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ENDO-WILD REPRESENTATION TYPE AND GENERIC REPRESENTATIONS OF FINITE POSETS

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Dedicated to the memory of Sheila Brenner

The category of finite-dimensional representations of a finite poset over an arbitrary field k is shown to have k-wild representation type if and only if it has k-endo-wild representation type. Included are some characterizations of representation type in terms of existence conditions on infinite-dimensional generic representations and their endomorphism rings.

1. Introduction

Let k be an arbitrary field and $(S, \leq)$ a finite partially ordered set (poset) with a partial order relation $\leq$. Throughout we write simply S instead of $(S, \leq)$, and we assume that S has a unique maximal element, denoted by ∞. We are interested in the category $\mathbf{fpr}(S, k)$ of S-filtered finite-dimensional k-linear representations of S with objects $U = (U_s;\ s \in S)$, where each U_s is a finite-dimensional k-vector space and $U_s \subseteq U_t \subseteq U_\infty$ if $s \leq t$ in S. A morphism from $U = (U_s;\ s \in S)$ to $V = (V_s;\ s \in S)$ is a k-linear map $f : U_\infty \longrightarrow V_\infty$ such that $f(U_s) \subseteq V_s$ for each $s \in S$.

Notice that $\mathbf{fpr}(S, k)$ is an additive Krull–Schmidt k-category, and it is the category S'-sp of S'-spaces over k studied in [Simson 1992], where $S' = S \setminus \{\infty\}$ is viewed as a subposet of S.

We recall that the category $\mathbf{fpr}(S, k)$ is equivalent to a full subcategory of the category $\operatorname{mod} kS$ of finitely generated modules over the incidence k-algebra kS of the poset S with coefficients in k; see [Simson 1992, Chapter 5]. Many properties of $\mathbf{fpr}(S, k)$, such as the existence of almost split sequences, are inherited from $\operatorname{mod} kS$. On the other hand, $\mathbf{fpr}(S, k)$ has properties not enjoyed by $\operatorname{mod} kS$, or more generally $\operatorname{mod} A$ for a finite-dimensional k-algebra A, without some restrictions. For example, the classical Kleiner–Nazarova characterizations of representation type and a tame/wild dichotomy theorem for $\mathbf{fpr}(S, k)$ are independent of k [Nazarova 1975; Zavadskij and Nazarova 1977; Zavadskij 1987; 1991; Arnold and

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Simson 2002], while the only known tame/wild dichotomy theorem for mod A, where A is a finite-dimensional k-algebra, has the assumption that k is an algebraically closed field [Drozd 1979; Crawley-Boevey 1988].

In this paper, we study two other properties of $\mathbf{fpr}(S, k)$. The first asserts that k-wild representation type is equivalent to k-endo-wild representation type for $\mathbf{fpr}(S, k)$, where k is an arbitrary field (Theorem 2.4).

We recall from [Han 2002] that k-wild representation type need not be equivalent to k-endo-wild representation type, for the category mod A of finite-dimensional A-modules, even under the assumption that A is a finite-dimensional algebra over an algebraically closed field k (see also [Han 2001a; 2001b]). It is shown in [Han 2002, Theorem 2] that for any radical square zero k-algebra A of finite dimension, the category mod A is k-endo-wild if and only if mod A is fully k-wild, and mod A is Corner type k-endo-wild if and only if mod A is wild. In [Han 2001a; 2001b; 2002], a connection between k-endo-wildness and controlled k-wildness for a k-algebra is also discussed; see also [Ringel 2002; Ringel and Schmidmeier 2004].

The second property asserts that representation type for $\mathbf{fpr}(S, k)$ can be characterized by existence conditions on generic representations and their endomorphism rings (Theorems 3.4 and 3.8). Similar existence conditions on generic modules for mod A, where A is a finite-dimensional k-algebra over an algebraically closed field k, are proved in [Crawley-Boevey 1991] using the theory of bocses. Since some of the reduction algorithms in the theory of bocses are not well developed, unless k is an algebraically closed field, our arguments rely on the special nature of $\mathbf{fpr}(S, k)$ for an arbitrary field k.

As demonstrated in [Arnold 2000; Arnold and Simson 2002; Simson 1992], properties of the categories $\mathbf{fspr}(S, R)$ for a commutative uniserial ring R and $\mathbf{fpr}(S, k)$ for an arbitrary field k have immediate application to categories of abelian groups and lattices over orders. In particular, Theorems 2.4, 3.4 and 3.8 answer some open questions stated in [Arnold 2000] for some quasi-homomorphism categories of torsion-free abelian groups of finite rank. See also [Arnold et al. 1993; Dugas and Rangaswamy 2002; Mader 2000; Nongxa and Vinsonhaler 1996; Richman and Walker 1999; Schmidmeier 2004].

There is a classical theorem in abelian group theory due to A. L. S. Corner [1963]: If R is a ring whose additive group is a reduced torsion-free group of finite rank, then there is a torsion-free abelian group G of finite rank with End $G \cong R$. The resulting diversity of endomorphism rings of torsion-free abelian groups of finite rank allows for the construction of examples of pathological direct sum decompositions [Fuchs 1973]. Since Corner's theorem appeared, a diversity of endomorphism rings has served as a de-facto deterrent to a search for complete sets of invariants of classes of torsion-free abelian groups of finite rank. A consequence of Theorem 2.4, together with a category equivalence of M. C. R. Butler [1968], is that, for the

quasi-isomorphism category of Butler groups with typesets in a fixed finite lattice of types, k-endo-wild representation type is equivalent to k-wild representation type. In this case, the imprecise notion that a diversity of endomorphism rings is a deterrent to classification of classes of torsion-free abelian groups of finite rank is made precise.

Throughout this paper we use the standard terminology and notation. In particular, given a ring R, we denote by $J(R)$ the Jacobson radical of R, by $\operatorname{Mod} R$ the category of all right R-modules and by $\operatorname{mod} R$ the full subcategory of $\operatorname{Mod} R$ formed by finitely generated modules.

2. k-wild and k-endo-wild representation type

Let k be a field and S a finite poset with a unique maximal element. Following [Drozd 1979; Simson 1992, Section 14.2; 1993; 2003], we define a full exact subcategory $\mathscr{C}$ of $\mathbf{fpr}(S, k)$ to have *k-wild representation type* if there is an exact k-linear functor

$$T : \operatorname{modf} k\langle t_1, t_2\rangle \longrightarrow \mathscr{C}$$

that preserves isomorphism classes (that is, $T(X) \cong T(Y)$ implies $X \cong Y$) and preserves indecomposables, where $k\langle t_1, t_2\rangle$ is the polynomial k-algebra in two non-commuting variables t_1 and t_2 and $\operatorname{modf} k\langle t_1, t_2\rangle$ is the category of finitely generated right $k\langle t_1, t_2\rangle$-modules with finite k-dimension. If, in addition, the functor T is full, then $\mathscr{C}$ has *fully k-wild representation type*.

It follows from [Simson 1993] and Corollary 2.3 below that $\mathscr{C}$ has fully k-wild representation type if and only if it has *strictly k-wild representation type* in the sense of [Crawley-Boevey 1992]; see also below.

Following [Crawley-Boevey 1992], [Ringel 1976] and [Simson 2003, Remark 2.5], we need to distinguish between k-wildness and wildness, because the two notions do not coincide for the category $\operatorname{mod} R$. We recall that the category $\operatorname{mod} R$ of finite-dimensional right modules over a finite dimensional k-algebra R is defined as *wild* if there exist a finite field extension k' of k and a faithful exact additive functor $T' : \operatorname{modf} k'\langle t_1, t_2\rangle \longrightarrow \operatorname{mod} R$ that preserves isomorphism classes and preserves indecomposables; moreover, $\operatorname{mod} R$ is *strictly wild* if there is such a T' that is fully faithful. If k is algebraically closed, $\operatorname{mod} R$ is k-wild if and only if it is wild, and it is fully k-wild if and only if it is strictly wild.

Following the definition of fully wild representation type and [Arnold 2000], [Simson 2002, Definition 5.1] and [Simson 2003, Definition 2.6], we introduce the following two useful notions for an arbitrary additive k-category.

Definition 2.1. Let k be a field. An additive k-category $\mathscr{C}$ is defined to have *k-endo-wild representation type* if for each finite-dimensional k-algebra A there is an object U of $\mathscr{C}$ and a k-algebra isomorphism $\operatorname{End} U \cong A$.

According to [Han 2002, Theorem 2], for the category $\mathcal{C} = \operatorname{mod} A$ of modules over a radical square zero k-algebra A of finite dimension, then k-endo-wild representation type and fully k-wild representation type coincide.

The following simple lemma is very useful (see [Simson 2003]).

Lemma 2.2. *Let k be a field and let $\mathcal{C}$ be a full exact k-subcategory of $\mathbf{fpr}(S, k)$.*

(a) *The category $\mathcal{C}$ has fully k-wild representation type if and only if, for each finite-dimensional k-algebra A, there is a fully faithful exact k-linear functor $H : \operatorname{mod} A \longrightarrow \mathcal{C}$.*

(b) *If, for each finite-dimensional k-algebra A, there exists a fully faithful k-linear functor $H : \operatorname{mod} A \longrightarrow \mathcal{C}$, then, for each finite-dimensional k-algebra A, there exists such a functor H that is exact.*

(c) *If, for each finite-dimensional k-algebra A, there exists a fully faithful k-linear functor $H : \operatorname{mod} A \longrightarrow \mathcal{C}$, then $\mathcal{C}$ has k-endo-wild representation type.*

Proof. (a) Sufficiency is easy, because for the k-algebra

$$A = \begin{pmatrix} k & k^3 \\ 0 & k \end{pmatrix}$$

of k-dimension five there exists a fully faithful exact k-linear functor

$$\operatorname{modf} k\langle t_1, t_2\rangle \longrightarrow \operatorname{mod} A$$

(see [Simson 1992, pp. 286–287]).

Necessity is a consequence of [Brenner 1974a] and [Simson 1993, Theorem 2.9], as follows. Suppose that $\mathcal{C}$ has fully k-wild representation type and A is a finite-dimensional k-algebra. The theorem just cited says that there exists a fully faithful exact k-linear functor $T : \operatorname{modf} k\langle t_1, t_2\rangle \longrightarrow \mathcal{C}$. On the other hand, by a well known result from Sheila Brenner [1974a], there is a fully faithful exact k-linear functor $F : \operatorname{mod} A \longrightarrow \operatorname{modf} k\langle t_1, t_2\rangle$. It follows that the composite k-linear functor $H = T \circ F : \operatorname{mod} A \longrightarrow \mathcal{C}$ is fully faithful and exact.

(b) The proof of is analogous to that of (a).

(c) Assume that A is a finite-dimensional k-algebra. Then there exists a fully faithful k-linear functor $H : \operatorname{mod} A \longrightarrow \mathcal{C}$. It follows that $U = H(A)$ is an object of $\mathcal{C} \subseteq \mathbf{fpr}(S, k)$ such that $\operatorname{End} U \cong A$. □

The proof of Lemma 2.2, together with [Simson 1993, Theorem 2.9], yields:

Corollary 2.3. *Let k be a field. A full exact k-subcategory $\mathcal{C}$ of $\mathbf{fpr}(S, k)$ has fully k-wild representation type if and only if there exists a fully faithful exact k-linear functor $H : \operatorname{modf} k\langle t_1, t_2\rangle \longrightarrow \mathcal{C}$. In this case H has the form $H =$*

$(-)\otimes_{k\langle t_1,t_2\rangle} N$, *where* ${}_{k\langle t_1,t_2\rangle}N_{kS}$ *is a* $k\langle t_1,t_2\rangle$-kS*-bimodule such that the left* $k\langle t_1, t_2\rangle$*-module* ${}_{k\langle t_1,t_2\rangle}N$ *is free of finite rank.*

The arguments applied above show that if R is a fully k-wild k-algebra, that is, mod R is of fully k-wild representation type, then the category mod R is k-endo-wild. Since not every local wild algebra over an algebraically closed field k is fully wild then, according to [Han 2002, Theorem 2], there exists a k-algebra R such that mod R is k-wild, but it is not k-endo-wild. The problem of determining all finite-dimensional k-algebras R for which mod R has fully k-wild representation type if and only if mod R has k-endo-wild representation type remains unsolved (see [Simson 2002, Problem 5.2]).

Let $U = (U_s;\ s \in S)$ be an object of $\mathbf{fpr}(S, k)$. The *coordinate vector* of U is defined in [Simson 1992] to be the integral vector

$$\mathbf{cdn}\, U = (u_s)_{s\in S} \in \mathbb{Z}^S,$$

where $u_s = \dim_k\big(U_s/\sum_{t<s} U_s\big)$ if $s \in S \setminus \{\infty\}$ and $u_\infty = \dim_k(U_\infty)$. We set

$$|\mathbf{cdn}\, U| = \sum_{s\in S} u_s.$$

The representation U is defined to be *sincere* if U is indecomposable and $u_s \neq 0$ for each $s \in S \setminus \{\infty\}$. The poset S is *sincere* if there is an indecomposable sincere object of $\mathbf{fpr}(S, k)$.

We recall from [Simson 1992, Proposition 5.14] that for any indecomposable object U of $\mathbf{fpr}(S, k)$, with $\mathbf{cdn}\, U = (u_s)_{s\in S}$, the subposet

$$S_U = \{s \in S;\ \ u_s \neq 0\}$$

of S, called the *coordinate support* of U, is sincere, the restriction V of U to S_U is an indecomposable sincere object of $\mathbf{fpr}(S_U, k)$, and there exists a fully faithful k-linear embedding $i : \mathbf{fpr}(S_U, k) \longrightarrow \mathbf{fpr}(S, k)$ such that $U \cong i(V)$. In particular, there is a k-algebra isomorphism $\operatorname{End} U \cong \operatorname{End} V$.

Now we show that, for the category $\mathbf{fpr}(S, k)$, the k-wild representation and k-endo-wild representation type coincide.

Theorem 2.4. *Assume that* k *is an arbitrary field and* S *is a finite poset with a unique maximal element. The following three statements are equivalent.*

(a) *The category* $\mathbf{fpr}(S, k)$ *has* k*-wild representation type.*

(b) *The category* $\mathbf{fpr}(S, k)$ *has* k*-endo-wild representation type.*

(c) *There exists a fully faithful* k*-linear functor* $T : \operatorname{modf} k\langle t_1, t_2\rangle \longrightarrow \mathbf{fpr}(S, k)$.

(d) *The category* $\mathbf{fpr}(S, k)$ *has fully* k*-wild representation type.*

Proof. (d) $\Rightarrow$ (c) is obvious, and (c) $\Rightarrow$ (b) follows from Lemma 2.2(c).

(a) $\Rightarrow$ (d): Assume that the category $\mathbf{fpr}(S,k)$ has k-wild representation type. The poset S contains a subposet $\mathcal{N}$ isomorphic to one of the six hypercritical posets of [Nazarova 1975]:

$\mathcal{N}_1 = (\cdot\cdot\cdot\cdot\cdot)$, $\mathcal{N}_2 = (\cdot\cdot\cdot\vdots)$, $\mathcal{N}_3 = (\vdots\vdots\vdots)$, $\mathcal{N}_4 = (\cdot\vdots\vdots)$, $\mathcal{N}_5 = (\vdots\vdots\vdots)$, $\mathcal{N}_6 = (\cdot\vdots\vdots)$

(see also [Simson 1992, Theorem 15.3]). There exists a pair of fully faithful exact k-linear functors

$$\operatorname{modf} k\langle t_1, t_2\rangle \xrightarrow{\widehat{F}_{\mathcal{N}}} \operatorname{prin} k\mathcal{N}^{\infty} \xrightarrow{\widehat{T}_{\mathcal{N}}} \operatorname{prin} kS,$$

where, for a poset T, $\operatorname{prin} kT$ denotes the category of prinjective right kT-modules and $T^{\infty} = T \cup \{\infty\}$, with $t < \infty$, for all $t \in T$. The functor $\widehat{F}_{\mathcal{N}}$ is constructed in [Simson 1992, pp. 310–312] (and corrected in Theorem A.1 of the Appendix), whereas $\widehat{T}_{\mathcal{N}}$ is the subposet induction functor ([Simson 1992, (11.85)]; see also [Kasjan and Simson 1995]). This shows that the subcategory $\operatorname{prin} kS$ of $\operatorname{mod} kS$ has fully k-wild representation type. By applying [Kasjan 2003, Proposition 5.4] to the poset $S' = S \setminus \{\infty\}$ and to the adjustment functor $\mathbf{\Theta} : \operatorname{prin} kS \longrightarrow S'\text{-sp} \cong \operatorname{mod}_{\mathrm{sp}} kS$ defined in [Simson 1992, 11.32–34], we conclude that the category $\mathbf{fpr}(S,k) \cong \operatorname{mod}_{\mathrm{sp}} kS$ has fully k-wild representation type, and (d) follows.

(b) $\Rightarrow$ (a): Assume that $\mathbf{fpr}(S,k)$ does not have k-wild representation type. As a consequence of Nazarova's theorem [1975] (see also [Zavadskij and Nazarova 1977; Simson 1992, Theorem 15.3]), the poset S does not contain, as a subposet, any of the six hypercritical posets $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5, \mathcal{N}_6$ of Nazarova [Nazarova 1975]. There are several cases to be considered in order to show that $\mathbf{fpr}(S,k)$ does not have k-endo-wild representation type. The analysis of these cases will occupy us until the end of this section.

Case I. $S = (\cdot\cdot\cdot\cdot) \cup \{\infty\}$.

By [Brenner 1974b], for each indecomposable object U of $\mathbf{fpr}(S,k)$, the algebra $\operatorname{End} U$ is isomorphic to a quotient algebra of the polynomial algebra $k[x]$. In particular, either U is preprojective or preinjective with $\operatorname{End} U \cong k$ or else U is regular with $\operatorname{End} U \cong k[x]/(g(x)^e)$, for some irreducible polynomial $g(x)$ (see also [Simson 1992, Section 15.6; Zavadskij 1987; 1991]). This shows that $\mathbf{fpr}(S,k)$ does not have k-endo-wild representation type, because the matrix k-subalgebra

$$A = \left\{ \begin{pmatrix} a & b & d \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} ;\ a,b,c,d \in k \right\}$$

of $\mathbb{M}_3(K)$ of dimension four has no nontrivial idempotents and it is not isomorphic to an algebra $k[x]/(g(x)^e)$, for some irreducible polynomial $g(x)$.

Case II. The poset S does not contain, as a subposet, any of the posets $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_4$, $\mathcal{N}_5$, $\mathcal{N}_6$, or

$$\mathcal{NL} = (\vdots\!\times\!\vdots \ \cdot\ \cdot).$$

It is sufficient to prove that the algebra $\operatorname{End} U$ is commutative, for each sincere indecomposable object U of $\mathbf{fpr}(S,k)$, recalling that each indecomposable object in $\mathbf{fpr}(S,k)$ may be identified with a sincere indecomposable object in $\mathbf{fpr}(S_U,k)$, for some subposet S_U of S. In this case, $\mathbf{fpr}(S,k)$ does not have k-endo-wild representation type since, for example, the k-algebra A of dimension four defined in Case I is noncommutative and has no nontrivial idempotents.

The proof that $\operatorname{End} U$ is commutative, for each sincere indecomposable object U of $\mathbf{fpr}(S,k)$, is an induction on

$$p(U) = (|\mathbf{cdn}\, U|, v(U)) \in \mathbb{N} \times \mathbb{N}^{op},$$

where $v(U)$ is the cardinality of the set $\{(s,t);\ s<t \text{ in } S,\ u_s \neq 0, u_t \neq 0\}$, $\mathbb{N}\times\mathbb{N}^{op}$ is ordered lexicographically and $\mathbf{cdn}\, U = (u_s)_{s\in S}$, see [Simson 1992, proof of Theorem 15.54].

If $S = (\cdot\cdot\cdot\cdot)\cup\{\infty\}$, then $\operatorname{End} U$ is commutative, by Case I.

Now assume that $S \neq (\cdot\cdot\cdot\cdot)\cup\{\infty\}$ and S does not contain any of the posets $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_4$, $\mathcal{N}_5$, $\mathcal{N}_6$, $\mathcal{NL}$, as a subposet. In view of [Simson 1992, Theorem 15.30], there exists an irreducible suitable pair (a,b) of elements of S. Let $\delta_{(a,b)} : \mathbf{fpr}(S,k) \longrightarrow \mathbf{fpr}(\delta_{(a,b)}S,k)$ denote the Zavadskij derivative (see [Zavadskij 1977; 1991; Simson 1992, Definition 9.2]) with respect to (a,b).

Define $S_{a<b}$ to be the poset obtained from S by adding the relation $a<b$ and let $u : \mathbf{fpr}(S_{a<b},k) \longrightarrow \mathbf{fpr}(S,k)$ be the natural fully faithful embedding. Given a sincere indecomposable object U of $\mathbf{fpr}(S,k)$, then either $U' = \delta_{(a,b)}(U)$ is a sincere indecomposable object of $\mathbf{fpr}(\delta_{(a,b)}S,k)$, with $p(U') < p(U)$, or else $U_a \subseteq U_b$ and $U = u(U')$, for some sincere indecomposable object U' of $\mathbf{fpr}(S_{a<b},k)$, with $p(U') < p(U)$, [Simson 1992, Lemma 15.52]. Since $\delta_{(a,b)}S$ does not contain any of the posets $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_4$, $\mathcal{N}_5$, $\mathcal{N}_6$, $\mathcal{NL}$, as a subposet, [Simson 1992, Proposition 15.27 and Theorem 15.30], then the algebra $\operatorname{End} U'$ is commutative, by induction on $p(U)$.

If $U = u(U')$, then $\operatorname{End} U \cong \operatorname{End} U'$ is commutative, since u is a fully faithful functor.

Now suppose that $U' = \delta_{(a,b)}(U)$. If U is isomorphic to P_a (an indecomposable projective based on a, with one-dimensional peak space) or $P \cong P_{a,c_i} = P_a + P_{c_i}$ (an indecomposable subobject of the injective envelope $E(P_\infty)$ of P_∞ generated by P_a and P_{c_i}, see [Simson 1992, Chapter 5]), then $\operatorname{End} U \cong k$.

Next, assume that U is sincere and indecomposable, but not isomorphic to P_a or any P_{a,c_i}. By applying Lemma 9.13 and Corollary 9.18 of [Simson 1992], we show that $\operatorname{End} U' \cong \operatorname{End} W$.

To see this, we note that, by Proposition 9.16 of [Simson 1992], the functor $\delta'_{(a,b)}$ induces a k-algebra isomorphism $\phi : \operatorname{End} U \to (\operatorname{End} W)/\mathscr{A}_W$, where

$$\mathscr{A}_W = \{f \in \operatorname{End} W : f \text{ factors through } P_a^t \text{ for some } t\}.$$

Now $\mathscr{A}_W \subseteq \operatorname{Hom}(W, W_{(\geq a)})$, where $W_{(\geq a)} = (U_s : s \in \delta_{(a,b)}S)$, with $U_s = W_s$, if $s \geq a$, and $U_s = 0$, if $s \not\geq a$.

Assume $\mathscr{A}_W \neq 0$. Then $W_a \neq 0$, because $W_a = 0$ would imply $\operatorname{Hom}(P_a, W) \cong W_a = 0$. Since $\mathscr{A}_W \neq 0$ and $W_a \neq 0$, there is a morphism $W \to W_{(\geq a)}$ with image isomorphic to P_a. This is a contradiction to the assumption that W is indecomposable and not isomorphic to P_a and the fact that P_a is projective. Consequently, $\mathscr{A}_W = 0$ and $\operatorname{End} U \cong \operatorname{End} W \cong \operatorname{End} U'$ is commutative, as desired.

Case III. S does not contain any one of $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5, \mathcal{N}_6$, but does contain $\mathcal{NL}$ as a subposet.

As noted in Case II, it is sufficient to assume that S is a sincere poset. Then S is a union of two garlands [Simson 1992, Proposition 15.61], where a *garland* is a poset of width 2 that does not contain the poset $(1, 2)$ as a subposet.

Define $\mathcal{G}_{m,n}$ to be the disjoint union of two garlands $\mathcal{G}_m$ and $\mathcal{G}_n$, where $\mathcal{G}_m$ is a garland with exactly m pairs of incomparable elements, that is, it has the form

$$\mathcal{G}_m : \quad \begin{array}{ccccccccc} 1 & \to & 3 & \to & 5 & \to & \cdots & \to & 2m-1 \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ 2 & \to & 4 & \to & 6 & \to & \cdots & \to & 2m. \end{array}$$

As in [Simson 1992, Corollary 15.63], $\mathbf{fpr}(S, k)$ does not have k-endo-wild representation type if $\mathbf{fpr}(\mathcal{G}^{\infty}_{m,n}, k)$ does not have k-endo-wild representation type.

It remains to prove that $\mathbf{fpr}(\mathcal{G}^{\infty}_{m,n}, k)$ does not have k-endo-wild representation type. The category $\mathbf{fpr}(\mathcal{G}^{\infty}_{m,n}, k)$ is equivalent to the category $\operatorname{rep}_k(\mathcal{C}_{m,n})$ of k-linear representations of a nonsingular separably reducible clan $\mathcal{C}_{m,n}$ associated to $\mathcal{G}_{m,n}$ in [Simson 1992, Lemma 15.67]. Indecomposable objects W of $\operatorname{rep}_k(\mathcal{C}_{m,n})$ are classified in terms of k-linear functors $S_w : \operatorname{modf} A_w \longrightarrow \operatorname{rep}_k(\mathcal{C}_{m,n})$ indexed by words w as follows, see [Crawley-Boevey 1989]:

(a) W is an asymmetric string isomorphic to $S_w(M)$ for some indecomposable M in $\operatorname{modf} A_w$, where $A_w = k$;

(b) W is a symmetric string isomorphic to $S_w(M)$ for some indecomposable module M in $\operatorname{modf} A_w$, where $A_w = k[x]/(q(x))$ and $q(x)$ is a quadratic polynomial with nonzero constant term and distinct roots in k;

(c) W is an asymmetric band isomorphic to $S_w(M)$ for some indecomposable M in modf A_w, where $A_w = k[x, x^{-1}]$;

(d) W is a symmetric band isomorphic to $S_w(M)$ for some indecomposable M in modf A_w, where $A_w = k\langle t_1, t_2\rangle/(p(t_1), q(t_2))$ and $p(t_1)$ and $q(t_2)$ are quadratic polynomials with nonzero constant terms and distinct roots in k.

Moreover, there are k-linear functors $F_w : \mathrm{rep}_k(\mathscr{C}_{m,n}) \longrightarrow \mathrm{modf}\, A_w$, with $F_w \circ S_w$ naturally equivalent to the identity functor on modf A_w.

This is proved in [Crawley-Boevey 1989], when the field k has at least three elements, and in [Deng 2000], when the field k is arbitrary.

It now follows that, for each indecomposable object W of $\mathrm{rep}_k(\mathscr{C}_{m,n})$, there is some A_w, an indecomposable module M in modf A_w, and an ideal I_w of End W with $(\mathrm{End}\, W)/I_w \cong \mathrm{End}_{A_w} M$ and $F_{w*}(I_w) = 0$, where the map $F_w^* : \mathrm{End}\, W \longrightarrow \mathrm{End}_{A_w} M$ is induced by the functor F_w. Because W is indecomposable, End W is a local ring so that $I_w \subseteq J(\mathrm{End}\, W)$. Hence, there is a k-algebra isomorphism

$$\mathrm{End}\, W/J(\mathrm{End}\, W) \cong \mathrm{End}_{A_w} M/J(\mathrm{End}_{A_w} M).$$

In each case, $\mathrm{End}_{A_w} M$ is isomorphic to a quotient algebra of $k[x]$. This is because indecomposable modules in modf A_w are cyclic $k[x]$-modules in the case of types (a)–(c), as we can see by observing that $k[x, x^{-1}]$ is a localization of $k[x]$.

Next, we show that (d) is a consequence of Case I.

Let $S = (\cdots\cdot) \cup \{\infty\}$ and $A_w = k\langle t_1, t_2\rangle/(p(t_1), q(t_2))$, where $p(t_1)$ and $q(t_2)$ are quadratic polynomials with nonzero constant terms and distinct roots in k. Assume that $p(t_1) = (t_1 - \lambda_1)(t_1 - \lambda_2)$ and $q(t_2) = (t_2 - \lambda_3)(t_2 - \lambda_4)$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are nonzero distinct elements of k. Define a functor F : modf $A_w \longrightarrow \mathbf{fpr}(S, k)$ by the formula $F(M) = (M, M_1, M_2, M_3, M_4)$, where M_j is the eigenspace of λ_j. Since F induces the k-algebra isomorphism $F_* : \mathrm{End}_{A_w} M \longrightarrow \mathrm{End}\, F(M)$, we conclude as in Case I that if M is an indecomposable module in modf A_w, then $\mathrm{End}_{A_w} M$ is isomorphic to a quotient k-algebra of the polynomial algebra $k[x]$.

In summary, End $W/J(\mathrm{End}\, W)$ is a finite-dimensional field extension of k, for each indecomposable object W of $\mathrm{rep}_k(\mathscr{C}_{m,n})$.

Case III(i). There is a noncommutative finite dimensional division k-algebra D.

Since D is noncommutative, there is no indecomposable object W of $\mathrm{rep}_k(\mathscr{C}_{m,n})$ with $\mathrm{End}\, W/J(\mathrm{End}\, W) \cong D$. Therefore, $\mathbf{fpr}(\mathscr{G}_{m,n}^{\infty}, k)$, and consequently $\mathbf{fpr}(S, k)$, does not have k-endo-wild representation type.

Case III(ii). Every finite-dimensional division k-algebra is a field (e.g., k is finite or algebraically closed).

It is sufficient to prove that the category $\mathrm{rep}_k(\mathscr{C}_{m,n})$ is not k-endo-wild.

• Assume that W is a band representation in $\mathrm{rep}_k(\mathcal{C}_{m,n})$ (types (c) or (d) above).

The functor $S_w : \mathrm{mod}\, A_w \to \mathrm{rep}_k(\mathcal{C}_{m,n})$ preserves irreducible morphisms, hence Auslander–Reiten sequences, in (c) and (d); see [Geiß 1999, Proposition 4]. Since the functor $S_w : \mathrm{mod}\, A_w \to \mathrm{rep}_k(\mathcal{C}_{m,n})$ preserves irreducible morphisms, S_w induces an isomorphism

$$\mathrm{Irr}(M, M) = J(M, M)/J^2(M, M) \longrightarrow \mathrm{Irr}(W, W) = J(W, W)/J^2(W, W),$$

with notation as in [Simson 1992], where M is a module in mod A_w, with $S_w(M) = W$. Since End M and End W are local rings, we have isomorphisms

$$\mathrm{Irr}(M, M) = J(\mathrm{End}\, M)/J(\mathrm{End}\, M)^2 \text{ and } \mathrm{Irr}(W, W) = J(\mathrm{End}\, W)/J(\mathrm{End}\, W)^2.$$

For (c) and (d), $J(\mathrm{End}\, M)/J(\mathrm{End}\, M)^2 \cong k$. Possibility (c) is obvious, and (d) follows as above by reducing to Case I and showing that End W is a factor ring of $k[x]$, for each indecomposable W in $\mathbf{fpr}(\mathcal{G}^{\infty}_{m,n}, k)$.

This shows that $J(\mathrm{End}\, W)/J(\mathrm{End}\, W)^2 \cong k$ for each band representation W in $\mathrm{rep}_k(\mathcal{C}_{m,n})$. Because the algebra A defined in Case I is of dimension four, has no nontrivial idempotents and $J(A)/J(A)^2$ is noncommutative, then A is not isomorphic to End W, for any indecomposable band representation W in $\mathrm{rep}_k(\mathcal{C}_{m,n})$.

• Now assume instead that W is a string representation in $\mathrm{rep}_k(\mathcal{C}_{m,n})$.

It is sufficient to prove that End W is not isomorphic to the k-algebra A defined in Case I. The problem reduces to the case $m = 2$, $n = 1$, where $\mathcal{G}_{m,n} = \mathcal{G}_{2,1} = \mathcal{NL}$. The clan $\mathcal{C}_{2,1}$ corresponding to the poset $\mathcal{NL}$ and the category $\mathrm{rep}_k(\mathcal{C}_{2,1})$ are presented in [Simson 1992, p. 360]. By applying this description and the arguments given [Geiß 1999] and [Krause 1991], one can show, by a detailed technical argument (not included), that the k-algebra A defined in Case I is not of the form End W, up to isomorphism, where W is a string representation in $\mathrm{rep}_k(\mathcal{C}_{2,1})$. This finishes the proof of Case III(ii), and hence also of Theorem 2.4. □

3. Generic representations and representation type

Let $(S, \leq)$ be a finite poset with a unique maximal element ∞ and R a unitary ring. Following [Arnold and Simson 2002], we define the category $\mathrm{rep}_{fg}(S, R)$ to be the category of filtered finitely generated R-representations of S with objects $U = (U_s;\ s \in S)$ such that each U_s is a finitely generated left R-module and $U_s \subseteq U_t$ if $s \leq t$ in S. A morphism from $U = (U_s;\ s \in S)$ to $V = (V_s;\ s \in S)$ is an R-module homomorphism $f : U_\infty \to V_\infty$ such that $f(U_s) \subseteq V_s$, for each $s \in S$.

Let $\mathbf{fspr}(S, R)$ denote the full subcategory of filtered subprojective representations of $\mathrm{rep}_{fg}(S, R)$ with objects $U = (U_s;\ s \in S)$ such that U_∞ is a projective R-module. Define $\mathbf{fpr}(S, R)$ to be the full subcategory of $\mathbf{fspr}(S, R)$ with

$U = (U_s;\ s \in S)$ such that U_s and U_∞ / U_s are free R-modules, for each $s \in S$. If k is a field, then $\mathbf{fspr}(S, k) = \mathbf{fpr}(S, k) = \mathrm{rep}_{fg}(S, k)$.

For a field k, we define $\mathbf{Fpr}(S, k)$ to be the category with objects $U = (U_s;\ s \in S)$ such that each U_s is a k-vector space and $U_s \subseteq U_t$ if $s \leq t$ in S. A morphism from $U = (U_s;\ s \in S)$ to $V = (V_s;\ s \in S)$ is a k-linear transformation $f : U_\infty \to V_\infty$ with $f(U_s) \subseteq V_s$ for each $s \in S$. The *peak R-rank* of an object $U = (U_s;\ s \in S)$ of $\mathbf{Fpr}(S, R)$ is defined to be the R-rank of the free R-module U_∞. The category $\mathbf{Fpr}(S, k)$ is an extension of $\mathbf{fpr}(S, k)$ in that U_∞ may be infinite-dimensional.

We write $\mathrm{End}\, U$ for the endomorphism ring of an object U of $\mathbf{Fpr}(S, k)$.

We recall from [Simson 1992, Section 5.1] that the category $\mathbf{Fpr}(S, k)$ may be viewed as a full exact subcategory of the module category $\mathrm{Mod}\, kS$ over the incidence k-algebra kS of S, because there is a fully faithful k-linear exact functor

$$\boldsymbol{\rho} : \mathbf{Fpr}(S, k) \longrightarrow \mathrm{Mod}\, kS$$

which associates to any object $M = (M_s;\ s \in S)$ of $\mathbf{Fpr}(S, k)$ the k-vector space $\boldsymbol{\rho}(M) = \bigoplus_{s \in S} M_s$ equipped with the right multiplication $\cdot : \boldsymbol{\rho}(M) \times kS \to \boldsymbol{\rho}(M)$ defined by attaching to every element $m = (m_s)_{s \in S} \in \bigoplus_{s \in S} M_s = \boldsymbol{\rho}(M)$ and any matrix $\lambda = [\lambda_{pq}]_{p,q \in S} \in kS \subseteq \mathbb{M}_S(k)$ the vector $m \cdot \lambda = (m_s) \cdot [\lambda_{pq}] = (m'_s)_{s \in S} \in \bigoplus_{s \in S} M_s = \boldsymbol{\rho}(M)$, where $m'_s = \sum_{i \leq s} m_i \lambda_{is}$. If $f : M \to M'$ is a morphism in $\mathbf{Fpr}(S, k)$ defined by a k-linear map $f : M_\infty \to M'_\infty$, we define the kS-module homomorphism $\boldsymbol{\rho}(f) : \boldsymbol{\rho}(M) \to \boldsymbol{\rho}(M')$ by the formula $\boldsymbol{\rho}(f)(m) = (f(m_s))_{s \in S}$, for every $m = (m_s)_{s \in S} \in \boldsymbol{\rho}(M)$.

The functor $\boldsymbol{\rho}$ allows us to identify the category $\mathbf{Fpr}(S, k)$ with a full exact subcategory of the module category $\mathrm{Mod}\, kS$. By identifying any object $M = (M_s;\ s \in S)$ of $\mathbf{Fpr}(S, k)$ with the right kS-module $\boldsymbol{\rho}(M)$ we can view M as a left module over the endomorphism k-algebra $\mathrm{End}\, M \cong \mathrm{End}_{kS}\ \boldsymbol{\rho}(M)$.

Lemma 3.1. *Let k be a field and S a finite poset with a unique maximal element. The category $\mathbf{fpr}(S, k)$ has k-wild representation type if and only if there exists an object N of $\mathbf{fpr}(S, k\langle t_1, t_2\rangle)$ such that the k-linear functor*

$$- \otimes_{k\langle t_1, t_2\rangle} N : \mathrm{Mod}\, k\langle t_1, t_2\rangle \longrightarrow \mathbf{Fpr}(S, k)$$

is exact, faithful, preserves indecomposables, preserves isomorphism classes and restricts to the functor $- \otimes_{k\langle t_1, t_2\rangle} N : \mathrm{modf}\, k\langle t_1, t_2\rangle \longrightarrow \mathbf{fpr}(S, k)$.

Proof. The proof of sufficiency is obvious. To prove necessity, we assume that $\mathbf{fpr}(S, k)$ has k-wild representation type, that is, there exists an exact k-linear functor

$$T' : \mathrm{modf}\, k\langle t_1, t_2\rangle \longrightarrow \mathbf{fpr}(S, k)$$

that preserves isomorphism classes and preserves indecomposables. By a well known result of Brenner [1974a], there is a fully faithful exact k-linear functor

$$T'' : \operatorname{mod} A \longrightarrow \operatorname{modf} k\langle t_1, t_2\rangle, \quad \text{where } A = \begin{pmatrix} k & k^3 \\ 0 & k \end{pmatrix}.$$

It follows that the composite k-linear functor $T' \circ T'' : \operatorname{mod} A \longrightarrow \mathbf{fpr}(S, k) \subseteq \operatorname{Mod} kS$ is exact, preserves isomorphism classes and preserves indecomposables. By the Wildness Correction Lemma [Simson 1993, Lemma 2.6] applied to $T' \circ T''$, there exists a finite-dimensional A-kS-bimodule ${}_A M'_{kS}$ such that the left A-module ${}_A M'$ is free of finite rank and the k-linear functor

$$H' = - \otimes_A M'_{kS} : \operatorname{mod} A \longrightarrow \mathbf{fpr}(S, k) \subseteq \operatorname{Mod} kS$$

is exact, faithful, preserves isomorphism classes and preserves indecomposables. Moreover, it follows from the proof of [Simson 1993, Lemma 2.6] that the functor H' extends to a unique k-linear exact faithful functor

$$H' = - \otimes_A M'_{kS} : \operatorname{Mod} A \longrightarrow \mathbf{Fpr}(S, k) \subseteq \operatorname{Mod} kS,$$

that preserves isomorphism classes and preserves indecomposables. By [Simson 1992, pp. 286–287] and the arguments used in the proof, there exists a fully faithful exact k-linear functor $H'' : \operatorname{Mod} k\langle t_1, t_2\rangle \longrightarrow \operatorname{Mod} A$, which restricts to $H'' : \operatorname{modf} k\langle t_1, t_2\rangle \longrightarrow \operatorname{mod} A$ and is of the form $H'' = - \otimes_{k\langle t_1, t_2\rangle} M''_A$, where ${}_{k\langle t_1, t_2\rangle} M''_A$ is an $k\langle t_1, t_2\rangle$-A-bimodule such that the left $k\langle t_1, t_2\rangle$-module M'' is free of finite rank. Since the right kS-module ${}_A M'_{kS} = A \otimes_A M'_{kS} = H'(A)$ is an object of $\mathbf{fspr}(S, k)$ of the form $({}_A M'_s;\ s \in S)$, then the $k\langle t_1, t_2\rangle$-kS-bimodule

$${}_{k\langle t_1, t_2\rangle} N_{kS} = {}_{k\langle t_1, t_2\rangle} M'' \otimes_A M'_{kS}$$

can be identified with the object $({}_{k\langle t_1, t_2\rangle} N_s;\ s \in S)$ of the category $\mathbf{fpr}(S, k\langle t_1, t_2\rangle)$, where ${}_{k\langle t_1, t_2\rangle} N_s = {}_{k\langle t_1, t_2\rangle} M'' \otimes_A M'_s$ is a free left $k\langle t_1, t_2\rangle$-module of finite rank, for each $s \in S$. It follows that the composite k-linear functor $H' \circ H''$ is fully faithful, exact, preserves isomorphism classes, preserves indecomposables, is of the form

$$- \otimes_{k\langle t_1, t_2\rangle} N_{kS} : \operatorname{Mod} k\langle t_1, t_2\rangle \longrightarrow \mathbf{Fpr}(S, k)$$

and restricts to the functor $- \otimes_{k\langle t_1, t_2\rangle} N_{kS} : \operatorname{modf} k\langle t_1, t_2\rangle \longrightarrow \mathbf{fpr}(S, k)$. □

To define generic representations of S, we define the *endolength* of M to be the (composition) length of the left $\operatorname{End} U$-module M, compare with [Crawley-Boevey 1991]. Moreover, we define the object $M = (M_s;\ s \in S)$ of $\mathbf{Fpr}(S, k)$ to be a *generic representation* of S, if M is indecomposable, the k-dimension of M_∞ is infinite, M has finite endolength, and the k-algebra

$$D_M = \operatorname{End} M / J(\operatorname{End} M)$$

contains, as a k-subalgebra, the quotient field $k(x)$ of the polynomial k-algebra $k[x]$ in one indeterminate x.

By definition, the object $M = (M_s;\ s \in S)$ of $\mathbf{Fpr}(S, k)$ is a generic representation of S if and only if $\boldsymbol{\rho}(M)$ is a *generic kS-module* (in the sense of [Crawley-Boevey 1991], that is, $\boldsymbol{\rho}(M)$ is indecomposable of infinite length as a kS-module, but is of finite endolength) with the additional property that $k(x) \subseteq D_M$.

If the field k is algebraically closed, then the condition that $k(x) \subseteq D_M$ is redundant. This is because, by Lemma 3.2, D_M is a division k-algebra of infinite k-dimension (since M has finite D_M-dimension equal to the endolength of M). Under the assumption that k is algebraically closed, the division k-algebra D_M contains an element x transcendental over k and therefore $k(x) \subseteq D_M$.

The standard properties of generic modules proved in [Crawley-Boevey 1991] yield the following useful lemma.

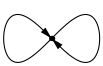
Lemma 3.2. *Let k be a field and S a finite poset with a unique maximal element. If an object $M = (M_s;\ s \in S)$ of $\mathbf{Fpr}(S, k)$ is a generic representation of S, then*:

(a) *the k-dimension of M_∞ is countably infinite*;

(b) *the k-algebra $D_M = \operatorname{End} M / J(\operatorname{End} M)$ is a division algebra with $J(\operatorname{End} M)$ nilpotent*; *and*

(c) *the representation M is an object of $\mathbf{fpr}(S, D_M)$ and the D_M-dimension of M_∞ equals the endolength of M.*

Example 3.3. Assume that T is any of the hypercritical posets $\mathcal{N}_1, \ldots, \mathcal{N}_6$ of Nazarova and let $T^\infty = T \cup \{\infty\}$. In the proof of Theorem 2.4, we have constructed a fully faithful k-linear exact functor $F'_T : \operatorname{modf} k\langle t_1, t_2\rangle \longrightarrow \mathbf{fpr}(T^\infty, k\langle t_1, t_2\rangle)$. We recall from the construction that the modules in $\operatorname{Mod} k\langle t_1, t_2\rangle$ are identified with k-linear representations of the two-loop quiver



consisting of a single point and two loops a and b. Any such representation is a triple $\mathbb{V} = (V, \tilde{a}, \tilde{b})$, where V is a k-vector space and $\tilde{a}, \tilde{b} : V \to V$ are k-linear endomorphisms of V.

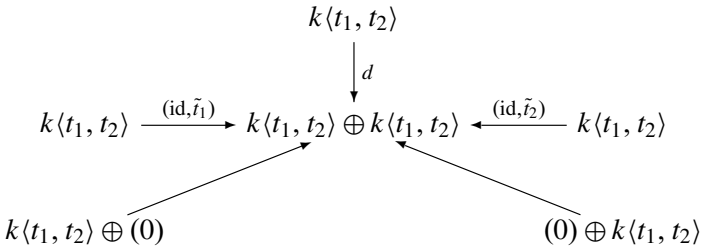
For $j = 1, \ldots, 6$, the functor $F'_{\mathcal{N}_j}$ associates to the representation $\mathbb{V} = (V, \tilde{a}, \tilde{b})$ the object

$$U^{(j)}_{\mathbb{V}} = F'_{\mathcal{N}_j}(\mathbb{V})$$

of the category $\mathbf{Fpr}(\mathcal{N}^\infty_j, k\langle t_1, t_2\rangle)$ defined by the corresponding diagram $U^{(j)}_{\mathbb{V}}$ from the proof of Theorem 2.4.

Note that the module $k\langle t_1, t_2\rangle$ in $\operatorname{Mod} k\langle t_1, t_2\rangle$ is identified with the triple $\mathbb{V} = (k\langle t_1, t_2\rangle, \tilde{t}_1, \tilde{t}_2)$, where $\tilde{t}_1, \tilde{t}_2$ are the k-linear endomorphisms of $k\langle t_1, t_2\rangle$ defined by the multiplication by t_1 and t_2, respectively. Given $T = \mathcal{N}_j$, we define the object

$U^T_{k\langle t_1,t_2\rangle}$ of $\mathbf{fpr}(T^\infty, k\langle t_1,t_2\rangle)$ to be the object $U^{(j)}_{\mathbb{V}}$, with $\mathbb{V} = (k\langle t_1,t_2\rangle, \tilde{t}_1, \tilde{t}_2)$. For instance, if $T = \mathcal{N}_1$, then $U^T_{k\langle t_1,t_2\rangle}$ is the object of $\mathbf{fpr}(T^\infty, k\langle t_1,t_2\rangle)$ defined by the diagram of $k\langle t_1,t_2\rangle$-modules



where d is the diagonal embedding, given by $d(x) = (x,x)$, for $x \in k\langle t_1,t_2\rangle$.

For each hypercritical poset T, clearly $U^T_{k\langle t_1,t_2\rangle}$ is an object of $\mathbf{fpr}(T^\infty, k\langle t_1,t_2\rangle)$ such that $(U^T_{k\langle t_1,t_2\rangle})_\infty$ has infinite k-dimension and finite $k\langle t_1,t_2\rangle$-rank. Since the k-linear functor F'_T is fully faithful, then there are k-algebra isomorphisms

$$k\langle t_1,t_2\rangle \cong \mathrm{End}_{k\langle t_1,t_2\rangle} k\langle t_1,t_2\rangle \cong \mathrm{End}\, F'_T(k\langle t_1,t_2\rangle) \cong \mathrm{End}\, U^T_{k\langle t_1,t_2\rangle}.$$

This shows that $U^T_{k\langle t_1,t_2\rangle}$ is an indecomposable object of $\mathbf{fpr}(T^\infty, k\langle t_1,t_2\rangle)$ of finite endolength. Therefore $U^T_{k\langle t_1,t_2\rangle}$ is a generic object of $\mathbf{Fpr}(T^\infty, k\langle t_1,t_2\rangle)$.

The main theorem of this section (Theorem 3.8) gives several characterizations of tame and k-wild representation type of the category $\mathbf{fpr}(S,k)$ in terms of existence conditions on generic representations. This theorem parallels that of the main theorem in [Crawley-Boevey 1991] for mod A for a finite dimensional algebra A over an algebraically closed field k. The proof is different, however, because of the assumption that k is an arbitrary field.

In particular, condition (c) of Theorem 3.8 is an existence condition on generic representations M with $\mathrm{End}\, M/J(\mathrm{End}\, M) \cong k(x)$. It is proved in [Crawley-Boevey 1991], using properties of minimal bocses, that if A is a finite-dimensional algebra over an algebraically closed field k and mod A has tame representation type, then

$$\mathrm{End}\, M/J(\mathrm{End}\, M) \cong k(x)$$

for each generic A-module M. Since the theory of minimal bocses is not available in our context, this condition is included as an assumption in condition (c). The question of whether it is necessary remains unresolved.

The remainder of the paper is devoted to a proof of Theorem 3.8. We start with the following result, proved in [Arnold 2000, Theorem 1.5.3].

Theorem 3.4. *Let k be a field and S a finite poset with a unique maximal element. Then $\mathbf{fpr}(S,k)$ has finite representation type if and only if there are no generic representations in $\mathbf{Fpr}(S,k)$.*

We recall from [Arnold and Simson 2002] that the category $\mathbf{fpr}(S,k)$ has *tame representation type* if for each vector $w=(w_s)_{s\in S}\in\mathbb{Z}^S$, with each w_s a non-negative integer, there are finitely many indecomposable objects $N_1,\ldots,N_{m_w}$ of $\mathbf{fpr}(S,k[x])$ such that if U is an indecomposable object of $\mathbf{fpr}(S,k)$ with $\mathbf{cdn}\,U=w$, then $U\cong Z\otimes_{k[x]}N_j$, for some $1\le j\le m_w$ and a cyclic $k[x]$-module Z.

Let N be an object of $\mathbf{fpr}(S,k[x])$ and define

$$\widehat{N}^x=k(x)\otimes_{k[x]}N,$$

an object of $\mathbf{fpr}(S,k(x))$ containing N. It follows that the natural embedding $N\hookrightarrow\widehat{N}^x$ induces a k-algebra embedding $\operatorname{End}N\subseteq\operatorname{End}\widehat{N}^x$. We define the *nil radical* of $\operatorname{End}N$ to be the nilpotent ideal $\operatorname{Nil}(\operatorname{End}N)=\operatorname{End}N\cap J(\operatorname{End}\widehat{N}^x)$ of $\operatorname{End}N$.

The first step in the proof of Theorem 3.8 is to relate generic representations to certain objects of $\mathbf{fpr}(S,k[x])$. We call an object N of $\mathbf{fpr}(S,k[x])$ *pregeneric* if N is indecomposable as an object of $\mathbf{Fpr}(S,k)$. Notice that a pregeneric object of $\mathbf{fpr}(S,k[x])$ is also an indecomposable object of $\mathbf{fpr}(S,k[x])$. The terminology, pregeneric representation, is motivated by the following lemma.

Lemma 3.5. *Let k be a field and S a finite poset with a unique maximal element.*

(a) *If $N=(N_s;\ s\in S)$ is a pregeneric object of the category $\mathbf{fpr}(S,k[x])$ such that $\operatorname{End}N/\operatorname{Nil}(\operatorname{End}N)\cong k[x]$, then $\widehat{N}^x=k(x)\otimes_{k[x]}N$ is a generic object in $\mathbf{Fpr}(S,k)$, the endolength of $\widehat{N}^x$ equals the $k[x]$-rank of N_∞ and*

$$\operatorname{End}\widehat{N}^x/J(\operatorname{End}\widehat{N}^x)\cong k(x).$$

(b) *If $M=(M_s;\ s\in S)$ is a generic object of $\mathbf{Fpr}(S,k)$ such that*

$$\operatorname{End}M/J(\operatorname{End}M)\cong k(x),$$

there is a pregeneric object $N=(N_s;\ s\in S)$ of $\mathbf{fpr}(S,k[x])$ such that $M\cong\widehat{N}^x$, $\operatorname{End}N/\operatorname{Nil}(\operatorname{End}N)\cong k[x]$, and the endolength of M equals the $k[x]$-rank of the free module N_∞.

Proof. (a) Clearly, $\widehat{N}^x=k(x)\otimes_{k[x]}N$ is an object of $\mathbf{fpr}(S,k(x))$, and hence an object of $\mathbf{Fpr}(S,k)$ with infinite k-dimension and finite endolength, since $k(x)\subseteq\operatorname{End}\widehat{N}^x$. Obviously, there is a k-algebra embedding

$$\operatorname{End}N/\operatorname{Nil}(\operatorname{End}N)\longrightarrow\operatorname{End}\widehat{N}^x/J(\operatorname{End}\widehat{N}^x).$$

Because $k(x)$ is the quotient field of $k[x]\subseteq\operatorname{End}N$, N is finitely generated and free as a $k[x]$-module, and $\operatorname{End}N/\operatorname{Nil}(\operatorname{End}N)\cong k[x]$, then $\operatorname{End}\widehat{N}^x/J(\operatorname{End}\widehat{N}^x)\cong k(x)$. This shows that M is an indecomposable object of $\mathbf{Fpr}(S,k)$.

(b) By Lemma 3.2, M_∞ is a $k(x)$-module, the endolength of the object M equals $\dim_{k(x)}M_\infty$ and M is an object of $\mathbf{fpr}(S,k(x))$. Consequently, $M\cong k(x)\otimes_{k[x]}N$

for some object N of $\mathbf{fpr}(S, k[x])$ such that the endolength of M equals the $k[x]$-rank of N_∞. Since there is an obvious k-algebra embedding

$$\operatorname{End} N/\operatorname{Nil}(\operatorname{End} N) \longrightarrow \operatorname{End} M/J(\operatorname{End} M) \cong k(x),$$

then the algebra $\operatorname{End} N$ has no nontrivial idempotents and therefore N is an indecomposable object of $\mathbf{Fpr}(S, k)$. It follows that $\operatorname{End} N/\operatorname{Nil}(\operatorname{End} N) \cong k[x]$. □

The next lemma is a characterization of tame representation type in terms of pregeneric representations. The general framework of the proof of the lemma is like that of the proof of Theorem 2.4, in that Case 2 is an induction proof beginning with Case 1 and Case 3 is proved by using properties of representations of clans. However, in the proof of the following lemma, the goal is to construct pregeneric representations in $\mathbf{fpr}(S, k[x])$ rather than finding endomorphism rings of indecomposable objects of $\mathbf{fpr}(S, k)$. We write "almost all" for "finitely many exceptions, up to isomorphism".

Lemma 3.6. *Let k be a field and S a finite poset with a unique maximal element. The category $\mathbf{fpr}(S, k)$ has tame representation type if and only if for each vector $w = (w_s)_{s\in S} \in \mathbb{Z}^S$, with each w_s a nonnegative integer, there are finitely many pregeneric objects $N_1, \dots, N_{m_w}$ of $\mathbf{fpr}(S, k[x])$, with $\operatorname{End} N_i/\operatorname{Nil}(\operatorname{End} N_i) \cong k[x]$ for each i, such that for almost all indecomposable objects U of $\mathbf{fpr}(S, k)$ with $\mathbf{cdn}\, U = w$, there is an isomorphism $U \cong Z \otimes_{k[x]} N_j$, for some $1 \le j \le m_w$ and a cyclic $k[x]$-module Z.*

Proof. It is sufficient to assume that $\mathbf{fpr}(S, k)$ has tame representation type and confirm the conditions of the lemma (the converse is clear). The proof follows that of Theorem 2.4.

Case 1. $S = (\cdot\cdot\cdot\cdot) \cup \{\infty\}$.

Define an object N of $\mathbf{fpr}(S, k[x])$ by the formula

$$N = (k[x] \oplus (0),\ (0) \oplus k[x],\ (1,1)k[x],\ (1,x)k[x],\ k[x] \oplus k[x]).$$

A simple calculation shows that $\operatorname{End} N \cong k[x]$, and the endomorphism k-algebra of N, viewed as an object of $\mathbf{fpr}(S, k)$, is also isomorphic to $k[x]$. It follows that N is a pregeneric representation in $\mathbf{fpr}(S, k[x])$.

It follows from the classification given in [Brenner 1974b] (see also [Simson 1992, Section 15.6]) that, for each $w = (w_s)_{s\in S} \in \mathbb{Z}^S$, either

(i) there are only finitely many isomorphism classes of indecomposable objects U of $\mathbf{fpr}(S, k)$ with $\mathbf{cdn}\, U = w$, or

(ii) each indecomposable object U of $\mathbf{fpr}(S, k)$ is isomorphic to $Z \otimes_{k[x]} N$, for some cyclic $k[x]$-module Z.

Case 2. The poset S does not contain, as a subposet, any of the hypercritical posets $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5, \mathcal{N}_6$ of Nazarova, or

$$\mathcal{NL} = (\begin{smallmatrix}\cdot\\ \cdot\\ \cdot\\ \cdot\end{smallmatrix}\!\!\times\!\!\begin{smallmatrix}\cdot\\ \cdot\\ \cdot\\ \cdot\end{smallmatrix}\ \cdot\ \cdot).$$

This case is proved by an induction on $p(U) = (|\mathbf{cdn}\, U|, v(U)) \in \mathbb{N} \times \mathbb{N}^{op}$, as in [Simson 1992, proof of Theorem 15.54], beginning with Case 1. As summarized in Case II of the proof of Theorem 2.4, if U is an indecomposable sincere object of $\mathbf{fpr}(S, k)$, then either $p(U') < p(U)$, for $U' = \delta_{(a,b)}(U)$, $\mathbf{cdn}\, U' = \mathbf{cdn}\, U$, and $\delta_{(a,b)}S$ does not contain $\mathcal{NL}$ as a subposet; or else $U_a \subseteq U_b$ and $U = u(U')$, for some object U' of $\mathbf{fpr}(S_{a<b}, k)$ with $p(U') < p(U)$ and $\mathbf{cdn}\, U' = \mathbf{cdn}\, U$. In either case, by induction on $p(U)$ beginning with Case 1, there are finitely many pregeneric objects $N_1, \ldots, N_{m_w}$ of $\mathbf{fpr}(S, k[x])$ with $\operatorname{End} N_i / \operatorname{Nil}(\operatorname{End} N_i) \cong k[x]$ for each i such that for almost all indecomposable objects U of $\mathbf{fpr}(S, k)$ with $\mathbf{cdn}\, U = w$, $U \cong Z \otimes_{k[x]} N_j$ for some $1 \le j \le m_w$ and cyclic $k[x]$-module Z.

Case 3. The poset S does not contain, as a subposet, any of the posets $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5$, or $\mathcal{N}_6$, but does contain $\mathcal{NL}$.

This case reduces to the category $\operatorname{rep}_k(\mathcal{C}_{m,n})$ of k-linear representations of a clan $\mathcal{C}_{m,n}$, as outlined in Case III of the proof of Theorem 2.4. There we classified the indecomposable representations in $\operatorname{rep}_k(\mathcal{C}_{m,n})$ explicitly, into possibilities (a)–(d). (See page 8.)

In (a)–(c), each indecomposable object U of $\operatorname{rep}_k(\mathcal{C}_{m,n})$ is isomorphic to $S_w(M)$, for some cyclic A_w-module M, where A_w is a quotient algebra of $k[x]$ and

$$\operatorname{End} U / J(\operatorname{End} U) \cong \operatorname{End} M / J(\operatorname{End} M) \cong k[x]/(g(x))$$

is a finite-dimensional field extension of k. There is a natural extension of the correspondences

$$\mathbf{fpr}(S, k) \longrightarrow \mathbf{fpr}(\mathcal{G}^{\infty}_{m,n}, k) \longrightarrow \operatorname{rep}_k(\mathcal{C}_{m,n})$$

to correspondences $\mathbf{Fpr}(S, k) \longrightarrow \mathbf{Fpr}(\mathcal{G}^{\infty}_{m,n}, k) \longrightarrow \operatorname{Rep}_k(\mathcal{C}_{m,n})$, where $\operatorname{Rep}_k(\mathcal{C}_{m,n})$ is the category of all k-linear representations of the clan $\mathcal{C}_{m,n}$ (including also infinite dimensional representations). For (a), (b), and (c), $S_w(k[x])$ is an indecomposable object of $\operatorname{Rep}_k(\mathcal{C}_{m,n})$. Let N be the preimage of $S_w(k[x])$ in $\mathbf{Fpr}(S, k)$, an indecomposable object in $\mathbf{fpr}(S, k[x])$ with $\operatorname{End} N / \operatorname{Nil}(\operatorname{End} N) \cong k[x]$. Since each indecomposable object U of $\operatorname{rep}_k(\mathcal{C}_{m,n})$ is isomorphic to $S_w(M)$ for some cyclic A_w-module M of $\mathbf{fpr}(S, k)$ and A_w is a factor k-algebra of $k[x]$, it follows that if V is an indecomposable object of $\mathbf{fpr}(S, k)$, then U is isomorphic to $Z \otimes_{k[x]} N$ for some cyclic $k[x]$-module Z. Thus (d) reduces to Case I. In any case, the conditions of the lemma are satisfied. □

Lemma 3.7. *Let S be a finite poset with a unique maximal element, k a field, N a pregeneric object of $\mathbf{fpr}(S, k[x])$ with $\operatorname{End} N/\operatorname{Nil}(\operatorname{End} N) \cong k[x]$ and d_N the $k[x]$-rank of the free $k[x]$-module N_∞. Assume that $\mathbf{fpr}(S, k)$ has tame representation type.*

(a) *There is a torsion cyclic $k[x]$-module Z_N such that $U^N = Z_N \otimes_{k[x]} N = (U_s^N;\ s \in S)$ is an indecomposable object of $\mathbf{fpr}(S, k)$ and the $k[x]$-rank of U_∞^N is equal to d_N.*

(b) *Assume that N' is another pregeneric object of the category $\mathbf{fpr}(S, k[x])$ such that $\operatorname{End} N'/\operatorname{Nil}(\operatorname{End} N') \cong k[x]$ and Z is a torsion cyclic $k[x]$-module such that there are $k[x]$-module embeddings $Z_N \subseteq Z$ and $Z_{N'} \subseteq Z$. Then*

$$Z_N \otimes_{k[x]} N \quad \text{and} \quad Z_{N'} \otimes_{k[x]} N'$$

are isomorphic objects of $\mathbf{fpr}(S, k)$ if and only if N and N' are isomorphic objects of $\mathbf{fpr}(S, k[x])$.

Proof. (a) Write $N = (N_s;\ s \in S)$. Pick a $k[x]$-basis $B_s = \{b_{s,i}\}$ of N_s, for each $s \in S$, and express each element $b_{s,i}$ of B_s as a $k[x]$-linear combination $b_{s,i} = \sum_j f_{s,i,j}(x) b_{\infty,j}$ of elements of B_∞. Let $m = \max\{\deg f_{s,i,j}(x)\}$ and let $Z_N = k[x]/(x^{m+1})$. We set $U^N = Z_N \otimes_{k[x]} N$. It is clear that $U^N = (U_s^N;\ s \in S)$ is an object of $\mathbf{fpr}(S, k)$, because $N = (N_s;\ s \in S)$ is an object of $\mathbf{fpr}(S, k[x])$ and $U_s^N = (N_s + x^{m+1}N_\infty)/x^{m+1}N_\infty$, for each $s \in S$. Since N_∞ is a finitely generated free $k[x]$-module and each N_s is a finitely generated free summand of N_∞, the choice of m guarantees that any $k[x]$-automorphism of U_∞^N lifts to a $k[x]$-automorphism of N_∞. Therefore, there is a k-algebra isomorphism

$$\operatorname{End} U^N / J(\operatorname{End} U^N) \cong \operatorname{End} N / J(\operatorname{End} N).$$

Let $N^\vee$ be the representation N viewed as an object of the category $\mathbf{Fpr}(S, k)$. Then there exists a k-algebra isomorphism $\operatorname{End} N^\vee / J(\operatorname{End} N^\vee) \cong \operatorname{End} N / J(\operatorname{End} N)$, because $\operatorname{End} N^\vee / \operatorname{Nil}(\operatorname{End} N^\vee) \cong k[x]$ and $\operatorname{Nil}(\operatorname{End} N^\vee) \subseteq J(\operatorname{End} N^\vee)$. It now follows that $\operatorname{End} U^N / J(\operatorname{End} U^N) \cong \operatorname{End} N^\vee / J(\operatorname{End} N^\vee)$. Since N is an indecomposable object of $\mathbf{Fpr}(S, k)$, U^N is an indecomposable object of $\mathbf{fpr}(S, k)$ and the $k[x]$-rank of U_∞^N equals d_N, because $Z_N = k[x]/(x^{m+1})$.

(b) Clearly, $Z \otimes_{k[x]} N$ and $Z \otimes_{k[x]} N'$ are isomorphic objects of $\mathbf{fpr}(S, k)$ if N and N' are isomorphic objects of $\mathbf{fpr}(S, k[x])$. An argument similar to that of (a) shows that an isomorphism $Z \otimes_{k[x]} N \simeq Z \otimes_{k[x]} N'$ lifts to an isomorphism $N \simeq N'$. □

Now we are able to prove the main result of this section containing a characterization of tame and k-wild representation type of the category $\mathbf{fpr}(S, k)$.

Theorem 3.8. *Let k be a field and S a finite poset with a unique maximal element. The following statements are equivalent.*

(a) *The category* $\mathbf{fpr}(S,k)$ *has tame representation type.*

(b) *The category* $\mathbf{fpr}(S,k)$ *does not have* k*-wild representation type.*

(c) *For each integer* $d \in \mathbb{N}$*, there are only finitely many isoclasses of generic representations* M *in the category* $\mathbf{Fpr}(S,k)$ *with endolength* d *and*

$$\operatorname{End} M/J(\operatorname{End} M) \cong k(x).$$

(d) *For each generic representation* M *in* $\mathbf{Fpr}(S,k)$*, the algebra*

$$\operatorname{End} M/J(\operatorname{End} M)$$

does not contain a k*-subalgebra isomorphic to* $k\langle t_1, t_2\rangle$*.*

Proof. (a) $\Longleftrightarrow$ (b): See, for example, [Arnold and Simson 2002, Theorem 2.1; Zavadskij and Nazarova 1977; Zavadskij 1987; 1991].

(a) $\Rightarrow$ (c): Assume that the category $\mathbf{fpr}(S,k)$ has tame representation type and there exist an integer $d \in \mathbb{N}$ and infinitely many pairwise nonisomorphic generic representations $M_1, M_2, \ldots, M_m, \ldots$ in $\mathbf{Fpr}(S,k)$ with endolength d,

$$\operatorname{End} M_i/J(\operatorname{End} M_i) \cong k(x)$$

for each i, and $M_i \cong M_j$ if and only if $i = j$. By Lemma 3.2, each M_i is a $k(x)$-module such that the endolength of $M_i = (M_{i,s}\,;\ s \in S)$ equals $\dim_{k(x)} M_{i,\infty}$, and equals d as well. In addition,

$$M_i \cong k(x) \otimes_{k[x]} N_i$$

for some pregeneric object $N_i = (N_{i,s}\,;\ s \in S)$ of $\mathbf{fpr}(S,k[x])$ such that the $k[x]$-rank of $N_{i,\infty}$ equals d and $\operatorname{End} N_i/\operatorname{Nil}(\operatorname{End} N_i) \cong k[x]$ (see Lemma 3.5). Moreover, if $N_i \cong N_j$, then $M_i \cong M_j$ and so $i = j$. Consequently, there must be infinitely many isomorphism classes of the pregeneric objects $N_1, N_2, \ldots, N_m, \ldots$.

If $N = (N_s;\ s \in S)$ is a pregeneric object of $\mathbf{fpr}(S,k[x])$ such that

$$\operatorname{End} N/\operatorname{Nil}(\operatorname{End} N) \cong k[x]$$

and the $k[x]$-rank of N_∞ equals d then, by Lemma 3.5, $\widehat{N}^x = k(x) \otimes_{k[x]} N$ is a generic representation in $\mathbf{Fpr}(S,k)$ with endolength d and $\operatorname{End} \widehat{N}^x/J(\operatorname{End} \widehat{N}^x) \cong k(x)$. Hence, $\widehat{N}^x \cong M_i$ for some i. It now follows from Lemma 3.7 that there is some $w = (w_s)_{s\in S}$ with $w_\infty = d$ such that, up to isomorphism, infinitely many pregeneric objects N of $\mathbf{fpr}(S,k[x])$ with $\operatorname{End} N/\operatorname{Nil}(\operatorname{End} N) \cong k[x]$ are required so that each indecomposable U of $\mathbf{fpr}(S,k)$, with $w = \mathbf{cdn}\, U$, is isomorphic to $Z \otimes_{k[x]} N$ for some cyclic $k[x]$-module Z. This is a contradiction to Lemma 3.6 and the assumption that $\mathbf{fpr}(S,k)$ has tame representation type.

(c) $\Rightarrow$ (b): We follow [Crawley-Boevey 1991, Theorem 4.4]. Assume that $\mathbf{fpr}(S, k)$ has k-wild representation type. In view of Lemma 3.1, there is some

$$N \in \mathbf{fpr}(S, k\langle t_1, t_2\rangle)$$

such that the k-linear functor $- \otimes_{k\langle t_1,t_2\rangle} N^\vee : \operatorname{Mod} k\langle t_1, t_2\rangle \longrightarrow \mathbf{Fpr}(S, k)$ is exact, preserves isomorphism classes, preserves indecomposables and restricts to the functor $-\otimes_{k\langle t_1,t_2\rangle} N^\vee : \operatorname{modf} k\langle t_1, t_2\rangle \longrightarrow \mathbf{fpr}(S, k)$, where $N^\vee$ is the representation N viewed as an object of $\mathbf{Fpr}(S, k)$. For each $\lambda \in k(x)$, we define

$$M_\lambda = (k(x)[y]/(y - \lambda)) \otimes_{k\langle t_1,t_2\rangle} N^\vee,$$

an indecomposable object of $\mathbf{Fpr}(S, k)$. Then M_λ has infinite k-dimension and d is equal to the $k\langle t_1, t_2\rangle$-rank of N_∞ as well as to the endolength of M_λ, and this number is finite, for each $\lambda \in k(x)$. Consequently, each M_λ is a generic representation in $\mathbf{Fpr}(S, k)$. Since $\operatorname{End} M_\lambda / J(\operatorname{End} M_\lambda)) \cong k(x)$, for each λ, and $M_\lambda \cong M_\sigma$ if and only if $\lambda = \sigma$, the proof is complete.

(d) $\Rightarrow$ (b): Assume that $\mathbf{fpr}(S, k)$ has k-wild representation type. Then S contains a subposet T isomorphic to one of the 6 hypercritical posets $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_4$, $\mathcal{N}_5$, or $\mathcal{N}_6$ of Nazarova [Nazarova 1975], see also [Simson 1992, Theorem 15.3]. As noted in Example 3.3, for each T, there is an object $U^T_{k\langle t_1,t_2\rangle}$ of $\mathbf{fpr}(T^\infty, k\langle t_1, t_2\rangle)$ such that $\operatorname{End} U^T_{k\langle t_1,t_2\rangle} \cong k\langle t_1, t_2\rangle$. It was shown in the proof of Theorem 2.4 that there exists a fully faithful k-linear functor

$$H_T : \mathbf{fpr}(T^\infty, k\langle t_1, t_2\rangle) \longrightarrow \mathbf{fspr}(S, k\langle t_1, t_2\rangle).$$

It follows that $N_T = H_T(U^T_{k\langle t_1,t_2\rangle})$ is an object of $\mathbf{fspr}(S, k\langle t_1, t_2\rangle)$ such that

$$\operatorname{End} N_T \cong \operatorname{End} N_T^\vee \cong \operatorname{End} U^T_{k\langle t_1,t_2\rangle} \cong k\langle t_1, t_2\rangle.$$

It follows from [Kasjan 2003, Corollary 6.1] that there exists such fully faithful k-linear functor H_T that satisfies the following three conditions:

- $N_T = H_T(U^T_{k\langle t_1,t_2\rangle})$ is an object of $\mathbf{fpr}(S, k\langle t_1, t_2\rangle)$,
- there are k-algebra isomorphisms $\operatorname{End} N_T \cong \operatorname{End} N_T^\vee \cong \operatorname{End} U^T_{k\langle t_1,t_2\rangle} \cong k\langle t_1, t_2\rangle$,
- the k-linear functor $-\otimes_{k\langle t_1,t_2\rangle} N_{kS} : \operatorname{Mod} k\langle t_1, t_2\rangle \longrightarrow \mathbf{Fpr}(S, k)$ is full, faithful and exact.

Let D be the universal division algebra of fractions of $k\langle t_1, t_2\rangle$. Then the representation $\widehat{N}^D_T = D \otimes_{k\langle t_1,t_2\rangle} N_T$ is an indecomposable object in $\mathbf{Fpr}(S, k)$ such that

$$\operatorname{End} \widehat{N}^D_T \cong D \otimes_{k\langle t_1,t_2\rangle} \operatorname{End} N_T \cong D \otimes_{k\langle t_1,t_2\rangle} k\langle t_1, t_2\rangle \cong D.$$

Moreover, the representation $\widehat{N}^D_T = ((\widehat{N}^D_T)_s;\ s \in S)$ has infinite k-dimension and finite endolength, since the endolength of $\widehat{N}^D_T$ is equal to the D-dimension of

$(\widehat{N}_T^D)_\infty$, as well as to the $k\langle t_1, t_2\rangle$-rank of $(N_T)_\infty$. Therefore $\widehat{N}_T^D$ is a generic representation in $\mathbf{Fpr}(S, k)$ and $\operatorname{End} \widehat{N}_T^D$ contains a subalgebra isomorphic to $k\langle t_1, t_2\rangle$.

(a) $\Rightarrow$ (d): Let M be a generic representation in the category $\mathbf{Fpr}(S, k)$ such that d is the endolength of M and the k-algebra $\operatorname{End} M/J(\operatorname{End} M)$ contains as a subalgebra a universal division k-algebra D of fractions of $k\langle t_1, t_2\rangle$. Then $M \supseteq D \otimes_{k\langle t_1,t_2\rangle} N$ for some indecomposable object $N = (N_s;\ s \in S)$ of $\mathbf{fpr}(S, k\langle t_1, t_2\rangle)$ such that the $k\langle t_1, t_2\rangle$-rank of N_∞ equals d and there is a k-algebra isomorphism $\operatorname{End} N/J(\operatorname{End} N) \cong k\langle t_1, t_2\rangle$. Given irreducible polynomials $f(t_1) \in k[t_1]$ and $g(t_2) \in k[t_2]$ and positive integers i and j, we consider the quotient k-algebra

$$R = k\langle t_1, t_2\rangle/(t_1t_2,\ t_2t_1,\ f(t_1)^i, g(t_2)^j)$$

of $k\langle t_1, t_2\rangle$ and the indecomposable object $U = R \otimes_{k\langle t_1,t_2\rangle} N$ of $\mathbf{fpr}(S, k)$ with endomorphism ring R. Then $\mathbf{fpr}(S, k)$ has k-endo-wild representation type, because each finite-dimensional k-algebra A is isomorphic to the centralizer of two matrices M and N with minimal polynomials $f(t_1)^i$ and $g(t_2)^j$, respectively, see [Brenner 1974a]. Therefore, there are k-algebra isomorphisms $A \cong R \cong \operatorname{End} U$ and, according to Theorem 2.4, the category $\mathbf{fpr}(S, k)$ has k-wild representation type. □

The following corollary shows that if $\mathbf{fpr}(S, k)$ has tame representation type, then (finite-dimensional) indecomposable objects of $\mathbf{fpr}(S, k)$ can be constructed from (infinite-dimensional) generic representations in $\mathbf{Fpr}(S, k)$.

Corollary 3.9. *Let k be a field and S a finite poset with a unique maximal element. If $\mathbf{fpr}(S, k)$ has tame representation type, then all indecomposable objects U of $\mathbf{fpr}(S, k)$ can be constructed as follows.*

(a) *Choose a generic object M in $\mathbf{Fpr}(S, k)$ with $\operatorname{End} M/J(\operatorname{End} M) \cong k(x)$.*

(b) *Write $M = k(x) \otimes_{k[x]} N$, where N is some pregeneric object of the category $\mathbf{fpr}(S, k[x])$ with $\operatorname{End} N/J(\operatorname{End} N) \cong k[x]$.*

(c) *There is an object U of $\mathbf{fpr}(S, k)$ such that $U \cong Z \otimes_{k[x]} N$, for some cyclic $k[x]$-module Z.*

Proof. Apply Lemmas 3.6 and 3.7. □

Appendix

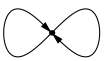
In the proof of Theorem 2.4, we essentially used the following theorem proved in [Simson 1992, pp. 311–312]. Since there are some inaccuracies in the proof given in [Simson 1992], we present here, for convenience of the reader, a correct construction of the functors $\widehat{F}_{\mathcal{N}_1}, \ldots, \widehat{F}_{\mathcal{N}_6}$.

Theorem A.1. *If $\mathcal{N}$ is any of the hypercritical posets $\mathcal{N}_1, \dots, \mathcal{N}_6$ of Nazarova presented in Section 2, then there exists a fully faithful exact k-linear functor $\widehat{F}_{\mathcal{N}}$: $\operatorname{Mod} k\langle t_1, t_2\rangle \longrightarrow \mathbf{Fpr}(\mathcal{N}^\infty, k)$, which restricts to the functor*

$$\widehat{F}_{\mathcal{N}} : \operatorname{modf} k\langle t_1, t_2\rangle \longrightarrow \mathbf{fpr}(\mathcal{N}^\infty, k) \cap \operatorname{prin} k\mathcal{N}^\infty.$$

Proof. We follow the proof given in [Simson 1992, pp. 310–312]. We recall that $\mathbf{fpr}(\mathcal{N}^\infty, k)$ is the category $\mathcal{N}$-sp of $\mathcal{N}$-spaces over k studied in [Simson 1992].

We define a functor $\widehat{F}_{\mathcal{N}} : \operatorname{Mod} k\langle t_1, t_2\rangle \longrightarrow \mathbf{Fpr}(\mathcal{N}^\infty, k)$, for any $\mathcal{N} \in \{\mathcal{N}_1, \dots, \mathcal{N}_6\}$. For this purpose, we recall from [Simson 1992, Section 14.1] that the modules in $\operatorname{Mod} k\langle t_1, t_2\rangle$ are identified with k-linear representations of the two-loop quiver



consisting of a single point and two loops a and b. Any such representation is a triple $\mathbb{V} = (V, \tilde{a}, \tilde{b})$, where V is a k-vector space and $\tilde{a}, \tilde{b} : V \to V$ are k-linear endomorphisms of V. For $j = 1, \dots, 6$, the functor $\widehat{F}_{\mathcal{N}_j}$ associates to the representation $\mathbb{V} = (V, \tilde{a}, \tilde{b})$ the object $U_{\mathbb{V}}^{(j)} = \widehat{F}_{\mathcal{N}_j}(\mathbb{V})$ of the category $\mathbf{Fpr}(\mathcal{N}_j^\infty, k)$ defined by the following diagrams, where O is the zero vector space and we set $O^i \oplus V^j = \underbrace{O \oplus \cdots \oplus O}_{i} \oplus \underbrace{V \oplus \cdots \oplus V}_{j}$:

$$U_{\mathbb{V}}^{(1)}: \quad \begin{array}{ccccc} & & V & & \\ & & \downarrow{\scriptstyle d} & & \\ V & \xrightarrow{(\mathrm{id},\tilde{a})} & V \oplus V & \xleftarrow{(\mathrm{id},\tilde{b})} & V, \\ & \nwarrow & & \nearrow & \\ V \oplus O & & & & O \oplus V \end{array}$$

where $d(x) = (x, x)$;

$$U_{\mathbb{V}}^{(2)}: \quad \begin{array}{ccccccc} & & & & O^2 \oplus V^2 & & \\ & & & & \downarrow & & \\ V \oplus O & \longrightarrow & V^2 & \xrightarrow{h} & V^4 & \xleftarrow{d} & V^2, \\ & & & & \uparrow & & \\ & & & & V^2 \oplus O^2 & & \end{array}$$

where $d(x, y) = (x, y, x, y)$ and $h(x, y) = (x, y, \tilde{a}(x) + \tilde{b}(y), x)$;

$$U_{\mathbb{V}}^{(3)}: \quad \begin{array}{ccccccc} & & & & O^2 \oplus V^2 & \longrightarrow & V^4 \\ & & & & & \searrow^{A} & \\ & & O^4 \oplus V^2 & \longrightarrow & O^2 \oplus V^4 & \longrightarrow & V^6, \\ & \nearrow & & & & & \\ V \oplus O^5 & \longrightarrow & V^2 \oplus O^4 & \longrightarrow & V^4 \oplus O^2 & & \end{array} \quad \text{where } A = \begin{bmatrix} 1 & 0 & \tilde{b} & 0 \\ 0 & 1 & \tilde{a} & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix};$$

$$U_{\mathbb{V}}^{(4)}: \qquad \begin{array}{ccccccccc} & & & & & & V^4 & & \\ & & & & & \overset{B}{\nearrow} & & & \\ V^2\oplus O^6 & \hookrightarrow & V^4\oplus O^4 & \hookrightarrow & V^6\oplus O^2 & \hookrightarrow & & & V^8, \\ & & & & & \underset{}{\nearrow} & & & \\ O^6\oplus V\oplus O & \hookrightarrow & O^6\oplus V^2 & \hookrightarrow & O^4\oplus V^4 & \hookrightarrow & O^2\oplus V^6 & & \end{array}$$

where $B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ \tilde{a} & \tilde{b} & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$;

$$U_{\mathbb{V}}^{(5)}: \qquad \begin{array}{ccccccc} & & & & O^2\oplus V^4\oplus O^2 & \hookrightarrow & V^8 \\ & & & \nearrow & & & \searrow^{C} \\ O^6\oplus V^2 & \hookrightarrow & O^2\oplus V^4\oplus V^2 & & \xrightarrow{\Phi} & & V^{10}, \\ & \nearrow & & & & & \\ O\oplus V\oplus O^8 & \hookrightarrow & V^2\oplus O^8 & \hookrightarrow & V^4\oplus O^6 & \hookrightarrow & V^6\oplus O^4 \hookrightarrow V^8\oplus O^2 \end{array}$$

where $C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ and $\Phi = O^2 \oplus D \oplus \mathrm{id}_{V^2}$ with $D = \begin{bmatrix} 0 & 1 & 1 & 0 \\ \tilde{a} & \tilde{b} & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$;

and finally

$$U_{\mathbb{V}}^{(6)}: \qquad \begin{array}{ccccccc} & & & & & & V^6\oplus O^6 \\ & & & & & \nearrow & \\ U & \hookrightarrow & W & & \hookrightarrow & & V^{12}, \\ & \searrow & & & & & \\ O^{10}\oplus V^2 & \hookrightarrow & O^8\oplus V^4 & \hookrightarrow & O^7\oplus V^5 \hookrightarrow O^5\oplus V^7 \hookrightarrow O^4\oplus V^8 & \hookrightarrow & O^2\oplus V^{10} \end{array}$$

where

$$W = \big\{(w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_5, w_6, w_3, w_4);\ w_i \in V \text{ for } 1 \le i \le 8\big\},$$

$$U = \big\{w \in W;\ w_5 = w_1+w_3,\ w_6 = w_2+\tilde{a}(w_4),\ w_7 = w_2+w_3+\tilde{b}(w_4),$$
$$w_8 = w_1+w_4,\ w_9 = w_1+w_3,\ w_{10} = w_2+\tilde{a}(w_4),\ w_{11} = w_3,\ w_{12} = w_4\big\}.$$

A straightforward calculation shows that, for each $j = 1, \dots, 6$, we have defined a fully faithful exact k-linear functor $\widehat{F}_{\mathcal{N}_j} : \operatorname{Mod} k\langle t_1, t_2\rangle \longrightarrow \mathbf{Fpr}(\mathcal{N}_j^\infty, k)$, which restricts to the functor $\widehat{F}_{\mathcal{N}_j} : \operatorname{modf} k\langle t_1, t_2\rangle \longrightarrow \mathbf{fpr}(\mathcal{N}_j^\infty, k) = \mathcal{N}_j\text{-sp}$. The arguments given in [Simson 1992, p. 312] show that $\operatorname{Im} \widehat{F}_{\mathcal{N}_j} \subseteq \operatorname{prin} k\mathcal{N}_j^\infty$, for $j = 1, \dots, 6$. This finishes the proof. □

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