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We set up a noncommutative version of the Bernšteĭn–Gel'fand–Gel'fand (BGG) correspondence and apply it to periodic injective resolutions.

Introduction

The Bernšteĭn–Gel'fand–Gel'fand (BGG) correspondence is surprising. Originally established in [Bernšteĭn et al. 1978, Theorem 2], it gives an equivalence of categories

$$\overline{\mathsf{gr}}(E) \simeq \mathsf{D}^{\mathsf{b}}(\mathsf{coh}\,\mathbb{P}^e).$$

Let me explain this formula: On the left, E is the exterior algebra $\bigwedge(Y_1, \dots, Y_{e+1})$ and $\overline{\mathsf{gr}}(E)$ is the category of finitely generated graded E-left-modules modulo morphisms which factor through injectives. On the right, $\mathbb{P}^e$ is e-dimensional projective space, $\mathsf{coh}\,\mathbb{P}^e$ is the category of coherent sheaves on $\mathbb{P}^e$, and $\mathsf{D}^{\mathsf{b}}(\mathsf{coh}\,\mathbb{P}^e)$ is the derived category of bounded complexes of such sheaves.

The surprising thing about the correspondence is that the geometric object on the right-hand side is equivalent to the purely algebraic object on the left-hand side. Put differently, if one did not know about the BGG correspondence, it would really not be obvious that it is possible to recover $\mathsf{D}^{\mathsf{b}}(\mathsf{coh}\,\mathbb{P}^e)$ purely algebraically!

In this paper, I will generalize the BGG correspondence to noncommutative projective geometry. Noncommutative projective geometry is well established; one of the seminal papers is [Artin and Zhang 1994] but many have been published since, showing how a range of projective geometry can be generalized in a noncommutative way. This turns out also to be true of the BGG correspondence, which is generalized in Theorem 3.1 below and now takes the form

$$\overline{\mathsf{Gr}}(A^!) \simeq \mathsf{D}(\mathsf{QGr}\,A).$$

Here A is a suitable noncommutative graded algebra with Koszul dual algebra $A^!$, and the category $\mathsf{QGr}(A)$ is a noncommutative analogue of the category $\mathsf{QCoh}(\mathbb{P}^e)$ of quasi-coherent sheaves on $\mathbb{P}^e$.

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After proving this, I consider an application to periodic injective resolutions. The background is a result by Eisenbud [2002, Theorem 2.2]: Let M be a finitely generated graded module without injective direct summands over the exterior algebra E, for which the Bass numbers

$$\mu^i(M) = \dim_k \operatorname{Ext}^i_E(k, M)$$

are bounded for $i \geq 0$. Then the minimal injective resolution I of M is periodic with period one: All the modules I^i and all the differentials ∂^i_I are the same, up to isomorphism and degree shift. (In fact, Eisenbud worked with minimal free resolutions, but using the Matlis duality functor $\operatorname{Hom}_k(-, k)$ on his result gives the above.)

I will show that this phenomenon can be understood geometrically in a very simple way: Using the BGG correspondence, the module M can be translated to a geometric object on $\mathbb{P}^e$. Since the Bass numbers of M are bounded, this object turns out to have zero-dimensional support, so is stable under twisting, that is, tensoring by $\mathcal{O}_{\mathbb{P}^e}(1)$. Translating back, this means that M is its own first syzygy, and periodicity of the minimal injective resolution follows.

Next, I consider the noncommutative case where a similar procedure yields remarkably different results: Let A be a noncommutative graded algebra, and let M be a finitely generated graded module over the Koszul dual $A^!$, for which the Bass numbers $\mu^i(M)$ are bounded for $i \geq 0$. Then, choosing A and M suitably, it is possible to make the minimal injective resolution of M periodic with any finite period, or to make it aperiodic.

The reason is that when translating M through the noncommutative BGG correspondence, one still obtains a geometric object with zero-dimensional support. However, due to the noncommutative (hence nonlocal) nature of the situation, it is no longer true that such an object is invariant under twisting. Rather, the object can have an orbit of any finite length, or an infinite orbit. Translating back gives the above results on periodicity of the minimal injective resolution.

The concrete example I will give of this behaviour is already known from [Smith 1996]. But the present geometric view through the BGG correspondence is new.

Here is a synopsis of the paper. Section 1 exhibits $\mathsf{D}(\mathsf{QGr}\, A)$ as a full subcategory of $\mathsf{D}(\mathsf{Gr}\, A)$. Section 2 considers a version of Koszul duality. Section 3 combines these results into the noncommutative BGG correspondence, and shows that under the correspondence, the simple module k over $A^!$ corresponds to the "structure sheaf" $\mathcal{O}$ in $\mathsf{D}(\mathsf{QGr}\, A)$.

Section 4 does a few computations that are put to use in Section 5, where the BGG correspondence is applied to periodicity of minimal injective resolutions.

To avoid a lengthy section on nomenclature, hints on notation are given along the way. The reader should rest assured that no new, let alone revolutionary, notation is introduced. The paper remains firmly on classical ground, and differs notationally only in minor details from such papers as [Artin and Zhang 1994], [Jørgensen 1999], and [Smith 1996]. However, I do need the following blanket items, which apply throughout.

Setup 0.1. k is a field, and $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ is a connected $\mathbb{N}$-graded noetherian k-algebra which is AS regular and Koszul (see [Jørgensen 1999, p. 206] and [Beilinson et al. 1996, def. 1.2.1], or Remark 0.2). I assume gldim $A = d \geq 2$.

Remark 0.2.

(i) For A to be AS regular means that gldim $A = d$ is finite, and that the graded A-bi-module $k = A/A_{\geq 1}$ satisfies

$$\operatorname{Ext}^i_A(k, A) \cong \operatorname{Ext}^i_{A^{\mathrm{op}}}(k, A) \cong \begin{cases} 0 & \text{for } i \neq d, \\ k(\ell) & \text{for } i = d \end{cases}$$

for some ℓ. As usual, $(-)(\ell)$ denotes ℓ-th degree shift of graded modules, so $M(\ell)_i = M_{i+\ell}$.

(ii) For A to be Koszul means that the minimal free resolution L of the graded A-left-module $k = A/A_{\geq 1}$ is linear. That is, the i-th module L_i has all its generators in graded degree i, so has the form $\coprod A(-i)$.

(iii) It is easy to see that since A is Koszul, the constant ℓ in (i) must be d.

(iv) By [Beilinson et al. 1996, Cor. 2.3.3], the algebra A is quadratic, that is, it has the form

$$A \cong \mathrm{T}(V)/(R)$$

where V is a finite-dimensional vector space, $\mathrm{T}(V)$ the tensor algebra, and (R) the two sided ideal generated by a space of relations R in $V \otimes_k V$. Let $(-)'$ denote $\operatorname{Hom}_k(-, k)$ and define $R^\perp$ by the exact sequence

$$0 \to R^\perp \longrightarrow V' \otimes_k V' \longrightarrow R' \to 0.$$

Then the Koszul dual algebra of A is

$$A^! = \mathrm{T}(V')/(R^\perp);$$

see [Beilinson et al. 1996, Def. 2.8.1].

(v) By [Beilinson et al. 1996, Theorem 2.10.1] there is an isomorphism $(A^!)^{\mathrm{op}} \cong \operatorname{Ext}_A(k, k)$. Combining this with gldim $A = d$ gives that $A^!$ is concentrated in graded degrees $0, \ldots, d$.

(vi) The algebra $A^!$ is graded Frobenius by [Smith 1996, Proposition 5.10]. This means that $\dim_k A^!$ is finite, and that there is an isomorphism of graded $A^!$-left-modules $(A^!)' \cong A^!(m)$, where $(A^!)' = \operatorname{Hom}_k(A^!, k)$ is the Matlis dual module of $A^!$.

(vii) Since $A^!$ is concentrated in graded degrees $0, \dots, d$, the constant m in (vi) must be d. So there is an isomorphism of graded $A^!$-left-modules $(A^!)' \cong A^!(d)$.

1. The categories $\mathsf{Gr}(A)$ and $\mathsf{QGr}(A)$

Remark 1.1. Let me first recapitulate a few items from [Artin and Zhang 1994], to which I refer for further details and proofs.

The category $\mathsf{Gr}(A)$ has as objects all $\mathbb{Z}$-graded A-left-modules and as morphisms all homomorphisms of A-left-modules which preserve graded degree.

A module M in $\mathsf{Gr}(A)$ is called torsion if each m in M is annihilated by $A_{\geq n}$ for some n. The torsion modules form a dense subcategory $\mathsf{Tors}(A)$ of $\mathsf{Gr}(A)$, and the quotient category is

$$\mathsf{QGr}(A) = \mathsf{Gr}(A)/\mathsf{Tors}(A).$$

This category behaves like the category of quasi-coherent sheaves on the space $\mathrm{Proj}(A)$, although $\mathrm{Proj}(A)$ itself may not make sense. For instance, if A is commutative, $\mathsf{QGr}(A)$ is in fact equivalent to the category of quasi-coherent sheaves on $\mathrm{Proj}(A)$ by Serre's theorem, as given in [Artin and Zhang 1994, Theorem, p. 229].

The degree shifting functor $(-)(1)$ on $\mathsf{Gr}(A)$ induces a functor on $\mathsf{QGr}(A)$ which I will also denote $(-)(1)$.

The category $\mathsf{Gr}(A)$ has the full subcategory $\mathsf{gr}(A)$ consisting of finitely generated modules. Induced by this, $\mathsf{QGr}(A)$ has the full subcategory $\mathsf{qgr}(A)$ which behaves like the category of coherent sheaves on $\mathrm{Proj}(A)$.

The projection functor $\mathsf{Gr}(A) \xrightarrow{\pi} \mathsf{QGr}(A)$ has a right-adjoint functor

$$\mathsf{QGr}(A) \xrightarrow{\omega} \mathsf{Gr}(A)$$

by [Artin and Zhang 1994, p. 234], so there is an adjoint pair

$$\mathsf{Gr}(A) \underset{\omega}{\overset{\pi}{\rightleftarrows}} \mathsf{QGr}(A).$$

As follows from [Artin and Zhang 1994, Proposition 7.1], these functors send injective objects to injective objects, and restrict to a pair of quasi-inverse equivalences

$$\mathsf{Inj}_{\,\mathrm{tf}}(A) \underset{\omega}{\overset{\pi}{\rightleftarrows}} \mathsf{QInj}(A) \tag{1}$$

between the subcategory of torsion-free injective objects of $\mathsf{Gr}(A)$ and the subcategory of all injective objects of $\mathsf{QGr}(A)$.

Let me next turn to derived categories. The projection functor π is exact and so extends to a triangulated functor

$$\mathsf{D}(\mathsf{Gr}\,A) \xrightarrow{\pi} \mathsf{D}(\mathsf{QGr}\,A)$$

between derived categories. Moreover, since A has finite global dimension, each object of the category $\mathsf{Gr}(A)$ has a bounded resolution by injective objects. The same therefore holds for $\mathsf{QGr}(A)$, as one sees using ω and π. So right-derived functors can be defined on the unbounded derived categories $\mathsf{D}(\mathsf{Gr}\,A)$ and $\mathsf{D}(\mathsf{QGr}\,A)$ by [Weibel 1994, Section 10.5].

In particular, $\mathsf{D}(\mathsf{QGr}\,A) \xrightarrow{\mathrm{R}\omega} \mathsf{D}(\mathsf{Gr}\,A)$ exists, and it is not hard to see that

$$\mathsf{D}(\mathsf{Gr}\,A) \underset{\mathrm{R}\omega}{\overset{\pi}{\rightleftarrows}} \mathsf{D}(\mathsf{QGr}\,A) \tag{2}$$

is an adjoint pair of functors.

Definition 1.2. Let

$$k^{\perp} = \{N \in \mathsf{D}(\mathsf{Gr}\,A) \mid \mathrm{RHom}_A(k, N) = 0\}.$$

Proposition 1.3. *The functors in equation* (2) *restrict to a pair of quasi-inverse equivalences of triangulated categories*

$$k^{\perp} \underset{\mathrm{R}\omega}{\overset{\pi}{\rightleftarrows}} \mathsf{D}(\mathsf{QGr}\,A). \tag{3}$$

Proof. First observe that diagram (1) extends to a pair of quasi-inverse equivalences

$$\mathsf{K}(\mathsf{Inj}_{\mathrm{tf}}\,A) \underset{\omega}{\overset{\pi}{\rightleftarrows}} \mathsf{K}(\mathsf{QInj}\,A) \tag{4}$$

between the homotopy category of complexes of torsion free injective objects of $\mathsf{Gr}(A)$, and the homotopy category of complexes of injective objects of $\mathsf{QGr}(A)$.

Next, the finite global dimension of A implies that $\mathsf{K}(\mathsf{Inj}\,A)$, the homotopy category of complexes of injective objects of $\mathsf{Gr}(A)$, is equivalent to $\mathsf{D}(\mathsf{Gr}\,A)$. Under the equivalence, the restriction of a functor F to $\mathsf{K}(\mathsf{Inj}\,A)$ corresponds to the right derived functor $\mathrm{R}F$ on $\mathsf{D}(\mathsf{Gr}\,A)$. See [Weibel 1994, Section 10.5], for example. A similar remark applies to $\mathsf{K}(\mathsf{QInj}\,A)$ and $\mathsf{D}(\mathsf{QGr}\,A)$. So forming

$$\mathsf{K}(\mathsf{Inj}\,A) \underset{\omega}{\overset{\pi}{\rightleftarrows}} \mathsf{K}(\mathsf{QInj}\,A)$$

gives a diagram which, up to equivalence, is just diagram (2).

This shows that diagram (4) gives an equivalence between some subcategory of $\mathsf{D}(\mathsf{Gr}\,A)$ and the whole category $\mathsf{D}(\mathsf{QGr}\,A)$. To finish the proof, I must show that the subcategory in question is $k^{\perp}$. That is, I must show that the subcategory $\mathsf{K}(\mathsf{Inj}_{\mathrm{tf}}\,A)$ of $\mathsf{K}(\mathsf{Inj}\,A)$ corresponds to the subcategory $k^{\perp}$ of $\mathsf{D}(\mathsf{Gr}\,A)$. For this, note that by the above, the functor $\mathrm{Hom}_A(k,-)$ on $\mathsf{K}(\mathsf{Inj}\,A)$ corresponds to the derived functor $\mathrm{RHom}_A(k,-)$ on $\mathsf{D}(\mathsf{Gr}\,A)$, so I must show that $\mathsf{K}(\mathsf{Inj}_{\mathrm{tf}}\,A)$ is the subcategory of $\mathsf{K}(\mathsf{Inj}\,A)$ annihilated by $\mathrm{Hom}_A(k,-)$.

In fact, this is not quite true, but it is true and easy to see that the subcategory of $\mathsf{K}(\mathsf{Inj}\,A)$ annihilated by $\mathrm{Hom}_A(k,-)$ consists exactly of the complexes isomorphic to complexes in $\mathsf{K}(\mathsf{Inj}_{\mathrm{tf}}\,A)$, and this is enough. □

2. Koszul duality

Remark 2.1. Let me recapitulate the version of Koszul duality set up by Beilinson, Ginzburg and Soergel, and elaborated on by Fløystad.

In [Beilinson et al. 1996, proof of Theorem 2.12.1] and [Fløystad 2003, Section 3.2] we find a construction for an adjoint pair of functors between categories of complexes of graded modules,

$$\mathsf{Ch}(\mathsf{Gr}\,A^!) \underset{\mathrm{G}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathsf{Ch}(\mathsf{Gr}\,A). \tag{5}$$

These functors are defined as follows: Given M in $\mathsf{Ch}(\mathsf{Gr}\,A^!)$, one constructs a double complex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \to & A(1)\otimes M_1^{-1} & \to & A(1)\otimes M_1^{0} & \to & A(1)\otimes M_1^{1} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \to & A\otimes M_0^{-1} & \to & A\otimes M_0^{0} & \to & A\otimes M_0^{1} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \to & A(-1)\otimes M_{-1}^{-1} & \to & A(-1)\otimes M_{-1}^{0} & \to & A(-1)\otimes M_{-1}^{1} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

with certain differentials, and the total complex $\mathrm{Tot}^{\coprod}$, defined using coproducts, is $\mathrm{F}(M)$. In the diagram, superscripts indicate cohomological degree and subscripts indicate graded degree. Also, $\otimes$ denotes tensor product over k.

And given N in $\mathsf{Ch}(\mathsf{Gr}\,A)$, one constructs a double complex

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \mathrm{Hom}(A^!(-1), N_1^{-1}) & \longrightarrow & \mathrm{Hom}(A^!(-1), N_1^{0}) & \longrightarrow & \mathrm{Hom}(A^!(-1), N_1^{1}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \mathrm{Hom}(A^!, N_0^{-1}) & \longrightarrow & \mathrm{Hom}(A^!, N_0^{0}) & \longrightarrow & \mathrm{Hom}(A^!, N_0^{1}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \mathrm{Hom}(A^!(1), N_{-1}^{-1}) & \longrightarrow & \mathrm{Hom}(A^!(1), N_{-1}^{0}) & \longrightarrow & \mathrm{Hom}(A^!(1), N_{-1}^{1}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

with certain differentials, and the total complex $\mathrm{Tot}^{\prod}$, defined using products, is $\mathrm{G}(N)$. In the diagram, Hom denotes homomorphisms over k.

Now consider $\mathsf{CoFree}(A^!)$, the full subcategory of $\mathsf{Gr}(A^!)$ consisting of modules which have the form $\prod_j (A^!)'(m_j)$, and $\mathsf{Free}(A)$, the full subcategory of $\mathsf{Gr}(A)$ consisting of modules which have the form $\coprod_i A(n_i)$. On the corresponding homotopy categories of complexes, the functors F and G induce functors which, abusively, I will denote by the same letters,

$$\mathsf{K}(\mathsf{CoFree}\,A^!) \underset{\mathrm{G}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathsf{K}(\mathsf{Free}\,A). \tag{6}$$

According to [Fløystad 2003, Proposition 5.11], this is a pair of quasi-inverse equivalences of triangulated categories.

Finite global dimension of A implies that $\mathsf{D}(\mathsf{Gr}\,A)$ is equivalent to $\mathsf{K}(\mathsf{Free}\,A)$ (see [Weibel 1994, Section 10.5]), so the equivalences (6) can also be read as

$$\mathsf{K}(\mathsf{CoFree}\,A^!) \underset{\mathrm{G}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathsf{D}(\mathsf{Gr}\,A). \tag{7}$$

Remark 2.2. The name Koszul duality is potentially confusing: "duality" might lead one to think of contravariant functors, while F and G are in fact covariant.

For the following lemma, note that I use $\Sigma^i(-)$ for the i-th suspension, so if M is a complex then $(\Sigma^i M)^\ell = M^{i+\ell}$.

Lemma 2.3. *The functors* F *and* G *in equation* (7) *satisfy the following.*

(i) $\mathrm{F}(M(i)) \cong \Sigma^i(\mathrm{F}M)(-i)$.

(ii) $\mathrm{G}(N(j)) \cong \Sigma^j(\mathrm{G}N)(-j)$.

(iii) $\mathrm{F}((A^!)')$ *is isomorphic to the A-left-module k.*

Proof. (i) and (ii) can be seen by playing with the double complexes which define F and G. (iii) follows from [Beilinson et al. 1996, Theorem 2.12.5(iii)]. □

Remark 2.4. The injective stable category over a ring is defined as the module category modulo the ideal of morphisms which factor through an injective module.

The present paper uses the graded version of this, so the injective stable category $\overline{\mathsf{Gr}}(A^!)$ is defined as $\mathsf{Gr}(A^!)$ modulo the ideal of morphisms which factor through an injective object of $\mathsf{Gr}(A^!)$.

Since $A^!$ is graded Frobenius by Remark 0.2(vi), the category $\mathsf{Gr}(A^!)$ is Frobenius by the graded version of [Happel 1987, Section 9.2], and so the category $\overline{\mathsf{Gr}}(A^!)$ is triangulated by [Happel 1987, Section 9.4]. For M in $\overline{\mathsf{Gr}}(A^!)$, the suspension ΣM is the first syzygy in an injective resolution of M. So ΣM is the cokernel of an injective pre-envelope, that is, an injective homomorphism $M \longrightarrow I$ in $\mathsf{Gr}(A^!)$, where I is an injective object of $\mathsf{Gr}(A^!)$. Any injective pre-envelope can be used; changing the injective pre-envelope does not change the isomorphism class of ΣM in $\overline{\mathsf{Gr}}(A^!)$.

The degree shifting functor $(-)(1)$ on $\mathsf{Gr}(A^!)$ induces a functor on $\overline{\mathsf{Gr}}(A^!)$ which I will also denote $(-)(1)$.

Since $\mathsf{Gr}(A^!)$ is Frobenius, the methods of [Keller 1994, Section 4.3] show that the category $\overline{\mathsf{Gr}}(A^!)$ is equivalent to the full subcategory of exact complexes in $\mathsf{K}(\mathsf{CoFree}\, A^!)$. Under the equivalence, a module M corresponds to a complete cofree resolution C of M, that is, a complex C in $\mathsf{K}(\mathsf{CoFree}\, A^!)$ which is exact and has its zeroth cycle module $\mathrm{Z}^0(C)$ isomorphic to M.

Under the equivalence between $\overline{\mathsf{Gr}}(A^!)$ and the full subcategory of exact complexes in $\mathsf{K}(\mathsf{CoFree}\, A^!)$, the suspension Σ on $\overline{\mathsf{Gr}}(A^!)$ corresponds to the ordinary suspension Σ on $\mathsf{K}(\mathsf{CoFree}\, A^!)$, given by moving complexes one step to the left and switching signs of differentials. Also, the functor $(-)(1)$ on $\overline{\mathsf{Gr}}(A^!)$ corresponds to the functor $(-)(1)$ on $\mathsf{K}(\mathsf{CoFree}\, A^!)$ induced by degree shifting of $A^!$-left-modules.

Proposition 2.5. *The functors in* (7) *induce a pair of quasi-inverse equivalences of triangulated categories*

$$\overline{\mathsf{Gr}}(A^!) \rightleftarrows k^{\perp}.$$

Proof. Remark 2.4 identifies $\overline{\mathsf{Gr}}(A^!)$ with the full subcategory of exact complexes in $\mathsf{K}(\mathsf{CoFree}\, A^!)$, and Definition 1.2 defines $k^{\perp}$ as a full subcategory of $\mathsf{D}(\mathsf{Gr}\, A)$. To prove the proposition, I must show that these subcategories are mapped to each other by the functors F and G of equation (7).

However, let N be in $\mathsf{D}(\mathsf{Gr}\,A)$. Then the j-th graded component of the i-th cohomology module of the complex $\mathrm{G}N$ is

$$\begin{aligned} \mathrm{h}^i(\mathrm{G}N)_j &\overset{\text{(a)}}{\cong} \mathrm{Hom}_{\mathsf{K}(\mathsf{Gr}\,A^!)}(A^!, \Sigma^i(\mathrm{G}N)(j)) \\ &\overset{\text{(b)}}{\cong} \mathrm{Hom}_{\mathsf{K}(\mathsf{CoFree}\,A^!)}((A^!)'(-d), \Sigma^i(\mathrm{G}N)(j)) \\ &\cong \mathrm{Hom}_{\mathsf{K}(\mathsf{CoFree}\,A^!)}(\Sigma^{-i}(A^!)'(-d-j), \mathrm{G}N) \\ &= (*), \end{aligned}$$

where (a) is classical and (b) holds because of $A^! \cong (A^!)'(-d)$; see Remark 0.2(vii). Adjointness between F and G gives (c) in

$$\begin{aligned} (*) &\overset{\text{(c)}}{\cong} \mathrm{Hom}_{\mathsf{D}(\mathsf{Gr}\,A)}(\mathrm{F}(\Sigma^{-i}(A^!)'(-d-j)), N) \\ &\overset{\text{(d)}}{\cong} \mathrm{Hom}_{\mathsf{D}(\mathsf{Gr}\,A)}(\Sigma^{-i-d-j}\,\mathrm{F}((A^!)')(d+j), N) \\ &\cong \mathrm{Hom}_{\mathsf{D}(\mathsf{Gr}\,A)}(\mathrm{F}((A^!)'), \Sigma^{i+d+j}N(-d-j)) \\ &\overset{\text{(e)}}{\cong} \mathrm{Hom}_{\mathsf{D}(\mathsf{Gr}\,A)}(k, \Sigma^{i+j+d}N(-j-d)) \\ &\cong \mathrm{h}^{i+j+d}\,\mathrm{RHom}_A(k, N)_{-j-d}, \end{aligned}$$

and (d) and (e) are by Lemma 2.3, parts (i) and (iii).

But now it is clear that $\mathrm{G}N$ is exact if and only if N is in $k^{\perp}$, as desired. □

3. The BGG correspondence

Composing the equivalences of categories from Propositions 1.3 and 2.5 gives the following main theorem of the paper.

Theorem 3.1 (The BGG correspondence). *There are quasi-inverse equivalences of triangulated categories*

$$\overline{\mathsf{Gr}}(A^!) \underset{\gamma}{\overset{\varphi}{\rightleftarrows}} \mathsf{D}(\mathsf{QGr}\,A).$$

Example 3.2. If A is the polynomial algebra $k[X_1, \dots, X_d]$ then it is classical that A satisfies the conditions of Setup 0.1, and the definition of $A^!$ in Remark 0.2(iv) makes it easy to see that $A^!$ is the exterior algebra $E = \bigwedge(Y_1, \dots, Y_d)$. Also, $\mathsf{QGr}(A)$ is equivalent to the category $\mathsf{QCoh}(\mathbb{P}^{d-1})$ of quasi-coherent sheaves on $(d-1)$-dimensional projective space by Serre's theorem, [Artin and Zhang 1994, Theorem, p. 229]. So Theorem 3.1 gives an equivalence of categories

$$\overline{\mathsf{Gr}}(E) \simeq \mathsf{D}(\mathsf{QCoh}\,\mathbb{P}^{d-1}).$$

This is the classical BGG correspondence, originally established in [Bernšteĭn et al. 1978, Theorem 2], with the slight improvement of dealing with the stable category of all modules and the unbounded derived category of quasi-coherent sheaves rather than the finite subcategories in [Bernšteĭn et al. 1978, Theorem 2].

Remark 3.3. The quasi-inverse equivalences φ and γ from Theorem 3.1 are constructed by composing some other functors. Untangling the construction gives the following concrete descriptions.

To get $\varphi(M)$, take a complete cofree resolution C of M. Then $\varphi(M) = \pi\mathrm{F}(C)$, where F is one of the functors from equation (7) and π is one of the functors from equation (2).

To get $\gamma(\mathcal{M})$, consider $\mathrm{G}(\mathrm{R}\omega(\mathcal{M}))$, where $\mathrm{R}\omega$ is one of the functors from equation (2) and G is one of the functors from equation (7). This is an object of $\mathsf{K}(\mathsf{CoFree}\,A^!)$ and in fact, it is even in the full subcategory of exact complexes of $\mathsf{K}(\mathsf{CoFree}\,A^!)$. Now $\gamma(\mathcal{M}) = \mathrm{Z}^0\mathrm{G}(\mathrm{R}\omega(\mathcal{M}))$, where Z^0 takes the zeroth cycle module.

The next lemma follows immediately from Lemma 2.3, parts (i) and (ii).

Lemma 3.4. *The functors φ and γ satisfy the following.*

(i) $\varphi(M(i)) \cong \Sigma^i(\varphi M)(-i)$.

(ii) $\gamma(\mathcal{M}(j)) \cong \Sigma^j(\gamma\mathcal{M})(-j)$.

For the following lemma, let L be the minimal free resolution of the graded $A^!$-left-module k. Each L^i is finitely generated free and hence cofree because Remark 0.2(vii) implies $A^! \cong (A^!)'(-d)$. So L is a complex in $\mathsf{K}(\mathsf{CoFree}\,A^!)$, and I can apply the functor F from equation (7) and get a complex $\mathrm{F}(L)$ in $\mathsf{D}(\mathsf{Gr}\,A)$.

Lemma 3.5. *The cohomology of* $\mathrm{F}(L)$ *is torsion.*

Proof. The version of F from equation (5) respects small colimits because it is constructed using tensor products and small coproducts. In the category of complexes $\mathsf{Ch}(\mathsf{Gr}\,A^!)$, the object L is the colimit of the objects

$$L\langle j\rangle = \cdots \longrightarrow 0 \longrightarrow L^{-j} \longrightarrow \cdots \longrightarrow L^0 \longrightarrow 0 \longrightarrow \cdots,$$

so

$$\mathrm{F}(L) \cong \mathrm{F}(\operatorname{colim} L\langle j\rangle) \cong \operatorname{colim} \mathrm{F}(L\langle j\rangle). \tag{8}$$

Now, $A^!$ is Koszul by [Beilinson et al. 1996, Proposition 2.9.1], and L is the minimal free resolution of k over $A^!$, and so

$$L^{-i} \cong \coprod A^!(-i). \tag{9}$$

This implies that L^{-i} is concentrated in graded degrees $i, \ldots, d+i$ because $A^!$ is concentrated in graded degrees $0, \ldots, d$ by Remark 0.2(v). So the construction in Remark 2.1 says that $\mathrm{F}(L\langle j\rangle)$ is $\mathrm{Tot}^{\coprod}$ of a double complex whose nonzero part can be sketched as

$$
\begin{array}{ccccccc}
A(d+j)\otimes L^{-j}_{d+j} & & & & & & \\
\uparrow & & & & & & \\
A(d+j-1)\otimes L^{-j}_{d+j-1} & \longrightarrow & \cdots & & & & \\
\uparrow & & & & & & \\
\vdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & A(d)\otimes L^{0}_{d} \\
\uparrow & & & & & & \uparrow \\
A(j)\otimes L^{-j}_{j} & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \vdots \\
 & & & & & & \uparrow \\
 & & & & \cdots & \longrightarrow & A(1)\otimes L^{0}_{1} \\
 & & & & & & \uparrow \\
 & & & & & & A\otimes L^{0}_{0}.
\end{array}
\tag{10}
$$

Also, combining equation (9) with $A^! \cong (A^!)'(-d)$, which holds by Remark 0.2(vii), gives $L^{-i} \cong \coprod (A^!)'(-d-i)$. So, up to degree shift and suspension, the $(-i)$-th column of (10) is just a coproduct of copies of the column obtained from $(A^!)'$. This column has nonzero part

$$
\begin{array}{c}
A\otimes (A^!)'_0 \\
\uparrow \\
\vdots \\
\uparrow \\
A(-d)\otimes (A^!)'_{-d},
\end{array}
$$

and is a free resolution of the A-left-module k, as follows from [Beilinson et al. 1996, Theorem 2.12.5(iii)]. So the columns of (10) have cohomology only at the top ends, and the cohomology in the $(-i)$-th column is $\coprod k(d+i)$.

Now consider the first spectral sequence of the double complex (10) (see [Weibel 1994, Section 5.6]). The previous part of the proof shows that the E_2-term of the spectral sequence is nonzero only at the top ends of the columns of (10), where

$$E_2^{0d} \cong \coprod k(d), \quad \ldots, \quad E_2^{-j,d+j} \cong \coprod k(d+j).$$

Since the double complex is bounded in all directions, the spectral sequence converges towards the cohomology of $\mathrm{Tot}^{\coprod}$. Consequently, $\mathrm{Tot}^{\coprod}$ of the double complex has cohomology only in cohomological degree d, and this cohomology sits in graded degrees $-d, \dots, -d-j$.

But this $\mathrm{Tot}^{\coprod}$ is $\mathrm{F}(L\langle j\rangle)$. So (8) now shows that $\mathrm{F}(L)$ has cohomology only in cohomological degree d, and that this cohomology can be nonzero only in graded degrees $-d, -d-1, \dots$. In particular, the cohomology of $\mathrm{F}(L)$ is torsion. □

Now consider the graded $A^!$-left-module k viewed as an object of $\overline{\mathsf{Gr}}(A^!)$, and consider $\mathcal{O}$, the "structure sheaf" in $\mathsf{QGr}(A)$ defined by $\mathcal{O} = \pi(A)$. Then $\mathcal{O}$ can also be viewed as a complex in $\mathsf{D}(\mathsf{QGr}\,A)$ concentrated in cohomological degree zero, and the following result holds.

Theorem 3.6. *The functor φ satisfies $\varphi(k) \cong \mathcal{O}$.*

Proof. To get $\varphi(k)$, I must take $\pi\mathrm{F}(C)$, where C is a complete cofree resolution of the $A^!$-left-module k, while F and π are the functors from equations (7) and (2); see Remark 3.3.

For this, consider first the functors F and G from (7). Let X in $\mathsf{K}(\mathsf{CoFree}\,A^!)$ be a cofree resolution of k. From [Beilinson et al. 1996, Theorem 2.12.5(iii)] there follows $\mathrm{F}(X) \cong A$. Hence $\mathrm{GF}(X) \cong \mathrm{G}(A)$, and as F and G are quasi-inverse equivalences of categories, this implies $X \cong \mathrm{G}(A)$. But k is quasi-isomorphic to X, so this shows that k is quasi-isomorphic to $\mathrm{G}(A)$. However, it is clear from the construction of G in Remark 2.1 that $\mathrm{G}(A)$ is a complex of cofree modules placed in nonnegative cohomological degrees. All in all, $\mathrm{G}(A)$ must be a cofree resolution of k, so there is a canonical morphism $k \longrightarrow \mathrm{G}(A)$.

Now let L be a minimal free resolution of k as in Lemma 3.5, so there is a canonical morphism $L \longrightarrow k$. Composing the morphisms $L \longrightarrow k$ and $k \longrightarrow \mathrm{G}(A)$ gives a morphism $L \longrightarrow \mathrm{G}(A)$ whose mapping cone C is easily seen to be a complete cofree resolution of k.

The distinguished triangle $L \longrightarrow \mathrm{G}(A) \longrightarrow C \longrightarrow$ in $\mathsf{K}(\mathsf{CoFree}\,A^!)$ gives a distinguished triangle

$$\pi\mathrm{F}(L) \longrightarrow \pi\mathrm{FG}(A) \longrightarrow \pi\mathrm{F}(C) \longrightarrow$$

in $\mathsf{D}(\mathsf{QGr}\,A)$. Let me compute the three complexes here: The cohomology of $\mathrm{F}(L)$ is torsion by Lemma 3.5, so $\pi\mathrm{F}(L) \cong 0$. And F and G are quasi-inverse equivalences, so $\mathrm{FG}(A)$ is isomorphic to A, so $\pi\mathrm{FG}(A) \cong \pi(A) = \mathcal{O}$.

Finally, $\pi\mathrm{F}(C)$ is $\varphi(k)$ as mentioned above. So the distinguished triangle reads

$$0 \longrightarrow \mathcal{O} \longrightarrow \varphi(k) \longrightarrow ,$$

proving $\varphi(k) \cong \mathcal{O}$. □

4. Computations

This section contains computations, some involving the BGG correspondence, which will be used on periodic injective resolutions in Section 5.

The following lemma is just a graded version of [Benson 1998, Cor. 2.5.4(ii)].

Lemma 4.1. *Let M be in $\overline{\mathsf{Gr}}(A^!)$. There are canonical isomorphisms*

$$\operatorname{Hom}_{\overline{\mathsf{Gr}}(A^!)}(k, \Sigma^i M) \longrightarrow \operatorname{Ext}^i_{\mathsf{Gr}(A^!)}(k, M)$$

for $i \geq 1$.

Lemma 4.2. *Let M be in $\overline{\mathsf{Gr}}(A^!)$ and consider $\mathcal{M} = \varphi(M)$ in $\mathsf{D}(\mathsf{QGr}\, A)$. Then*

$$\operatorname{Ext}^i_{\mathsf{Gr}(A^!)}(k, M(-i+j)) \cong \operatorname{Ext}^j_{\mathsf{QGr}(A)}(\mathcal{O}, \mathcal{M}(i-j))$$

for $i \geq 1$ and each j.

Proof. This is a simple computation,

$$\begin{aligned}
\operatorname{Ext}^i_{\mathsf{Gr}(A^!)}(k, M(-i+j)) &\overset{\text{(a)}}{\cong} \operatorname{Hom}_{\overline{\mathsf{Gr}}(A^!)}(k, \Sigma^i M(-i+j)) \\
&\overset{\text{(b)}}{\cong} \operatorname{Hom}_{\mathsf{D}(\mathsf{QGr}\, A)}(\varphi k, \varphi(\Sigma^i M(-i+j))) \\
&\overset{\text{(c)}}{\cong} \operatorname{Hom}_{\mathsf{D}(\mathsf{QGr}\, A)}(\mathcal{O}, \Sigma^j \mathcal{M}(i-j)) \\
&= \operatorname{Ext}^j_{\mathsf{QGr}(A)}(\mathcal{O}, \mathcal{M}(i-j)),
\end{aligned}$$

where (a) is by Lemma 4.1 and (b) is by the BGG correspondence, Theorem 3.1, while (c) is by Theorem 3.6 and Lemma 3.4(i). □

For the following lemma, observe that the finitely generated graded modules form a full subcategory $\overline{\mathsf{gr}}(A^!)$ of $\overline{\mathsf{Gr}}(A^!)$, and that the complexes which have bounded cohomology consisting of objects from the category $\mathsf{qgr}(A)$ form a full subcategory $\mathsf{D}^{\mathrm{f}}(\mathsf{QGr}\, A)$ of $\mathsf{D}(\mathsf{QGr}\, A)$.

Lemma 4.3. *The subcategories $\overline{\mathsf{gr}}(A^!)$ and $\mathsf{D}^{\mathrm{f}}(\mathsf{QGr}\, A)$ map to each other under the BGG correspondence*

$$\overline{\mathsf{Gr}}(A^!) \underset{\gamma}{\overset{\varphi}{\rightleftarrows}} \mathsf{D}(\mathsf{QGr}\, A).$$

Proof. It is not hard to check that $\overline{\mathsf{gr}}(A^!)$ consists of the objects of $\overline{\mathsf{Gr}}(A^!)$ which are finitely built from objects of the form $k(i)$.

Similarly, $\mathsf{D}^{\mathrm{f}}(\mathsf{QGr}\, A)$ consists of the objects of $\mathsf{D}(\mathsf{QGr}\, A)$ which are finitely built from objects of the form $\mathcal{O}(j)$.

But under the BGG correspondence, $k(i)$ corresponds to $\Sigma^i \mathcal{O}(-i)$ by Theorem 3.6 and Lemma 3.4(i), so the present lemma follows. □

Lemma 4.4. *Let $\mathcal{M}$ be in $\mathsf{D}^{\mathrm{f}}(\mathsf{QGr}\, A)$. Then for $i \gg 0$ I have*

$$\operatorname{Ext}^j_{\mathsf{QGr}(A)}(\mathcal{O}, \mathcal{M}(i-j)) \cong \operatorname{Hom}_{\mathsf{QGr}(A)}(\mathcal{O}, \mathrm{h}^j(\mathcal{M})(i-j))$$

for each j, where $\mathrm{h}^j(\mathcal{M})$ is the j-th cohomology of $\mathcal{M}$.

Proof. The algebra A has global dimension d by assumption, so $\mathsf{qgr}(A)$ has cohomological dimension at most $d-1$ by [Artin and Zhang 1994, Proposition 7.10(3)], so $\operatorname{Ext}^{\geq d}_{\mathsf{QGr}(A)}(\mathcal{O}, \mathcal{N}) = 0$ holds for each $\mathcal{N}$ in $\mathsf{qgr}(A)$.

Moreover, A is even AS regular by assumption, so $\mathsf{qgr}(A)$ satisfies Serre vanishing by [Artin and Zhang 1994, Theorems 8.1(1) and 7.4]. That is, given $\mathcal{N}$ in $\mathsf{qgr}(A)$ and given p with $1 \leq p \leq d-1$, I have $\operatorname{Ext}^p_{\mathsf{QGr}(A)}(\mathcal{O}, \mathcal{N}(r)) = 0$ for $r \gg 0$.

So given $\mathcal{N}$, I can kill all the $\operatorname{Ext}^p_{\mathsf{QGr}(A)}(\mathcal{O}, \mathcal{N}(r))$ with $p \geq 1$ by choosing r large enough. That is, given $\mathcal{N}$ in $\mathsf{qgr}(A)$, I have

$$r \gg 0 \;\Rightarrow\; \operatorname{Ext}^p_{\mathsf{QGr}(A)}(\mathcal{O}, \mathcal{N}(r)) = 0 \text{ for } p \geq 1. \tag{11}$$

There is a convergent spectral sequence

$$E_2^{pq} = \operatorname{Ext}^p_{\mathsf{QGr}(A)}(\mathcal{O}, \mathrm{h}^q(\mathcal{M})(i-j)) \Rightarrow \operatorname{Ext}^{p+q}_{\mathsf{QGr}(A)}(\mathcal{O}, \mathcal{M}(i-j))$$

by [Weibel 1994, 5.7.9] (convergence because the cohomology $\mathrm{h}(\mathcal{M})$ is bounded). By assumption on $\mathcal{M}$, the finitely many nonzero $\mathrm{h}^q(\mathcal{M})$'s are in $\mathsf{qgr}(A)$. So equation (11) implies that for $i-j \gg 0$, the term E_2^{pq} is concentrated on the line $p=0$. So the spectral sequence collapses and gives

$$\operatorname{Hom}_{\mathsf{QGr}(A)}(\mathcal{O}, \mathrm{h}^q(\mathcal{M})(i-j)) \cong \operatorname{Ext}^q_{\mathsf{QGr}(A)}(\mathcal{O}, \mathcal{M}(i-j)) \tag{12}$$

for $i-j \gg 0$ and each q.

Now observe that the isomorphism (12) also holds for $q \gg 0$, simply because both sides are then zero. For the left-hand side, this is true because $\mathrm{h}(\mathcal{M})$ is bounded. For the right-hand side, use that $\mathrm{h}(\mathcal{M})$ is bounded and that $\mathsf{qgr}(A)$ has cohomological dimension at most $d-1$.

So setting q equal to j, the isomorphism (12) holds for $j \gg 0$, and for other values of j I can force $i-j \gg 0$ by picking $i \gg 0$, and then the isomorphism also holds. That is,

$$\operatorname{Hom}_{\mathsf{QGr}(A)}(\mathcal{O}, \mathrm{h}^j(\mathcal{M})(i-j)) \cong \operatorname{Ext}^j_{\mathsf{QGr}(A)}(\mathcal{O}, \mathcal{M}(i-j))$$

for $i \gg 0$ and each j, proving the lemma. □

5. Periodic injective resolutions

This section shows how the BGG correspondence can be used to understand the periodicity of certain injective resolutions over exterior algebras as a geometric phenomenon.

I also show an analogous noncommutative example with much more complicated behaviour, due to the more intricate nature of noncommutative geometry.

The commutative case. Denote by E the exterior algebra $\bigwedge(Y_1, \dots, Y_d)$ over k, and recall that $\mathsf{gr}(E)$ is the category of finitely generated graded E-left-modules. The following result appears in [Eisenbud 2002, Theorem 2.2].

Theorem 5.1 (Eisenbud). *Let M in $\mathsf{gr}(E)$ be without injective direct summands, and suppose that the Bass numbers*

$$\mu^i(M) = \dim_k \operatorname{Ext}^i_E(k, M)$$

are bounded for $i \geq 0$.

Then the minimal injective resolution I of M is periodic with period one in the following sense: Up to isomorphism, I^i is $I^0(i)$ and ∂^i_I is $\partial^0_I(i)$.

In other words, up to isomorphism and degree shift, all the I^i and all the ∂^i_I are the same. (In fact, Eisenbud worked with minimal free resolutions, but using Matlis duality on his result gives Theorem 5.1.)

This phenomenon can be understood geometrically in a very simple way, using the BGG correspondence: The module M can be translated to a geometric object on $\mathbb{P}^{d-1}$, and since the Bass numbers of M are bounded, this object turns out to have zero-dimensional support. Therefore the object is stable under twisting, that is, tensoring by $\mathcal{O}_{\mathbb{P}^{d-1}}(1)$, and translating back, this gives that M is its own first syzygy, and periodicity of the minimal injective resolution follows.

In more detail, let A be the polynomial algebra $k[X_1, \dots, X_d]$ so I am in the situation of Example 3.2. In particular, $A^!$ is the exterior algebra $E = \bigwedge(Y_1, \dots, Y_d)$, and $\mathsf{QGr}(A)$ is equivalent to $\mathsf{QCoh}(\mathbb{P}^{d-1})$, with the subcategory $\mathsf{qgr}(A)$ corresponding to the subcategory $\mathsf{coh}(\mathbb{P}^{d-1})$ of coherent sheaves. Let M be in $\mathsf{gr}(E)$, and suppose that the Bass numbers

$$\mu^i(M) = \dim_k \operatorname{Ext}^i_E(k, M)$$

are bounded for $i \geq 0$.

The BGG correspondence associates to M the object

$$\mathcal{M} = \varphi(M) \in \mathsf{D}(\mathsf{QCoh}\, \mathbb{P}^{d-1}).$$

In fact, Lemma 4.3 even says that only finitely many of the cohomologies $\mathrm{h}^\ell(\mathcal{M})$ are nonzero, and that each $\mathrm{h}^\ell(\mathcal{M})$ is coherent.

For $i \geq 1$ I have

$$\begin{aligned} \mu^i(M) &= \dim_k \operatorname{Ext}^i_E(k, M) \\ &\overset{(a)}{=} \sum_j \dim_k \operatorname{Ext}^i_{\mathsf{Gr}(E)}(k, M(-i+j)) \\ &\overset{(b)}{=} \sum_j \dim_k \operatorname{Ext}^j_{\mathsf{QCoh}(\mathbb{P}^{d-1})}(\mathcal{O}_{\mathbb{P}^{d-1}}, \mathcal{M}(i-j)) \\ &= (*), \end{aligned} \tag{13}$$

where in (a), I am being clever by using the degree shift $-i + j$ instead of simply j, and where (b) is by Lemma 4.2. And for $i \gg 0$ I have

$$(*) = \sum_j \dim_k \operatorname{Hom}_{\mathsf{QCoh}(\mathbb{P}^{d-1})}(\mathcal{O}_{\mathbb{P}^{d-1}}, \mathrm{h}^j(\mathcal{M})(i-j)) \tag{14}$$

by Lemma 4.4.

It follows that if $\mu^i(M)$ is bounded for $i \geq 0$, then for each j,

$$\dim_k \operatorname{Hom}_{\mathsf{QCoh}(\mathbb{P}^{d-1})}(\mathcal{O}_{\mathbb{P}^{d-1}}, \mathrm{h}^j(\mathcal{M})(i-j))$$

is bounded for $i \geq 0$. Hence for each j,

$$\dim_k \operatorname{Hom}_{\mathsf{QCoh}(\mathbb{P}^{d-1})}(\mathcal{O}_{\mathbb{P}^{d-1}}, \mathrm{h}^j(\mathcal{M})(\ell)) \tag{15}$$

is bounded for $\ell \gg 0$. However, this is now a geometric statement: For $\ell \gg 0$, the polynomial growth rate of the numbers in equation (15) equals the dimension of the support of $\mathrm{h}^j(\mathcal{M})$ on $\mathbb{P}^{d-1}$, as follows from [Hartshorne 1977, Theorem I.7.5]. So it follows that each of the finitely many nonzero $\mathrm{h}^j(\mathcal{M})$ has zero-dimensional support; in other words, the support is a finite collection of points.

Now suppose that the ground field k is infinite. Then it is possible to pick a hyperplane H in $\mathbb{P}^{d-1}$ which is disjoint from the support of each $\mathrm{h}^j(\mathcal{M})$. To H corresponds an injection $\mathcal{O}_{\mathbb{P}^{d-1}}(1) \hookrightarrow \mathcal{O}_{\mathbb{P}^{d-1}}$ which is an isomorphism away from H. Tensoring over $\mathcal{O}_{\mathbb{P}^{d-1}}$ with $\mathcal{M}$ gives a morphism

$$\mathcal{M} \otimes \mathcal{O}_{\mathbb{P}^{d-1}}(1) \xrightarrow{\mu} \mathcal{M} \otimes \mathcal{O}_{\mathbb{P}^{d-1}},$$

and $\mathrm{h}^j(\mu)$ is $\mathrm{h}^j(\mathcal{M}) \otimes \mathcal{O}_{\mathbb{P}^{d-1}}(1) \longrightarrow \mathrm{h}^j(\mathcal{M}) \otimes \mathcal{O}_{\mathbb{P}^{d-1}}$. However, this is an isomorphism for each j because $\mathcal{O}_{\mathbb{P}^{d-1}}(1) \hookrightarrow \mathcal{O}_{\mathbb{P}^{d-1}}$ is an isomorphism away from H and hence an isomorphism on the support of each $\mathrm{h}^j(\mathcal{M})$. So μ is an isomorphism in $\mathsf{D}(\mathsf{QCoh}\,\mathbb{P}^{d-1})$, proving

$$\mathcal{M}(1) \cong \mathcal{M}.$$

Under the BGG correspondence this gives $\gamma(\mathcal{M}(1)) \cong \gamma(\mathcal{M})$, and using $\gamma(\mathcal{M}) = \gamma\varphi(M) \cong M$ and Lemma 3.4(ii) this can be rearranged as

$$\Sigma M \cong M(1) \tag{16}$$

in $\overline{\mathsf{Gr}}(E)$.

In $\overline{\mathsf{Gr}}(E)$, the suspension ΣM is computed as the first syzygy of M in an injective resolution; see Remark 2.4. So equation (16) shows that in $\overline{\mathsf{Gr}}(E)$, this first syzygy is just M itself, with a degree shift of one. It is possible to improve this with a few remarks: First, if M is without injective direct summands, then it is not hard to show that the isomorphism (16) lifts to hold in $\mathsf{Gr}(E)$, if ΣM is obtained as the first syzygy in a *minimal* injective resolution of M. Secondly, the assumption that k is infinite can be dropped using [Grothendieck 1965, Proposition 2.5.8].

Iterating equation (16) now shows that in the minimal injective resolution I of M, the syzygy $\Sigma^i M$ is simply $M(i)$. Hence the module I^i must be $I^0(i)$, and the differential ∂_I^i must be $\partial_I^0(i)$. So I have recovered Theorem 5.1.

The noncommutative case. In the above argument, the minimal injective resolution is periodic with period one because points in $\mathbb{P}^{d-1}$ are invariant under twisting. It is known that this invariance breaks down when one passes to noncommutative analogues of $\mathbb{P}^{d-1}$.

Here the twist can move points, and it is possible to have orbits of length n, for any finite n, and orbits of infinite length. So it is natural to expect that suitable noncommutative analogues of the above argument might give examples of algebras $A^!$, analogous to E, and modules M where $\mu^i(M)$ is bounded for $i \geq 0$, and yet where the minimal injective resolution of M is periodic with period n, or aperiodic. Indeed, this turns out to hold.

Note that the following example of this behaviour is already known from [Smith 1996]. But the present geometric view through the BGG correspondence is new.

Setup 5.2. Assume that the ground field k is algebraically closed. Let C be an elliptic curve over k with a line bundle $\mathcal{L}$ of degree d, and an automorphism τ given by translation by a point of C. Let A be the Sklyanin algebra associated to these data in [Smith 1996, Sec. 8].

Remark 5.3. Note that A satisfies the standing assumptions from Setup 0.1. In fact, A is a noncommutative analogue of the polynomial algebra on d variables $k[X_1, \ldots, X_d]$, and hence the Koszul dual $A^!$ is a noncommutative analogue of the exterior algebra $\bigwedge(Y_1, \ldots, Y_d)$.

Remark 5.4. The construction of A in [Smith 1996, Sec. 8] is so that the curve C sits inside $\mathbb{P}(A_1')$. So each point p on C is also a point in $\mathbb{P}(A_1')$, that is, a one dimensional subspace of A_1'. This subspace has an annihilator $p^\perp$ in A_1, and the graded A-left-module $P\langle p\rangle = A/Ap^{\perp}$ is a so-called point module. That is, it is cyclic, and each graded piece in nonnegative degrees is one dimensional.

Let me now use the functor π from Remark 1.1 to write

$$\mathcal{M}\langle p\rangle = \pi(P\langle p\rangle).$$

This is an object of $\mathsf{qgr}(A)$, and I view it as a complex concentrated in cohomological degree zero. This complex is an object of $\mathsf{D}(\mathsf{QGr}\,A)$, so finally the BGG correspondence gives the object

$$M\langle p\rangle = \gamma(\mathcal{M}\langle p\rangle)$$

in $\overline{\mathsf{Gr}}(A^!)$. In fact, $\mathcal{M}\langle p\rangle$ viewed as an object of $\mathsf{D}(\mathsf{QGr}\,A)$ is in the subcategory $\mathsf{D}^{\mathrm{f}}(\mathsf{QGr}\,A)$, so Lemma 4.3 says that $M\langle p\rangle$ is even in $\overline{\mathsf{gr}}(A^!)$.

Observe that $M\langle p\rangle$ is only well-defined up to isomorphism in $\overline{\mathsf{Gr}}(A^!)$, so when looking at $M\langle p\rangle$ as a graded $A^!$-left-module, I can drop any injective direct summands, and so assume that $M\langle p\rangle$ is without injective direct summands.

Let me start by pointing out the following property of the modules $M\langle p\rangle$.

Proposition 5.5. *The Bass numbers $\mu^i(M\langle p\rangle)$ are bounded for $i \geq 0$.*

Proof. By a computation like the one in equations (13) and (14), it follows that for $i \gg 0$ I have

$$\mu^i(M\langle p\rangle) = \sum_j \dim_k \mathrm{Hom}_{\mathsf{QGr}(A)}(\mathcal{O}, \mathrm{h}^j(\mathcal{M}\langle p\rangle)(i-j)) = (*).$$

However, the complex $\mathcal{M}\langle p\rangle$ is just the object $\mathcal{M}\langle p\rangle$ placed in cohomological degree zero, so

$$(*) = \dim_k \mathrm{Hom}_{\mathsf{QGr}(A)}(\mathcal{O}, \mathcal{M}\langle p\rangle(i)) = (**),$$

and since $\mathcal{M}\langle p\rangle$ is $\pi(P\langle p\rangle)$ and i is large, this is

$$(**) = \dim_k P\langle p\rangle_i = 1$$

by [Artin and Zhang 1994, Theorem 8.1(1) and Proposition 3.13(2)], because the algebra A is AS regular. □

Now some computations with the $M\langle p\rangle$'s.

Lemma 5.6. *The module $M\langle p\rangle$ determines p.*

Proof. It is certainly true that $M\langle p\rangle$ determines $\mathcal{M}\langle p\rangle \cong \varphi(M\langle p\rangle)$. In turn, $\mathcal{M}\langle p\rangle$ determines the tail $P\langle p\rangle_{\geq n}$ for $n \gg 0$, because when viewing $\mathcal{M}\langle p\rangle$ as an object of $\mathsf{qgr}(A)$, I have

$$P\langle p\rangle_{\geq n} \cong \omega\pi(P\langle p\rangle)_{\geq n} = \omega(\mathcal{M}\langle p\rangle)_{\geq n} \tag{17}$$

for $n \gg 0$ by [Artin and Zhang 1994, Theorem 8.1(1) and Proposition 3.13(2)]. But $P\langle p\rangle_{\geq n}$ determines p by [Smith 1996, Sec. 8]. □

Lemma 5.7. *Recall d and τ from Setup 5.2. The modules $M\langle p\rangle$ satisfy*

$$\Sigma(M\langle p\rangle) \cong M\langle \tau^{2-d} p\rangle(1)$$

in $\overline{\mathsf{Gr}}(A^!)$.

Proof. In [Smith 1996, Example 9.5] is proved

$$P\langle p\rangle_{\geq 1}(1) \cong P\langle \tau^{2-d} p\rangle,$$

and applying π shows

$$\mathcal{M}\langle p\rangle(1) \cong \mathcal{M}\langle \tau^{2-d} p\rangle$$

because π only sees the tail of a module. Applying γ and Lemma 3.4(ii), this can be rearranged to the lemma's isomorphism

$$\Sigma(M\langle p\rangle) \cong M\langle \tau^{2-d} p\rangle(1).$$

□

Lemma 5.8. *If $\Sigma^i(M\langle p\rangle) \cong M\langle q\rangle(j)$ holds in $\overline{\mathsf{Gr}}(A^!)$ for some points p and q on C, then $i = j$.*

Proof. The lemma's isomorphism implies $\varphi(\Sigma^i(M\langle p\rangle)) \cong \varphi(M\langle q\rangle(j))$, and using Lemma 3.4(i) and $\varphi(M\langle p\rangle) = \mathcal{M}\langle p\rangle$, this becomes $\Sigma^i(\mathcal{M}\langle p\rangle) \cong \Sigma^j(\mathcal{M}\langle q\rangle)(-j)$. Since the cohomologies of $\mathcal{M}\langle p\rangle$ and $\mathcal{M}\langle q\rangle$ are concentrated in cohomological degree zero, this is only possible with $i = j$. □

Finally, these lemmas can be used as follows. If there is to be periodicity in the sense

$$\Sigma^i(M\langle p\rangle) \cong M\langle p\rangle(j) \tag{18}$$

in $\overline{\mathsf{Gr}}(A^!)$ for some i and j, then $i = j$ by Lemma 5.8. Moreover, $\Sigma^i(M\langle p\rangle) \cong M\langle \tau^{(2-d)i} p\rangle(i)$ holds by Lemma 5.7. Substituting into equation (18) gives

$$M\langle \tau^{(2-d)i} p\rangle(i) \cong M\langle p\rangle(i),$$

hence $M\langle \tau^{(2-d)i} p\rangle \cong M\langle p\rangle$, and as $M\langle p\rangle$ determines p by Lemma 5.6, this implies

$$\tau^{(2-d)i}(p) = p.$$

Conversely, $\tau^{(2-d)i}(p) = p$ gives

$$\Sigma^i(M\langle p\rangle) \cong M\langle \tau^{(2-d)i} p\rangle(i) \cong M\langle p\rangle(i)$$

in $\overline{\mathsf{Gr}}(A^!)$ by Lemma 5.7.

Summing up, if d, τ and p are so that $\tau^{(2-d)i}(p) \neq p$ for $i = 1, \ldots, n-1$ but $\tau^{(2-d)n}(p) = p$, then in $\overline{\mathsf{Gr}}(A^!)$ the suspension $\Sigma^i(M\langle p\rangle)$ is not a degree shift of $M\langle p\rangle$ for $i = 1, \ldots, n-1$, but $\Sigma^n(M\langle p\rangle)$ is $M\langle p\rangle(n)$.

And if d, τ and p are so that $\tau^{(2-d)i}(p) \neq p$ for $i \geq 1$, then in $\overline{\mathsf{Gr}}(A^!)$ the suspension $\Sigma^i(M\langle p\rangle)$ is not a degree shift of $M\langle p\rangle$ for $i \geq 1$.

Using that $M\langle p\rangle$ contains no injective direct summands, this easily lifts to give the same result in $\mathsf{Gr}(A^!)$ for syzygies in *minimal* injective resolutions. So I get the following example which shows the promised contrast to Theorem 5.1 with respect to periodicity of minimal injective resolutions.

Example 5.9. (1) Let d, τ and p be so that $\tau^{(2-d)i}(p) \neq p$ for $i = 1, \ldots, n-1$ but $\tau^{(2-d)n}(p) = p$.

Then the minimal injective resolution I of $M\langle p\rangle$ is periodic with period n, in the sense that in the resolution, the i-th syzygy $\Sigma^i(M\langle p\rangle)$ is not isomorphic to a degree shift of $M\langle p\rangle$ for $i = 1, \ldots, n-1$, but the n-th syzygy $\Sigma^n(M\langle p\rangle)$ is isomorphic to $M\langle p\rangle(n)$.

Hence up to isomorphism, I^n is $I^0(n)$ and ∂_I^n is $\partial_I^0(n)$, while the same is not true with any smaller value of n.

(2) Let d, τ and p be so that $\tau^{(2-d)i}(p) \neq p$ for $i \geq 1$.

Then the minimal injective resolution I of $M\langle p\rangle$ is aperiodic, in the sense that in the resolution, no syzygy $\Sigma^i(M\langle p\rangle)$ is a degree shift of $M\langle p\rangle$ for $i \geq 1$.

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References

[Artin and Zhang 1994] M. Artin and J. J. Zhang, "Noncommutative projective schemes", *Adv. Math.* **109**:2 (1994), 228–287. MR 96a:14004 Zbl 0833.14002

[Beilinson et al. 1996] A. Beilinson, V. Ginzburg, and W. Soergel, "Koszul duality patterns in representation theory", *J. Amer. Math. Soc.* **9**:2 (1996), 473–527. MR 96k:17010 Zbl 0864.17006

[Benson 1998] D. J. Benson, *Representations and cohomology, I: Basic representation theory of finite groups and associative algebras*, Second ed., Cambridge Stud. Adv. Math. **30**, Cambridge University Press, Cambridge, 1998. MR 99f:20001a Zbl 0908.20001

[Bernšteĭn et al. 1978] I. N. Bernšteĭn, , I. M. Gel'fand, and S. I. Gel'fand, "Algebraic vector bundles on $\mathbb{P}^n$ and problems of linear algebra", *Funktsional. Anal. i Prilozhen.* **12**:3 (1978), 66–67. MR 80c:14010a Zbl 0424.14002

[Eisenbud 2002] D. Eisenbud, "Periodic resolutions over exterior algebras", *J. Algebra* **258**:1 (2002), 348–361. MR 2004a:16009 Zbl 01868092

[Fløystad 2003] G. Fløystad, "Koszul duality and equivalences of categories", revised version, 2003. math.RA/0012264

[Grothendieck 1965] A. Grothendieck, "Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, II", *Inst. Hautes Études Sci. Publ. Math.* **24** (1965), 5–231. MR 33 #7330 Zbl 0135.39701

[Happel 1987] D. Happel, "On the derived category of a finite-dimensional algebra", *Comment. Math. Helv.* **62**:3 (1987), 339–389. MR 89c:16029 Zbl 0626.16008

[Hartshorne 1977] R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math. **52**, Springer, New York, 1977. MR 57 #3116 Zbl 0367.14001

[Jørgensen 1999] P. Jørgensen, "Non-commutative Castelnuovo–Mumford regularity", *Math. Proc. Cambridge Philos. Soc.* **125**:2 (1999), 203–221. MR 2000h:13010 Zbl 0920.14002

[Keller 1994] B. Keller, "Deriving DG categories", *Ann. Sci. École Norm. Sup.* (4) **27**:1 (1994), 63–102. MR 95e:18010 Zbl 0799.18007

[Smith 1996] S. P. Smith, "Some finite-dimensional algebras related to elliptic curves", pp. 315–348 in *Representation theory of algebras and related topics* (Mexico City, 1994), edited by R. Bautista et al., CMS Conf. Proc. **19**, Amer. Math. Soc., Providence, RI, 1996. MR 97e:16053 Zbl 0856.16009

[Weibel 1994] C. A. Weibel, *An introduction to homological algebra*, Cambridge Stud. Adv. Math. **38**, Cambridge University Press, Cambridge, 1994. MR 95f:18001 Zbl 0834.18001

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