapter VIII].

Definition 1.1. Let G be a group. An *N-series* is a series $\{H_n\}$ of normal subgroups with $H_1=G$, $H_{n+1}\le H_n$ and $[H_m,H_n]\le H_{m+n}$ for all $m,n\ge 1$. The associated Lie ring is

$$\mathscr{L}(G)=\bigoplus_{n=1}^{\infty}\mathscr{L}_n,$$

with $\mathscr{L}_n=H_n/H_{n+1}$ and the bracket operation $\mathscr{L}_n\otimes\mathscr{L}_m\to\mathscr{L}_{m+n}$ induced by commutation in G.

For p a prime, an *N_p-series* is an N-series $\{H_n\}$ such that $\mho_p(H_n)\le H_{pn}$, and the associated Lie ring is a restricted Lie algebra over $\mathbb{F}_p$:

$$\mathscr{L}_{\mathbb{F}_p}(G)=\bigoplus_{n=1}^{\infty}\mathscr{L}_n,$$

with the p-mapping $\mathscr{L}_n\to\mathscr{L}_{pn}$ induced by raising to the power p in H_n.

We recall that $\mathscr{L}$ is a *restricted* Lie algebra (see [Jacobson 1941] or [Strade and Farnsteiner 1988, Section 2.1]) if it is over a field $\Bbbk$ of characteristic p, and there exists a mapping $x \mapsto x^{[p]}$ such that $\mathsf{ad}\, x^{[p]} = \mathsf{ad}(x)^p$, $(\alpha x)^{[p]} = \alpha^p x^{[p]}$ and $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$, where the s_i are obtained by expanding $\mathsf{ad}(x \otimes \hbar + y \otimes 1)^{p-1}(a \otimes 1) = \sum_{i=1}^{p-1} s_i(x, y) \otimes i\hbar^{i-1}$ in $\mathscr{L} \otimes \Bbbk[\hbar]$. Equivalently:

Proposition 1.2 (Jacobson). *Let (e_i) be a basis of $\mathscr{L}$ such that, for some $y_i \in \mathscr{L}$, we have $\mathsf{ad}(e_i)^p = \mathsf{ad}(y_i)$. Then $\mathscr{L}$ is restricted; more precisely, there exists a unique p-mapping such that $e_i^{[p]} = y_i$.*

The standard examples of an N-series are the *lower central series*, $\{\gamma_n(G)\}_{n=1}^{\infty}$, given by $\gamma_1(G) = G$ and $\gamma_n(G) = [G, \gamma_{n-1}(G)]$, and the *lower exponent-p central series* or *Frattini series* given by $P_1(G) = G$ and

$$P_n(G) = [G, P_{n-1}(G)]\,\mho_p(P_{n-1}(G)).$$

The Frattini series differs from the lower central series in that its successive quotients are all elementary p-groups.

The standard example of an N_p-series is the *dimension series*, also known as the p-lower central, Zassenhaus [1940], Jennings [1941], Lazard [1954] or Brauer series, given by $G_1 = G$ and $G_n = [G, G_{n-1}]\,\mho_p(G_{\lceil n/p \rceil})$, where $\lceil n/p \rceil$ is the least integer no less than n/p. It can alternatively be described, by a result of Lazard [1954], as

$$G_n = \prod_{i \cdot p^j \geq n} \mho_{p^j}(\gamma_i(G)),$$

or as

$$G_n = \{g \in G \mid g - 1 \in \varpi^n\},$$

where ϖ is the augmentation (or fundamental) ideal of the group algebra $\mathbb{F}_p G$. Note that this last definition extends to characteristic 0, giving a graded Lie algebra $\mathscr{L}_{\mathbb{Q}}(G)$ over $\mathbb{Q}$. In that case, the subgroup G_n is the isolator of $\gamma_n(G)$:

$$G_n = \sqrt{\gamma_n(G)} = \big\{g \in G \mid \langle g \rangle \cap \gamma_n(G) \neq \{1\}\big\}.$$

A good reference for these results is [Passi 1979, Chapter VIII].

We mention finally for completeness another N_p-series, the *Lie dimension series*

$$L_n(G) = \{g \in G \mid g - 1 \in \varpi^{(n)}\},$$

where $\varpi^{(n)}$ is the n-th Lie power of $\varpi \leq \Bbbk G$, given by $\varpi^{(1)} = \varpi$ and $\varpi^{(n+1)} = [\varpi^{(n)}, \varpi] = \{xy - yx \mid x \in \varpi^{(n)}, y \in \varpi\}$. As shown in [Passi and Sehgal 1975],

$$L_n(G) = \prod_{(i-1) \cdot p^j \geq n} \mho_{p^j}(\gamma_i(G))$$

if $\Bbbk$ is of characteristic p, and

$$L_n(G) = \sqrt{\gamma_n(G)} \cap [G, G]$$

if $\Bbbk$ is of characteristic 0.

In the sequel we will only consider the N-series $\{\gamma_n(G)\}$ and $\{P_n(G)\}$ and the N_p-series $\{G_n\}$ of dimension subgroups. We reserve the symbols $\mathscr{L}$ and $\mathscr{L}_{\mathbb{F}_p}$ for their respective Lie algebras.

Definition 1.3. Let $\{H_n\}$ be an N-series for G. The *degree* of $g \in G$ is the maximal $n \in \mathbb{N} \cup \{\infty\}$ such that g belongs to H_n.

Recall that the rank of an abelian group A is the minimal number of elements that generate A. A series $\{H_n\}$ has *finite width* if there is a constant W such that $\ell_n := \operatorname{rank}[H_n : H_{n+1}] \leq W$ for all n. A group has *finite width* if its lower central series has finite width; this definition comes from [Klaas et al. 1997].

Definition 1.4. Let $a = \{a_n\}$ and $b = \{b_n\}$ be sequences of real numbers. We write $a \precsim b$ if there is an integer $C > 0$ such that $a_n < C b_{Cn+C} + C$ for all $n \in \mathbb{N}$, and write $a \sim b$ if $a \precsim b$ and $b \precsim a$.

In the sense of this definition, a group has finite width if and only if $\{\ell_n\} \sim \{1\}$.

Question 1. If the rank of $\gamma_n(G)/\gamma_{n+1}(G)$ is bounded, does it follow that the rank of G_n/G_{n+1}, $P_n(G)/P_{n+1}(G)$ or $L_n(G)/L_{n+1}(G)$ is bounded? How about a converse?

More generally, say an N-series $\{H_n\}$ has *finite width* if $\operatorname{rank}(H_n/H_{n+1})$ is bounded over $n \in \mathbb{N}$. If G has a finite-width N-series intersecting to $\{1\}$, are all N-series of G of finite width?

I do not know the answer to these natural questions.

The following result is well-known, and shows that sometimes the Lie ring $\mathscr{L}(G)$ is actually a Lie algebra over $\mathbb{F}_p$.

Lemma 1.5. *Let G be a group generated by a set S. Let $\mathscr{L}(G)$ be the Lie ring associated to the lower central series.*

(1) *If S is finite, $\mathscr{L}_n$ is a finite-rank $\mathbb{Z}$-module for all n.*

(2) *If there is a prime p such that all generators $s \in S$ have order p, then $\mathscr{L}_n$ is a vector space over $\mathbb{F}_p$ for all n. It follows that the Frattini series (for that prime p) and the lower central series coincide.*

Proof. First, $\mathscr{L}_1$ is generated by $\bar{S}$, the image of S in G/G'. Since $\mathscr{L}$ is generated by $\mathscr{L}_1$, in particular $\mathscr{L}_n$ is generated by the finitely many $(n-1)$-fold products of elements of $\bar{S}$; this proves the first point.

Actually, far fewer generators are required for $\mathscr{L}_n$; in the extremal case when G is a free group, a basis of $\mathscr{L}_n$ is given in terms of "standard monomials" of degree n. See Section 3.2 or [Hall 1950].

For the second claim, assume more generally that $s^p \in G'$ for all $s \in S$, so that G/G' is an $\mathbb{F}_p$-vector space. We use the identity $[x, y]^p \equiv [x, y^p] \bmod \gamma_3\langle x, y\rangle$, due to Philip Hall. Let $g = [x, y]$ be a generator of $\gamma_n(G)$, with $x \in G$ and $y \in$

$\gamma_{n-1}(G)$. Then $y^p \in \gamma_n(G)$ by induction, so $g^p \in \gamma_{n+1}(G)$ and $\mathscr{L}_n$ is an $\mathbb{F}_p$-vector space. □

Anticipating, we note that the groups $\ddot{\Gamma}$ and Γ we shall consider satisfy these hypotheses for $p = 3$, and $\mathfrak{G}$ satisfies them for $p = 2$.

1.4. *Growth of groups and vector spaces.* Let G be a group generated by a finite set S. The *length* $|g|$ of an element $g \in G$ is the minimal number n such that g can be written as $s_1 \dots s_n$ with $s_i \in S$. The *growth series* of G is the formal power series

$$\mathsf{growth}(G) = \sum_{g \in G} \hbar^{|g|} = \sum_{n \geq 0} f_n \hbar^n,$$

where $f_n = \#\{g \in G \mid |g| = n\}$. The *growth function* of G is the $\sim$-equivalence class of the sequence $\{f_n\}$. Note that although $\mathsf{growth}(G)$ depends on S, this equivalence class is independent of the choice of S.

Let X be a transitive G-set and $x_0 \in X$ be a fixed base point. The *length* $|x|$ of an element $x \in X$ is the minimal length of a $g \in G$ moving x_0 to x. The *growth series* of X is the formal power series

$$\mathsf{growth}(X, x_0) = \sum_{x \in X} \hbar^{|x|} = \sum_{n \geq 0} f_n \hbar^n,$$

where $f_n = \{x \in X \mid \min_{gx_0 = x} |g| = n\}$. The *growth function* of X is the equivalence class under $\sim$ of the sequence $\{f_n\}$. It is again independent of the choice of x_0 and of generators of G.

Let $V = \bigoplus_{n \geq 0} V_n$ be a graded vector space. The *Hilbert–Poincaré* series of V is the formal power series

$$\mathsf{growth}(V) = \sum_{n \geq 0} v_n \hbar^n = \sum_{n \geq 0} \dim V_n \hbar^n.$$

We return to the dimension series of G. Consider the graded algebra

$$\overline{\mathbb{F}_p G} = \bigoplus_{n=0}^{\infty} \varpi^n / \varpi^{n+1}.$$

Here a fundamental result connecting $\mathscr{L}_{\mathbb{F}_p}(G)$ and $\overline{\mathbb{F}_p G}$:

Theorem 1.6 [Quillen 1968]. *$\overline{\mathbb{F}_p G}$ is the restricted enveloping algebra of the Lie algebra $\mathscr{L}_{\mathbb{F}_p}(G)$ associated to the dimension series.*

The Poincaré–Birkhoff–Witt Theorem then gives a basis of $\overline{\mathbb{F}_p G}$ consisting of monomials over a basis of $\mathscr{L}_{\mathbb{F}_p}(G)$, with exponents at most $p - 1$. Therefore:

Proposition 1.7 [Jennings 1941]. *Let G be a group, and let $\sum_{n\geq 1} \ell_n \hbar^n$ be the Hilbert–Poincaré series of $\mathscr{L}_{\mathbb{F}_p}(G)$. Then*

$$\mathsf{growth}(\overline{\mathbb{F}_p G}) = \prod_{n=1}^{\infty} \left(\frac{1-\hbar^{pn}}{1-\hbar^n} \right)^{\ell_n}.$$

Approximations from analytical number theory [Li 1996] and complex analysis then give:

Proposition 1.8 [Bartholdi and Grigorchuk 2000a, Proposition 2.2; Petrogradsky 1999, Theorem 2.1]. *Let G be a group and expand the power series*

$$\mathsf{growth}(\mathscr{L}_{\mathbb{F}_p}(G)) = \sum_{n\geq 1} \ell_n \hbar^n \quad \textit{and} \quad \mathsf{growth}(\overline{\mathbb{F}_p G}) = \sum_{n\geq 0} f_n \hbar^n.$$

(1) *$\{f_n\}$ grows exponentially if and only if $\{\ell_n\}$ does, and*

$$\limsup_{n\to\infty} \frac{\ln \ell_n}{n} = \limsup_{n\to\infty} \frac{\ln f_n}{n}.$$

(2) *If $\ell_n \sim n^d$, then $f_n \sim e^{n^{(d+1)/(d+2)}}$.*

The Lie algebras we consider have polynomial growth, i.e., finite Gelfand–Kirillov dimension. This notion is more commonly studied for associative rings [Gelfand and Kirillov 1966]:

Definition 1.9. The *Gelfand–Kirillov dimension* of a graded Lie algebra $\mathscr{L} = \bigoplus \mathscr{L}_n$ is

$$\dim_{GK}(\mathscr{L}) = \limsup_{n\to\infty} \frac{\log\left(\dim \mathscr{L}_1 + \cdots + \dim \mathscr{L}_n\right)}{\log n}.$$

If $\ell_n \sim n^d$, then $\mathscr{L}$ has Gelfand–Kirillov dimension $d+1$. However, the converse is not true, since the sequence $\log(\ell_1 + \cdots + \ell_n)/\log n$ need not converge. If the group G has finite width, its algebra $\mathscr{L}(G)$ has Gelfand–Kirillov dimension 1.

Note also that if A is any algebra generated in degree 1, then $\dim_{GK}(A) = 0$ or $\dim_{GK}(A) \geq 1$. Furthermore, George Bergman [1978] has shown that if A is associative, then $\dim_{GK}(A) = 1$ or $\dim_{GK}(A) \geq 2$; see [Krause and Lenagan 1985, Theorem 2.5] for a proof. Victor Petrogradsky [1997] showed that there exist Lie algebras of any Gelfand–Kirillov dimension ≥ 1.

Finally, we recall a connection between the growth of G and that of $\overline{\mathbb{F}_p G}$. We use the notation $\sum f_n \hbar^n \geq \sum g_n \hbar^n$ to mean $f_n \geq g_n$ for all $n \in \mathbb{N}$.

Proposition 1.10 [Grigorchuk 1989, Lemma 8]. *Let G be a group generated by a finite set S. Then*

$$\frac{\mathsf{growth}(G)}{1-\hbar} \geq \mathsf{growth}(\overline{\Bbbk G}).$$

2. Branch groups

Branch groups were introduced by Rostislav Grigorchuk [2000], where he developed a general theory of groups acting on rooted trees. We shall content ourselves with a restricted definition; recall that $G \wr \mathfrak{S}_d$ is the *wreath product* $G^{\times d} \rtimes \mathfrak{S}_d$, the action of $\mathfrak{S}_d$ on the direct product induced by the permutation action of $\mathfrak{S}_d$ on $\Sigma = \{1, \dots, \mathsf{d}\}$.

Definition 2.1. A group G is a *regular branch group* if for some $d \in \mathbb{N}$ there is

(1) an embedding $\psi : G \hookrightarrow G \wr \mathfrak{S}_d$ such that the image of $\psi(G)$ in $\mathfrak{S}_d$ acts transitively on Σ. Define for $n \in \mathbb{N}$ the subgroups $\mathsf{Stab}_G(n)$ of G by $\mathsf{Stab}_G(0) = G$, and inductively

$$\mathsf{Stab}_G(n) = \psi^{-1}(\mathsf{Stab}_G(n-1)^{\times d})$$

where $\mathsf{Stab}_G(n-1)^{\times d}$ is seen as a subgroup of $G \wr \mathfrak{S}_d$. One requires then that $\bigcap_{n \in \mathbb{N}} \mathsf{Stab}_G(n) = \{1\}$;

(2) a subgroup $K \le G$ of finite index with $\psi(K) \le K^{\times d}$.

To avoid ambiguous bracket notations, we write the decomposition map

$$\psi(g) = \langle\!\langle g_1, \dots, g_d \rangle\!\rangle \pi,$$

with π expressed as a permutation in disjoint cycle notation.

We shall abbreviate "regular branch group" to "branch group", since all branch groups in this paper are actually regular branch. We shall usually omit d from the description, and say that "G branches over K".

Lemma 2.2. *If G is a branch group, then G branches over a subgroup K of G such that K is normal in G, and $K^{\times d}$ is normal in $\psi(K)$.*

Proof. Let G be branch over L of finite index, and set $K = \bigcap_{g \in G} L^g$, the *core* of L. Then obviously $L \lhd G$; and since $(L^{\times d})^{\psi g} \le \psi(K^g)$ for all $g \in G$, we have, writing $\psi(g) = \langle\!\langle g_1, \dots, g_d \rangle\!\rangle \pi$,

$$K^{\times d} \le \bigcap_{g \in G} (L^{g_{1\pi}} \times \cdots \times L^{g_{d\pi}}) = \bigcap_{g \in G} (L^{\times d})^{\psi g} \le K,$$

and $(K^{\times d})^{\psi(g)} = K^{g_{1\pi}} \times \cdots \times K^{g_{d\pi}} = K^{\times d}$, so $K^{\times d} \lhd \psi(G)$. □

Let G be a branch group, with d, Σ and K as in the definition. The *rooted tree* on Σ is the free monoid Σ^*, with root the empty sequence $\varnothing$; it is a metric space for the distance

$$\operatorname{dist}(\sigma, \tau) = |\sigma| + |\tau| - 2 \max \{n \in \mathbb{N} \mid \sigma_n = \tau_n\}.$$

The *natural action* of G is an action on Σ^* defined inductively by

$$(2\text{–}1)\qquad (\sigma_1\sigma_2\ldots\sigma_n)^g = (\sigma_1)^\pi(\sigma_2\ldots\sigma_n)^{g_{\sigma_1}} \text{ for } \sigma_1,\ldots,\sigma_n\in\Sigma,$$

where $\psi(g) = \ll g_1,\ldots,g_d\gg\pi$. By the condition $\bigcap \mathsf{Stab}_G(n) = \{1\}$, this action is faithful and G is residually finite. Note that $\mathsf{Stab}_G(n)$ is the fixator of Σ^n in this action.

Note that the action (2–1) gives a geometrical meaning to the branch structure of G that closely parallels the structure of the tree Σ^*. Indeed one may consider G as a group acting on the tree Σ^*; then the choice of a vertex σ of Σ^* and of a subgroup J of K determines a subgroup L_σ of K, namely the group of tree automorphisms of Σ^* that fix $\Sigma^*\setminus\sigma\Sigma^*$ and whose action on $\sigma\Sigma^*$ is that of an element of J on Σ^*. The choice of a subgroup J_σ for all $\sigma\in\Sigma^*$ determines a subgroup M of K, namely the closure of the L_σ associated to σ and J_σ when σ ranges over Σ^*.

This geometrical vision can also give pictorial descriptions of group elements:

Definition 2.3. Suppose G branches over K; let T be a transversal of K in G, and let U be a transversal of $\psi^{-1}(K^{\times d})$ in K. The *branch portrait* of an element $g\in G$ is a labeling of Σ^*, as follows: the root vertex $\varnothing$ is labeled by an element of TU, and all other vertices are labeled by an element of U.

Given $g\in G$, write first $g = kt$ with $k\in K$ and $t\in T$, then write

$$k = \psi^{-1}(k_1,\ldots,k_{\mathsf{d}})u_\varnothing,$$

and inductively $k_\sigma = \psi^{-1}(k_{\sigma 1},\ldots,k_{\sigma\mathsf{d}})u_\sigma$ for all $\sigma\in\Sigma^*$. Label the root vertex by $tu_\varnothing$ and then label each vertex $\sigma\neq\varnothing$ by u_σ.

There are uncountably many branch portraits, even for a countable branch group. We therefore introduce the following notion:

Definition 2.4. Let G be a branch group. Its *completion* $\overline{G}$ is the inverse limit

$$\operatorname*{proj\,lim}_{n\to\infty} G/\mathsf{Stab}_G(n).$$

This is also the closure in $\mathsf{Aut}\,\Sigma^*$ of G seen through its natural action (2–1).

Since $\overline{G}$ is closed in $\mathsf{Aut}\,\Sigma^*$ it is a profinite group, and thus is compact, and totally disconnected. If G has the *congruence subgroup property* [Grigorchuk 2000], meaning that all finite-index subgroups of G contain $\mathsf{Stab}_G(n)$ for some n, then $\overline{G}$ is also the profinite completion of G.

Lemma 2.5. *Let G be a branch group and $\overline{G}$ its completion. Then Definition 2.3 yields a bijection between the set of branch portraits and $\overline{G}$.*

We shall often simplify notation by omitting ψ from subgroup descriptions, as in statements like $\mathsf{Stab}_G(n)\le\mathsf{Stab}_G(n-1)^{\times d}$.

2.1. ***The Grigorchuk group*** $\boldsymbol{\mathfrak{G}}$**.** We shall consider more carefully three examples of branch groups in the sequel. The first example of a branch group was considered by Grigorchuk in 1980, and has appeared innumerable times in recent mathematics — the entire chapter VIII of [de la Harpe 2000] is devoted to it. It is defined as follows: it is a 4-generated group $\mathfrak{G}$ (with generators a, b, c, d), its map ψ is given by

$$\psi : \mathfrak{G} \hookrightarrow (\mathfrak{G} \times \mathfrak{G}) \rtimes \mathfrak{S}_2,$$
$$a \mapsto \ll 1, 1 \gg (1, 2), \quad b \mapsto \ll a, c \gg, \quad c \mapsto \ll a, d \gg, \quad \mapsto \ll 1, b \gg,$$

and its subgroup K is the normal closure of $[a, b]$, of index 16. Grigorchuk [1980; 1983] proved that $\mathfrak{G}$ is an intermediate-growth, infinite-torsion group. Its lower central series was computed in [Bartholdi and Grigorchuk 2000a], along with a description of its Lie algebra. We shall reproduce that result using a more general method.

2.2. ***The Gupta–Sidki group*** $\boldsymbol{\ddot{\Gamma}}$**.** This 2-generated group was introduced by Narain Gupta and Said Sidki in [Gupta and Sidki 1983], where they proved it to be an infinite torsion group. Later Sidki obtained a complete description of its automorphism group [Sidki 1987], along with information on its subgroups. It is a branch group with generators a, t, its map ψ is given by

$$\psi : \ddot{\Gamma} \hookrightarrow (\ddot{\Gamma} \times \ddot{\Gamma} \times \ddot{\Gamma}) \rtimes \mathfrak{A}_3,$$
$$a \mapsto \ll 1, 1, 1 \gg (1, 2, 3), \quad t \mapsto \ll a, a^{-1}, t \gg,$$

and its subgroup K is $\ddot{\Gamma}'$, of index 9.

It was recently proved in [Bartholdi 2000] that $\ddot{\Gamma}$ has intermediate growth, which increases its analogy with the Grigorchuk group mentioned above. An outstanding question was whether $\ddot{\Gamma}$ has finite width. Ana Cristina Vieira [1998; 1999] computed the first 9 terms of the lower central series and showed that there are all of rank at most 2. We shall shortly see, however, that $\ddot{\Gamma}$ has unbounded width.

The following lemma is straightforward:

Lemma 2.6. $\ddot{\Gamma}'/(\ddot{\Gamma}' \times \ddot{\Gamma}' \times \ddot{\Gamma}')$ *is isomorphic to* $C_3 \times C_3$*, generated by* $c = [a, t]$ *and* $u = [a, c]$*.*

Note finally that the notations in [Sidki 1987] are slightly different: his x is our a, and his y is our t. In [Vieira 1998] her $y^{[1]}$ is our u, and more generally her g_1 is our $\mathbb{0}(g)$ and her $g^{[1]}$ is our $\mathbb{2}(g)$. In [Bartholdi and Grigorchuk 2002], where a great deal of information on $\ddot{\Gamma}$ is gathered, the group is called $\overline{\overline{\overline{\Gamma}}}$.

2.3. ***The Fabrykowski–Gupta group*** $\boldsymbol{\Gamma}$**.** This other group is at first sight close to $\ddot{\Gamma}$: it is also a branch group, generated by two elements a, t. Its map ψ is given by

$$\psi : \Gamma \hookrightarrow (\Gamma \times \Gamma \times \Gamma) \rtimes \mathfrak{A}_3,$$
$$a \mapsto \ll 1, 1, 1 \gg (1, 2, 3), \quad t \mapsto \ll a, 1, t \gg,$$

and its subgroup K is Γ', of index 9.

This group was first considered in [Fabrykowski and Gupta 1991], where its growth was studied. In [Bartholdi and Grigorchuk 2002] it was proved that it is a branch group, and that its subgroup $L = \langle at, ta \rangle$ has index 3 and is torsion-free. In [Bartholdi 2000] another proof of the subexponential growth of Γ is given.

3. Lie algebras

We now describe the Lie algebras associated to the groups $\mathfrak{G}$, $\ddot{\Gamma}$ and Γ defined in the previous section. We start by considering a group G, and make the following hypotheses on G, which will be satisfied by $\mathfrak{G}$, $\ddot{\Gamma}$ and Γ:

(1) G is finitely generated by a set S;

(2) there is a prime p such that all $s \in S$ have order p.

Under these conditions, it follows from Lemma 1.5 that $\gamma_n(G)/\gamma_{n+1}(G)$ is a finite-dimensional vector space over $\mathbb{F}_p$, and therefore that $\mathscr{L}(G)$ is a Lie algebra over $\mathbb{F}_p$ that is finite at each dimension. Clearly the same property holds for the restricted algebra $\mathscr{L}_{\mathbb{F}_p}(G)$.

We propose the following notation for such algebras:

Definition 3.1. Let

$$\mathscr{L} = \bigoplus_{n \geq 1} \mathscr{L}_n$$

be a graded Lie algebra over $\mathbb{F}_p$, and choose a basis B_n of $\mathscr{L}_n$ for all $n \geq 1$. For $x \in \mathscr{L}_n$ and $b \in B_n$ denote by $\langle x | b \rangle$ the b-coefficient of x in the basis B_n.

The *Lie graph* associated to these choices is an abstract graph. Its vertex set is $\bigcup_{n \geq 1} B_n$, and each vertex $x \in B_n$ has degree n. Its edges are labeled as αx, with $x \in B_1$ and $\alpha \in \mathbb{F}_p$, and may only connect a vertex of degree n to a vertex of degree $n+1$. For all $x \in B_1$, $y \in B_n$ and $z \in B_{n+1}$, there is an edge labeled $\langle [x, y] | z \rangle x$ from y to z.

If $\mathscr{L}$ is a restricted algebra of $\mathbb{F}_p$, there are additional edges, labeled $\alpha \cdot p$ with $\alpha \in \mathbb{F}_p$, from vertices of degree n to vertices of degree pn. For all $x \in B_n$ and $y \in B_{pn}$, there is an edge labeled $\langle x^p | y \rangle \cdot p$ from x to y.

Edges labeled $0x$ are naturally omitted. Edges labeled $1x$ are simply written x.

There is some analogy between this definition and that of a Cayley graph — this topic will be developed in Section 4. The generators (in the Cayley sense) are simply chosen to be the $\mathtt{ad}(x)$ with x running through B_1, a basis of $G/[G, G]$.

A presentation for the $\mathscr{L}$ can also be read off its Lie graph. For every n, consider the set $\mathscr{W}$ of all words of length n over B_1. For a path π in the Lie graph, define its weight as the product of the labels on its edges. Each $w \in \mathscr{W}$ defines an element of $\mathscr{L}_n$, by summing the weights of all paths labeled w in the Lie graph. Let $\mathscr{R}_n$ be the set of all linear dependence relations among these words. Then $\mathscr{L}$ admits a presentation by generators and relations as

$$\mathscr{L} = \langle B_1 \mid \mathscr{R}_1, \mathscr{R}_2, \dots \rangle.$$

We give a few examples of Lie graphs. First, if G is abelian, its Lie graph has $\operatorname{rank}(G)$ vertices of weight 1 and no other vertices. If G is the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, its Lie ring is an algebra over $\mathbb{F}_2$, and the Lie graph of $\mathscr{L}(Q_8) = \mathscr{L}_{\mathbb{F}_2}(Q_8)$ is

$$\begin{tikzcd}
i \arrow[dr, "j"] & \\
& -1 \\
j \arrow[ur, "i"'] &
\end{tikzcd}$$

3.1. ***The infinite dihedral group.*** As another example, let G be the infinite dihedral group $D_\infty = \langle a, b \mid a^2, b^2 \rangle$. Then $\gamma_n(G) = \left\langle (ab)^{2^{n-1}} \right\rangle$ for all $n \geq 2$, and its Lie ring is again a Lie algebra over $\mathbb{F}_2$, with Lie graph

$$\begin{tikzcd}
a \arrow[dr, "b"] & & & & \\
& (ab)^2 \arrow[r, "{a,b}"] & (ab)^4 \arrow[r, "{a,b}"] & (ab)^8 \arrow[r, dotted, "{a,b}"] & {} \\
b \arrow[ur, "a"'] & & & &
\end{tikzcd}$$

The lower 2-central series of G is different: $G_{2^n} = G_{2^n+1} = \cdots = G_{2^{n+1}-1} = \gamma_{n+1}(G)$, so the Lie graph of $\mathscr{L}_{\mathbb{F}_2}(G)$ is

$$\begin{tikzcd}
a \arrow[dr, "b"] & & & & \\
& (ab)^2 \arrow[r, "\cdot 2"] & (ab)^4 \arrow[r, "\cdot 2"] & (ab)^8 \arrow[r, dotted, "\cdot 2"] & {} \\
b \arrow[ur, "a"'] & & & &
\end{tikzcd}$$

3.2. ***The free group.*** Consider, as an example producing exponential growth, the free group F_r and its Lie algebra $\mathscr{L}$; this is a free Lie algebra of rank r. Using Theorem 1.6 and Möbius inversion, we get

$$\dim_{\mathbb{Q}}\big(\gamma_n(F_r)/\gamma_{n+1}(F_r)\otimes\mathbb{Q}\big)=\#\{u\in\mathcal{M}\mid\deg u=n\}=\frac{1}{n}\sum_{d|n}\mu_{n/d}r^d\precsim r^n,$$

where μ is the Möbius function; therefore $\mathsf{growth}(\overline{\mathbb{Q}F_r})\leq 1/(1-r\hbar)$. Recall that

$$\mathsf{growth}(F_r)=\frac{1+\hbar}{1-(2r-1)\hbar},$$

so the group growth rate can be strictly larger than the algebra growth rate in Proposition 1.10.

It is an altogether different story to find explicitly a basis of $\mathscr{L}$. Pick a basis X of F_r; its image in $\mathscr{L}_1\cong\mathbb{Z}^r$ is a generating set of $\mathscr{L}$, still written X. A *Hall set* is a linearly ordered set of nonassociative words $\mathcal{M}$ with $X\subset\mathcal{M}$ and

$$[u,v]\in\mathcal{M}\text{ if and only if }u<v\in\mathcal{M}\text{ and }(u\in X\text{ or }u=[p,q],q\geq v);$$

furthermore one requires $[u,v]<v$. Note that an order on the nonassociative words uniquely defines a corresponding Hall set.

There are many examples of Hall sets, and for each Hall set $\mathcal{M}$ the set $\{u\in\mathcal{M}\mid |u|=r\}$ is a basis of the abelian group $\gamma_n(F_r)/\gamma_{n+1}(F_r)$. For example, the *Hall basis* [Hall 1950] is the linearly ordered set $\mathcal{M}$ having as maximal elements X in an arbitrary order, and such that $u<v$ in $\mathcal{M}$ whenever $\deg u>\deg v$. It contains then all $[x,y]$ with $x,y\in X$ and $x>y$; then all $\big[[u,v],w\big]$ whenever $[u,v]<w\leq v$ and $u,v,w\in\mathcal{M}$.

Another basis, more computationally efficient (it is a Lie algebra equivalent of Gröbner bases), is the *Lyndon–Shirshov basis* [Širšov 1962; Lothaire 1990; Reutenauer 1993]. It is defined as follows: order X arbitrarily; on the free monoid X^* put the lexicographical ordering: $u\leq uv$, and $uxv<uyw$ for all $u,v,w\in X^*$ and $x<y\in X$. A nonempty word $w\in X^*$ is a *Lyndon–Shirshov word* if for any nontrivial factorization $w=uv$ we have $w<v$. If furthermore we insist that v be $<$-minimal, then u and v are again Lyndon–Shirshov words. For a Lyndon–Shirshov word w, define its *bracketing* $B(w)$ inductively as follows: if $w\in X$ then $B(w)=w$. If $w=uv$ with v minimal then $B(w)=[B(u),B(v)]$. Then $\{B(w)\}$ is a basis of $\mathscr{L}$.

From our perspective, an optimal basis B would consist only of left-ordered commutators, and be prefix-closed, i.e., be such that $[u,x]\in B$ implies $u\in B$; then indeed the Lie algebra structure of an arbitrary Lie algebra would be determined $\mathsf{ad}(u)$ for all $u\in B$, and therefore would be a tree in the case of a free Lie algebra. Kukin announced in [Kukin 1978] a construction of such bases, but his proof does not appear to be altogether complete [Blessenohl and Laue 1993], and the problem of construction of a left-ordered basis seems to be considered open.

3.3. ***The lamplighter group.*** As another example, consider the *lamplighter group* $G = C_2 \wr \mathbb{Z}$, with a generating C_2 and t generating $\mathbb{Z}$. Define the elements

$$a_n = \prod_{i=0}^{n-1} a^{(-1)^i \binom{n-1}{i} t^i} = at^{-1}a^{-(n-1)}t^{-1} \dots a^{(-1)^{n-1}} t^{n-1}$$

of G. The Lie algebra $\mathscr{L}_{\mathbb{F}_2}(G)$ is as follows:

$$\begin{tikzcd}
a \arrow[r, "t"] & a_2 \arrow[r, "t"] & a_3 \arrow[r, "t"] & a_4 \arrow[r, "t"] & a_5 \arrow[r, "t"] & a_6 \arrow[r, "t"] & a_7 \arrow[r, "t"] & a_8 \arrow[r, "t"] & a_9 \arrow[r, dotted, "t"] & {} \\
t \arrow[r, "\cdot 2"'] \arrow[ur, "a"] & t^2 \arrow[rr, "\cdot 2"'] \arrow[ur, "a"] & & t^4 \arrow[rrrr, "\cdot 2"'] \arrow[ur, "a"] & & & & t^8 \arrow[rr, dotted, "\cdot 2"'] \arrow[ur, "a"] & & {}
\end{tikzcd}$$

Note that $\mathscr{L}_{\mathbb{F}_2}(G)$ has bounded width, while G has exponential growth! This shows that in Proposition 1.10 the group growth rate can be exponential while the algebra growth rate is polynomial.

3.4. ***The Nottingham group.*** As a final example, we give the Lie graph of the Nottingham group's Lie algebra [Jennings 1954; Camina 2000]. Recall that for odd prime p the Nottingham group $J(p)$ is the group of all formal power series

$$\hbar + \sum_{i>1} a_i \hbar^i \in \mathbb{F}_p[[\hbar]],$$

with composition (i.e., substitution) as binary operation. The lower central series is given by

$$J_n = \{\hbar + \sum_{i > \lceil \frac{np-1}{p-1} \rceil} a_i \hbar^i\},$$

and a basis of $\mathscr{L}$ is $\{f_i = \hbar(1+\hbar^i)\}_{i\geq 1}$, where f_i has degree $\lfloor ((p-1)i+1)/p \rfloor$. As a basis of J_1/J_2, we take $B_1 = \{x = \hbar + \hbar^2 + \hbar^3,\ y = \hbar + \hbar^3\}$. The commutations are given by

$$[f_i, x] = (i-1) f_{i+1}, \qquad [f_i, y] = \begin{cases} -2 f_{i+2} & \text{if } i \equiv 0 \bmod p \\ -f_{i+2} & \text{if } i \equiv 1 \bmod p \\ 0 & \text{otherwise,} \end{cases}.$$

This gives a Lie graph with a diamond structure [Caranti 1997]:

$$\begin{tikzcd}
1 & 2 & 3 & \cdots & p-2 & p-1 & p & p+1 & p+2 & \\
x \arrow[dr, "-y"] & & & & & & f_{p+1} \arrow[dr, "-y"] & & & \\
& f_3 \arrow[r, "2x"] & f_4 \arrow[rr, dotted] & & f_{p-1} \arrow[r, "-2x"] & f_p \arrow[ur, "-x"] \arrow[dr, "-2y"'] & & f_{p+3} \arrow[r, "2x"] & f_{p+4} \arrow[r, dotted] & {} \\
y \arrow[ur, "x"'] & & & & & & f_{p+2} \arrow[ur, "x"'] & & &
\end{tikzcd}$$

3.5. *The tree automorphism group's pro-p-Sylow* $\mathsf{Aut}_p(\Sigma^*)$. We start by considering a typical example of branch group. Let p be prime; write $p' = p - 1$ for notational simplicity. Let Σ be the p-letter alphabet $\{1, \dots, \mathsf{p}\}$, and let x_n, for $n \in \mathbb{N}$, be the p-cycle permuting the first p branches at level $n+1$ in the tree Σ^*. Therefore x_0 acts just below the root vertex, and $x_{n+1} = \ll x_n, 1, \dots, 1 \gg$ for all n.

For all $n \in \mathbb{N}$ we define $G_n = \mathsf{Aut}_p(\Sigma^*)$ as the group generated by $\{x_0, \dots, x_{n-1}\}$, and $G = \langle x_0, x_1, \dots \rangle$. Clearly $G = \operatorname{inj\,lim} G_n$, while its closure is $\overline{G} = \operatorname{proj\,lim} G_n$. Note that G_n is a p-Sylow of $\mathfrak{S}_{p^n}$, and $\overline{G}$ is a pro-p-Sylow of $\mathsf{Aut}(\Sigma^*)$.

Lemma 3.2. *$G = G \wr C_p$; therefore G is a regular branch group over itself.*

Proof. The subgroup $\langle x_1, x_2, \dots \rangle$ of G is isomorphic to G through $x_i \mapsto x_{i-1}$, and its p conjugates under powers of x_0 commute, since they act on disjoint subtrees. □

Lev Kaloujnine [1948] described the lower central series of G_n, using his notion of a *tableau*. Our purpose here shall be to describe the Lie algebra of G_n (and therefore G and $\overline{G}$) using our more geometric approach. Let us just mention that in Kaloujnine's theory of tableaux his polynomials $x_1^{e_1} \dots x_n^{e_n}$ correspond to our $\mathbb{e}_1 \dots \mathbb{e}_n(x_0)$.

Lemma 3.3. *For $u, v \in G$ and $X, Y \in \{\mathbb{0}, \dots, \mathbb{p}'\}^n$ we have*

$$[X(u), Y(v)] \equiv (X_1 + Y_1 - \mathbb{p}') \dots (X_n + Y_n - \mathbb{p}')([u, v])^{\prod_{i=1}^n (-1)^{p'-Y_i} \binom{X_i}{p'-Y_i}},$$

modulo terms in $\big[[X(u), Y(v)], G\big]$.

Proof. The proof follows by induction, and we may suppose $n = 1$ without loss of generality. Multiplying by terms in $\big[[X(u), Y(v)], G\big]$, we may assume $Y(v)$ by some element acting only on the last Y_1 subtrees below the root vertex. Then

$$\begin{aligned}
[X(u), Y(v)] &\equiv [\ll u, \dots, u^{(-1)^{X_1}}, 1, \dots, 1 \gg, \ll 1, \dots, 1, v, \dots, v^{(-1)^{Y_1}} \gg] \\
&= \ll [u, 1], \dots, [u^{(-1)^{p'-Y_1}\binom{p}{p'-Y_1}}, v], \dots, \\
&\qquad\qquad [u^{(-1)^{X_1}}, v^{(-1)^{X_1}\binom{p}{X_1}}], \dots, [1, v] \gg \\
&\equiv (X + Y - \mathbb{p}')([u, v])^{(-1)^{p'-Y_1}\binom{p}{p'-Y_1}}.
\end{aligned}$$

□

Theorem 3.4. *Consider the following Lie graph: its vertices are the symbols X for all words $X \in \{\mathbb{0}, \dots, \mathbb{p}'\}^*$, including the empty word λ. Their degrees are given by*

$$\deg X_1 \dots X_n = 1 + \sum_{i=1}^{n} X_i p^{i-1}.$$

For all $m > n \geq 0$ and all choices of X_i, there is an arrow labeled $\mathbb{0}^n$ from

$$\mathbb{p}'^n X_{n+1} \dots X_m$$

to

$$\mathbb{0}^n (X_{n+1} + \mathbb{1}) X_{n+2} \dots X_m,$$

and an arrow labeled $\mathbb{0}^m$ from $\mathbb{p}'^n$ to $\mathbb{0}^n \mathbb{1} \mathbb{0}^{m-n-1}$.

Then the resulting graph is the Lie graph of $\mathscr{L}(G)$ and of $\mathscr{L}_{\mathbb{F}_p}(G)$.

The subgraph spanned by all words of length up to $n-1$ is the Lie graph of $\mathscr{L}(G_n)$ and of $\mathscr{L}_{\mathbb{F}_p}(G_n)$.

Proof. We interpret X in the Lie graph as $X(x_0)$ in G. The generator x_n is then $\mathbb{0}^n(x_0)$. By Lemma 3.3, the adjoint operators $\mathsf{ad}(x_n)$ correspond to the arrows

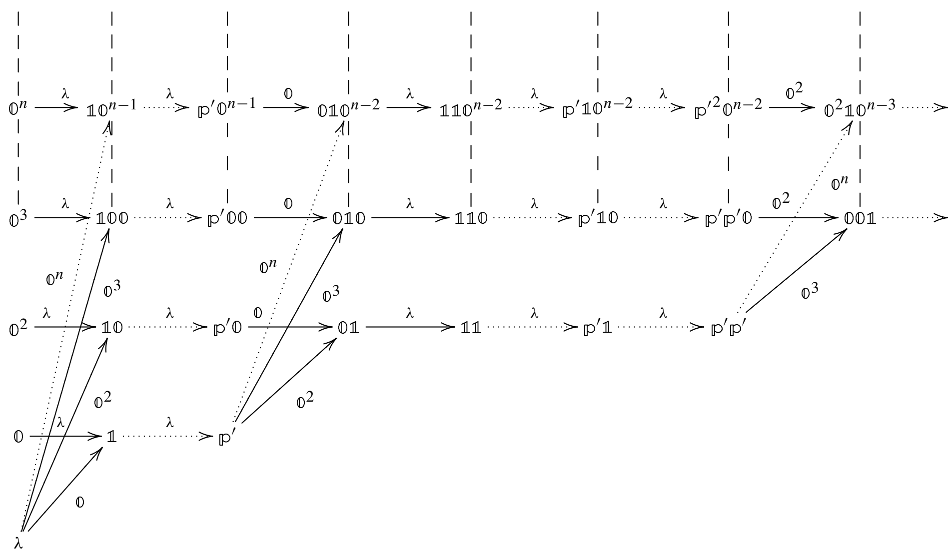


Figure 1. The beginning of the Lie graph of $\mathscr{L}(G)$ for G the p-Sylow of $\mathsf{Aut}(\Sigma^*)$.

labeled $\mathbb{0}^n$. The arrows connect elements whose degree differ by 1, so the degree of the element $X(x_0)$ is $\deg X$ as claimed.

The power maps $g \mapsto g^p$ are all trivial on the elements $X(x_0)$, so the Lie algebra and restricted Lie algebra coincide.

The elements $X(x_0)$ for $|X| \geq n$ belong to $\mathsf{Stab}_G(n)$, hence are trivial in G_n. $\square$

3.6. ***The Grigorchuk group $\mathfrak{G}$.*** We give an explicit description of the Lie algebra of $\mathfrak{G}$, and compute its Hilbert–Poincaré series. These results were obtained in [Bartholdi and Grigorchuk 2000a], and partly before in [Rozhkov 1996].

Set $x = [a, b]$. Then $\mathfrak{G}$ is branch over $K = \langle x\rangle^{\mathfrak{G}}$, and $K/(K \times K)$ is cyclic of order 4, generated by x.

Extend the generating set of $\mathfrak{G}$ to a formal alphabet

$$S = \left\{a, b, c, d, \left\{\begin{smallmatrix} b \\ c \end{smallmatrix}\right\}, \left\{\begin{smallmatrix} c \\ d \end{smallmatrix}\right\}, \left\{\begin{smallmatrix} d \\ b \end{smallmatrix}\right\}\right\}.$$

Define the transformation σ on words in S^* by

$$\sigma(a) = a\left\{\begin{smallmatrix} b \\ c \end{smallmatrix}\right\}a, \quad \sigma(b) = d, \quad \sigma(c) = b, \quad \sigma(d) = c,$$

extended to subsets by

$$\sigma\left\{\begin{smallmatrix} x \\ y \end{smallmatrix}\right\} = \left\{\begin{smallmatrix} \sigma x \\ \sigma y \end{smallmatrix}\right\}.$$

Note that for any fixed $g \in G$, all elements $h \in \mathsf{Stab}_{\mathfrak{G}}(1)$ such that $\psi(h) = \langle\!\langle g, *\rangle\!\rangle$ are obtained by picking a letter from each set in $\sigma(g)$. This motivates the definition of S.

Theorem 3.5. *Consider the following Lie graph: its vertices are the symbols $X(x)$ and $X(x^2)$, for words $X \in \{\mathbb{0}, \mathbb{1}\}^*$. Their degrees are given by*

$$\deg X_1 \ldots X_n(x) = 1 + \sum_{i=1}^{n} X_i 2^{i-1} + 2^n,$$

$$\deg X_1 \ldots X_n(x^2) = 1 + \sum_{i=1}^{n} X_i 2^{i-1} + 2^{n+1}.$$

There are four additional vertices: a, b, d of degree 1, and $[a, d]$ of degree 2.

Define the arrows as shown below, where an arrow labeled $\left\{\begin{smallmatrix} x \\ y \end{smallmatrix}\right\}$ or x, y stands for two arrows, labeled x and y, and the arrows labeled c are there to expose the *symmetry of the graph* (*indeed* $c = bd$ *is not in our chosen basis of* $G/[G, G]$):

$$\begin{aligned}
&a \xrightarrow{b,c} x &\qquad& a \xrightarrow{c,d} [a, d]\\
&b \xrightarrow{a} x && d \xrightarrow{a} [a, d]\\
&x \xrightarrow{a,b,c} x^2 && x \xrightarrow{c,d} \mathbb{0}(x)\\
&[a, d] \xrightarrow{b,c} \mathbb{0}(x) && \mathbb{0}* \xrightarrow{a} \mathbb{1}*\\
&\mathbb{1}^n(x) \xrightarrow{\sigma^n\left\{\substack{c\\d}\right\}} \mathbb{0}^{n+1}(x) && \mathbb{1}^n(x) \xrightarrow{\sigma^n\left\{\substack{b\\d}\right\}} \mathbb{0}^n(x^2)\\
&\mathbb{1}^n\mathbb{0}* \xrightarrow{\sigma^n\left\{\substack{c\\d}\right\}} \mathbb{0}^n\mathbb{1}* \quad \textit{if } n \geq 1.
\end{aligned}$$

Then the resulting graph is the Lie graph of $\mathscr{L}(\mathfrak{G})$. *A slight modification gives the Lie graph of* $\mathscr{L}_{\mathbb{F}_2}(\mathfrak{G})$: *the degree of* $X_1 \dots X_n(x^2)$ *is then* $2 \deg X_1 \dots X_n(x)$; *and the* 2*-mappings are given by*

$$X(x) \xrightarrow{\cdot 2} X(x^2),$$

$$\mathbb{1}^n(x^2) \xrightarrow{\cdot 2} \mathbb{1}^{n+1}(x^2).$$

The subgraph spanned by $a, t, X_1 \dots X_i(x)$ *for* $i \leq n-2$ *and* $X_1 \dots X_i(x^2)$ *for* $i \leq n-4$ *is the Lie graph associated to the finite quotient* $\mathfrak{G}/\operatorname{Stab}_{\mathfrak{G}}(n)$.

Figure 2 describes as Lie graphs the top of the Lie algebras associated to $\mathfrak{G}$. Note the infinite path, labeled by

$$\begin{aligned}
&\left\{\substack{c\\d}\right\}a\sigma(\left\{\substack{c\\d}\right\}a)\sigma^2(\left\{\substack{c\\d}\right\}a)\dots\\
&\qquad = \left\{\substack{c\\d}\right\}a\left\{\substack{b\\c}\right\}a\left\{\substack{b\\c}\right\}a\left\{\substack{b\\d}\right\}a\left\{\substack{b\\c}\right\}a\left\{\substack{b\\d}\right\}a\left\{\substack{b\\c}\right\}a\left\{\substack{c\\d}\right\}a\left\{\substack{b\\c}\right\}a\dots;
\end{aligned}$$

it is the same as the labeling of the parabolic space of $\mathfrak{G}$ — see Section 4 and [Bartholdi and Grigorchuk 2002].

The proof requires the computation, given a term N of a central series and a generator $s \in \{a, b, c, d\}$, of $[N, x]$ modulo $[N, G, G]$. We do slightly better in the following lemma — this will be useful in Section 5, where we describe all normal subgroups of G. For that purpose we introduce a symbol ${}^{\mathbb{0}}_{\mathbb{1}}(x) = \mathbb{0}(x)\mathbb{1}(x)^{-1}$. We then have

$$\mathbb{0}(x) = \ll x, 1\gg, \quad \mathbb{1}(x) = \ll x, x^{-1}\gg, \quad {}^{\mathbb{0}}_{\mathbb{1}}(x) = \ll 1, x\gg.$$

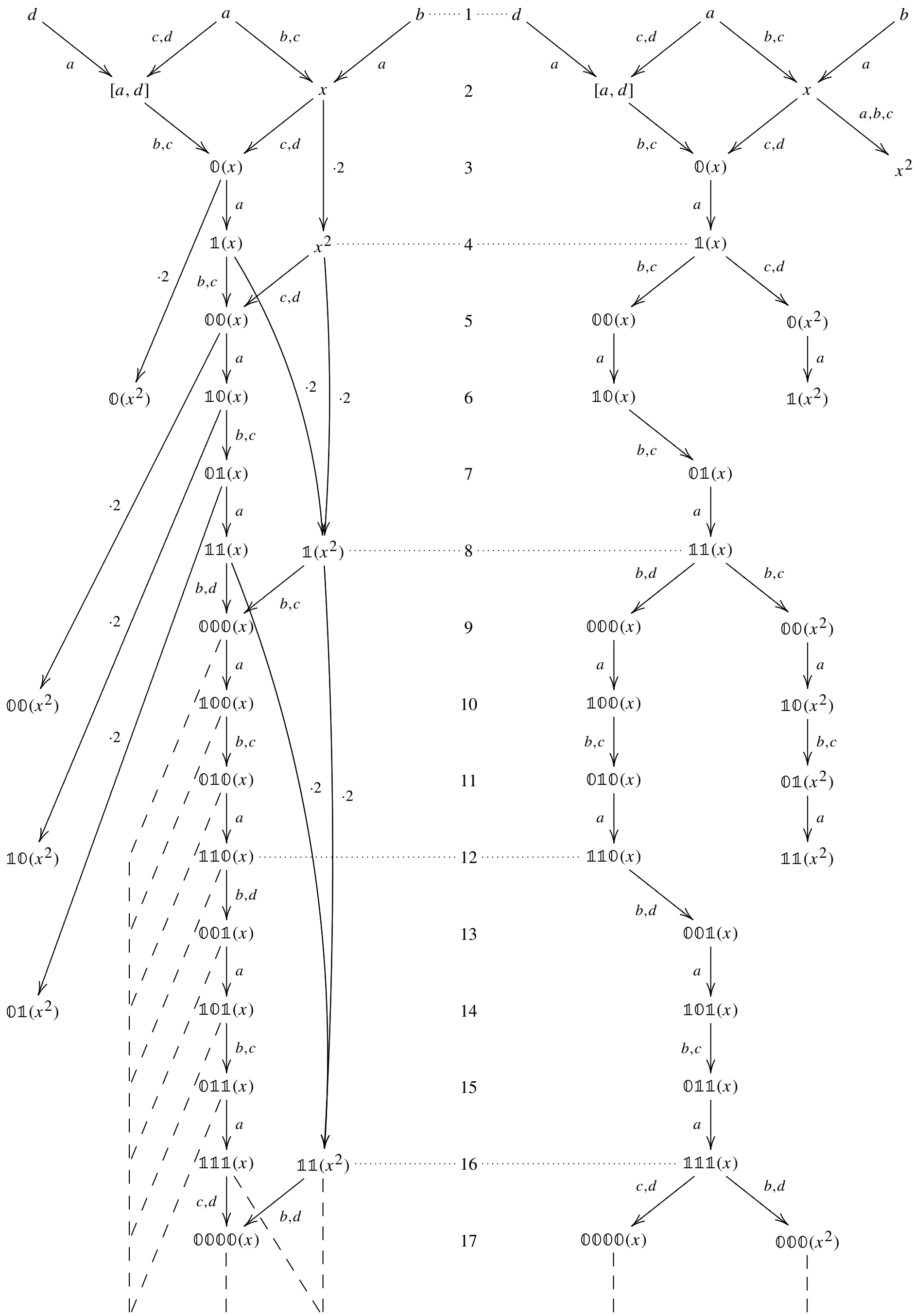


Figure 2. The beginning of the Lie graphs of $\mathscr{L}_{\mathbb{F}_2}(\mathfrak{G})$ (left) and $\mathscr{L}(\mathfrak{G})$ (right).

Lemma 3.6. *Assume N is a normal subgroup containing the left-hand operand of the commutators below. Then modulo $[N,G]'$ we have*

$$
\begin{aligned}
&[\mathbb{0}X,a]=\mathbb{1}X, && [\mathbb{1}X,a]=\mathbb{1}X^2,\\
&[\mathbb{0}X,b]=\mathbb{0}[X,a], && [\mathbb{1}X,b]=\mathbb{0}[X,a]+{}^{\mathbb{0}}_{\mathbb{1}}[X,c],\\
&[\mathbb{0}X,c]=\mathbb{0}[X,a], && [\mathbb{1}X,c]=\mathbb{0}[X,a]+{}^{\mathbb{0}}_{\mathbb{1}}[X,d],\\
&[\mathbb{0}X,d]=1, && [\mathbb{1}X,d]={}^{\mathbb{0}}_{\mathbb{1}}[X,b],\\
&[x,a]=x^2, && [x^2,a]=x^4=\mathbb{1}(x^2+\mathbb{1}x),\\
&[x,b]=x^2, && [x^2,b]=\mathbb{1}(x^2+\mathbb{1}x),\\
&[x,c]=\mathbb{0}(x)+x^2, && [x^2,c]=\mathbb{0}(x^2+\mathbb{0}x)+\mathbb{1}(x^2+\mathbb{1}x),\\
&[x,d]=\mathbb{0}(x), && [x^2,d]=\mathbb{0}(x^2+\mathbb{0}x).
\end{aligned}
$$

Proof. Direct computation, using the decompositions $\psi(b)=(a,c)=\mathbb{0}(a)\cdot{}^{\mathbb{0}}_{\mathbb{1}}(c)$ etc. and linearizing. □

Proof of Theorem 3.5. The proof proceeds by induction on length of words, or, what amounts to the same, on depth in the lower central series.

First, the assertion is checked "manually" up to degree 3. The details of the computations are the same as in [Bartholdi and Grigorchuk 2000a].

We claim that for all words X, Y with $\deg Y(x) > \deg X(x)$ we have $Y(x) \in \langle X(x)\rangle^{\mathfrak{G}}$, and similarly $Y(x^2) \in \langle X(x^2)\rangle^{\mathfrak{G}}$. The claim is verified by induction on $\deg X$.

We then claim that for any nonempty word X, either $\mathsf{ad}(a)X(*)=0$ (if X starts with $\mathbb{1}$) or $\mathsf{ad}(v)X(*)=0$ for $v\in\{b,c,d\}$ (if X starts with $\mathbb{0}$). Again this holds by induction.

We then prove that the arrows are as described above; this follows from Lemma 3.6. For instance,

$$
\mathsf{ad}(\sigma^n\left\{\begin{smallmatrix}c\\d\end{smallmatrix}\right\})\mathbb{1}^n\mathbb{0}*=\begin{cases}\big(\mathsf{ad}(\sigma^n\left\{\begin{smallmatrix}d\\b\end{smallmatrix}\right\})\mathbb{1}^{n-1}\mathbb{0}*,\ \mathsf{ad}(\left\{\begin{smallmatrix}a\\1\end{smallmatrix}\right\})\mathbb{1}^{n-1}\mathbb{0}*\big) & \\ \qquad =\mathbb{0}\,\mathsf{ad}(\sigma^{n-1}\left\{\begin{smallmatrix}c\\d\end{smallmatrix}\right\})\mathbb{1}^{n-1}\mathbb{0}*=\mathbb{0}^n\mathbb{1}* & \text{if } n\geq 2,\\ \big(\mathsf{ad}(\left\{\begin{smallmatrix}b\\c\end{smallmatrix}\right\})\mathbb{0}*,\ \mathsf{ad}(a)\mathbb{0}*\big)=\mathbb{0}\mathbb{1}* & \text{if } n=1.\end{cases}
$$

Finally we check that the degrees of all basis elements are as claimed. For that purpose, we first check that the degree of an arrow's destination is always one more than the degree of its source. Then fix a word $X(*)$ and consider the largest n such that $X(*)$ belongs to $\gamma_n(\mathfrak{G})$. There is an expression of $X(*)$ as a product of n-place commutators on elements of $\mathfrak{G}\setminus[\mathfrak{G},\mathfrak{G}]$, and therefore in the Lie graph there is a family of paths starting at some element of B_1 and following $n-1$ arrows to reach $X(*)$. This implies that the degree of $X(*)$ is n, as required.

The modification giving the Lie graph of $\mathscr{L}_{\mathbb{F}_2}(\mathfrak{G})$ is justified by the fact that in $\mathscr{L}(\mathfrak{G})$ we always have $\deg X(x^2) \leq 2\deg X(x)$, so the element $X(x^2)$ appears always last as the image of $X(x)$ through the square map. The degrees are modified accordingly. Now $X(x^2) = X\mathbb{1}(x^2)$, and $2\deg X\mathbb{1}(x) \geq 4\deg X(x)$, with equality only when $X = \mathbb{1}^n$. This gives an additional square map from $\mathbb{1}^n(x^2)$ to $\mathbb{1}^{n+1}(x^2)$, and requires no adjustment of the degrees. □

Corollary 3.7. *Define the polynomials*

$$Q_2 = -1 - \hbar,$$
$$Q_3 = \hbar + \hbar^2 + \hbar^3,$$
$$Q_n(\hbar) = (1+\hbar)Q_{n-1}(\hbar^2) + \hbar + \hbar^2 \quad \textit{for } n \geq 4.$$

Then Q_n is a polynomial of degree $2^{n-1}-1$, and the first $2^{n-3}-1$ coefficients of Q_n and Q_{n+1} coincide. The termwise limit $Q_\infty = \lim_{n\to\infty} Q_n$ therefore exists.

The Hilbert–Poincaré series of $\mathscr{L}(\mathfrak{G}/\operatorname{Stab}_{\mathfrak{G}}(n))$ is $3\hbar + \hbar^2 + \hbar Q_n$, and the Hilbert–Poincaré series of $\mathscr{L}(\mathfrak{G})$ is $3\hbar + \hbar^2 + \hbar Q_\infty$.

The Hilbert–Poincaré series of $\mathscr{L}_{\mathbb{F}_2}(\mathfrak{G})$ is $3 + (2\hbar + \hbar^2)/(1-\hbar^2)$.

As a consequence, $\mathfrak{G}/\operatorname{Stab}_{\mathfrak{G}}(n)$ is nilpotent of class 2^{n-1}, and $\mathfrak{G}$ has finite width.

Proof. Consider the sequence of coefficients of Q_n. They are, in condensed form,

$$1, 2^{2^0}, 1^{2^0}, 2^{2^1}, 1^{2^1}, \ldots, 2^{n-4}, 1^{n-4}, 1^{n-2}.$$

The i-th coefficient is 2 if there are $X(x)$ and $Y(x^2)$ of degree i in $\mathfrak{G}/\operatorname{Stab}_{\mathfrak{G}}(n)$, and is 1 if there is only $X(x)$. All conclusions follow from this remark. □

3.7. ***The Gupta–Sidki group $\ddot{\Gamma}$.*** We now give an explicit description of the Lie algebra of $\ddot{\Gamma}$, and compute its Hilbert–Poincaré series.

Introduce the sequence of integers

$$\alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_n = 2\alpha_{n-1} + \alpha_{n-2} \quad \text{for } n \geq 3,$$

and set $\beta_n = \sum_{i=1}^n \alpha_i$. One has

$$\alpha_n = \frac{1}{2\sqrt{2}}\big((1+\sqrt{2})^n - (1-\sqrt{2})^n\big),$$
$$\beta_n = \frac{1}{4}\big((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1} - 2\big).$$

The first few values are

n	1	2	3	4	5	6	7	8
α_n	1	2	5	12	29	70	169	398
β_n	1	3	8	20	49	119	288	686

Theorem 3.8. *In $\ddot{\Gamma}$ write $c=[a,t]$ and $u=[a,c]=\mathbb{2}(t)$. Consider the following Lie graph: its vertices are the symbols $X_1 \dots X_n(x)$ with $X_i \in \{\mathbb{0},\mathbb{1},\mathbb{2}\}$ and $x \in \{c,u\}$. Their degrees are given by*

$$\deg X_1 \dots X_n(c) = 1 + \sum_{i=1}^{n} X_i\alpha_i + \alpha_{n+1},$$

$$\deg X_1 \dots X_n(u) = 1 + \sum_{i=1}^{n} X_i\alpha_i + 2\alpha_{n+1}.$$

There are two additional vertices, labeled a and t, of degree 1.

Define the arrows as follows:

$$a \xrightarrow{-t} c \qquad c \xrightarrow{t} \mathbb{0}(c)$$

$$t \overset{a}{\dashrightarrow} c \qquad c \overset{a}{\dashrightarrow} u$$

$$u \xrightarrow{t} \mathbb{1}(c)$$

$$\mathbb{0}* \overset{a}{\dashrightarrow} \mathbb{1}* \qquad \mathbb{1}* \overset{a}{\dashrightarrow} \mathbb{2}*$$

$$\mathbb{2}* \xrightarrow{t} \mathbb{0} \quad \text{whenever } * \xrightarrow{t} \#$$

$$\mathbb{2}(c) \xrightarrow{t} \mathbb{1}(u) \qquad \mathbb{1}(c) \xrightarrow{-t} \mathbb{0}(u)$$

$$\mathbb{1}\mathbb{0}* \xrightarrow{-t} \mathbb{0}\mathbb{1}* \qquad \mathbb{1}\mathbb{1}* \xrightarrow{-t} \mathbb{0}\mathbb{2}*$$

$$\mathbb{2}\mathbb{0}* \xrightarrow{t} \mathbb{1}\mathbb{1}* \qquad \mathbb{2}\mathbb{1}* \xrightarrow{t} \mathbb{1}\mathbb{2}*$$

(The last three lines can be replaced by the rules $\mathbb{2}* \xrightarrow{t} \mathbb{1}\#$ and $\mathbb{1}* \xrightarrow{-t} \mathbb{0}\#$ for all arrows $* \xrightarrow{a} \#$.)

Then the resulting graph is the Lie graph of $\mathscr{L}(\ddot{\Gamma})$. It is also the Lie graph of $\mathscr{L}_{\mathbb{F}_3}(\ddot{\Gamma})$, with the only nontrivial cube maps given by

$$\mathbb{2}^n(c) \xrightarrow{\cdot 3} \mathbb{2}^n\mathbb{0}\mathbb{0}(c), \qquad \mathbb{2}^n(c) \xrightarrow{\cdot 3} \mathbb{2}^n\mathbb{1}(u).$$

The subgraph spanned by a, t, the $X_1 \dots X_i(c)$ for $i \le n-2$ and the $X_1 \dots X_i(u)$ for $i \le n-3$ is the Lie graph associated to the finite quotient $\ddot{\Gamma}/\mathsf{Stab}_{\ddot{\Gamma}}(n)$.

Proof. We perform the computations in the completion of $\ddot{\Gamma}$, still written $\ddot{\Gamma}$. With Lemma 2.5 in mind, $\ddot{\Gamma}'$ is the subgroup generated by all $X(c)$ and $X(u)$, for $X \in \{\mathbb{0},\mathbb{1},\mathbb{2}\}^*$.

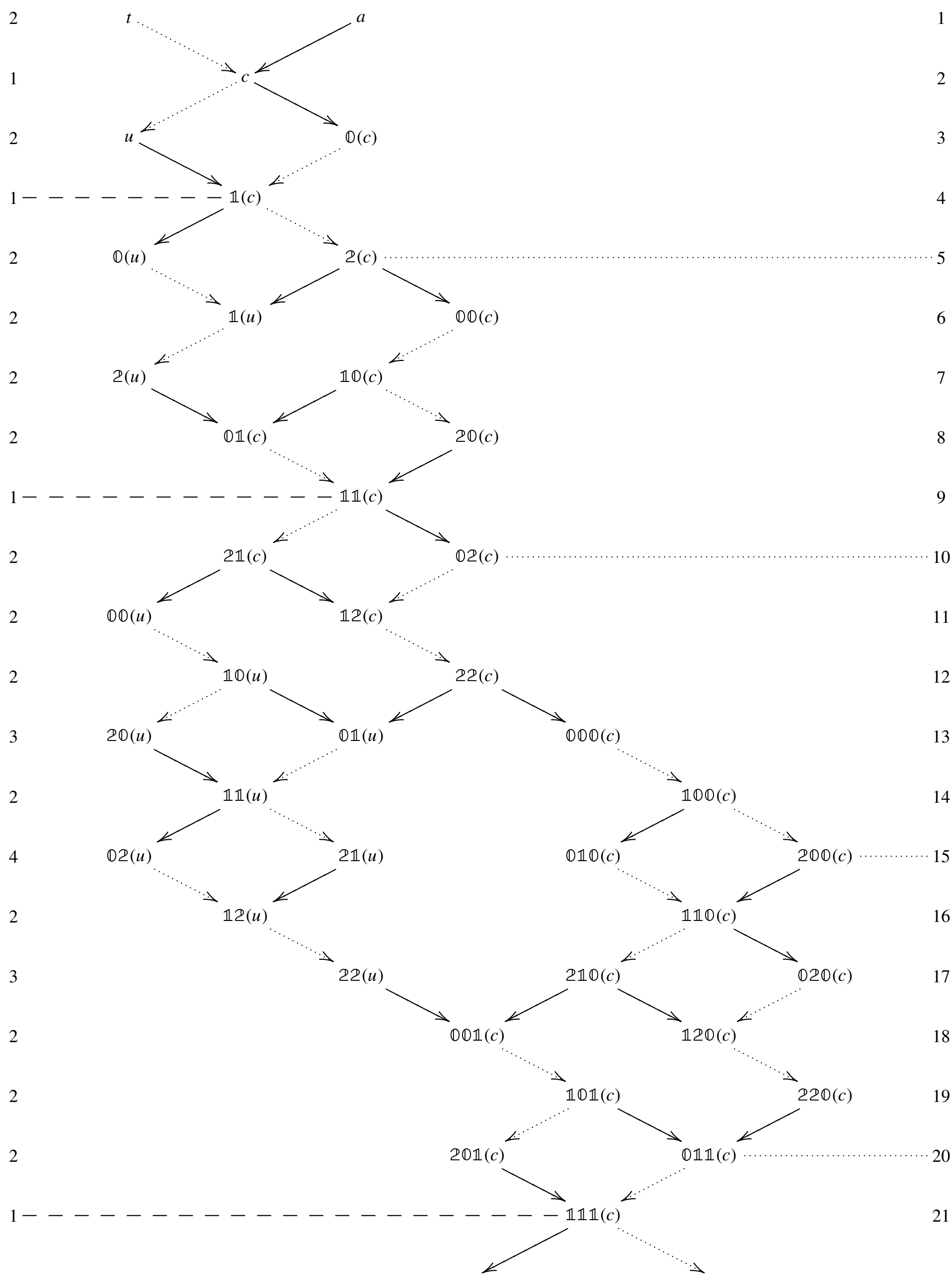


Figure 3. The beginning of the Lie graph of $\mathscr{L}(\ddot{\Gamma})$. The generator $\mathsf{ad}(t)$ is shown by plain arrows, and the generator $\mathsf{ad}(a)$ is shown by dotted arrows. The left column indicates the dimensions of $\mathscr{L}_n$.

We claim inductively that if $X_i \geq Y_i$ at all positions i, then $X(c) \in \langle Y(c)\rangle^{\ddot{\Gamma}}$, and similarly for u. Therefore some terms may be neglected in the computations of brackets.

Now we compute $\mathsf{ad}(x)y$ for $x, y \in \{a, t, c, u\}$. Here $\equiv$ means some terms of greater degree have been neglected:

$$\begin{aligned}
&[a, \mathbb{0}*] = \mathbb{1}*, \quad [a, \mathbb{1}*] = \mathbb{2}*, \quad [a, \mathbb{2}*] = 1 \text{ by definition,}\\
&[t, \mathbb{0}*] = [\ll a, a^{-1}, t\gg, \ll *, 1, 1\gg] = \ll [a, *], 1, 1\gg = \mathbb{0}[a, *]\\
&\qquad \equiv [\ll a^{-1}, t, a\gg, \ll *, 1, 1\gg] = -\mathbb{0}[a, *], \qquad \text{so } [t, \mathbb{0}*] = 1,\\
&[t, \mathbb{1}*] = [\ll a, a^{-1}, t\gg, \ll *, *^{-1}, 1\gg] \equiv -\mathbb{0}[a, *],\\
&[t, \mathbb{2}*] = [\ll a, a^{-1}, t\gg, \ll *, *, *\gg] \equiv \mathbb{1}[a, *] + \mathbb{0}[t, *].
\end{aligned}$$

All asserted arrows follow from these equations.

Finally, we prove that the degrees of $X(c)$ and $X(u)$ are as claimed, by remarking that $\deg c = 3$ and $\deg u = 4$, that $\deg \mathsf{ad}(s)* \geq \deg(*)$ for $s = a, t$ and all words $*$ (so the claimed degrees smaller of equal to their actual value), and that each word of claimed degree n appears only as $\mathsf{ad}(s)*$ for words $*$ of degree at most $n-1$ (so the claimed degrees are greater or equal to their actual value).

The last point to check concerns the cube map; we skip the details. $\square$

Corollary 3.9. *Define the polynomials*

$$\begin{aligned}
Q_1 &= 0,\\
Q_2 &= \hbar + \hbar^2,\\
Q_3 &= \hbar + \hbar^2 + 2\hbar^3 + \hbar^4 + \hbar^5,\\
Q_n &= (1 + \hbar^{\alpha_n - \alpha_{n-1}})Q_{n-1} + \hbar^{\alpha_{n-1}}(\hbar^{-\alpha_{n-2}} + 1 + \hbar^{\alpha_{n-2}})Q_{n-2} \quad \textit{for } n \geq 3.
\end{aligned}$$

Then Q_n is a polynomial of degree α_n, and the polynomials Q_n and Q_{n+1} coincide on their first $2\alpha_{n-1}$ terms. Thus the coefficientwise limit $Q_\infty = \lim_{n\to\infty} Q_n$ exists.

The largest coefficient in Q_{2n+1} is 2^n, at position $\frac{1}{2}(\alpha_{2n+1}+1)$, so the coefficients of Q_∞ are unbounded. The integers k such that $\hbar^k$ has coefficient 1 in Q_∞ are precisely the $\beta_n + 1$.

The Hilbert–Poincaré series of $\mathscr{L}(\ddot{\Gamma}/\mathsf{Stab}_{\ddot{\Gamma}}(n))$ is $\hbar + Q_n$, and the Hilbert–Poincaré series of $\mathscr{L}(\ddot{\Gamma})$ is $\hbar + Q_\infty$. The same holds for the Lie algebras

$$\mathscr{L}_{\mathbb{F}_3}(\ddot{\Gamma}/\mathsf{Stab}_{\ddot{\Gamma}}(n)) \quad \textit{and} \quad \mathscr{L}_{\mathbb{F}_3}(\ddot{\Gamma}).$$

As a consequence, $\ddot{\Gamma}/\mathsf{Stab}_{\ddot{\Gamma}}(n)$ is nilpotent of class α_n, and $\ddot{\Gamma}$ does not have finite width.

Proof. Define polynomials

$$R_n = \sum_{w\in\{\mathbb{0},\mathbb{1},\mathbb{2}\}^n} \hbar^{\deg w(c)} + \sum_{w\in\{\mathbb{0},\mathbb{1},\mathbb{2}\}^{n-1}} \hbar^{\deg w(u)} + \hbar.$$

One checks directly that the polynomials R_n satisfy the same initial values and recurrence relation as Q_n, hence are equal. All convergence properties also follow from the definition of R_n.

The words of degree $\frac{1}{2}(\alpha_{2n+1}+1)$ are $(\mathbb{0}\mathbb{1})^{n-1}\mathbb{0}(c)$, $(\mathbb{0}\mathbb{1})^{n-2}\mathbb{0}\mathbb{2}(u)$, and all the words that can be obtained from these by iterating the substitutions $\mathbb{0}\mathbb{0}\mathbb{1}\mapsto\mathbb{1}\mathbb{2}\mathbb{0}$, $\mathbb{1}\mathbb{0}\mathbb{1}\mapsto\mathbb{2}\mathbb{2}\mathbb{0}$, $\mathbb{0}\mathbb{0}\mathbb{2}\mapsto\mathbb{1}\mathbb{2}\mathbb{1}$, $\mathbb{1}\mathbb{0}\mathbb{2}\mapsto\mathbb{2}\mathbb{2}\mathbb{1}$ along with $\mathbb{0}\mathbb{1}\mapsto\mathbb{2}\mathbb{0}$ and $\mathbb{0}\mathbb{2}\mapsto\mathbb{2}\mathbb{1}$ at the beginning of the word. This gives 2^n words in total, half of the form $X(c)$ and half $X(u)$.

There is a unique word of degree β_n+1, and that is $\mathbb{1}^n(c)$.

Note that these last two claims have a simple interpretation: there are 2^{n-1} ways of writing $\frac{1}{2}(\alpha_{2n+1})-1-\alpha_{n+1}$ in base α using only the digits 0, 1, 2; there is a unique way of writing β_n in base α using these digits. □

We note as an immediate consequence that

$$\left[\ddot{\Gamma} : \gamma_{\beta_n+1}(\ddot{\Gamma})\right] = 3^{(3^n+1)/2},$$

so that the asymptotic growth of $\ell_n = \dim \gamma_n(\ddot{\Gamma})/\gamma_{n+1}(\ddot{\Gamma})$ is polynomial of degree $d = \log 3/\log(1+\sqrt{2})-1$:

Corollary 3.10. *The Gelfand–Kirillov dimension of* $\mathscr{L}(\ddot{\Gamma})$ *is* $\log 3/\log(1+\sqrt{2})-1$.

We then deduce:

Corollary 3.11. *The growth of* $\ddot{\Gamma}$ *is at least* $e^{n^{\frac{\log 3}{\log(1+\sqrt{2})+\log 3}}} \cong e^{n^{0.554}}$.

Proof. Apply Proposition 1.10 to the series $\sum n^d \hbar^n$, which is comparable to the Hilbert–Poincaré series of $\mathscr{L}(\ddot{\Gamma})$. □

Turning to the derived series, we may also improve the general result $\ddot{\Gamma}^{(k)} \le \gamma_{2^k}(\ddot{\Gamma})$ to the following:

Theorem 3.12. *For all* $k \in \mathbb{N}$ *we have*

$$\ddot{\Gamma}^{(k)} \le \gamma_{\alpha_{k+1}}(\ddot{\Gamma}).$$

Proof. Clearly true for $k=0,1$; then a direct consequence of $\ddot{\Gamma}^{(k)} = \gamma_5(\ddot{\Gamma})^{\times 3^{k-2}}$ (obtained in [Vieira 1998]) and $\gamma_{\alpha_j}(\ddot{\Gamma})^{\times 3} \le \gamma_{\alpha_{j+1}}(\ddot{\Gamma})$ for $j=3,\dots,k$. □

3.8. ***The Fabrykowski–Gupta group* Γ.** We now give an explicit description of the Lie algebra of Γ, and compute its Hilbert–Poincaré series.

Theorem 3.13. *In Γ write $c = [a, t]$ and $u = [a, c] \equiv \mathbb{2}(at)$. For words $X = X_1 \dots X_n$ with $X_i \in \{\mathbb{0}, \mathbb{1}, \mathbb{2}\}$ define symbols $\overline{X_1 \dots X_n}(c)$ (representing elements of Γ) by*

$$\begin{aligned} \overline{\mathbb{i}\mathbb{0}}(c) &= \mathbb{i}\mathbb{0}(c)/\mathbb{i}(u),\\ \overline{\mathbb{i}\mathbb{2}^{m+1}\mathbb{1}^n}(c) &= \mathbb{i}\Big(\overline{\mathbb{2}^{m+1}\mathbb{1}^n}(c)\cdot \mathbb{0}\mathbb{1}^m\mathbb{0}^n(u)^{(-1)^n}\Big),\\ \overline{\mathbb{i}X}(c) &= \mathbb{i}\overline{X}(c) \textit{ for all other } X. \end{aligned}$$

Consider the following Lie graph: its vertices are the symbols $\overline{X}(c)$ and $X(u)$. Their degrees are given by

$$\deg \overline{X_1 \dots X_n}(c) = 1 + \sum_{i=1}^{n} X_i 3^{i-1} + \frac{1}{2}(3^n + 1),$$

$$\deg X_1 \dots X_n(u) = 1 + \sum_{i=1}^{n} X_i 3^{i-1} + (3^n + 1).$$

There are two additional vertices, labeled a and t, of degree 1.

Define the arrows as follows, for all $n \geq 1$:

$$\begin{array}{ll} a \xrightarrow{-t} c & t \overset{a}{\dashrightarrow} c \\ c \xrightarrow{-t} \mathbb{0}(c) & c \xrightarrow{a} u \\ u \xrightarrow{-t} \mathbb{1}(c) & \overline{\mathbb{2}^n}(c) \xrightarrow{-t} \overline{\mathbb{0}^{n+1}}(c) \\ \mathbb{0}* \overset{a}{\dashrightarrow} \mathbb{1}* & \mathbb{1}* \overset{a}{\dashrightarrow} \mathbb{2}* \\ \mathbb{2}^n\mathbb{0}* \xrightarrow{t} \mathbb{0}^n\mathbb{1}* & \mathbb{2}^n\mathbb{1}* \xrightarrow{t} \mathbb{0}^n\mathbb{2}* \end{array}$$

$$\overline{X_1 \dots X_n}(c) \xrightarrow{-(-1)^{\sum X_i} t} (X_1-1)\dots(X_n-1)(u)$$

Then the resulting graph is the Lie graph of $\mathscr{L}(\Gamma)$.

The subgraph spanned by a, t, the $\overline{X_1 \dots X_i}(c)$ for $i \leq n-2$ and the $X_1 \dots X_i(u)$ for $i \leq n-3$ is the Lie graph associated to the finite quotient $\Gamma/\operatorname{Stab}_\Gamma(n)$.

Proof. The proof is similar to that of Theorems 3.5 and 3.8, but a bit more tricky. Again we perform the computations in the completion of Γ, still written Γ. Again Γ' is the subgroup generated by all $\overline{X}(c)$ and $X(u)$, for $X \in \{\mathbb{0}, \mathbb{1}, \mathbb{2}\}^*$.

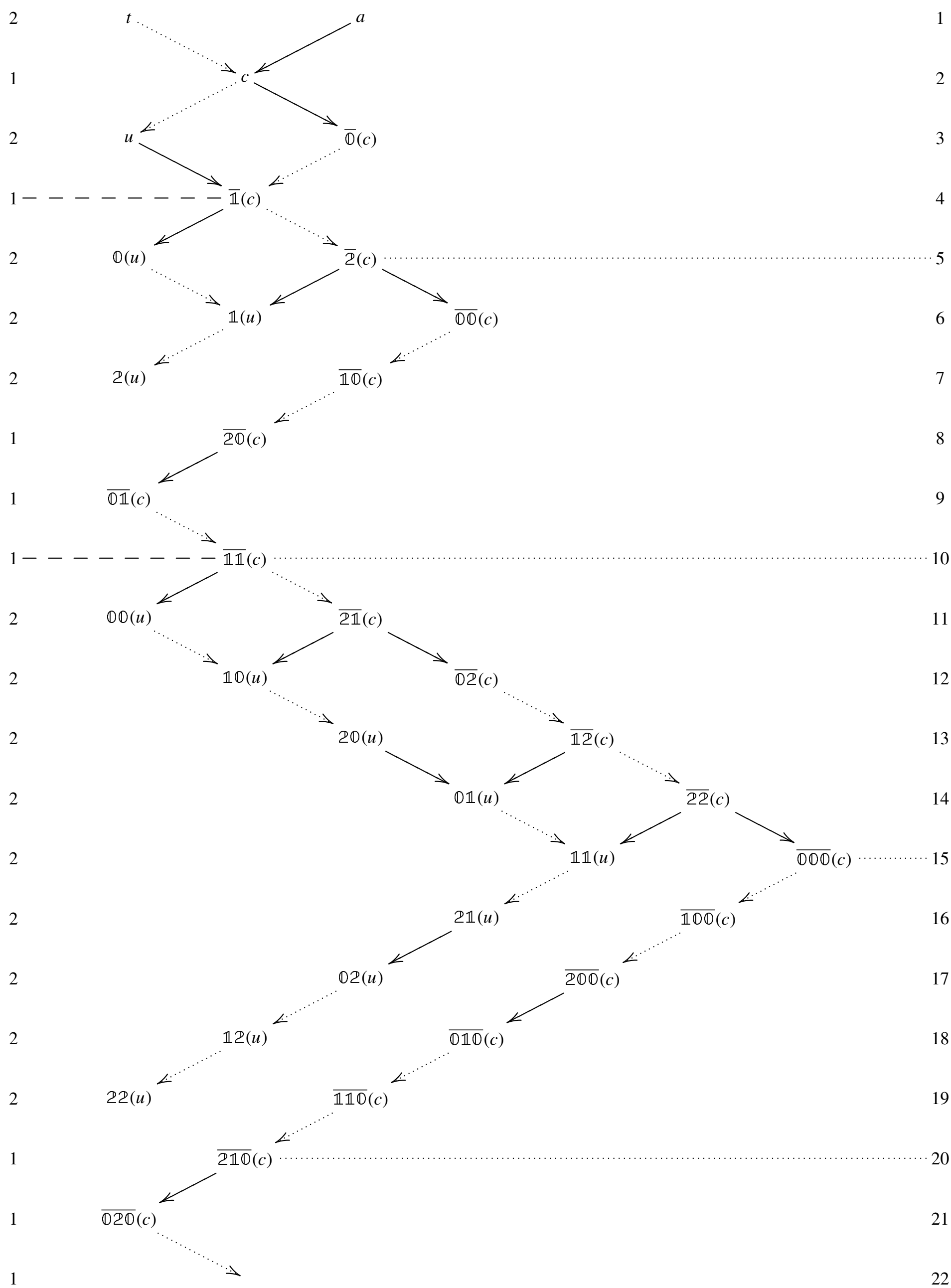


Figure 4. The beginning of the Lie graph of $\mathscr{L}(\Gamma)$. The generator $\mathsf{ad}(t)$ is shown by plain arrows, and the generator $\mathsf{ad}(a)$ is shown by dotted arrows. The left row indicates the dimensions of $\mathscr{L}_n$.

We claim inductively that if $X_i \geq Y_i$ at all positions i, then $X(c) \in \langle Y(c)\rangle^\Gamma$, and similarly for u. Therefore some terms may be neglected in the computations of brackets.

Now we compute $\mathsf{ad}(x)y$ for $x, y \in \{a, t, c, u\}$. Here $\equiv$ means some terms of greater degree have been neglected:

$$\begin{aligned}
&[a, \mathbb{0}*] = \mathbb{1}*, \quad [a, \mathbb{1}*] = \mathbb{2}*, \quad [a, \mathbb{2}*] = 1 \text{ by definition,}\\
&[t, \mathbb{0}*] \equiv [\ll 1, t, a\gg, \ll *, 1, 1\gg] = 1,\\
&[t, \mathbb{1}*] = [\ll a, 1, t\gg, \ll *, *^{-1}, 1\gg] = \mathbb{0}[a, *]\\
&\qquad\quad \equiv [\ll 1, t, a\gg, \ll *, *^{-1}, 1\gg] \equiv -\mathbb{0}[t, *],\\
&[t, \mathbb{2}*] = [\ll a, 1, t\gg, \ll *, *, *\gg] \equiv \mathbb{0}[a, *] + \mathbb{0}[t, *] + \mathbb{1}[t, *].
\end{aligned}$$

Note that in the last line the "negligible" term $\mathbb{1}[t, *]$ has been kept; this is necessary since sometimes the $\mathbb{0}[t, *]$ term cancels out.

Now we check each of the asserted arrows against the relations described above. First the a arrows are clearly as described, and so are the t arrows on $X(u)$; for instance,

$$\begin{aligned}
\mathsf{ad}(t)\mathbb{2}^n\mathbb{1}*(u) &= \mathbb{0}\,\mathsf{ad}(a)\mathbb{2}^{n-1}\mathbb{1}*(u) + \mathbb{0}\,\mathsf{ad}(t)\mathbb{2}^{n-1}\mathbb{1}*(u) + \mathbb{1}\,\mathsf{ad}(t)\mathbb{2}^{n-1}\mathbb{1}*(u)\\
&\equiv \mathbb{0}^n\big(\mathsf{ad}(a)\mathbb{1}*(u) + \mathsf{ad}(t)\mathbb{1}*(u)\big) \equiv \mathbb{0}^n\mathbb{2}*(u),
\end{aligned}$$

which holds by induction on the length of $*$. Next, the t arrows on $\bar{X}(c)$ agree; for instance,

$$\begin{aligned}
\mathsf{ad}(t)\overline{\mathbb{2}\mathbb{1}^n}(c) &= \mathbb{0}\,\mathsf{ad}(a)\mathbb{1}^n(c) + \mathbb{0}\,\mathsf{ad}(t)\mathbb{1}^n(c) + \mathbb{1}\,\mathsf{ad}(t)\mathbb{1}^n(c)\\
&= \mathbb{0}\mathbb{2}\mathbb{1}^{n-1}(c) + (-1)^n \cdot \mathbb{0}^{n+1}(u) + (-1)^n \cdot \mathbb{1}\mathbb{0}^n(u)\big)\\
&= \overline{\mathbb{0}\mathbb{2}\mathbb{1}^{n-1}}(c) + (-1)^n \cdot \mathbb{1}\mathbb{0}^n(u) \text{ by induction on } n,\\
\mathsf{ad}(t)\overline{\mathbb{2}^n}(c) &= \mathsf{ad}(t)\mathbb{2}\big(\overline{\mathbb{2}^{n-1}}(c) \cdot \mathbb{0}\mathbb{1}^{n-2}(u)\big)\\
&\equiv \mathbb{0}\mathbb{1}^{n-1}(u) + \mathbb{0}\big(-\overline{\mathbb{0}^n}(c) - \mathbb{1}^{n-1}(u)\big) + \mathbb{1}\big(-\overline{\mathbb{0}^n}(c) - \mathbb{1}^{n-1}(u)\big)\\
&\equiv -\overline{\mathbb{0}^{n+1}}(c) - \mathbb{1}^n(u).
\end{aligned}$$

All other cases are similar. Note how the calculation for $\overline{\mathbb{2}\mathbb{1}^n}(c)$ explains the definition of $\bar{X}(c)$: both $\mathbb{0}\mathbb{2}\mathbb{1}^{n-1}(c)$ and $\mathbb{0}^{n+1}(u)$ have degree smaller than $d = \deg\overline{\mathbb{2}\mathbb{1}^n}(c)$ in $\mathscr{L}(\Gamma)$, but they are linearly dependent in $\gamma_{d-1}(\Gamma)/\gamma_d(\Gamma)$.

Finally, we prove that the degrees of $X(c)$ and $X(u)$ are as claimed, by remarking that $\deg c = 3$ and $\deg u = 4$, that $\deg \mathsf{ad}(s)* \geq \deg(*)$ for $s = a, t$ and all words $*$ (so the claimed degrees smaller of equal to their actual value), and that each word of claimed degree n appears only as $\mathsf{ad}(s)*$ for words $*$ of degree at most $n-1$ (so the claimed degrees are greater or equal to their actual value). $\square$

Corollary 3.14. *Define the integers $\alpha_n = \frac{1}{2}(5\cdot 3^{n-2}+1)$ and the polynomials*

$$\begin{aligned} Q_2 &= 1,\\ Q_3 &= 1+2\hbar+\hbar^2+\hbar^3+\hbar^4+\hbar^5+\hbar^6,\\ Q_n(\hbar) &= (1+\hbar+\hbar^2)Q_{n-1}(\hbar^3)+\hbar+\hbar^{\alpha_n-2} \quad \text{for } n\geq 4. \end{aligned}$$

Then Q_n is a polynomial of degree α_n-2, and the first $3^{n-2}+1$ coefficients of Q_n and Q_{n+1} coincide. The termwise limit $Q_\infty = \lim_{n\to\infty} Q_n$ therefore exists.

The Hilbert–Poincaré series of $\mathscr{L}(\Gamma/\mathsf{Stab}_\Gamma(n))$ is $2\hbar+\hbar^2 Q_n$, and the Hilbert–Poincaré series of $\mathscr{L}(\Gamma)$ is $2\hbar+\hbar^2 Q_\infty$.

As a consequence, $\Gamma/\mathsf{Stab}_\Gamma(n)$ is nilpotent of class α_n, and Γ has finite width.

Proof. Consider the sequence of coefficients of $2\hbar+\hbar^2 Q_n$. They are, in condensed form,

$$2,\,1,\,2^{3^0},\,1^{3^0},\,2^{3^1},\,1^{3^1},\,\dots,\,2^{3^{n-3}},\,1^{3^{n-3}},\,1^{(3^{n-1}+1)/2}.$$

The i-th coefficient is 2 if there are $\overline{X}(c)$ and $Y(u)$ of degree i in $\Gamma/\mathsf{Stab}_\Gamma(n)$, and is 1 if there is only $\overline{X}(c)$. All conclusions follow from this remark. $\square$

In quite the same way as for $\ddot{\Gamma}$, we may improve the general result $\Gamma^{(k)}\leq\gamma_{2^k}(\Gamma)$:

Theorem 3.15. *The derived series of Γ satisfies $\Gamma'=\gamma_2(\Gamma)$ and $\Gamma^{(k)}=\gamma_5(\Gamma)^{\times 3^{k-2}}$ for $k\geq 2$. We have*

$$\Gamma^{(k)}\leq\gamma_{2+3^{k-1}}(\Gamma) \quad \textit{for all } k\in\mathbb{N}.$$

Proof. It is a general fact for a 2-generated group Γ that $\Gamma''\leq\gamma_5(\Gamma)$. Since $[c,\mathbb{0}(c)]\equiv\mathbb{0}(u)^{-1}$ and $[c,u]\equiv\mathbb{2}(c)^{-1}$ (modulo $\gamma_6(\Gamma)$), we have $[\gamma_2(\Gamma),\gamma_3(\Gamma)] = \gamma_5(\Gamma)$ and therefore $\Gamma''=\gamma_5(\Gamma)$.

Next, $\gamma_5(\Gamma)=\gamma_3(\Gamma)^{\times 3}\cdot\mathbb{2}(c)$, so $\Gamma^{(3)}=[\gamma_3(\Gamma),c]^{\times 3}=\gamma_5^{\times 3}$, and the claimed formula holds for all $\Gamma^{(k)}$ by induction. Finally $\gamma_{2+3^{j-2}}(\Gamma)^{\times 3}\leq\gamma_{2+3^{j-1}}(\Gamma)$ for all $j=3,\dots,k$. $\square$

We omit altogether the proofs of the next two results, since they are completely analogous to that of Theorem 3.13.

Theorem 3.16. *Keep the notations of Theorem 3.13. Define furthermore symbols $\overline{X_1\dots X_n}(u)$ (representing elements of Γ) by*

$$\begin{aligned} \overline{\mathbb{2}^n}(u) &= \mathbb{2}^n(u)\cdot\mathbb{2}^{n-1}\mathbb{0}(c)\cdot\mathbb{2}^{n-2}\mathbb{0}\mathbb{1}(c)\cdots\mathbb{2}\mathbb{0}\mathbb{1}^{n-2}(c),\\ \overline{X}(u) &= X(u) \quad \textit{for all other } X. \end{aligned}$$

Consider the following Lie graph: its vertices are the symbols $\overline{X}(c)$ and $\overline{X}(u)$. Their degrees are given by

$$
\begin{aligned}
\deg \overline{X_1 \dots X_n}(c) &= 1 + \sum_{i=1}^{n} X_i 3^{i-1} + \frac{1}{2}(3^n+1),\\
\deg \mathbb{2}^n(u) &= 3^{n+1},\\
\deg X_1 \dots X_n(u) &= \max\left\{1 + \sum_{i=1}^{n} X_i 3^{i-1} + (3^n+1),\ \frac{1}{2}(9-3^n) + 3\sum_{i=1}^{n} X_i 3^{i-1}\right\}.
\end{aligned}
$$

There are two additional vertices, labeled a and t, of degree 1.

Define the arrows as follows, for all $n \geq 1$:

$$
\begin{array}{cc}
a \xrightarrow{-t} c & t \xrightarrow{a} c\\[6pt]
c \xrightarrow{-t} \mathbb{0}(c) & c \xrightarrow{a} u\\[6pt]
u \xrightarrow{-t} \mathbb{1}(c) & \overline{\mathbb{2}^n}(c) \xrightarrow{-t} \overline{\mathbb{0}^{n+1}}(c)\\[6pt]
\mathbb{0}* \xrightarrow{a} \mathbb{1}* & \mathbb{1}* \xrightarrow{a} \mathbb{2}*\\[6pt]
\mathbb{2}^n\mathbb{0}* \xrightarrow{t} \mathbb{0}^n\mathbb{1}* & \mathbb{2}^n\mathbb{1}* \xrightarrow{t} \mathbb{0}^n\mathbb{2}*
\end{array}
$$

$$
\overline{X_1 \dots X_n}(c) \xrightarrow{-(-1)^{\sum X_i} t} (X_1-1)\dots(X_n-1)(u)
$$

$$
\begin{array}{cc}
c \xrightarrow{\cdot 3} \overline{\mathbb{0}\mathbb{0}}(c) & \overline{\mathbb{2}^n}(u) \xrightarrow{\cdot 3} \overline{\mathbb{2}^{n+1}}(u)
\end{array}
$$

$$
*\mathbb{0}(c) \xrightarrow{\cdot 3} *\mathbb{2}(u) \quad \text{if } 3\deg *\mathbb{0}(c) = \deg *\mathbb{2}(u)
$$

Then the resulting graph is the Lie graph of $\mathscr{L}_{\mathbb{F}_3}(\Gamma)$.

The subgraph spanned by a, t, the $\overline{X_1 \dots X_i}(c)$ for $i \leq n-2$ and the $X_1 \dots X_i(u)$ for $i \leq n-3$ is the Lie graph of the Lie algebra $\mathscr{L}_{\mathbb{F}_3}(\Gamma/\operatorname{Stab}_\Gamma(n))$.

As a consequence, the dimension series of $\Gamma/\operatorname{Stab}_\Gamma(n)$ has length 3^{n-1} (the degree of $\overline{\mathbb{2}^n}(u)$), and Γ has finite width.

Proposition 1.8 then implies:

Corollary 3.17. *The growth of Γ is at least $e^{\sqrt{n}}$.*

4. Parabolic space

In the natural action of a branch group G on the tree Σ^*, consider a "parabolic subgroup" P, the stabilizer of an infinite ray in Σ^*. (The terminology comes from geometry, where a parabolic subgroup is the stabilizer of a point on the boundary of an appropriate G-space.) Such a parabolic subgroup may be defined directly as follows: let $\omega = \omega_1\omega_2 \cdots \in \Sigma^\infty$ be an infinite sequence. Set $P_0^\omega = G$ and inductively set

$$P_n^\omega = \psi^{-1}(G \times \cdots \times P_{n-1}^\omega \times \cdots \times G),$$

with the P_{n-1}^ω in position ω_n. Set $P^\omega = \bigcap_{n\geq 0} P_n^\omega$.

In the natural tree action (2–1) of G on Σ^* or on Σ^∞ its boundary, P_n^ω is the stabilizer of the point $\omega_1 \dots \omega_n$, and P^ω is the stabilizer of the infinite sequence ω.

The following facts easily follow from the definitions:

Lemma 4.1. $\bigcap_{\omega\in\Sigma^\infty} P^\omega = 1$. *The index of P_n^ω in G is d^n, and that of P^ω is infinite.*

Definition 4.2. Let G be a branch group. A *parabolic space* for G is a homogeneous space G/P, where P is a parabolic subgroup.

Suppose now that G is finitely generated by a set S.

Proposition 4.3 [Bartholdi and Grigorchuk 2000b]. *Suppose that the length $|\cdot|$ on the branch group G is such that, for certain constants λ, μ and for all $g \in \mathsf{Stab}_G(1)$, one has $|g_i| < \lambda|g| + \mu$, where we have written $\psi(g) = (g_1, \dots, g_d)$. Then all parabolic spaces of G have polynomial growth of degree at most* $\log_{1/\lambda}(d)$.

Theorem 4.4. *Let G be a finitely generated branch group. There exists a constant C such that, for any $x_0 \in G$,*

$$\frac{C\,\mathsf{growth}(G/P, x_0P)}{1-\hbar} \geq \frac{\mathsf{growth}\,\mathscr{L}(G)}{1-\hbar}.$$

Proof. Assume G acts on a d-regular tree, and write as before $d' = d - 1$. The proof relies on an identification of the Lie action on group elements and the natural action on tree levels. We first claim that for any $u \in K$ and $W \in \{{}^{\mathbb{0}}_{\mathbb{1}}, \dots, \mathbb{d}'\}^*$

$$\deg W(u) \geq \deg(\mathbb{0}^{|W|}(u)) + d_{G/P}(\mathbb{0}^{|W|}, W),$$

where $d(W, X)$ is the length of a minimal word moving W to X in the tree Σ^*.

Therefore the growth of $\mathscr{L}(G)$ and G/P may be compared just by considering the degrees of elements of the form $\mathbb{0}^n(u)$ for some fixed $u \in K$; indeed the other $W(u)$ will contribute a smaller growth to the Lie growth series than the corresponding vertices to the parabolic growth series, and the N finitely many values u may take in a branch portrait description will be taken care of by the constant C.

Now there is a constant $\ell \in \mathbb{N}$ such that $\mathbb{0}^{\ell+m}(u)$ has greater degree than $(\mathbb{d}')^m(u)$ for all $m \in \mathbb{N}$. Indeed there exists $k \in K$ and $\ell \in \mathbb{N}$ such that $[k, u] = \mathbb{0}^\ell(u)$, and then $[\mathbb{0}^m k, \mathbb{d}'^m(u)] = \mathbb{0}^{\ell+m}(u)$, proving the claim.

We may now take $C = \ell N$. The Lie growth series is the sum over all $n \in \mathbb{N}$ and coset representatives $u \in T$ of the power series counting the growth of $W(u)$ over words W of length n. There are N choices for u, and for given u at most ℓ of these power series overlap. $\square$

Note that this result is valid even if the action on the rooted tree is not cyclic, i.e., even if in the decomposition map $G \to G \wr A$ the finite group A is not cyclic. If A is not nilpotent, the Lie algebra $\mathscr{L}$ is no longer isomorphic to G, so the best we can hope for is an inequality bounding the growth of $\mathscr{L}$ by that of G/P.

5. Normal subgroups

Using the notion of a branch portrait, it is not too difficult to determine the exact structure of normal subgroups in a branch group. Consider a p-group G and its p-Lie algebra $\mathscr{L}$ over $\mathbb{F}_p$. Normal subgroups of G correspond to ideals of $\mathscr{L}$, just as subgroups of G correspond to subalgebras of $\mathscr{L}$; and the index of $H \leq G$ is $p^{\dim \mathscr{L}/\mathscr{M}}$, where the subgroup H corresponds to the subalgebra $\mathscr{M}$. This correspondence is not exact, and we shall neither use it nor make it explicit; however it serves as a motivation for relating subgroup growth and the study of Lie algebras. In all cases, sufficient knowledge of $\mathscr{L}$, as well as its finiteness of width, allow an explicit description of the normal subgroup lattice of G.

We focus on the first and most important example, $\mathfrak{G}$, for which we obtain an explicit answer. The computations presented here clearly extend, *mutatis mutandis*, to any regular branch group.

Set $\mathscr{W} = \{\mathbb{0}, \mathbb{1}\}^*$, and order words $X \in \mathscr{W}$ by reverse shortlex: the rank of $X_1 \dots X_n$ is

$$\#X_1 \dots X_n = 1 + \sum_{i=1}^{n} X_i 2^{i-1} + 2^n.$$

(Note that $\#X = \deg X(x)$ according to the definition in Section 3.6.) We write $<$ the order induced by rank.

Theorem 5.1. *The nontrivial normal subgroups of* $\mathfrak{G}$ *are as follows*:

- *there are respectively* 1, 7, 7, 1 *subgroups of index* 1, 2, 4, 8 *corresponding to the lifts to* $\mathfrak{G}$ *of subgroups of* $\mathfrak{G}/[\mathfrak{G}, \mathfrak{G}] = C_2^{\times 3}$;
- *there are* 12 *other subgroups of* $\mathfrak{G}$ *not contained in* K: *six of index* 8, *namely* $\langle [a, c], d^a b\rangle^{\mathfrak{G}}$, $\langle c\rangle^{\mathfrak{G}}$, $\langle x, c^a d\rangle^{\mathfrak{G}}$, $\langle b\rangle^{\mathfrak{G}}$, $\langle [a, d], b^a c\rangle^{\mathfrak{G}}$, *and* $\langle d, x^2\rangle^{\mathfrak{G}}$; *four of index* 16, *namely* $\langle [a, c]\rangle^{\mathfrak{G}}$, $\langle [a, d], x^2\rangle^{\mathfrak{G}}$, $\langle d\rangle^{\mathfrak{G}}$, *and* $\langle [a, d], x^2 d\rangle^{\mathfrak{G}}$; *and two of index* 32, *namely* $\langle [a, d]x^2\rangle^{\mathfrak{G}}$ *and* $\langle [a, d]\rangle^{\mathfrak{G}}$;
- *all normal subgroups* $N \triangleleft \mathfrak{G}$ *contained in* K *are of the form*

$$(*) \qquad W(A; B_1, \dots, B_m; C) := \big\langle A(x) B_1(x^2) \dots B_m(x^2), C(x^2)\big\rangle^{\mathfrak{G}},$$

for words $A, B_i, C \in \mathscr{W}$. *There are functions* $M(A, \{B_i\}, C)$ *and* $S(A, \{B_i\}, C)$ (*defined in the proof*), *with values in* $\mathscr{W}$, *such that there is a unique description* *of N in the form $(*)$ satisfying*

$$B_1 < B_2 < \cdots < B_m \leq S(A, \{B_i\}, C) < C \leq M(A, \{B_i\}).$$

The index of N is $2^{\#A+\#S(A,\{B_i\},C)}$. The groups can furthermore be subdivided into three types:

I: *$C \leq \mathbb{0}^{|A|}$ and $A \leq \mathbb{0}^{|C|}\mathbb{1}\mathbb{0}$. Then all B_i are optional, i.e., there are 2^m groups with these A and C, obtained by choosing any subset of the B_i's*;

II: *$C > \mathbb{0}^{|A|}$ and $C \leq \mathbb{0}^{|A|+1}$. Then $A = B_1\mathbb{1}$ and all other B_i's are optional*;

III: *$A = \mathbb{0}^n$ and some $B_i = \mathbb{0}^{n-1}$. Then in fact an alternate description exists, obtained by suppressing A and B_i from the description.*

Note that we have only described finite-index subgroups of $\mathfrak{G}$. Since $\mathfrak{G}$ is just-infinite, all its nontrivial normal subgroups have finite index.

We depict the top of the lattice in Figure 5, which shows all normal subgroups of index at most 2^{13} (there are never more than 7 subgroups of a given lesser index).

The first few subgroups of K are described in Table 1, sorted by their index in $\mathfrak{G}$, and identified by their type in $\{(\mathrm{I}), (\mathrm{II}), (\mathrm{III})\}$. We write λ for the empty sequence. An argument $[B_i]$ means that term is optional, and therefore stands for two groups, one with that term and one without.

Among the remarkable subgroups are: $K^{\times 2^n} = \langle \mathbb{0}^n(x)\rangle^{\mathfrak{G}}$, written K_n in [Bartholdi and Grigorchuk 2002]; the subgroup $K^{\times 2^n}\mho_2(K)^{\times 2^{n-1}} = \langle \mathbb{0}^n(x), \mathbb{0}^{n-1}(x^2)\rangle$, written N_n in the same reference; and $\mathsf{Stab}_G(n) = \big\langle \mathbb{0}^{n-3}(\mathbb{1}(x)x^2), \mathbb{0}^{n-2}(x^2)\big\rangle$.

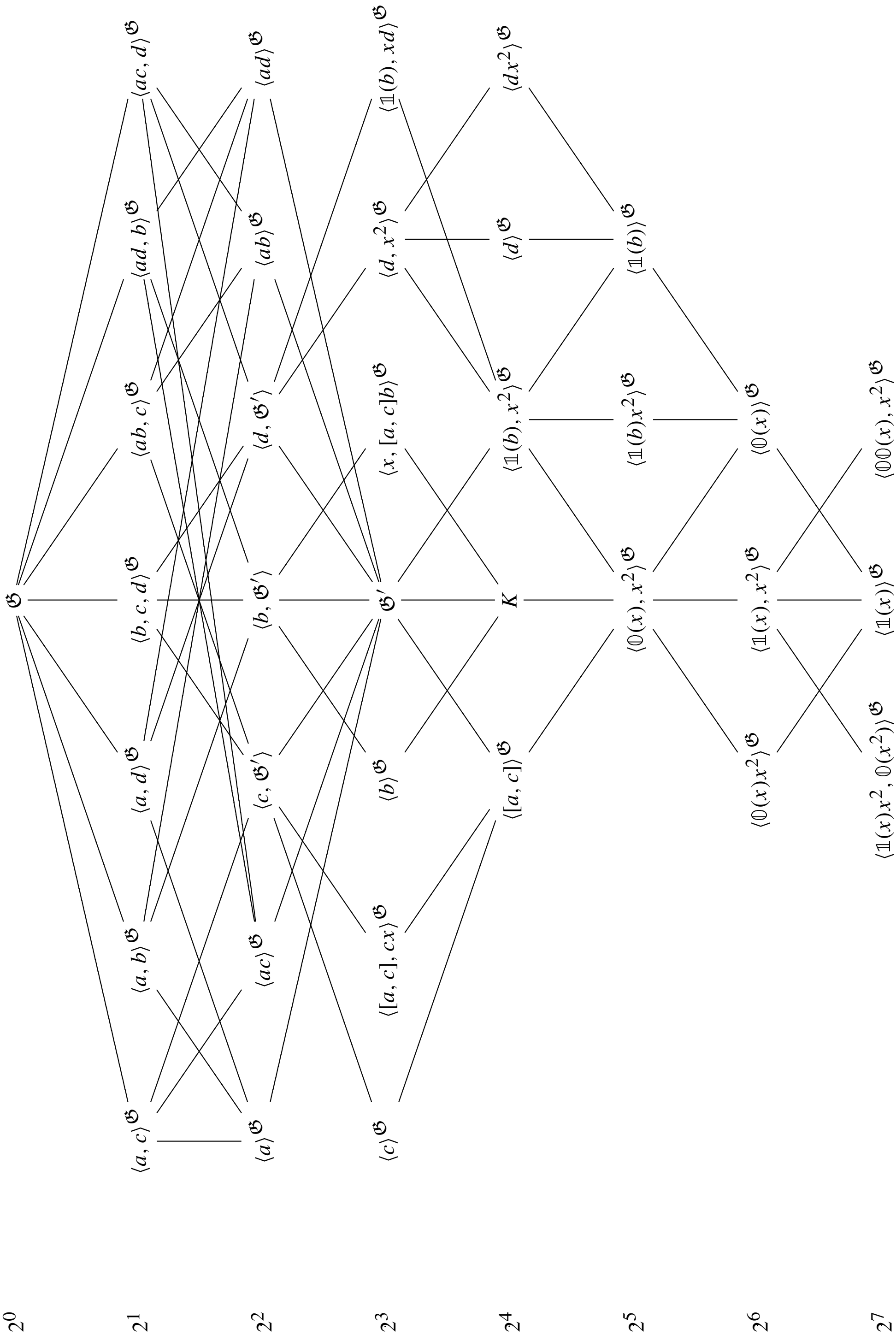
The lattice of normal subgroups of $\mathfrak{G}$ is described in Figure 5. Even though I do not understand completely the lattice's structure, some remarks can be made: the lattice has a fractal appearance; all its nodes have 1 or 3 descendants, and 1 or 3 ascendants. Large portions of it have a grid-like structure. This can be explained by the construction $N \rightsquigarrow N \times N$ of normal subgroups, lending the lattice some self-similarity.

Proof of Theorem 5.1. The first two assertions are checked directly as follows. Let $\mathcal{F}$ be the set of finite-index subgroups of $\mathfrak{G}$ not in K. Consider the finite quotient $Q = \mathfrak{G}/\mathsf{Stab}_6(\mathfrak{G})$, and the preimage P of $\mathfrak{G}$ defined as

$$P = \big\langle a, b, c, d \bigm| a^2, b^2, c^2, d^2, bcd, \sigma^i(ad)^4, \sigma^i(adacac)^4 \quad (i = 0\dots 5)\big\rangle.$$

Clearly the image of $\mathcal{F}$ in Q is at most as large as $\mathcal{F}$, and the preimage of $\mathcal{F}$ in P is at least as large as $\mathcal{F}$. Now we use the algorithms in GAP [GAP 2002] computing the top of the lattice of normal subgroups for finite groups (Q) and finitely presented groups (P). The number of subgroups not contained in K agree in P and Q, so give the structure of the lattice not below K in $\mathfrak{G}$.

Let now N be a normal subgroup of $\mathfrak{G}$, contained in K. If N is nontrivial, then it has finite index [Grigorchuk 2000, Corollary to Proposition 9]. It is easy to


2^0
$\mathfrak{G}$
2^1
$\langle ac, d\rangle^{\mathfrak{G}}$
$\langle ad, b\rangle^{\mathfrak{G}}$
$\langle ab, c\rangle^{\mathfrak{G}}$
$\langle b, c, d\rangle^{\mathfrak{G}}$
$\langle a, d\rangle^{\mathfrak{G}}$
$\langle a, b\rangle^{\mathfrak{G}}$
$\langle a, c\rangle^{\mathfrak{G}}$
2^2
$\langle ad\rangle^{\mathfrak{G}}$
$\langle ab\rangle^{\mathfrak{G}}$
$\langle d, \mathfrak{G}'\rangle$
$\langle b, \mathfrak{G}'\rangle$
$\langle c, \mathfrak{G}'\rangle$
$\langle ac\rangle^{\mathfrak{G}}$
$\langle a\rangle^{\mathfrak{G}}$
2^3
$\langle \mathbb{1}(b), xd\rangle^{\mathfrak{G}}$
$\langle d, x^2\rangle^{\mathfrak{G}}$
$\langle x, [a, c]b\rangle^{\mathfrak{G}}$
$\mathfrak{G}'$
$\langle b\rangle^{\mathfrak{G}}$
$\langle [a, c], cx\rangle^{\mathfrak{G}}$
$\langle c\rangle^{\mathfrak{G}}$
2^4
$\langle dx^2\rangle^{\mathfrak{G}}$
$\langle d\rangle^{\mathfrak{G}}$
$\langle \mathbb{1}(b), x^2\rangle^{\mathfrak{G}}$
K
$\langle [a, c]\rangle^{\mathfrak{G}}$
2^5
$\langle \mathbb{1}(b)\rangle^{\mathfrak{G}}$
$\langle \mathbb{1}(b)x^2\rangle^{\mathfrak{G}}$
$\langle \mathbb{0}(x), x^2\rangle^{\mathfrak{G}}$
2^6
$\langle \mathbb{0}(x)\rangle^{\mathfrak{G}}$
$\langle \mathbb{1}(x), x^2\rangle^{\mathfrak{G}}$
$\langle \mathbb{0}(x)x^2\rangle^{\mathfrak{G}}$
2^7
$\langle \mathbb{0}\mathbb{0}(x), x^2\rangle^{\mathfrak{G}}$
$\langle \mathbb{1}(x)\rangle^{\mathfrak{G}}$
$\langle \mathbb{1}(x)x^2, \mathbb{0}(x^2)\rangle^{\mathfrak{G}}$

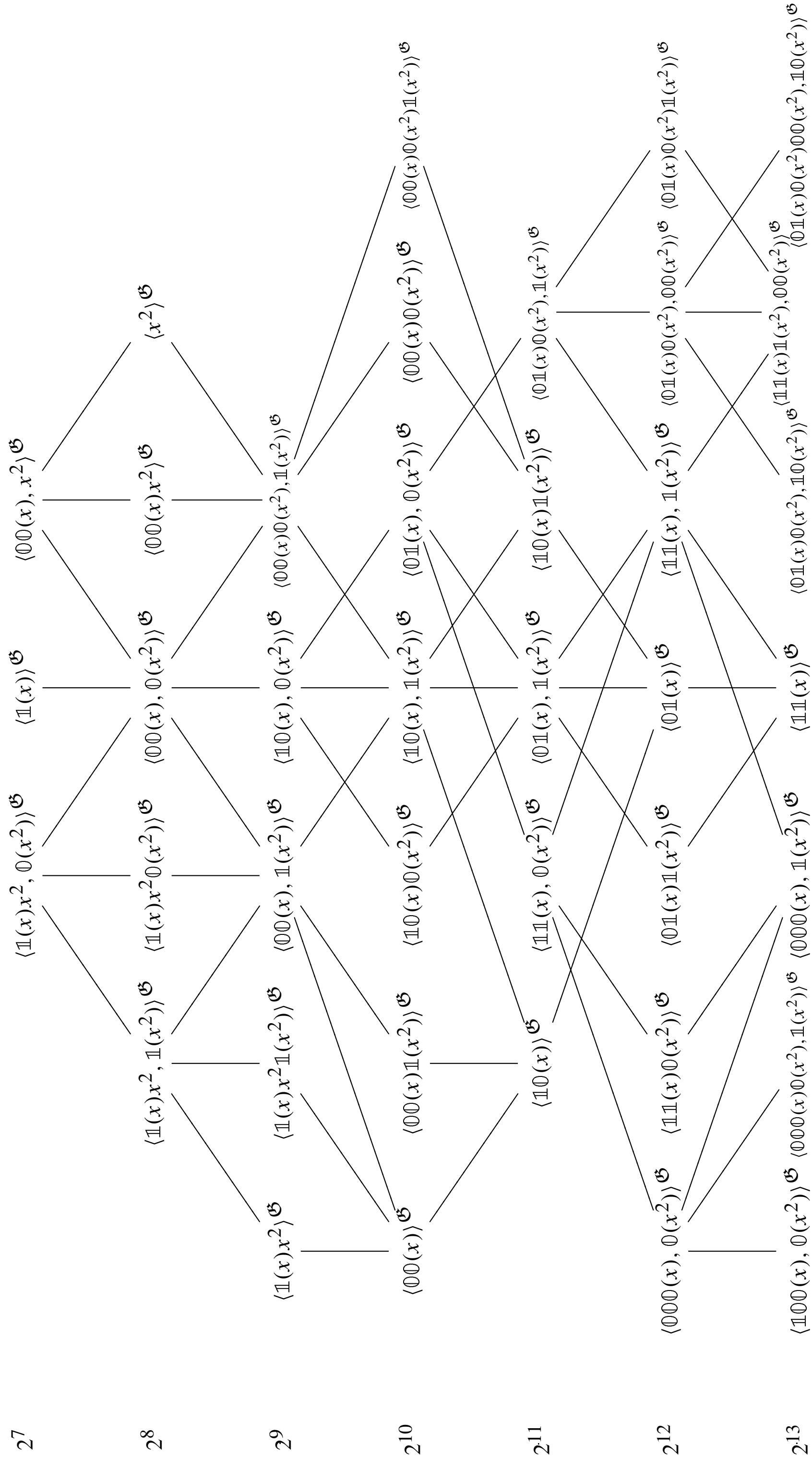


Figure 5. The top of the lattice of normal subgroups of $\mathfrak{G}$, of index at most 2^{13}.

Index	Count	Description		
2^4	1	$W(\lambda;\,;\lambda)_{\mathrm{I}} = K$		
2^5	1	$W(\mathbb{0};\,;\lambda)_{\mathrm{I}}$		
2^6	3	$W(\mathbb{1};\,;\lambda)_{\mathrm{I}}$	$W(\mathbb{0};[\lambda];\mathbb{0})_{\mathrm{I}}$	
2^7	3	$W(\mathbb{00};\,;\lambda)_{\mathrm{I}}$	$W(\mathbb{1};[\lambda];\mathbb{0})_{\mathrm{I}}$	
2^8	5	$W(\mathbb{10};\,;\lambda)_{\mathrm{I}}$	$W(\mathbb{00};\,;\mathbb{0})_{\mathrm{I}}$	
			$W(\mathbb{1};\lambda,[\mathbb{0}];\mathbb{1})_{\mathrm{II}}$	$W(\infty;\lambda,\mathbb{0};\mathbb{1})_{\mathrm{III}}$
2^9	5	$W(\mathbb{10};\,;\mathbb{0})_{\mathrm{I}}$	$W(\mathbb{00};[\mathbb{0}];\mathbb{1})_{\mathrm{I}}$	$W(\mathbb{1};\lambda,[\mathbb{1}];\mathbb{00})_{\mathrm{II}}$
2^{10}	7	$W(\mathbb{01};\,;\mathbb{0})_{\mathrm{I}}$	$W(\mathbb{10};[\mathbb{0}];\mathbb{1})_{\mathrm{I}}$	$W(\mathbb{00};[\mathbb{0}],[\mathbb{1}];\mathbb{00})_{\mathrm{I}}$
2^{11}	5	$W(\mathbb{11};\,;\mathbb{0})_{\mathrm{I}}$	$W(\mathbb{01};[\mathbb{0}];\mathbb{1})_{\mathrm{I}}$	$W(\mathbb{10};[\mathbb{1}];\mathbb{00})_{\mathrm{I}}$
2^{12}	7	$W(\mathbb{000};\,;\mathbb{0})_{\mathrm{I}}$	$W(\mathbb{11};[\mathbb{0}];\mathbb{1})_{\mathrm{I}}$	$W(\mathbb{01};[\mathbb{0}],[\mathbb{1}];\mathbb{00})_{\mathrm{I}}$
2^{13}	7	$W(\mathbb{100};\,;\mathbb{0})_{\mathrm{I}}$	$W(\mathbb{000};[\mathbb{0}];\mathbb{1})_{\mathrm{I}}$	$W(\mathbb{11};[\mathbb{1}];\mathbb{00})_{\mathrm{I}}$
				$W(\mathbb{01};\mathbb{0},[\mathbb{00}];\mathbb{10})_{\mathrm{II}}$
2^{14}	13	$W(\mathbb{010};\,;\mathbb{0})_{\mathrm{I}}$	$W(\mathbb{100};[\mathbb{0}];\mathbb{1})_{\mathrm{I}}$	$W(\mathbb{000};\,;\mathbb{00})_{\mathrm{I}}$
			$W(\mathbb{11};\mathbb{1},[\mathbb{00}];\mathbb{10})_{\mathrm{II}}$	$W(\mathbb{01};\mathbb{0},[\mathbb{00}],[\mathbb{10}];\mathbb{01})_{\mathrm{II}}$
			$W(\infty;\mathbb{1},\mathbb{00};\mathbb{10})_{\mathrm{III}}$	$W(\infty;\mathbb{0},[\mathbb{1}],\mathbb{00};\mathbb{01})_{\mathrm{III}}$
2^{15}	9	$W(\mathbb{010};\,;\mathbb{1})_{\mathrm{I}}$	$W(\mathbb{100};\,;\mathbb{00})_{\mathrm{I}}$	$W(\mathbb{000};[\mathbb{00}];\mathbb{10})_{\mathrm{I}}$
		$W(\mathbb{11};\mathbb{1},[\mathbb{10}];\mathbb{01})_{\mathrm{II}}$	$W(\mathbb{01};\mathbb{0},[\mathbb{01}];\mathbb{11})_{\mathrm{II}}$	$W(\infty;\mathbb{1},\mathbb{10};\mathbb{01})_{\mathrm{III}}$
2^{16}	13	$W(\mathbb{010};\,;\mathbb{00})_{\mathrm{I}}$	$W(\mathbb{100};[\mathbb{00}];\mathbb{10})_{\mathrm{I}}$	$W(\mathbb{000};[\mathbb{00}],[\mathbb{10}];\mathbb{01})_{\mathrm{I}}$
			$W(\mathbb{11};\mathbb{1},[\mathbb{01}];\mathbb{11})_{\mathrm{II}}$	$W(\mathbb{01};\mathbb{0},[\mathbb{01}],[\mathbb{11}];\mathbb{000})_{\mathrm{II}}$
2^{17}	11	$W(\mathbb{110};\,;\mathbb{00})_{\mathrm{I}}$	$W(\mathbb{010};[\mathbb{00}];\mathbb{10})_{\mathrm{I}}$	$W(\mathbb{100};[\mathbb{10}];\mathbb{01})_{\mathrm{I}}$
		$W(\mathbb{000};[\mathbb{00}],[\mathbb{01}];\mathbb{11})_{\mathrm{I}}$		$W(\mathbb{11};\mathbb{1},[\mathbb{11}];\mathbb{000})_{\mathrm{II}}$
2^{18}	19	$W(\mathbb{001};\,;\mathbb{00})_{\mathrm{I}}$	$W(\mathbb{110};[\mathbb{00}];\mathbb{10})_{\mathrm{I}}$	$W(\mathbb{010};[\mathbb{00}],[\mathbb{10}];\mathbb{01})_{\mathrm{I}}$
		$W(\mathbb{100};[\mathbb{10}],[\mathbb{01}];\mathbb{11})_{\mathrm{I}}$	$W(\mathbb{000};[\mathbb{00}],[\mathbb{01}],[\mathbb{11}];\mathbb{000})_{\mathrm{I}}$	

Table 1. Normal subgroups of index up to 2^{18} in $\mathfrak{G}$, contained in K.

see that N contains $C(x^2)$ and $D(x)$ for some words C, D, using for instance the congruence property [Grigorchuk 2000, Proposition 10]; therefore the generators of N may be chosen as

$$\big\{A_1(x)\cdots A_n(x)B_1(x^2)\cdots B_m(x^2),\ A'_1(x)\cdots A'_{n'}(x)B'_1(x^2)\cdots B'_{m'}(x^2),\ \dots,\ C(x^2),\ D(x)\big\},$$

with $A_i^{(j)} < D$ and $B_i^{(j)} < C$ for all i, j.

Taking the commutators of these generators with the appropriately chosen generator among $\{a, b, c, d\}$, we shift the ranks of the A-terms up by 1, and multiplying a generator by another we may get rid of all generators except $C(x^2)$ and the one with A_1 of smallest rank.

We therefore consider all subgroups $W(A; B_1, \dots, B_m; C)$, and seek conditions on A, $\{B_i\}$ and C so that to each normal subgroup in K there corresponds a unique expression of the form $W(A; B_1, \dots, B_m; C)$.

Let first C be minimal such that $C(x^2) \in N$; then take A minimal such that for some $B_1 < \cdots < B_m < C$ we have $A(x)B_1(x^2)\cdots B_m(x^2) \in N$. Take also B'_1 minimal such that $B'_1(x^2)\cdots B'_{m'}(x^2) \in N$ for some B'_i.

Define the functions $M, S : \mathcal{W} \times 2^{\mathcal{W}} \times \mathcal{W} \to \mathcal{W}$ as follows (M stands for "monomial" and S stands for "squares"): Consider $A(x)B_1(x^2)\dots B_m(x^2)$ as an element of $\mathcal{L}_{\mathbb{F}_2}(\mathfrak{G})$, truncated at degree C. Successive commutations with generators $s \in \{a, b, c, d\}$, *according the the rules of Lemma 3.6*, give rise to other elements of $\mathcal{L}_{\mathbb{F}_2}(\mathfrak{G})$. We stress that we use the complete computations of commutators, and not just those in the filtered Lie algebra. Define $M(A, \{B_i\})$ as the minimal word D such that $D(x^2)$ that arises in this process; if no such word occurs, $M(A, \{B_i\}, C) = C$. Define $S(A, \{B_i\})$ as the minimal $B'_{m'}$ such that $B'_1(x^2)\cdots B'_{m'}(x^2)$ occurs in this process; if no such product occurs, $S(A, \{B_i\}, C) = C - 1$.

Now, since $M(A, \{B_i\}, C)(x^2) \in N$, we necessarily have $C \le M(A, \{B_i\})$. Also, all B_i of degree at least $B'_{m'}$ can be replaced by terms of lower degree $B'_1, \dots, B'_{m-1}$. This proves the claimed inequalities. Conversely, if there existed another description $A(x)\tilde{B}_1(x^2)\dots\tilde{B}_m(x^2) \in N$ for another choice of $\tilde{B}$'s, then by dividing we would obtain a product of $B_i(x^2)$ in N, contradicting $B_m < S(A, \{B_i\}, C)$. The data $(A; B_1, \dots, B_m; C)$ subjected to the theorem's constraints therefore correspond bijectively to N's.

The index of N can be computed in $\mathcal{L}_{\mathbb{F}_2}(\mathfrak{G})$. Seeing elements of N as inside $\mathcal{L}$, a vector-space complement of N is spanned by all $\tilde{A}(x)$ of rank less than A, and all $\tilde{B}(x^2)$ of rank less than $S(A, \{B_i\}, C)$.

We consider finally three cases: first assume $C \le \mathbb{0}^{|A|}$ and $|B_1| \ge |A| - 1$. Then $C(x^2)$ gives $\mathbb{0}^{|C|+1}(x^2)\mathbb{0}^{|C|+2}(x)$ by commutation with $\sigma^{|A|}(d)$, which itself gives $\mathbb{0}^{|C|}\mathbb{1}\mathbb{0}(x)$ by commutation with a, so we may suppose $A \le \mathbb{0}^{|C|}1\mathbb{0}$. Various B_i's can be added, giving the description (I).

Now assume $C > \mathbb{0}^{|A|}$. Then since $A(x)$ would produce $\mathbb{0}^{|A|}(x^2)$ by commutation with an appropriate conjugate of $\sigma^{|A|}(b)$, we must have $A = B_1\mathbb{1}$ so that the same commutation vanishes, giving the description (II).

Finally assume $C \le \mathbb{0}^{|A|}$ and $|B_1| < |A| - 1$. Then necessarily $A = \mathbb{0}^n$; taking appropriate commutations we see that the normal subgroup in question contains $\mathbb{0}^n(x)\mathbb{0}^{n-1}(x^2)$. We may then replace the generator $A(x)B_1(x^2)\dots B_m(x^2)$ by $\mathbb{0}^{n-1}(x^2)B_1(x^2)\dots B_m(x^2)$, and obtain the description (III). □

Corollary 5.2. *Let N be a normal subgroup of $\mathfrak{G}$. Then $N/[N, \mathfrak{G}]$ is an elementary 2-group of rank* 1 *or* 2, *unless it is $N = \mathfrak{G}$ (of rank* 3).

Corollary 5.3. *Every normal subgroup of G is characteristic.*

Proof. The automorphism group of $\mathfrak{G}$ is determined in [Bartholdi and Sidki 2003]: it also acts on the binary tree, and is

$$\operatorname{Aut}\mathfrak{G} = \langle G, \mathbb{1}^j\mathbb{0}[a,d] \text{ for all } j \in \mathbb{N}\rangle.$$

It then follows that $[K, \operatorname{Aut}\mathfrak{G}] = \langle \mathbb{0}(x), x^2\rangle^{\mathfrak{G}}$ is a strict subgroup of K; and hence $[N, \operatorname{Aut}\mathfrak{G}] < N$ for any normal subgroup that is generated by expressions in $W(x)$ and $W(x^2)$ for words $W \in \{\mathbb{0}, \mathbb{1}\}^*$. The theorem asserts that all normal subgroups of $\mathfrak{G}$ below K have this form; it then suffices to check, for instance using the algorithms in GAP, that the finitely many normal subgroups of $\mathfrak{G}$ not in K are characteristic. □

Corollary 5.4. *The number b_n of normal subgroups of $\mathfrak{G}$ of index 2^n starts as follows, and is asymptotically $n^{\log_2 3}$. More precisely, we have* $\liminf b_n/n^{\log_2(3)} = 5^{-\log_2 3} \approx 0.078$ *and* $\limsup b_n/n^{\log_2 3} = \frac{2}{9} \approx 0.222$.

index 2^n	2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}
$\lvert\{N \triangleleft \mathfrak{G}\}\rvert$	1	7	7	7	5	3	3	3	5	5	7	5
	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}	2^{17}	2^{18}	2^{19}	2^{20}	2^{21}	2^{22}	2^{23}
	7	7	13	9	13	11	19	11	13	11	19	15
	2^{24}	2^{25}	2^{26}	2^{27}	2^{28}	2^{29}	2^{30}	2^{31}	2^{32}	2^{33}	2^{34}	
	25	21	37	23	31	23	37	25	37	31	55	

Proof. The number of subgroups of index 2^n behaves in a somewhat erratic way, but is greater when n is of the form 2^k+2, so that there is a maximal number of choices for A and C, and is smaller when n is of the form $5\cdot 2^k+1$. We compute the numbers F_k and f_k of normal subgroups of $\mathfrak{G}$ contained in K of index 2^n, with respectively $n = 2^k+2$ and $n = 5\cdot 2^k+1$, yielding the upper and lower bounds. The computations are simplified by the fact that for these two values of n there are only subgroups of type I.

We start with the upper bound, when $n = 2^k+2$. First, for $k = 2$, the subgroups of index 2^n are $W(\mathbb{0};;\mathbb{0})$, $W(\mathbb{0};\lambda;\mathbb{0})$ and $W(\mathbb{1};;\lambda)$, giving $F_2 = 3$. Then, for $k > 2$, the subgroups can of index 2^n can be described as follows:

(1) $W(A\mathbb{1}\mathbb{0};\mathscr{B}\mathbb{0};C\mathbb{0})$ for all $W(A\mathbb{0};\mathscr{B};C)$ counted in F_{k-1}, except when $C = \mathbb{0}^{k-3}$, when no subgroup appears in F_k, and when $C = \mathbb{0}^{k-2}$, when $C\mathbb{0}$ should be replaced by $\mathbb{0}^{k-3}\mathbb{1}$;

(2) $W(A\mathbb{0};\mathscr{B}\mathbb{1};C\mathbb{1})$ for all $W(A;\mathscr{B};C)$ counted in F_{k-1}, except when $C = \mathbb{0}^{k-3}$, when no subgroup appears in F_k, and when $C = \mathbb{0}^{k-2}$, when $C\mathbb{1}$ should be replaced by $\mathbb{0}^{k-1}$;

(3) $W(A\mathbb{0};\{A\}\cup\mathscr{B}\mathbb{1};C\mathbb{1})$, with the same qualifications as above;

(4) $W(\mathbb{0}^{k-2}\mathbb{1};;\mathbb{0}^{k-2})$.

It follows that $F_k = 3(F_{k-1}-1)+1$, so $F_k = \frac{2}{9}3^k+1$ for all $k \geq 2$.

For the lower bound, we have $f_0 = F_2 = 3$; and for $k > 0$, when $n = 5\cdot 2^k+1$, the subgroups can of index 2^n can be described as follows:

(1) $W(A\mathbb{1}\mathbb{1}; \mathcal{B}\mathbb{0}; C\mathbb{0})$ for all $W(A\mathbb{1}; \mathcal{B}; C)$ counted in f_{k-1};

(2) $W(A\mathbb{0}\mathbb{1}; \mathcal{B}\mathbb{1}; C\mathbb{1})$ for all $W(A\mathbb{1}; \mathcal{B}; C)$ counted in f_{k-1};

(3) $W(A\mathbb{0}\mathbb{1}; \{A\mathbb{0}\}\cup\mathcal{B}\mathbb{1}; C\mathbb{1})$, with the same qualifications as above;

(4) $W(\mathbb{1}^k\mathbb{0}; ; \mathbb{0}^{k+1})$ and $W(\mathbb{1}^k\mathbb{0}; \mathbb{1}^k; \mathbb{0}^{k+1})$.

It follows that $f_k = 3(f_{k-1}-2)+2$, so $F_k = 3^k+2$ for all $k \geq 0$.

In summary, the number of normal subgroups of index 2^n oscillates between $3^{\log_2((n-1)/5)}+2$ and $\frac{2}{9}3^{\log_2(n-2)}+1$ for $n \geq 6$ (when all normal subgroups of $\mathfrak{G}$ are contained in K). These bounds give respectively

$$5^{-\log_2 3}(n-1)^{\log_2 3} \quad \text{and} \quad \frac{2}{9}(n-2)^{\log_2 3}. \qquad \square$$

Note also the following curiosity:

Corollary 5.5. *The number of normal subgroups of index r of G is odd for all r's a power of* 2, *and even* (*in fact*, 0) *for all other r.*

(The same congruence phenomenon holds for the group $C_2 * C_3$, as observed by Thomas Müller [1996].)

Proof. The proof follows from the description of Theorem 5.1. Assume $r = 2^k$. To determine the parity of the number of subgroups of index r, it suffices to consider which $W(A; \mathcal{B}; C)$ expressions have no choices for $\mathcal{B}$. These are precisely the $W(A; ; \mathbb{0}^n)_{\mathrm{I}}$ with $2^{n+1} < \#A \leq 5\cdot 2^n$, the $W(\mathbb{0}^n\mathbb{1}\mathbb{0}; ; C)_{\mathrm{I}}$ with $2^n < \#C \leq 2^{n+1}$ and the $W(\infty; \mathbb{1}^n, C-1; C)_{\mathrm{III}}$ with $2^{n+1}+1 < \#C \leq 3\cdot 2^n+1$.

Now these last two families yield a subgroup for precisely the same values of k, namely those satisfying $6\cdot 2^j+2 \leq k \leq 7\cdot 2^j+1$, and therefore contribute nothing modulo 2. The first family contributes a subgroup for all k. $\square$

5.1. ***Normal subgroups in $\ddot{\Gamma}$.*** The normal subgroup growth of $\ddot{\Gamma}$ is much larger. As a crude lower bound, consider the quotient $A = \gamma_k(\ddot{\Gamma})/\gamma_{k+1}(\ddot{\Gamma})$, where we take $k = \frac{1}{2}(\alpha_{2n+1}+1)$. It is abelian of rank 2^n; indeed, the index of $\gamma_k(\ddot{\Gamma})$ is $3^{3^{2n-1}-2^{n-1}+1}$, and that of $\gamma_{k+1}(\ddot{\Gamma})$ is $3^{3^{2n-1}+2^{n-1}+1}$.

In the vector space $\mathbb{F}_3{}^j$, there are roughly $3^{\binom{j}{2}}$ subspaces; so A has about 3^{4^n} subgroups $S = N/\gamma_{k+1}(\ddot{\Gamma})$, each of them giving rise to a subgroup N of index roughly 3^{9^n}.

It then follows that the number of normal subgroups of $\ddot{\Gamma}$ of index 3^n is at least $3^{n^{\log_3 2}}$, a function intermediate between polynomial and exponential growth. More precise estimations of the normal subgroup growth of $\ddot{\Gamma}$ will be the topic of a future paper.

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