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# MODULAR DIOPHANTINE INEQUALITIES AND NUMERICAL SEMIGROUPS

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**We study the set of integer solutions to the modular diophantine inequality $ax \bmod b \leq x$.**

## Introduction

Given $x \in \mathbb{Q}$, we set $\lceil x \rceil = \min\{z \in \mathbb{Z} \mid z \geq x\}$ and $\lfloor x \rfloor = \max\{z \in \mathbb{Z} \mid z \leq x\}$, as usual. Given integers $m, n$ with $n > 0$, we set $m \bmod n = m - n\lfloor m/n \rfloor$ and $m \bmod (-n) = m \bmod n$. A *modular diophantine inequality* is an expression of the form $ax \bmod b \leq x$ with $a, b$ integers such that $b \neq 0$. Since $ax \bmod b \geq 0$, the set $S$ of solutions to such an inequality is contained in the set $\mathbb{N}$ of nonnegative integers. $S$ is a *numerical semigroup*, that is, $S$ is closed under addition, $0 \in S$ and $\mathbb{N} \setminus S$ is finite. Not every numerical semigroup arises from a modular diophantine inequality, and Section 2 presents a procedure for testing numerical semigroups for this property. Theorem 12 is crucial for obtaining this algorithm, and thus Section 1 is devoted to it. One of the main consequences of this theorem is that if the inequalities $ax \bmod b \leq x$ and $cx \bmod d \leq x$ have the same solutions, then

$$b - (a, b) - (a-1, b) = d - (c, d) - (c-1, d),$$

where $(x, y)$ denotes the greatest common divisor of the integers $x$ and $y$.

A numerical semigroup $S$ is said to be *modular with modulus $b$ and factor $a$* if $S = \{x \in \mathbb{N} \mid ax \bmod b \leq x\}$. The preceding remark ensures that $b-(a, b)-(a-1, b)$ is an invariant of $S$, which we call the *weight* of $S$ and denote by $\mathrm{w}(S)$.

If $S$ is a numerical semigroup, the largest integer not in $S$ is called the *Frobenius number* of $S$ and is denoted by $\mathrm{g}(S)$. This integer has been widely studied; see for instance [Brauer 1942; Brauer and Shockley 1962; Johnson 1960; Selmer 1977; Sylvester 1884; Curtis 1990; Davison 1994; Djawadi and Hofmeister 1996]. In this direction it is worth highlighting [Ramírez Alfonsín 2000; ≥ 2005], where a review of this problem is given, with many references. In the literature one can also find a large number of publications devoted to the study of one-dimensional analytically

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irreducible local domains via their value semigroups, which are numerical semigroups; see, for instance, [Apéry 1946; Barucci et al. 1997; Bertin and Carbonne 1977; Delorme 1976; Fröberg et al. 1987; Kunz 1970; Teissier 1973; Watanabe 1973]. As a consequence of this study, some interesting kinds of numerical semigroups arise, such as symmetric and pseudo-symmetric numerical semigroups. In Section 1 we prove that a modular numerical semigroup $S$ is symmetric if and only if $\mathrm{w}(S) = \mathrm{g}(S)$, and pseudo-symmetric if and only if $\mathrm{g}(S) = \mathrm{w}(S) + 1$. Sections 3 and 4 are devoted to modular numerical semigroups with modulus equal to their weight plus two and three, respectively. We show that those of weight plus two are obtained from a symmetric numerical semigroup by adjoining its Frobenius number to it, and that those with weight plus three arise from a pseudo-symmetric numerical semigroup by adding to it its Frobenius number and this number divided by two.

In Section 5 we study those modular numerical semigroups $S$ such that the factor of $S$ divides the modulus. For these numerical semigroups we can explicitly give formulas for the multiplicity, the minimal generator set, the Apéry set and the Frobenius number, so the case $a \mid b$ is now well understood.

Section 6 addresses the problem of computing the Frobenius number in the complementary case $a \nmid b$, solving it when $(a-1)(a - (b \bmod a)) < b$. We have not been able to solve the general case.

## 1. Modular numerical semigroups

Let $a$ and $b$ be integers such that $b \neq 0$. Since $ax \bmod b = (a \bmod b)x \bmod b$ and $ax \bmod b = ax \bmod (-b)$, in order to study the solutions of $ax \bmod b \leq x$, we can assume that $b$ is a positive integer and that $0 \leq a < b$.

**Proposition 1.** *The set of integer solutions of a modular diophantine inequality is a numerical semigroup.*

*Proof.* Let $a$ and $b$ be two integers such that $0 \leq a < b$ and let $S = \{x \in \mathbb{N} \mid ax \bmod b \leq x\}$. Clearly $0 \in S$, and if $x$ is an integer greater than or equal to $b$, then $x \in S$. Hence $\mathbb{N} \setminus S$ is finite. For $x, y \in S$, we have $a(x+y) \bmod b \leq ax \bmod b + ay \bmod b \leq x + y$, whence $x + y \in S$, so $S$ is closed under addition. □

A numerical semigroup $S$ arising as in the proposition is said to be *modular*. The modular semigroup with modulus $b$ factor $a$ will be denoted by $\mathrm{S}(a, b)$; thus $\mathrm{S}(a, b) = \{x \in \mathbb{N} \mid ax \bmod b \leq x\}$. When we write $\mathrm{S}(a, b)$ we will generally assume tacitly that $a$ and $b$ are integers with $0 \leq a < b$.

**Example 2.** $\mathrm{S}(2, 3) = \mathrm{S}(2, 4) = \{0, 2, 3, \rightarrow\}$, where $\rightarrow$ means that all the elements beyond 3 are in the set. Thus $a$ and $b$ don't have to be unique.

The goal of this section is to prove Theorem 12, which counts the natural numbers *absent* from $\mathrm{S}(a,b)$. We prepare the ground with some simple results.

**Lemma 3.** *Let $a$ and $b$ be integers such that $0 \le a < b$. Then $ax \bmod b \le x$ if and only if $(b+1-a)x \bmod b \le x$.*

*Proof.* If $ax \bmod b \le x$, there exist $q, r \in \mathbb{N}$ such that $ax = qb + r$ with $0 \le r \le x$. Hence $(b+1-a)x = (b+1)x - ax = bx - qb + x - r$ and $(b+1-a)x \bmod b \le x - r \le x$. The converse follows by interchanging $a$ with $b+1-a$. □

**Lemma 4.** *Let $S$ be a modular numerical semigroup with modulus $b \ge 2$. Then there exists a positive integer $a$ such that $a \le \frac{1}{2}(b+1)$ and $S = \mathrm{S}(a,b)$.*

*Proof.* Write $S = \mathrm{S}(a,b)$ with $0 \le a < b$. By Lemma 3, $S = \mathrm{S}(b+1-a, b)$, so if $a > \frac{1}{2}(b+1)$ we can replace $a$ by $b+1-a \le \frac{1}{2}(b+1)$. Also if $a = 0$ we can replace it by $a = 1$, since $S = \mathbb{N}$ for both these values of $a$. □

**Lemma 5.** *Let $a$ and $b$ be integers such that $0 \le a < b$ and let $x \in \mathbb{N}$. Then*

$$a(b-x) \bmod b = \begin{cases} 0 & \text{if } ax \bmod b = 0, \\ b - (ax \bmod b) & \text{if } ax \bmod b \ne 0, \end{cases}$$

*and $ax \bmod b > x$ implies that $a(b-x) \bmod b < b - x$.* □

**Corollary 6.** *If $S = \mathrm{S}(a,b)$ and $x \in \mathbb{N} \setminus S$, then $b - x \in S$.* □

Given a subset $A$ of $\mathbb{N}$, we denote by $\mathrm{H}(A)$ the complement $\mathbb{N} \setminus A$, and by $\langle A \rangle$ the submonoid of $\mathbb{N}$ generated by $A$ (the set of finite sums of elements of $A$).

**Remark 7.** If $S = \mathrm{S}(a,b) \ne \mathbb{N}$ for positive $a$ and $b$, then $b-1 \notin \mathrm{H}(S)$, since otherwise $b - (b-1) = 1$ would be an element of $S$. Moreover $x \in S$ for all integers $x \ge b$. Therefore the Frobenius number $\mathrm{g}(S)$ is at most $b-2$.

We now characterize the case $\mathrm{g}(S) = b-2$. If $\mathrm{g}(S) = b-2$, Corollary 6 implies that $b - (b-2) = 2 \in S$. Hence $b$ is odd and $S = \langle 2, b \rangle$. In addition, since $2 \in S$, $2a \bmod b \le 2$ and this leads to $2a > b$, whence $a > \frac{1}{2}b$. But Lemma 4 says we can take $a \le \frac{1}{2}(b+1)$, which then means $a = \frac{1}{2}(b+1)$. Hence $S = \mathrm{S}(\frac{1}{2}(b+1), b)$. Conversely, if $S = \mathrm{S}(\frac{1}{2}(b+1), b)$ with $b$ odd, it is easy to check that $S = \langle 2, b \rangle$ and thus $\mathrm{g}(S) = b-2$.

**Example 8.** Suppose $b \ge 2$ and $S = \mathrm{S}(2,b)$. Then $S = \{0, \lfloor \frac{1}{2}(b+1) \rfloor, \rightarrow\}$. For clearly $\{b, \rightarrow\} \subseteq S$. Now take $0 < x < b$. Then $x \in S$ if and only if $2x \bmod b \le x$. However, $2x \bmod b = 2x$ if and only if $2x < b$, and thus in this case $x \notin S$. If $2x \ge b$, then $2x \bmod b = 2x - b \le x$, whence $x \in S$.

**Lemma 9.** *Let $S = \mathrm{S}(a,b)$ and let $x$ be an integer such that $0 \le x \le b$. Then $x$ and $b - x$ are both in $S$ if and only if $ax \bmod b \in \{0, x\}$.*

*Proof.* If $ax \bmod b \notin \{0, x\}$, Lemma 5 gives $a(b-x) \bmod b = b-(ax \bmod b)$. If $x \in S$, the right-hand side exceeds $b-x$ (since $ax \bmod b < x$). Thus $b-x \notin S$.

Conversely, if $ax \bmod b = 0$, clearly $x \in S$ and also $b-x \in S$ by Lemma 5; whereas if $ax \bmod b = x \neq 0$, again $x \in S$, and Lemma 5 gives $a(b-x) \bmod b = b-(ax \bmod b) = b-x$, so $b-x \in S$. □

**Lemma 10.** *Let a and b be positive integers and x an integer such that* $0 \le x < b$.

(1) $ax \bmod b = 0$ *if and only if x is a multiple of* $b/(a, b)$.

(2) $ax \bmod b = x$ *if and only if x is a multiple of* $b/(b, a-1)$. □

**Lemma 11.** *Let* $S = S(a, b)$ *with* $0 < a < b$. *Let* $\alpha = (b, a-1)$ *and* $\beta = (b, a)$, *and let x be an integer such that* $0 \le x \le b$. *Then*

$$\{x, b-x\} \subset S \iff x \in \left\{0, \frac{b}{\alpha}, 2\frac{b}{\alpha}, \ldots (\alpha-1)\frac{b}{\alpha}, \frac{b}{\beta}, 2\frac{b}{\beta}, \ldots, (\beta-1)\frac{b}{\beta}, b\right\} =: X.$$

*The cardinality of X is* $\alpha + \beta$.

*Proof.* The equivalence is just Lemmas 9 and 10 put together. To show there is no duplication in the elements of $X$ as written, note that $(\alpha, \beta) = 1$. If $sb/\alpha = tb/\beta$ for some $s, t \in \mathbb{N}$, then $s\beta = t\alpha = k\alpha\beta$ for some $k \in \mathbb{N}$. Hence $s = k\alpha$ and $t = k\beta$. □

**Theorem 12.** *Let* $S = \mathrm{S}(a, b)$ *for some integers* $0 \le a < b$. *Then*

$$\#\,\mathrm{H}(S) = \frac{b+1-(a, b)-(a-1, b)}{2}.$$

Here as usual # denotes cardinality.

*Proof.* Let $\alpha$, $\beta$ and $X$ be as in Lemma 11. By Corollary 6 and Lemma 11, for $0 \le x \le b$, at most one of $x$, $b-x$ lies in $\mathrm{H}(S)$, and it's exactly one unless $x \in X$. Thus $\#\,\mathrm{H}(S) = \frac{1}{2}(b+1-\#\,X) = \frac{1}{2}(b+1-\alpha-\beta)$. □

**Example 13.** If $p$ is an odd prime, $\#\,\mathrm{H}(\mathrm{S}(a, p)) = \frac{1}{2}(p-1)$ for all $a$ with $1 < a < p$.

As an immediate consequence of Theorem 12 we obtain:

**Corollary 14.** *Suppose* $\mathrm{S}(a, b) = \mathrm{S}(c, d)$. *Then*

$$b-(a, b)-(a-1, b) = d-(c, d)-(c-1, d).$$

**Example 15.** The converse of Corollary 14 is false. For instance, $\langle 4, 5, 6\rangle = \mathrm{S}(3, 12) \neq \mathrm{S}(2, 10) = \langle 5, 6, 7, 8, 9\rangle$.

Recall that we have defined the *weight* of $S = \mathrm{S}(a, b)$ as $\mathrm{w}(S) := b-(a, b)-(a-1, b)$; by Theorem 12, this number equals $2\#\,\mathrm{H}(S)-1$, and so is an invariant of $S$. Note that $\mathrm{w}(\mathbb{N}) = -1$. If $S \neq \mathbb{N}$, we can choose $a, b$ with $2 \le a < b$; hence $(a, b)+(a-1, b) \le \frac{1}{2}b+\frac{1}{3}b < b$, so $\mathrm{w}(S) \ge 1$. Thus, like the Frobenius number, the weight of a modular numerical semigroup is at least 1, except for the case $S = \mathbb{N}$, where $\mathrm{w}(S) = \mathrm{g}(S) = -1$.

Theorem 12 and the inequality $\# H(S) \geq \frac{1}{2}(\mathrm{g}(S) + 1)$, valid for *any* numerical semigroup $S$ (see [Fröberg et al. 1987], for instance), yield:

**Corollary 16.** *If $S$ is a modular numerical semigroup, then $\mathrm{w}(S)$ is odd and greater than or equal to $\mathrm{g}(S)$.* □

In view of this, modular numerical semigroups $S$ with $\mathrm{w}(S)=\mathrm{g}(S)$ and $\mathrm{g}(S)$ odd, or with $\mathrm{w}(S) = \mathrm{g}(S)+1$ and $\mathrm{g}(S)$ even, have minimal possible weight with respect to their Frobenius numbers. The next result characterizes this kind of numerical semigroup, but before proving it we need to recall some concepts.

A numerical semigroup $S$ is *symmetric* if $x \in \mathbb{N} \setminus S$ implies $\mathrm{g}(S) - x \in S$. It is straightforward to prove that a symmetric numerical semigroup has odd Frobenius number. A numerical semigroup is *pseudo-symmetric* if $\mathrm{g}(S)$ is even and $x \in \mathbb{N} \setminus S$ implies that either $x = \mathrm{g}(S)/2$ or $\mathrm{g}(S) - x \in S$. A numerical semigroup $S$ is symmetric if and only if $\# \mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + 1)$, and pseudo-symmetric if and only if $\# \mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + 2))$; see [Fröberg et al. 1987], for instance.

A numerical semigroup is *irreducible* if it cannot be expressed as the intersection of two numerical semigroups containing it properly. In [Rosales and Branco 2003] it is shown that $S$ is irreducible if and only if $S$ is symmetric or pseudo-symmetric (depending on the parity of $\mathrm{g}(S)$).

**Corollary 17.** *Let $S$ be a modular numerical semigroup.*

(1) *$S$ is symmetric if and only if* $\mathrm{w}(S) = \mathrm{g}(S)$.

(2) *$S$ is pseudo-symmetric if and only if* $\mathrm{w}(S) = \mathrm{g}(S) + 1$.

*Proof.* $S$ is symmetric if and only if $\# \mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + 1)$. By Theorem 12, $\# \mathrm{H}(S) = \frac{1}{2}(\mathrm{w}(S) + 1)$, whence $S$ is symmetric if and only if $\mathrm{g}(S) = \mathrm{w}(S)$. The proof of (2) is analogous. □

**Example 18.** If $b$ is an odd integer, there exists a modular numerical semigroup $S$ with $\mathrm{w}(S) = b$. It suffices to take $S = \mathrm{S}(2, b+2)$, since $\mathrm{w}(\mathrm{S}(2, b+2)) = b + 2 - (2, b + 2) - (1, b + 2) = b + 2 - 1 - 1 = b$.

## 2. Determining whether a numerical semigroup is modular

In this section we give a procedure for deciding whether a given numerical semigroup is a modular numerical semigroup, and if so to express it in the form $\mathrm{S}(a, b)$.

**Lemma 19.** *Let $S$ be a modular numerical semigroup with modulus $b$ and $S \neq \mathbb{N}$. Then $b \leq 12 \# \mathrm{H}(S) - 6$.*

*Proof.* As we saw right after Example 15, if $a \geq 2$ we have $(a, b) + (a-1, b) \leq \frac{5}{6}b$. By Theorem 12, $\# \mathrm{H}(S) \geq \frac{1}{2}(b + 1 - \frac{5}{6}b)$ and thus $b \leq 12 \# \mathrm{H}(S) - 6$. □

For a numerical semigroup $S$, the *multiplicity* of $S$, denoted by $\mathrm{m}(S)$, is the least positive integer in $S$. Here is an immediate consequence of Lemma 11:

**Lemma 20.** *For* $S = \mathrm{S}(a, b)$,

$$b - \mathrm{m}(S) \in S \Longleftrightarrow \mathrm{m}(S) = \min\left\{\frac{b}{(a, b)}, \frac{b}{(a-1, b)}\right\}.$$

**Lemma 21.** *Let $S$ be a modular numerical semigroup with modulus $b$. Then*

$$b \geq \mathrm{g}(S) + \mathrm{m}(S).$$

*Proof.* Since $1, 2, \dots, \mathrm{m}(S)-1$ are not in $S$, Corollary 6 ensures that $b-\mathrm{m}(S)+1, \dots, b-1$ are. But $\{b, \mathrm{m}(S)\} \subset S$, so $\{b - \mathrm{m}(S) + 1, \rightarrow\} \subseteq S$. This implies that $\mathrm{g}(S) \leq b - \mathrm{m}(S)$. □

**Lemma 22.** *For* $S = \mathrm{S}(a, b)$,

$$b = \mathrm{g}(S) + \mathrm{m}(S) \Longleftrightarrow \mathrm{m}(S) \neq \min\left\{\frac{b}{(a, b)}, \frac{b}{(a-1, b)}\right\}.$$

*Proof.* Follows from Lemmas 20 and 21. □

Now we have all the ingredients to give the algorithm announced at the start of this section, to decide whether a numerical semigroup is of the form $S(a, b)$, and if so, produce such a pair $(a, b)$ (or all such pairs with $a \leq \frac{1}{2}(b+1)$, if the algorithm is not stopped after the first pair is found).

**Algorithm 23.** Given a numerical semigroup $S$ distinct from $\mathbb{N}$:

(1) Compute $\#\mathrm{H}(S)$, $\mathrm{g}(S)$ and $\mathrm{m}(S)$.

(2) Set $b = \mathrm{g}(S) + \mathrm{m}(S)$.

(3) For every $a \in A := \left\{ a \in \mathbb{N} \;\middle|\; \begin{array}{l} 2 \leq a \leq \frac{1}{2}(b+1), \\ b = 2\#\mathrm{H}(S) + (a, b) + (a-1, b) - 1, \\ \mathrm{m}(S) < \min\{b/(a, b), b/(a-1, b)\} \end{array} \right\}$

compute $\mathrm{S}(a, b)$; if $S = \mathrm{S}(a, b)$, return this answer and stop.

(4) Compute $B = \big\{ b \in \{k \cdot \mathrm{m}(S) \mid k \in \mathbb{N}\} \;\big|\; 2\#\mathrm{H}(S) + 1 \leq b \leq 12\#\mathrm{H}(S) - 6 \big\}$.

(5) For every $b \in B$

for every $a \in A_b := \left\{ a \in \mathbb{N} \;\middle|\; \begin{array}{l} 2 \leq a \leq \frac{1}{2}(b+1), \\ b = 2\#\mathrm{H}(S) + (a, b) + (a-1, b) - 1, \\ \mathrm{m}(S) = \min\{b/(a, b), b/(a-1, b)\} \end{array} \right\}$

compute $\mathrm{S}(a, b)$; if $S = \mathrm{S}(a, b)$, return this answer and stop.

(6) Return “$S$ is not modular”.

We briefly justify the correctness of Algorithm 23. In Steps (2) and (3) we check whether $S$ is a modular numerical semigroup with modulus $\mathrm{g}(S)+\mathrm{m}(S)$, and the correct working of these steps relies on Lemmas 4 and 22 and Theorem 12. If $S$ is not a modular numerical semigroup with modulus $\mathrm{g}(S)+\mathrm{m}(S)$, Lemma 22 gives $\mathrm{m}(S)=\min\{b/(a,b), b/(a-1,b)\}$. This implies that $\mathrm{m}(S)$ divides $b$. Theorem 12 states that $b=2\#\mathrm{H}(S)+(a,b)+(a-1,b)-1$, so $b\geq 2\#\mathrm{H}(S)+1$; at the same time $b\leq 12\#\mathrm{H}(S)-6$ by Lemma 19. Therefore Steps (4) and (5) cover the case $b\neq \mathrm{g}(S)+\mathrm{m}(S)$.

**Example 24.** Let $S=\langle 3,5\rangle$. Then $\#\mathrm{H}(S)=4$, $\mathrm{g}(S)=7$ and $\mathrm{m}(S)=3$. In Step (2) we get $b=10$. Step (3) yields $A=\{2,3,4\}$, then $\mathrm{S}(2,10)=\langle 5,6,7,8,9\rangle$, $\mathrm{S}(3,10)=\langle 4,5,7\rangle$, and $\mathrm{S}(4,10)=\langle 3,5\rangle=S$, so the algorithm returns $S=S(4,10)$.

**Example 25.** Let $S=\langle 3,8,10\rangle$. In this case $\#\mathrm{H}(S)=5$, $\mathrm{g}(S)=7$ and $\mathrm{m}(S)=3$. In Step (2) we obtain $b=10$ and in Step (3), $A=\varnothing$. The only nonempty set $A_b$ with $b\in B$ is $A_{15}=\{5\}$. Since $S\neq \mathrm{S}(5,15)=\langle 3,7,11\rangle$, the algorithm returns No.

**Example 26.** Let $S=\langle 10,11,12\rangle$. Then $\#\mathrm{H}(S)=25$, $\mathrm{g}(S)=49$ and $\mathrm{m}(S)=10$. In Step (2) we obtain $b=59$ and $A$ is empty. Computing $B$, we obtain

$$B=\{60,70,80,90,100,110,120,130,140,150,160,170,180,\\ 190,200,210,220,230,240,250,260,270,280,290\}.$$

The only nonempty set $A_b$ with $b\in B$ is $A_{60}=\{6\}$. It turns out that $S=\mathrm{S}(6,60)$.

**Remark 27.** If the input to Algorithm 23 is known to be symmetric, the procedure can be improved, because if $S=\mathrm{S}(a,b)$ is symmetric then $b$ must be equal to $\mathrm{g}(S)+(a,b)+(a-1,b)$ (note that $\mathrm{w}(S)=\mathrm{g}(S)$ by Corollary 17). A similar argument applies to the pseudo-symmetric case.

**Remark 28.** The intersection $\bigcap_{i=1}^{n}\mathrm{S}(a_i,b_i)$ of $n\geq 1$ modular numerical semigroups is a numerical semigroup; it need not be modular, as can be seen from Example 25, since we can write $\langle 3,8,10\rangle=\langle 3,4\rangle\cap\langle 3,5\rangle=\mathrm{S}(3,8)\cap\mathrm{S}(4,10)$.

Nor can every numerical semigroup be written as such an intersection: for instance, $\langle 7,8,10,13\rangle$ is a symmetric, hence irreducible, numerical semigroup; thus it cannot be an intersection of modular numerical semigroups other than by being itself a modular numerical semigroup. Algorithm 23 says that it is not.

## 3. Modular numerical semigroups whose modulus is its weight plus two

We now study modular numerical semigroups $S=\mathrm{S}(a,b)$ whose modulus $b$ equals $\mathrm{w}(S)+2$. Since $b=\mathrm{w}(S)+(a,b)+(a-1,b)\geq \mathrm{w}(S)+2$, the condition $b=\mathrm{w}(S)+2$ is equivalent to $(a,b)=(a-1,b)=1$ (so $b$ is odd), and it characterizes modular numerical semigroups whose moduli are minimal with respect to their weights.

Every numerical semigroup $S$ is finitely generated (as an additive monoid). This is easy to see — for instance, start with two relatively prime $r, s \in S$ and then adjoin all elements of $S \cap \{0, 1, \ldots, rs-1\}$ as yet unaccounted for. Among all generating sets one can of course choose one that is *minimal*, say $\mathcal{M}(S)$. A minute's thought shows that $\mathcal{M}(S)$ is characterized by containing exactly those nonzero elements of $S$ that cannot be expressed as a sum of two nonzero elements of $S$:

$$\mathcal{M}(S) = (S \setminus \{0\}) \setminus \big((S \setminus \{0\}) + (S \setminus \{0\})\big).$$

In particular, $\mathcal{M}(S)$ is unique. We set $\mathrm{e}(S) = \#\mathcal{M}(S)$ and call this number the *embedding dimension* of $S$; the elements of $\mathcal{M}(S)$ are called *minimal generators*.

**Proposition 29.** *Let* $S = \mathrm{S}(a, b)$ *with* $2 \le a < b$ *and* $(a, b) = (a-1, b) = 1$. *Then*

(1) $b = \mathrm{g}(S) + \mathrm{m}(S)$,

(2) $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + \mathrm{m}(S) - 1)$,

(3) *$b$ is the largest minimal generator of $S$.*

*Proof.* (1) We already know that $b-1 \in S$ when $2 \le a < b$. Hence $\mathrm{m}(S) \ne b$. Using Lemma 22, we get $b = \mathrm{g}(S) + \mathrm{m}(S)$.

(2) Immediate from Theorem 12.

(3) First we prove that $b$ is a minimal generator of $S$. Assume to the contrary that $b = x + y$ with $x, y \in S \setminus \{0\}$. Then $ax \bmod b \le x$ and $ay \bmod b \le y$, and thus $(ax \bmod b) + (ay \bmod b) \le x + y = b$. Since $a(x + y) \bmod b = ab \bmod b = 0$, we deduce that $(ax \bmod b) + (ay \bmod b) \in \{0, b\}$. Thus either $ax \bmod b = x$ and $ay \bmod b = y$, or $ax \bmod b = 0$ and $ay \bmod b = 0$. These two cases contradict the two halves of Lemma 10.

To see that $b$ is the largest minimal generator, take $x \in S$ with $x > b$. By applying (1) we obtain $x > \mathrm{g}(S) + \mathrm{m}(S)$, which implies that $x - \mathrm{m}(S) > \mathrm{g}(S)$; this forces $x - \mathrm{m}(S) \in S$. Thus $x = \mathrm{m}(S) + (x - \mathrm{m}(S))$ cannot be a minimal generator of $S$. □

Proposition 29 allows us to relate the modular numerical semigroups in question with *unitary extensions of symmetric numerical semigroups* or *UESY-semigroups* in short. A numerical semigroup $S$ is a UESY-semigroup if there exists a symmetric numerical semigroup $S'$ such that $S' \subset S$ and $\#(S \setminus S') = 1$. In [Rosales ≥ 2005b] this condition is shown to be equivalent to the existence of a symmetric numerical semigroup $S'$ such that $S = S' \cup \{\mathrm{g}(S')\}$. The following result also appears there.

**Proposition 30.** *Let $S$ be a numerical semigroup*, $S \ne \mathbb{N}$. *The following conditions are equivalent*:

(1) *$S$ is a UESY-semigroup.*

(2) $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + \mathrm{m}(S) - 1)$ *and* $\mathrm{g}(S) + \mathrm{m}(S)$ *is a minimal generator of $S$.* □

A *pseudo-Frobenius number* [Rosales and Branco 2002] of a numerical semigroup $S$ is an integer $x \notin S$ such that $x+s \in S$ for all $s \in S \setminus \{0\}$. The set of pseudo-Frobenius numbers of $S$ is denoted by $\mathrm{Pg}(S)$, and its cardinality, called the *type* of $S$, is denoted by $\mathrm{t}(S)$. Clearly $\mathrm{g}(S) \in \mathrm{Pg}(S)$. Moreover $S$ is symmetric if and only if $\mathrm{Pg}(S) = \{\mathrm{g}(S)\}$, and $S$ is pseudo-symmetric if and only if $\mathrm{Pg}(S) = \{\mathrm{g}(S), \frac{1}{2}\mathrm{g}(S)\}$; see [Barucci et al. 1997; Fröberg et al. 1987], for instance.

In [Rosales ≥ 2005b] it is proved that if $S$ is a UESY-semigroup distinct from $\mathbb{N}$, then $\mathrm{t}(S) = \mathrm{e}(S)-1$. This, plus Propositions 29 and 30, gives:

**Corollary 31.** *Let* $S = \mathrm{S}(a, b)$ *be such that* $2 \le a < b$ *and* $(a, b) = (a-1, b) = 1$. *Then* $\mathrm{t}(S) = \mathrm{e}(S)-1$ *and there exists a symmetric numerical semigroup* $S'$ *such that* $S = S' \cup \{\mathrm{g}(S')\}$. □

**Theorem 32.** *Let* $S = \mathrm{S}(a, b)$. *Then* $b = \mathrm{w}(S) + 2$ *if and only if* $S$ *is a UESY-semigroup and* $b$ *is a minimal generator of* $S$.

*Proof.* If $b = \mathrm{w}(S)+2 = b-(a, b)-(a-1, b)+2$, we deduce $(a, b) = (a-1, b) = 1$. Corollary 31 then says that $S$ is a UESY-semigroup, and Proposition 29 that $b$ is a minimal generator of $S$.

Conversely, if $b$ is a minimal generator of $S$ it equals $\mathrm{g}(S) + \mathrm{m}(S)$, by Lemma 21 and the fact, shown in the proof of Proposition 29, that a minimal generator of $S$ cannot exceed $\mathrm{g}(S)+\mathrm{m}(S)$. If $S$ is a UESY, then, $\#\,\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + m(S)-1)$ by Proposition 30 and $\#\,\mathrm{H}(S) = \frac{1}{2}(\mathrm{w}(S) + 1)$ by Theorem 12. Thus $b = \mathrm{w}(S) + 2$. □

**Corollary 33.** *Let* $S$ *be a modular numerical semigroup with modulus* $b$. *Then* $b = \mathrm{w}(S) + 2$ *if and only if* $S \setminus \{b\}$ *is a symmetric numerical semigroup. Therefore, if* $b$ *is a prime integer,* $S \setminus \{b\}$ *is a symmetric numerical semigroup.*

*Proof.* If $b = \mathrm{w}(S)+2$, Theorem 32 says $b$ is a minimal generator of $S$, so $S' = S\setminus\{b\}$ is a numerical semigroup with $\mathrm{g}(S') = b$. By Corollary 6, $S'$ is symmetric.

Conversely, if $S \setminus \{b\}$ is a symmetric numerical semigroup, then $S$ is a UESY-semigroup with $b$ as a minimal generator. Now Theorem 32 gives $b = \mathrm{w}(S) + 2$.

Finally, $b$ prime implies $(a, b) = (a-1, b) = 1$, so $\mathrm{w}(S) = b - 2$. □

**Corollary 34.** *Let* $b \ge 3$ *be an integer. Then* $b$ *is prime if and only if* $b$ *is the largest minimal generator of* $\mathrm{S}(a, b)$ *for every* $a$ *such that* $2 \le a \le \sqrt{b}$.

*Proof.* If $b$ is prime Proposition 29 applies; this proves one direction. Conversely, suppose $b$ is not a prime — say $b = ac$ with integers $a, c \ge 2$ and $a \le \sqrt{b}$. For $S = \mathrm{S}(a, b)$, we have $ac \bmod b = 0$ and thus $c \in S$. But then $b = ac$ cannot be a minimal generator of $S$. □

## 4. Modular numerical semigroups whose modulus is its weight plus three

We now study modular numerical semigroups $S = \mathrm{S}(a, b)$ such that $b = \mathrm{w}(S) + 3$; this condition is equivalent to $(a, b) + (a-1, b) = 3$. There are two cases:

- $(a, b) = 1$ and $(a-1, b) = 2$.
- $(a, b) = 2$ and $(a-1, b) = 1$.

In both situations $b$ must be even and by Corollary 6 we deduce that $\frac{1}{2}b \in S$.

Let $S$ be a numerical semigroup with minimal generating set $\{n_1, \dots, n_p\}$. We say that $x \in S$ has a *unique expression* if the equality $x = a_1 n_1 + \dots + a_p n_p$, with $a_1, \dots, a_p \in \mathbb{N}$, determines $a_1, \dots, a_p$ uniquely.

**Proposition 35.** *Let* $S = \mathrm{S}(a, b)$ *be such that* $2 \leq a < b$ *and* $(a, b) + (a-1, b) = 3$.

(1) $\mathrm{m}(S) \neq \frac{1}{2}b \Leftrightarrow S \neq \{0, \frac{1}{2}b, \rightarrow\} \Leftrightarrow b = \mathrm{g}(S) + \mathrm{m}(S) \Leftrightarrow \#\mathrm{H}(S) = \dfrac{\mathrm{g}(S)+\mathrm{m}(S)-2}{2}$.

(2) $\frac{1}{2}b$ *is a minimal generator of* $S$.

(3) $b$ *has a unique expression in* $S$.

*Proof.* (1) Follows easily from Corollary 6, Lemma 22 and Theorem 12.

(2) Suppose $x + y = \frac{1}{2}b$ with $x, y \in S$. Then $ax \bmod b \leq x$ and $ay \bmod b \leq y$, whence $ax \bmod b + ay \bmod b \leq x + y = \frac{1}{2}b$. Thus $\frac{1}{2}ab \bmod b = a(x+y) \bmod b = ax \bmod b + ay \bmod b$. We must show that $x = 0$ or $y = 0$. We distinguish two cases. If $(a, b) = 2$, then $\frac{1}{2}ab \bmod b = 0$, so $ax \bmod b = 0$ and $ay \bmod b = 0$; then Lemma 10 shows that both $x$ and $y$ are multiples of $\frac{1}{2}b$, which leads to the desired conclusion. Similarly, if $(a-1, b) = 2$, then $\frac{1}{2}ab \bmod b = \frac{1}{2}b$, so $ax \bmod b = x$ and $ay \bmod b = y$; Lemma 10 again shows that $x$ and $y$ are multiples of $\frac{1}{2}b$.

(3) We prove that if $x, y \in S \setminus \{0\}$ are such that $x + y = b$, then $x = y = \frac{1}{2}b$. Arguing as in the proof of Proposition 29(3), we see that either $(ax \bmod b, ay \bmod y) = (x, y)$ or $ax \bmod b = ay \bmod y = 0$. Lemma 10 implies that $x$ and $y$ are both multiples of $\frac{1}{2}b$, and since $x \neq 0 \neq y$, we conclude that $x = y = \frac{1}{2}b$. □

Paralleling what we did in Section 3 for the case $b = \mathrm{w}(S) + 2$, we can use Proposition 35 to relate modular numerical semigroups such that $b = \mathrm{w}(S) + 3$ with a previous studied class of numerical semigroups. A numerical semigroup $S$ is called a *PESPY-semigroup* if there exists a pseudo-symmetric numerical semigroup $S'$ such that $S = S' \cup \left\{\frac{1}{2}\mathrm{g}(S'), \mathrm{g}(S')\right\}$ (the two additional elements are the pseudo-Frobenius numbers of $S'$; see [Barucci et al. 1997; Fröberg et al. 1987]).

Numerical semigroups of the form $\{0, x, \rightarrow\}$ with $x$ a positive integer are called *intervals*. The following result appears in [Rosales ≥ 2005a].

**Proposition 36.** *Let* $S$ *be a numerical semigroup that is not an interval. The following conditions are equivalent*:

(1) $S$ *is a PEPSY-semigroup.*

(2) $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + \mathrm{m}(S) - 2)$, $\frac{1}{2}(\mathrm{g}(S) + \mathrm{m}(S))$ *is a minimal generator of* $S$ *and* $\mathrm{g}(S) + \mathrm{m}(S)$ *is an element of unique expression of* $S$. □

The next result is an immediate consequence of Propositions 35 and 36.

**Corollary 37.** *Let* $S = \mathrm{S}(a, b)$ *be such that* $2 \le a < b$, $(a, b) + (a-1, b) = 3$ *and* $S$ *is not an interval. Then* $S$ *is a PEPSY-semigroup.*

In [Rosales ≥ 2005a] it is proved that if $S$ is a PEPSY-semigroup that is not an interval, then $\mathrm{t}(S) = \mathrm{e}(S)-1$. Thus:

**Corollary 38.** *Let* $S = \mathrm{S}(a, b)$ *be such that* $2 \le a < b$, $(a, b) + (a-1, b) = 3$ *and* $S$ *is not an interval. Then* $\mathrm{t}(S) = \mathrm{e}(S)-1$.

**Remark 39.** Among numerical semigroups, interval semigroups have maximal embedding dimension relative to multiplicity: $\mathrm{e}(S) = \mathrm{m}(S)$. For any numerical semigroup with maximal embedding dimension, $\mathrm{t}(S) = \mathrm{m}(S)-1 = \mathrm{e}(S)-1$ (see [Barucci et al. 1997], for instance). Hence the assumption "$S$ is not an interval" can be dropped from Corollary 38.

**Theorem 40.** *Assume that* $S = \mathrm{S}(a, b)$ *is not an interval. Then* $b = \mathrm{w}(S) + 3$ *if and only if* $S$ *is a PEPSY-semigroup,* $\frac{1}{2}b$ *is a minimal generator of* $S$ *and* $b$ *has a unique expression in* $S$.

*Proof.* Necessity follows from Corollary 37 and Proposition 35. Sufficiency: Lemma 21 says that $b \ge \mathrm{g}(S)+\mathrm{m}(S)$. If $b > \mathrm{g}(S)+\mathrm{m}(S)$, then $\mathrm{m}(S)+(b-\mathrm{m}(S))$ and $\frac{1}{2}b + \frac{1}{2}b$ are distinct expressions for $b$ in $S$ ($\mathrm{m}(S) \neq \frac{1}{2}b$ since otherwise $S$ is an interval, by Corollary 6). Therefore $b = \mathrm{g}(S) + \mathrm{m}(S)$. By Proposition 36, we know that $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + \mathrm{m}(S) - 2)$ and Theorem 12 ensures that $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{w}(S) + 1)$, whence $b = \mathrm{g}(S) + \mathrm{m}(S) = \mathrm{w}(S) + 3$. □

**Corollary 41.** *Let* $S$ *be a modular numerical semigroup with modulus* $b$. *Then* $b = \mathrm{w}(S)+3$ *if and only if* $S \setminus \left\{\frac{1}{2}b, b\right\}$ *is a pseudo-symmetric numerical semigroup. Therefore, if* $b = 2p$ *and* $a < p$ *for some positive prime* $p$, *then* $S \setminus \left\{\frac{1}{2}b, b\right\}$ *is a pseudo-symmetric numerical semigroup.*

*Proof.* Suppose $b = \mathrm{w}(S)+3$. By Theorem 40, $\frac{1}{2}b$ is a minimal generator of $S$ and $b$ has a unique expression in $S$. This implies that $S' = S \setminus \left\{\frac{1}{2}b, b\right\}$ is a numerical semigroup, and clearly $\mathrm{g}(S') = b$. Using Corollary 6 we can easily deduce that $S'$ is pseudo-symmetric.

Conversely, if $S \setminus \left\{\frac{1}{2}b, b\right\}$ is a pseudo-symmetric numerical semigroup, then $S$ is a PEPSY-semigroup by definition, $\frac{1}{2}b$ is a minimal generator of $S$ and $b = \frac{1}{2}b + \frac{1}{2}b$ is the unique expression of $b$ in $S$. Thus $b = \mathrm{w}(S) + 3$ by Theorem 40. □

## 5. When the factor divides the modulus

We next focus on numerical semigroups of the form $S = \mathrm{S}(a, ab)$, where we may as well assume $a, b > 1$. First a general definition: given a numerical semigroup

$S$ and $n \in S \setminus \{0\}$, the *Apéry set* of $n$ in $S$ [Apéry 1946] is

$$\mathrm{Ap}(S, n) = \{s \in S \mid s - n \notin S\}.$$

This set always has $n$ elements $w(0) = 0$, $w(1), \ldots, w(n-1)$, where $w(i)$ is the least element congruent to $i$ modulo $n$. Note also that $x \in \mathbb{Z}$ is an element of $S$ if and only if $x \geq w(x \bmod n)$. Consequently

$$(*) \qquad \mathrm{g}(S) = \max(\mathrm{Ap}(S, n)) - n.$$

The following result is a consequence of [Rosales 1996, Lemma 3.3] and gives a characterization of Apéry sets which will be useful later.

**Lemma 42.** *Let $m > 0$ be an integer and let $X = \{0 = w(0), w(1), \ldots, w(m-1)\}$ be a subset of $\mathbb{N}$ such that $i < w(i) \equiv i \bmod m$ for all $i \in \{1, \ldots, m-1\}$. Let $S$ be the submonoid of $\mathbb{N}$ generated by $X \cup \{m\}$. Then $S$ is a numerical semigroup with multiplicity $m$. Moreover, $\mathrm{Ap}(S, m) = X$ if and only if for all $i, j \in \{1, \ldots, m-1\}$ there exist $k \in \{0, \ldots, m-1\}$ and $t \in \mathbb{N}$ such that $w(i) + w(j) = w(k) + tm$.* □

Getting back to $S = \mathrm{S}(a, ab)$, with $a, b > 1$, we will give a description of the particular Apéry set $\mathrm{Ap}(S, \mathrm{m}(S))$ in terms of $a, b$, and this will lead to an explicit formula for the Frobenius number of $S$. We also show how the minimal generating set for such numerical semigroups can be computed from $a$ and $b$ as well as the corresponding sets of pseudo-Frobenius numbers.

**Lemma 43.** $\mathrm{m}(\mathrm{S}(a, ab)) = b$.

*Proof.* Let $S = \mathrm{S}(a, ab)$ and let $x \in \{1, \ldots, b-1\}$. Then $ax < ab$ and thus $ax \bmod ab = ax > x$, whence $x \notin S$. Clearly $b \in S$ and consequently $\mathrm{m}(S) = b$. □

**Theorem 44.** $\mathrm{Ap}(\mathrm{S}(a, ab), b) = \{0, k_1 b + 1, k_2 b + 2, \ldots, k_{b-1} b + b-1\}$, *where* $k_i = \lceil (a-1)i/b \rceil$ *for all* $i \in \{1, \ldots, b-1\}$.

*Proof.* Let $S'$ be the semigroup generated by $\{b, k_1 b + 1, \ldots, k_{b-1} b + b-1\}$. Since $k_i \geq 1$ for all $i \in \{1, \ldots, b-1\}$ we have $\mathrm{m}(S') = b$. Clearly $k_1 \leq \cdots \leq k_{b-1}$ and $k_i + k_j \geq k_{i+j}$ for all $i, j \in \{1, \ldots, b-1\}$ with $2 \leq i + j \leq b-1$. Using Lemma 42, we deduce that $\mathrm{Ap}(S', b) = \{0, k_1 b + 1, \ldots, k_{b-1} b + b-1\}$. Recall that $x \in \mathbb{Z}$ belongs to $S'$ if and only if $x \geq k_{x \bmod b} b + x \bmod b$, since this latter number is the element in $\mathrm{Ap}(S', b)$ that is congruent to $x$ modulo $b$. So, for $x$ an integer we have $x \in S' \iff \lfloor x/b \rfloor \geq k_{x \bmod b} \iff \lfloor x/b \rfloor \geq \lceil (a-1)(x \bmod b)/b \rceil \iff \lfloor x/b \rfloor \geq (a-1)(x \bmod b)/b \iff \lfloor x/b \rfloor b \geq (a-1)(x \bmod b) \iff x - (x \bmod b) \geq (a-1)(x \bmod b) \iff a(x \bmod b) \leq x \iff ax \bmod ab \leq x$. Thus $S' = \mathrm{S}(a, ab)$. □

Using this result and equality $(*)$ with $n = \mathrm{m}(S)$, we obtain:

**Corollary 45.** $\mathrm{g}(\mathrm{S}(a, ab)) = \left\lceil (b-1)(a-1)/b \right\rceil b - 1$.

Particularizing the formula given in Theorem 12 for the case at hand, we get

$$\#\mathrm{H}(\mathrm{S}(a,ab)) = \frac{a(b-1)-(a-1,b)+1}{2}.$$

***Minimal generators.*** We next turn our attention to the minimal generating set $\{n_0 < n_1 < \cdots < n_p\}$ of $\mathrm{S}(a,ab)$. We know that $n_0 = b$, by Lemma 43; our goal is to describe the remaining minimal generators.

**Lemma 46.** *Let $x$ and $y$ be positive integers. Then $\lceil x/b\rceil + \lceil y/b\rceil = \lceil (x+y)/b\rceil$ if and only if $x \equiv 0 \bmod b$ or $y \equiv 0 \bmod b$ or $(x \bmod b) + (y \bmod b) > b$.* □

**Remark 47.** If $S$ is any numerical semigroup and $m \in S\setminus\{0\}$, then $S$ is generated by $X = (\mathrm{Ap}(S,m)\setminus\{0\}) \cup \{m\} = \{m, w(1), \dots, w(m-1)\}$, and the minimal generating set of $S$ is $X \setminus (X+X)$. Now, in the case of $S = \mathrm{S}(a,ab)$, Theorem 44 says that $\mathrm{Ap}(S,b) = \{0, k_1 b+1, \dots, ak_{b-1}b+b-1\}$, with $k_i = \lceil (a-1)i/b\rceil$ for all $i \in \{1,\dots,b-1\}$. Thus $k_t b+t$ is a minimal generator of $S$ if and only if $k_t \neq k_i + k_{t-i}$ for all $i \in \{1,\dots,t-1\}$.

**Lemma 48.** *Let $S = \mathrm{S}(a,ab)$ with $a, b > 1$, set $k_i = \lceil (a-1)i/b\rceil$ for all $i \in \{1,\dots,b-1\}$ and take $t \in \{1,\dots,b-1\}$.*

(i) *If $t < b/(a-1,b)$, then $k_t b+t$ is a minimal generator of $S$ if and only if $(a-1)i \bmod b < (a-1)t \bmod b$ for all $i \in \{1,\dots,t-1\}$.*

(ii) *If $t > b/(a-1,b)$, then $k_t b+t$ is not a minimal generator of $S$.*

(iii) *If $t = b/(a-1,b)$, then $k_t b+t$ is a minimal generator of $S$.*

*Proof.* Using Lemma 46 and Remark 47, we see that $k_t b+t$ is a minimal generator of $S$ if and only if $(a-1)i \not\equiv 0 \bmod b$ and $(a-1)i \bmod b + (a-1)(t-i) \bmod b \le b$ for all $i \in \{1,\dots,t-1\}$. Observe that

$$(\dagger) \qquad \frac{b}{(a-1,b)} = \frac{\mathrm{lcm}(a-1,b)}{a-1} = \min\{i \mid (a-1)i \bmod b = 0\}.$$

(i) From the foregoing we deduce that if $t < b/(a-1,b)$, then $k_t b+t$ is a minimal generator of $S$ if and only if $(a-1)i \bmod b + (a-1)(t-i) \bmod b \le b$ for all $i \in \{1,\dots,t-1\}$. If $(a-1)i \bmod b + (a-1)(t-i) \bmod b = b$, then $(a-1)t \bmod b = 0$, which is impossible in view of (†), since $t < b/(a-1,b)$. Hence $k_t b+t$ is a minimal generator of $S$ if and only if for all $i \in \{1,\dots,t-1\}$ one has $(a-1)i \bmod b + (a-1)(t-i) \bmod b < b$, which is equivalent to $(a-1)i \bmod b + (a-1)(t-i) \bmod b = (a-1)t \bmod b$. Since $(a-1)(t-i) \bmod b \neq 0$, we conclude that $k_t b+t$ is a minimal generator of $S$ if and only if $(a-1)i \bmod b < (a-1)t \bmod b$ for all $i \in \{1,\dots,t-1\}$.

(ii) Let $i = b/(a-1,b)$. Then $(a-1)i \equiv 0 \bmod b$ and in view of Lemma 46 we get $k_i + k_{t-i} = k_t$, which implies that $k_t b+b$ is not a minimal generator of $S$.

(iii) In this setting $(a-1)t \bmod b=0$ and $(a-1)i \bmod b\neq 0$ for all $i\in\{1,\dots,t-1\}$. Hence for every $i\in\{1,\dots,t-1\}$ one gets $(a-1)i \bmod b+(a-1)(t-i) \bmod b=b$, and by Lemma 46 we deduce that $k_t\neq k_i+k_{t-i}$ for any $i\in\{1,\dots,t-1\}$. Therefore $k_tb+t$ is a minimal generator of $S$. □

Lemma 48 yields an explicit description of the minimal generating set of $S$:

**Theorem 49.** *Let $S=\mathrm{S}(a,ab)$ with $a,b>1$, and set $k_i=\lceil(a-1)i/b\rceil$ for $i\in\{1,\dots,b-1\}$.*

(1) *If $(b,a-1)=1$, the minimal generating set of $S$ is $\{b,k_{t_1}b+t_1,\dots,k_{t_r}b+t_r\}$, where $\{t_1,\dots,t_r\}=\big\{t\in\{1,\dots,b-1\}\bigm|(a-1)i \bmod b<(a-1)t \bmod b$ for all $i\in\{1,\dots,t-1\}\big\}$.*

(2) *If $(b,a-1)\neq 1$, let $t_{r+1}=b/(b,a-1)$. Then the minimal generating set of $S$ is $\{b,k_{t_1}b+t_1,\dots,k_{t_r}b+t_r,k_{t_{r+1}}b+t_{r+1}\}$, where $\big\{t_1,\dots,t_r\}=\{t\in\{1,\dots,t_{r+1}-1\}\bigm|(a-1)i \bmod b<(a-1)t \bmod b$ for all $i\in\{1,\dots,t-1\}\big\}$.* □

**Example 50.** Let $S=\mathrm{S}(5,35)$. Applying Theorem 49(1) with $a=5$ and $b=7$, we see that $\{t_1,\dots,t_r\}=\{1,3,5\}$ (observe that 1 is always in $\{t_1,\dots,t_r\}$), and that $S$ is minimally generated by $\{7,8,17,26\}$.

**Example 51.** Let $S=\mathrm{S}(5,30)$. Applying Theorem 49(2) with $a=5$ and $b=6$, we see that $t_{r+1}=3$, $\{t_1,\dots,t_r\}=\{1\}$, and $S$ is minimally generated by $\{6,7,15\}$.

**Corollary 52.** *Let $S=\mathrm{S}(a,ab)$ with $a,b>1$. Set $k_i=\lceil(a-1)i/b\rceil$ for $i\in\{1,\dots,b-1\}$, and*

$$t=\begin{cases}\min\{x\in\mathbb{N}\mid(a-1)x\equiv b-1 \bmod b\} & \text{if } (b,a-1)=1,\\ b/(b,a-1) & \text{if } (b,a-1)\neq 1.\end{cases}$$

*Then $k_tb+t$ is the greatest minimal generator of $S$.* □

**Corollary 53.** *Let $a\geq 3$ and let $b$ be a positive integer. Then $\mathrm{e}(\mathrm{S}(a,ab))\geq\lfloor b/(a-1)\rfloor+1$.*

*Proof.* The integer $b$ is always a minimal generator of $\mathrm{S}(a,ab)$. Also, if $(a-1)t\leq b$, then by Lemma 48, $k_tb+t$ is a minimal generator of $S$. □

***Pseudo-Frobenius numbers.*** For any numerical semigroup $S$, we define an order $\leq_S$ on $S$ as follows: $a\leq_S b$ if $b-a\in S$. Given a subset $A$ of $S$, denote by $\mathrm{Max}_{\leq_S}A$ the set of maximal elements of $A$ with respect to $\leq_S$. The following result appears in [Rosales and Branco 2002].

**Lemma 54.** *Let $S$ be any numerical semigroup with multiplicity $m$. If*

$$\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S,m))=\{w_{i_1},\dots,w_{i_t}\},$$

*the pseudo-Frobenius numbers of $S$ (page 387) are precisely $w_{i_1}-m,\dots,w_{i_t}-m$.*

Note that if $w, w' \in \mathrm{Ap}(S, m)$ and $w - w' \in S$, this forces $w - w'$ to be in $\mathrm{Ap}(S, m)$ as well. Hence

$$\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, m)) = \{w \in \mathrm{Ap}(S, m) \mid w + w' \notin \mathrm{Ap}(S, m) \text{ for all } 0 \neq w' \in \mathrm{Ap}(S, m)\}.$$

Let $S = \mathrm{S}(a, ab)$ with $a, b > 1$. Our aim is to compute the set $\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b))$ and thus, in view of Lemma 54, the pseudo-Frobenius set $\mathrm{Pg}(S)$.

**Remark 55.** By Theorem 44, $k_i b + i \notin \mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b))$ if and only if there exists $j \in \{1, \dots, b-1\}$ such that $i + j \leq b-1$ and $k_i + k_j = k_{i+j}$. Minimal generators are $\leq_S$-minimal elements of $\mathrm{Ap}(S, b)$, which is why the condition just stated is similar (dual) to the one presented on the previous page for minimal generators.

**Theorem 56.** *Let $a$ and $b$ be two integers greater than one, and let $S = \mathrm{S}(a, ab)$. Let $k_i = \lceil (a-1)i/b \rceil$ for $i \in \{1, \dots, b-1\}$. Then $k_i b + i \in \mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b))$ if and only if one of the following conditions hold*:

(i) $(a-1)i \equiv 0 \bmod b$ *and* $i = b-1$,

(ii) $(a-1)i \not\equiv 0 \bmod b$ *and for all* $t \in \{i+1, \dots, b-1\}$, *either* $(a-1)i \bmod b < (a-1)t \bmod b$ *or* $(a-1)t \bmod b = 0$.

*Proof.* Assume that $(a-1)i \equiv 0 \bmod b$ and $i < b-1$. Then by Lemma 46, we deduce that $k_i + k_1 = k_{i+1}$ and thus $k_i b + i \notin \mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b))$. If $(a-1)i \not\equiv 0 \bmod b$, then by Lemma 46 we have $k_i b_i + i \in \mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b))$ if and only if for all $t \in \{i+1, \dots, b-1\}$ we have $(a-1)(t-i) \not\equiv 0 \bmod b$ and $(a-1)i \bmod b + (a-1)(t-i) \bmod b \leq b$. If $(a-1)i \bmod b + (a-1)(t-i) \bmod b < b$, then $(a-1)i \bmod b + (a-1)(t-i) \bmod b = (a-1)t \bmod b$ and thus $(a-1)i \bmod b < (a-1)t \bmod b$. If $(a-1)i \bmod b + (a-1)(t-i) \bmod b = b$, then $(a-1)t \bmod b = 0$.

To prove the converse, assume $k_i b + i \notin \mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b))$. Then there exists $t \in \{1+i, \dots, b-1\}$ such that $k_i + k_{t-i} = k_t$. By using Lemma 46, we deduce that $(a-1)i \equiv 0 \bmod b$ or $(a-1)(t-i) \equiv 0 \bmod b$ or $(a-1)i \bmod b + (a-1)(t-i) \bmod b > b$. If $(a-1)i \equiv 0 \bmod b$, then $i$ must be equal to $b-1$, but this is impossible since $t \in \{i+1, \dots, b-1\}$. If $(a-1)(t-i) \equiv 0 \bmod b$, then $(a-1)i \bmod b = (a-1)t \bmod b$, which is also impossible by hypothesis. Finally if $(a-1)i \bmod b + (a-1)(t-i) \bmod b > b$, then $(a-1)t \bmod b = (a-1)i \bmod b + (a-1)(t-i) \bmod b - b < (a-1)i \bmod b$, leading again to a contradiction. □

**Example 57.** Let $S = \mathrm{S}(5, 30)$. Applying Theorem 56 we get $\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, 6)) = \{29\}$, which by Lemma 54 means that $\mathrm{Pg}(S) = \{23\}$. Thus $\mathrm{S}(5, 30)$ is symmetric.

**Proposition 58.** *Let $S = \mathrm{S}(a, ab)$ with $a, b > 1$.*

(1) *$S$ is symmetric if and only if* $(a-1, b) + (a-1) \bmod b = b$.

(2) *$S$ is pseudo-symmetric if and only if* $(a-1, b) + (a-1) \bmod b = b + 1$.

*Proof.* (1) Combining Corollaries 45 and 17(1), we see that $S$ is symmetric if and only if $\lceil (b-1)(a-1)/b \rceil b - 1 = ab - a - (a-1, b)$. The left-hand side can be written as $\big(a-1-\lfloor (a-1)/b \rfloor\big) b - 1 = (a-1)b - \lfloor (a-1)/b \rfloor b - 1 = ab - b - (a-1-(a-1) \bmod b) - 1$. Thus $S$ is symmetric if and only if $(a-1) \bmod b + (a-1, b) = b$.

(2) As above, but this time using Corollary 17(2). □

**Corollary 59.** *Let $k$ be a positive integer and let $b$ be a multiple of $k$. Then $\mathrm{S}(b-k+1+bn, (b-k+1+bn)b)$ is symmetric for all $n \in \mathbb{N}$.* □

The pseudo-symmetric case is completely different:

**Corollary 60.** $\mathrm{S}(a, ab)$ *is not pseudo-symmetric for any choice of $a, b > 1$.*

*Proof.* Set $q = \lfloor (a-1)/b \rfloor$ and choose $u, v \in \mathbb{Z}$ such that $(a-1, b) = u(a-1) + vb$. If $\mathrm{S}(a, ab)$ is pseudo-symmetric, we have $(a-1, b) + (a-1) \bmod b = b+1$, hence $u(a-1) + vb + (a-1) - qb = b + 1$, or yet $(u+1)(a-1) + (v-q-1)b = 1$. But this implies $(a-1, b) = 1$ and hence $1 + (a-1) \bmod b = b+1$, an impossibility. □

***Some families.*** We now present some families of numerical semigroups of the form $\mathrm{S}(a, ab)$ with $a, b > 1$ such that $(a-1, b) = 1$. For these families we can compute the minimal generating set and pseudo-Frobenius numbers explicitly. As a consequence of Theorems 49 and 56 one gets:

**Proposition 61.** *Let $S = \mathrm{S}(a, ab)$ with $a, b > 1$ and $(a-1, b) = 1$. Set $k_i = \lceil (a-1)i/b \rceil$ for $i \in \{1, \dots, b-1\}$ and take $t \in \{1, \dots, b-1\}$.*

1. *$k_t b + t$ is a minimal generator of $S$ if and only if $(a-1)i \bmod b < (a-1)t \bmod b$ for all $i \in \{1, \dots, t-1\}$.*
2. *$k_t b + t \in \mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b))$ if and only if $(a-1)t \bmod b < (a-1)i \bmod b$ for all $i \in \{t+1, \dots, b-1\}$.*

Let $\mathrm{S}_n$ be the symmetric group in $n$ elements $\{1, \dots, n\}$, and for $k$ relatively prime to $n+1$, define the permutation $\sigma_{k,n+1} \in \mathrm{S}_n$ by $\sigma(i) = ki \bmod (n+1)$ for $i = 1, \dots, n$. Such a permutation is called *modular*. Next, given any permutation $\sigma \in \mathrm{S}_n$, set

$$\mathrm{E}(\sigma) = \{t \in \{1, \dots, n\} \mid \sigma(i) < \sigma(t) \text{ for all } i \in \{1, \dots, t-1\}\},$$
$$\mathrm{T}(\sigma) = \{t \in \{1, \dots, n\} \mid \sigma(t) < \sigma(i) \text{ for all } i \in \{t+1, \dots, n\}\}.$$

With this notation we can rewrite Proposition 61 as follows.

**Corollary 62.** *Let $S = \mathrm{S}(a, ab)$ with $a, b > 1$ and $(a-1, b) = 1$. Then*

$$\mathrm{e}(S) = \#\mathrm{E}(\sigma_{a-1,b}) + 1 \quad \textit{and} \quad \mathrm{t}(S) = \#\mathrm{T}(\sigma_{a-1,b}).$$

*The minimal generating set of $S$ is $\{b\} \cup \big\{\lceil (a-1)i/b \rceil b + i \;\big|\; i \in \mathrm{E}(\sigma_{a-1,b})\big\}$, and*

$$\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b)) = \big\{\lceil (a-1)i/b \rceil b + i \;\big|\; i \in \mathrm{T}(\sigma_{a-1,b})\big\}.$$

**Example 63.** Let $S = \mathrm{S}(6, 42)$. Apply Corollary 62 with $a = 6$ and $b = 7$. Clearly $\sigma_{5,7} = (154623)$, $\mathrm{E}(\sigma_{5,7}) = \{1, 4\}$ and $\mathrm{T}(\sigma_{5,7}) = \{3, 6\}$. Hence $\mathrm{e}(S) = 3$ and $\mathrm{t}(S) = 2$. The set $\{7, \lceil (5\times 1)/7\rceil 7+1, \lceil (5\times 4)/7\rceil 7+4\} = \{7, 8, 25\}$ is a minimal generating set of $S$ and $\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, 7)) = \{\lceil (5\times 3)/7\rceil 7+3, \lceil (5\times 6)/7\rceil 7+6\} = \{24, 41\}$.

**Corollary 64.** *Let* $S = \mathrm{S}((b-1)+bn, ((b-1)+bn)b)$ *with* $n \in \mathbb{N}$ *and* $b \geq 5$ *odd. Then* $S$ *is minimally generated by* $\left\{b, (n+1)b+1, \left(\frac{b-1}{2}+n\frac{b+1}{2}\right)b+\frac{b+1}{2}\right\}$, *and*

$$\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b)) = \left\{\left(\tfrac{b-1}{2}+n\tfrac{b-1}{2}\right)b+\tfrac{b-1}{2}, ((b-2)+n(b-1))b+b-1\right\}.$$

*Proof.* Since $(b-2+bn, b) = (b-2, b) = 1$, we can apply Corollary 62. By inspection we see that $\mathrm{E}(\sigma_{b-2,b}) = \{1, (b+1)/2\}$ and $\mathrm{T}(\sigma_{b-2,b}) = \{(b-1)/2, b-1\}$. We can conclude the proof using Corollary 62, taking into account that

$$\left\lceil \frac{((b-2)+bn)1}{b} \right\rceil = n+1, \quad \left\lceil \frac{((b-2)+bn)(b\pm 1)/2}{b} \right\rceil = \frac{b-1}{2}+n\frac{b\pm 1}{2}, \quad \text{and}$$

$$\left\lceil \frac{((b-2)+bn)(b-1)}{b} \right\rceil = (b-2)+n(b-1). \qquad \square$$

**Corollary 65.** *Let* $b$ *be an integer greater than or equal to two and let* $n \in \mathbb{N}$. *Then* $S = \mathrm{S}((n+1)b, (n+1)b^2)$ *is minimally generated by* $\{b, (n+1)b+1\}$ *and* $\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b)) = \{(n+1)(b-1)b+b-1\}$.

*Proof.* Use Corollary 62 and the fact that $\sigma_{(n+1)b-1,b} = \sigma_{b-1,b}$ swaps $i$ and $b-i$. $\square$

**Corollary 66.** *Let* $S = \mathrm{S}(2+nb, (2+nb)b)$ *with* $n \in \mathbb{N}$ *and* $b \geq 2$. *Then* $S$ *is minimally generated by*

$$X = \{b, (n+1)b+1, (2n+1)b+2, \dots, ((b-1)n+1)b+b-1\}$$

*and* $\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b)) = X \setminus \{b\}$.

*Proof.* Use Corollary 62 and the fact that $\sigma_{1+nb,b} = \sigma_{1,b}$ is the identity. $\square$

**Corollary 67.** *Let* $S = \mathrm{S}(3+nb, (3+nb)b)$ *with and* $n \in \mathbb{N}$ $b \geq 3$ *odd. Then* $S$ *is minimally generated by* $\left\{b, (n+1)b+1, (2n+1)b+2, \dots, \left(\frac{b-1}{2}n+1\right)+\frac{b-1}{2}\right\}$ *and*

$$\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b)) = \left\{\left(\tfrac{b+1}{2}n+2\right)b+\tfrac{b+1}{2}, \dots, ((b-1)n+2)b+b-1\right\}.$$

*Proof.* By considering $\sigma_{2+bn,b} = \sigma_{2,b}$ we see that $\mathrm{E}(\sigma_{2,b}) = \left\{1, \dots, \frac{1}{2}(b-1)\right\}$ and $\mathrm{T}(\sigma_{2,b}) = \left\{\frac{1}{2}(b+1), \dots, b-1\right\}$. Using Corollary 62, the proof follows easily from

$$\left\lceil \frac{(2+bn)i}{b} \right\rceil b = \begin{cases} (ni+1)b+i & \text{if } i \leq \frac{1}{2}(b-1), \\ (ni+2)b+i & \text{if } i \geq \frac{1}{2}(b+1). \end{cases} \qquad \square$$

## 6. The Frobenius number in other special cases

In Section 5 we studied $\mathrm{S}(a, b)$ with $a \mid b$. We now give some partial results for the Frobenius number in the complementary case, $a \nmid b$. We are able to find the number when $(a-1)(a-(b \bmod a)) < b$. We use without further comment the fact that, for $q$ a rational number and $x$ a positive integer, $x < \lceil q \rceil$ implies $x < q$.

**Lemma 68.** *Let* $S = \mathrm{S}(a, b)$ *with* $0 < a < b$ *and* $b \bmod a \neq 0$. *Then*

$$\mathrm{g}(\mathrm{S}(a, b)) \leq b - \lceil b/a \rceil.$$

*Proof.* Let $x$ be a positive integer. If $x < \lceil b/a \rceil$, then $x < b/a$ and thus $ax \bmod b = ax > x$. Hence $x \notin S$ and in view of Corollary 6, this leads to $b - x \in S$. As $y \in S$ for all $y \geq b$, we conclude that $\mathrm{g}(S) \leq b - \lceil b/a \rceil$. □

**Lemma 69.** *Let* $a$ *and* $b$ *be positive integers such that* $a < b$ *and* $b \bmod a \neq 0$. *Then* $a\lceil b/a \rceil \bmod b = a - (b \bmod a)$. □

**Proposition 70.** *Let* $a$ *and* $b$ *be positive integers such that* $a < b$ *and* $b \bmod a \neq 0$. *Then* $\mathrm{g}(\mathrm{S}(a, b)) = b - \lceil b/a \rceil$ *if and only if* $(a-1)(a-(b \bmod a)) < b$.

*Proof.* Let $S = \mathrm{S}(a, b)$. From Lemma 68 we deduce that $\mathrm{g}(S) = b - \lceil b/a \rceil$ if and only if $b - \lceil b/a \rceil \notin S$, or in other words, $a(b - \lceil b/a \rceil) \bmod b > b - \lceil b/a \rceil$. This by Lemma 69 is equivalent to $((b \bmod a) - a) \bmod b > b - \lceil b/a \rceil$, and this condition holds if and only if $b + (b \bmod a) - a > b - \lfloor b/a \rfloor - 1$. Hence $\mathrm{g}(S) = b - \lceil b/a \rceil$ if and only if $\lfloor b/a \rfloor + 1 + (b \bmod a) > a$, or equivalently $(b - (b \bmod a))/a + 1 + (b \bmod a) > a$, and this holds if and only if $b > (a-1)(a-(b \bmod a))$. □

**Corollary 71.** *Let* $a$ *and* $b$ *be positive integers such that* $a < b$,$b \bmod a \neq 0$ *and* $(a-1)(a-(b \bmod a)) < b$. *Then* $\mathrm{m}(\mathrm{S}(a, b)) = \lceil b/a \rceil$.

*Proof.* Let $S = \mathrm{S}(a, b)$. By Proposition 70, we know that $\mathrm{g}(S) = b - \lceil b/a \rceil$. Thus $b - \lceil b/a \rceil \notin S$ and thus by Corollary 6, $\lceil b/a \rceil = b - (b - \lceil b/a \rceil) \in S$. Besides, if $x$ is a positive integer such that $x < \lceil b/a \rceil$, then $x < b/a$, whence $ax \bmod b = ax > x$ and thus $x \notin S$. Therefore $\mathrm{m}(S) = \lceil b/a \rceil$. □

Though we have given an explicit formula for $\mathrm{g}(\mathrm{S}(a, b))$ for several cases, we have not been able to find such a formula for arbitrary positive integers $a$ and $b$. We propose this as an open question.

**Problem 1.** Find a formula for $\mathrm{g}(\mathrm{S}(a, b))$ with $a$ and $b$ positive integers.

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J. C. ROSALES
DEPARTAMENTO DE ÁLGEBRA
UNIVERSIDAD DE GRANADA
E-18071 GRANADA
SPAIN
jrosales@ugr.es

P. A. GARCÍA-SÁNCHEZ
DEPARTAMENTO DE ÁLGEBRA
UNIVERSIDAD DE GRANADA
E-18071 GRANADA
SPAIN
pedro@ugr.es

J. M. URBANO-BLANCO
DEPARTAMENTO DE ÁLGEBRA
UNIVERSIDAD DE GRANADA
E-18071 GRANADA
SPAIN
jurbano@ugr.es