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Lebesgue measure on the linear dual of the Lie algebra of an exponential solvable Lie group is decomposed into semi-invariant orbital measures by means of a detailed analysis of orbital parameters and a natural measure on an explicit cross-section for generic coadjoint orbits. This decomposition yields a precise and explicit description of the Plancherel measure.

Introduction

For an exponential solvable Lie group G, the classical Plancherel formula for nonunimodular groups [Duflo and Moore 1976] is combined with the method of coadjoint orbits to construct an orbital Plancherel formula [Duflo and Raïs 1976]. Given a choice of a semi-invariant positive Borel function ψ on the linear dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$, measurable fields $\{\pi_{\mathbb{O}}, \mathcal{H}_{\mathbb{O}}\}_{\mathbb{O}\in\mathfrak{g}^*/G}$ of irreducible representations and $\{A_{\psi,\mathbb{O}}\}_{\mathbb{O}\in\mathfrak{g}^*/G}$ of positive self-adjoint, semi-invariant operators (transforming by the square root of the modular function) in $\mathcal{H}_{\mathbb{O}}$, and the Borel measure m_ψ on $\mathfrak{g}^*/G$ are constructed so that for the usual class of functions ϕ on G,

$$\phi(e) = \int_{\mathfrak{g}^*/G} \operatorname{Tr}\big(A^{-1}_{\psi,\mathbb{O}}\, \pi(\phi)\, A^{-1}_{\psi,\mathbb{O}}\big)\, dm_\psi(\mathbb{O}). \tag{0.1}$$

holds. Though each of the measurable fields above depends upon the choice of ψ, the object $\{A^{-2}_{\psi,\mathbb{O}}\, dm_\psi(\mathbb{O})\}$, which is interpreted as a measure on positive, semi-invariant operator fields over $\hat{G} = \mathfrak{g}^*/G$, is canonical, and is referred to as the Plancherel measure.

In the nilpotent case, where one takes $\psi \equiv 1$ and $A_{\psi,\mathbb{O}} \equiv \mathrm{Id}$, the measure $m_\psi = m$ is described precisely by L. Pukánszky [1967]. Let $\{Z_1, Z_2, \ldots, Z_n\}$ be a basis of $\mathfrak{g}$ where for each $1 \le j \le n$, the $\mathbb{R}$-span of $Z_1, \ldots, Z_j$ is an ideal in $\mathfrak{g}$. Let $\mathfrak{g}$ have Lebesgue measure dX obtained by its identification with $\mathbb{R}^n$ via this basis, let $\mathfrak{g}^*$ have the Lebesgue measure via its dual basis, and let G have the Haar measure $d(\exp X) = dX$. Given these initial choices, Pukánszky gives an algorithm for

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computing the Plancherel measure. For each $\ell \in \mathfrak{g}^*$, define the *jump index* set $\boldsymbol{e}(\ell)$ by

$$\boldsymbol{e}(\ell) = \{1 \le j \le n \mid \mathfrak{g}_j \not\subset \mathfrak{g}_{j-1} + \mathfrak{g}(\ell)\}$$

where $\mathfrak{g}(\ell)$ is the stabilizer subalgebra for ℓ. One has $|\boldsymbol{e}(\ell)| = \dim(\mathcal{O}_\ell)$; among those ℓ whose orbits have maximal dimension $2d$, where d is a nonnegative integer, view the sets $\boldsymbol{e}(\ell)$ as increasing sequences and order them lexicographically. Let $\boldsymbol{e} = \{e_1 < e_2 < \cdots < e_{2d}\}$ be the minimal jump index sequence, and $\Omega = \{\ell \in \mathfrak{g}^* \mid \boldsymbol{e}(\ell) = \boldsymbol{e}\}$. The set Ω is G-invariant and Zariski open in $\mathfrak{g}^*$. Also associated with $\boldsymbol{e}$ are the skew-symmetric matrices

$$M_{\boldsymbol{e}}(\ell) = \big[\ell([Z_{e_a}, Z_{e_b}])\big]_{1 \le a,b \le 2d}, \quad \ell \in \mathfrak{g}^*,$$

and the subspace $V = \{\ell \in \mathfrak{g}^* \mid \ell(Z_{e_a}) = 0 \text{ for } 1 \le a \le 2d\}$. One has $\Omega = \{\ell \in \mathfrak{g}^* \mid \det(M_{\boldsymbol{e}}(\ell)) \ne 0\}$ and $\Sigma = V \cap \Omega$ is a topological cross-section for Ω/G. In fact (see [Pukánszky 1967, Lemma 4]) there is an explicit rational map $P : \mathbb{R}^{2d} \times \Omega \to \Omega$ such that $P(z, s\ell) = P(z, \ell)$ for each $z \in \mathbb{R}^{2d}$ and $s \in G$, and that, for each $\ell \in \Omega$, $P(\,\cdot\,, \ell)$ is a polynomial bijection between $\mathbb{R}^{2d}$ and the coadjoint orbit of ℓ. The cross-section $\Sigma = P(0, \Omega)$ and the restriction of P to $\mathbb{R}^{2d} \times \Sigma$ is a rational bijection whose Jacobian is one. The basis of the Pukánszky algorithm for the Plancherel formula is the elementary decomposition of Lebesgue measure on $\mathfrak{g}^*$ [Pukánszky 1967, p. 279]:

$$\int_{\mathfrak{g}^*} h(\ell)\, d\ell = \int_\Sigma \int_{\mathbb{R}^{2d}} h(P(z, \lambda))\, dz\, d\lambda, \tag{0.2}$$

where $d\lambda$ is Lebesgue measure on V (when V is identified with $\mathbb{R}^{n-2d}$ via the dual basis $\{Z_j^* \mid j \notin \boldsymbol{e}\}$), and h is a positive Borel function on $\mathfrak{g}^*$. The inner integral in (0.2) is actually an integral over the coadjoint orbit $\mathcal{O}_\lambda$ of λ which is G-invariant, and hence is a multiple of the canonical measure β_λ on $\mathcal{O}_\lambda$. Precise computation of the Plancherel measure is simply a matter of computing this multiple $r(\lambda)$ for each λ and then plugging that into (0.2). The result is that $r(\lambda) = (2\pi)^{-d}\, |P_{\boldsymbol{e}}(\lambda)|$ where $P_{\boldsymbol{e}}(\lambda)$ is the Pfaffian of $M_{\boldsymbol{e}}(\lambda)$. Equivalently, the measure dm on $\mathfrak{g}^*/G$ is given on Σ by $(2\pi)^{-d}\, |P_{\boldsymbol{e}}(\lambda)|\, d\lambda$, and the formula

$$\phi(e) = \frac{1}{(2\pi)^{n+d}} \int_\Sigma \operatorname{Tr}(\pi_\lambda(\phi))\, |P_{\boldsymbol{e}}(\lambda)|\, d\lambda, \tag{0.3}$$

a simple version of (0.1), is obtained by combining the above with the Kirillov character formula and ordinary Fourier inversion. All this depends of course upon the choice of "Jordan–Hölder basis" made at the outset, but only upon this choice. Independent of this choice one sees that the Plancherel measure, as a measure on the orbit space, belongs to the family of rational measures on $\mathfrak{g}^*/G$.

Suppose now that G is exponential solvable. It is perhaps not surprising that the methods of Pukánszky can be extended to obtain a cross-section for generic coadjoint orbits. However, the execution of this method, and the orbit picture that emerges from it, are more complex. The jump sets $\boldsymbol{e}(\ell)$ are defined as before only now the basis $\{Z_j\}$ is a basis of the complexified Lie algebra $\mathfrak{s} = \mathfrak{g}_c$, for which $\operatorname{span}\{Z_1, Z_2, \ldots, Z_j\} = \mathfrak{s}_j$ is an ideal in $\mathfrak{s}$, and if $\mathfrak{s}_j \neq \bar{\mathfrak{s}}_j$ then $\mathfrak{s}_{j+1} = \bar{\mathfrak{s}}_{j+1}$ and $Z_{j+1} = \bar{Z}_j$. As shown in [Currey 1992; Currey and Penney 1989], the notion of generic orbits must be refined in order to complete the construction of an explicit topological cross-section for the generic orbits. Among other things this involves selecting an index subset φ of $\boldsymbol{e}$, which, roughly speaking, identifies directions in $\mathfrak{g}^*$ in which G acts "exponentially". Nevertheless, there is an explicit, G-invariant Zariski open subset $\Omega \subset \Omega_{\boldsymbol{e}}$, and for $\ell \in \Omega$, a precise generalization of the Pukánszky map $P(z, \ell)$ described above. One still has $P(z, s\ell) = P(z, \ell)$ for $s \in G$, but now some of the variables $z_1, z_2, \ldots, z_{2d}$ may be complex variables, and P is not necessarily rational but real analytic. Simultaneously there is an orbital cross-section Σ obtained by fixing the variables z_a in an appropriate way. Despite the highly nonalgebraic nature of the coadjoint action here, it is shown that the cross-section Σ is in fact a real algebraic submanifold of $\mathfrak{g}^*$.

For each $\ell \in \Omega$, there is a real analytic submanifold $T(\ell)$ of $\mathbb{C}^m$ (depending only on the orbit of ℓ) such that $P(\cdot, \ell)$ is an analytic bijection between $T(\ell)$ and the coadjoint orbit of ℓ. The result is that Ω has in a very explicit way the structure of a bundle over its orbital cross-section:

$$\bigcup_{\lambda \in \Sigma} T(\lambda) \xrightarrow{P} \bigcup_{\lambda \in \Sigma} \mathbb{O}_\lambda = \Omega \xrightarrow{P^*} \Sigma,$$

where $P^*(\ell) = P(z^*(\ell), \ell)$ for a particular (G-invariant) choice $z^*(\ell) \in T(\ell)$. The fiber of the bundle Ω is a cone $W \subset \mathbb{R}^{2d}$ that is naturally homeomorphic with each $T(\ell)$, and local trivializations are given over Zariski-open subsets E of Σ.

Given that these constructions are a natural generalization of the Pukánszky parametrization, the question now becomes: what is the appropriate generalization of (0.2) in the exponential case? There are at least two obvious complications:

(1) The description of Ω given by the Pukánszky map is not as a simple product, but rather as a bundle over the cross-section Σ; and

(2) Σ is not necessarily (a Zariski open subset of) a subspace V.

In [Currey 1992] it is shown that Σ is a smooth, real algebraic submanifold of $\mathfrak{g}^*$, determined by explicit polynomials. Letting S_t, for each $1 \leq t \leq n - 2d$, stand for any of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{S}^0 = \{-1, +1\}$, or $\mathbb{S}^1$, in this paper we show that there is a product

$$S = S_1 \times S_2 \times \cdots \times S_{n-2d}$$

such that each Zariski-open subset E of Σ over which Ω can be trivialized is naturally identified with a dense open subset of S. These identifications differ with the sets E, but only slightly; in particular, if A is a Borel subset of two trivializing subsets E_1 and E_2, then A is identified via E_1 and E_2 with sets in S of equal Lebesgue measure. Thus Σ carries a natural "Lebesgue" measure, which we denote by $d\lambda$. We then use the bundle structure of Ω to decompose Lebesgue measure $d\ell$ on $\mathfrak{g}^*$. We show that for each $\lambda \in \Sigma$ there is a *semi*-invariant measure ω_λ on the coadjoint orbit $\mathbb{O}_\lambda$ through λ, with multiplier Δ, such that

$$\int_{\mathfrak{g}^*} h(\ell)\, d\ell = \int_\Sigma \int_{\mathbb{O}_\lambda} h(\ell)\, d\omega_\lambda(\ell)\, d\lambda \tag{0.4}$$

for any positive Borel function h. If ψ is any positive semi-invariant function on $\mathfrak{g}^*$ with multiplier Δ^{-1}, then $d\omega_\lambda$ is given by

$$d\omega_\lambda = r_\psi(\lambda)\ \psi^{-1} d\beta_\lambda,$$

where β_λ is the canonical measure on $\mathbb{O}_\lambda$, and where $r_\psi(\lambda)$ is defined by

$$r_\psi(\lambda) = \frac{|P_{\boldsymbol{e}}(\lambda)\ \psi(\lambda)|}{(2\pi)^d \prod_{j\in\varphi} |1+i\alpha_j|}.$$

Here φ is the index subset of $\boldsymbol{e}$ referred to above (which is empty in the nilpotent case), and $1+i\alpha_j = \gamma_j/\Re(\gamma_j)$, where γ_j is the j-th root of the coadjoint action.

Just as in the nilpotent case, this yields a description of the Plancherel measure in precise terms. Take $(\pi_\lambda, \mathcal{H}_\lambda)$ to be the irreducible representation induced from the Vergne polarization at $\lambda \in \Sigma$ (corresponding to the Jordan–Hölder sequence already chosen). Since the Vergne polarization is contained in the kernel of Δ, the operator D_λ defined by $D_\lambda f(a) = \Delta(a) f(a)$ for $f \in \mathcal{H}_\lambda$ defines a positive self-adjoint semi-invariant operator of weight Δ^{-1}. Using this and the character formula for exponential solvable Lie groups, one has

$$\{A_{\psi,\mathbb{O}}^{-2} dm_\psi(\mathbb{O})\}_{\mathbb{O}\in\mathfrak{g}^*/G} = \{K_\lambda\, d\lambda\}_{\lambda\in\Sigma},$$

where

$$K_\lambda = \frac{|P_{\boldsymbol{e}}(\lambda)|}{(2\pi)^{n+d} \prod_{j\in\varphi} |1+i\alpha_j|} D_\lambda.$$

The Pukánszky version of the Plancherel formula becomes

$$\begin{aligned}\phi(e) &= \int_\Sigma \operatorname{Tr}\bigl(K_\lambda^{1/2}\, \pi_\lambda(\phi)\ K_\lambda^{1/2}\bigr)\, d\lambda \\ &= \frac{1}{(2\pi)^{n+d} \prod_{j\in\varphi} |1+i\alpha_j|} \int_\Sigma \operatorname{Tr}\bigl(D_\lambda^{1/2}\, \pi_\lambda(\phi)\, D_\lambda^{1/2}\bigr)\, |P_{\boldsymbol{e}}(\lambda)|\, d\lambda.\end{aligned}$$

In Section 1 of this paper we review the relevant results of [Currey 1992], and then proceed with an expansion of these results to obtain more detailed information about the bundle structure in general, and the cross-section Σ in the generic case. In Section 2 this information is used to define Lebesgue measure on Σ and then to deduce the decomposition (0.4) and the description of the Plancherel measure.

1. The Collective Orbit Structure

1.1. ***Preliminaries.*** Let $\mathfrak{g}$ be a solvable Lie algebra over $\mathbb{R}$ with $\mathfrak{s} = \mathfrak{g}_c$ its complexification, and choose a basis $\{Z_1, Z_2, \dots, Z_n\}$ for $\mathfrak{s}$ with the following properties.

(i) For each $1 \le j \le n$, the space $\mathfrak{s}_j = \mathbb{C}\text{-span}\{Z_1, Z_2, \dots, Z_j\}$ is an ideal in $\mathfrak{s}$.

(ii) If $\mathfrak{s}_j \neq \bar{\mathfrak{s}}_j$ then $\mathfrak{s}_{j+1} = \bar{\mathfrak{s}}_{j+1}$ and $Z_{j+1} = \bar{Z}_j$. Moreover, in this case, there is $A \in \mathfrak{g}$ such that $[A, Z_j] = (1 + i\alpha)Z_j \mod \mathfrak{s}_{j-1}$, where α is a nonzero real number.

(iii) If $\mathfrak{s}_j = \bar{\mathfrak{s}}_j$ and $\mathfrak{s}_{j-1} = \bar{\mathfrak{s}}_{j-1}$, then $Z_j \in \mathfrak{g}$.

As in [Currey 1992], it will be convenient to make the following notation: $I = \{1 \le i \le n \mid \mathfrak{s}_j = \bar{\mathfrak{s}}_j\}$, and for each $1 \le j \le n$ set

$$j' = \max\left(\{0, 1, \dots, j-1\} \cap I\right) \quad \text{and} \quad j'' = \min\left(\{j, j+1, \dots, n\} \cap I\right).$$

Thus for each j, $\mathfrak{s}_{j'} = \mathfrak{s}_{j-1} \cap \bar{\mathfrak{s}}_{j-1}$ and $\mathfrak{s}_{j''} = \mathfrak{s}_j + \bar{\mathfrak{s}}_j$. For $Z \in \mathfrak{s}$, denote the real part of Z by $\Re Z$, and the imaginary part of Z by $\Im Z$. (We also use these symbols to denote real and imaginary parts of a complex number.) Define a basis for $\mathfrak{g}$ as follows: let $X_j = Z_j$ if $Z_j \in \mathfrak{g}$, and if $\mathfrak{s}_j \neq \bar{\mathfrak{s}}_j$ then set $X_j = \Re Z_j$ and $X_{j+1} = \Im Z_j$. Using the ordered basis $\{X_j\}$ to identify $\mathfrak{g}$ with $\mathbb{R}^n$, let dX denote Lebesgue measure on $\mathfrak{g}$. Let $d\ell$ be Lebesgue measure on $\mathfrak{g}^*$ obtained via the ordered dual basis $\{X_j^*\}$. We regard $\mathfrak{g}^*$ as a real subspace of the complex vector space $\mathfrak{s}^*$, and for convenience we denote $\ell(Z) = \langle \ell, Z\rangle$ by ℓZ, for $Z \in \mathfrak{s}$ and $\ell \in \mathfrak{g}^*$. We identify an element $\ell \in \mathfrak{g}^*$ with the n-tuple $(\ell_1, \ell_2, \dots, \ell_n)$, where $\ell_j = \ell Z_j$.

For each $\ell \in \mathfrak{g}^*$ let $\mathfrak{s}(\ell) = \{Z \in \mathfrak{s} \mid \ell[Z, W] = 0, \text{ for all } Z \in \mathfrak{s}\}$, and let $\mathfrak{p}(\ell)$ be the complex Vergne polarization associated with the sequence $\{\mathfrak{s}_j\}$ chosen. For any $\ell \in \mathfrak{g}^*$ and any subset $\mathfrak{t}$ of $\mathfrak{s}$, we use the usual notation

$$\mathfrak{t}^\ell = \{Z \in \mathfrak{s} \mid \ell[Z, X] = 0 \text{ for all } X \in \mathfrak{t}\}.$$

Let G be the unique connected, simply connected Lie group with Lie algebra $\mathfrak{g}$; we assume in this paper that G is *exponential*, meaning that the exponential map $\exp : \mathfrak{g} \to G$ is a bijection. Let da be the left Haar measure on G defined by $d(\exp X) = j_G(X)dX$, where $j_G(X) = \left|\det(1 - e^{-adX})/\text{ad}\, X\right|$. Let Δ be the modular function: $d(ab) = \Delta(b)\, da$. The coadjoint action of G on $\mathfrak{g}^*$ extends to an action of G on $\mathfrak{s}^*$ and restricts to an action of G on each ideal $\mathfrak{s}_j$. We denote each such action multiplicatively. For each $1 \le j \le n$, set $\mathfrak{s}_j^\perp = \{\ell \in \mathfrak{g}^* \mid \ell(\mathfrak{s}_j) = \{0\}\}$, let $\mu_j : G \to \mathbb{C}^*$ be defined by $s \cdot Z_j^* = \mu_j(s) Z_j^* \bmod \mathfrak{s}_j^\perp$, and let $\gamma_j : \mathfrak{g} \to \mathbb{C}$ be the differential of μ_j. Since G is exponential, there is a real number α_j such that $\gamma_j = \Re(\gamma_j)(1 + i\alpha_j)$, for $1 \le j \le n$.

The results stated in [Currey 1992, Proposition 2.6, Theorem 2.8] provide us with a stratification of the linear dual $\mathfrak{g}^*$ of $\mathfrak{g}$ into $\mathrm{Ad}^*(G)$-invariant *layers* $\boldsymbol{\Omega}$ and in each layer an explicit description of the space of coadjoint orbits. We summarize the stratification procedure as follows.

(1) To each $\ell \in \mathfrak{g}^*$ there is associated an index set $\boldsymbol{e}(\ell) \subset \{1, 2, \dots, n\}$ defined by

$$\boldsymbol{e}(\ell) = \{1 \le j \le n \mid \mathfrak{s}_j \not\subset \mathfrak{s}_{j-1} + \mathfrak{s}(\ell)\}.$$

For a subset $\boldsymbol{e}$ of $\{1, 2, \dots, n\}$, the set $\boldsymbol{\Omega}_{\boldsymbol{e}} = \{\ell \in \mathfrak{g}^* \mid \boldsymbol{e}(\ell) = \boldsymbol{e}\}$ is algebraic and G-invariant, and we refer to the collection of nonempty $\boldsymbol{\Omega}_{\boldsymbol{e}}$ as the coarse stratification of $\mathfrak{g}^*$. The coarse stratification has had various applications; see for example [Pedersen 1984]. There is an ordering on the coarse stratification for which the minimal element is Zariski open in $\mathfrak{g}^*$ and consists of orbits having maximal dimension.

(2) To each ℓ there is associated a *polarizing sequence* of subalgebras

$$\mathfrak{s} = \mathfrak{h}_0(\ell) \supset \mathfrak{h}_1(\ell) \supset \cdots \supset \mathfrak{h}_d(\ell) = \mathfrak{p}(\ell),$$

and an index *sequence pair* $(\boldsymbol{i}(\ell), \boldsymbol{j}(\ell))$ having values $\boldsymbol{i}(\ell) = \{i_1 < i_2 < \cdots < i_d\}$ and $\boldsymbol{j}(\ell) = \{j_1, j_2, \dots, j_d\}$ in $\boldsymbol{e}(\ell)$, defined for $1 \le k \le d$ by the recursive equations

$$\begin{aligned} i_k &= \min\{1 \le j \le n \mid \mathfrak{s}_j \cap \mathfrak{h}_{k-1}(\ell) \not\subset \mathfrak{h}_{k-1}(\ell)^\ell\}, \\ \mathfrak{h}_k(\ell) &= \left(\mathfrak{h}_{k-1}(\ell) \cap \mathfrak{s}_{i_k}\right)^\ell \cap \mathfrak{h}_{k-1}(\ell), \\ j_k &= \min\{1 \le j \le n \mid \mathfrak{s}_j \cap \mathfrak{h}_{k-1}(\ell) \not\subset \mathfrak{h}_k(\ell)\}. \end{aligned}$$

Then $i_k < j_k$ for each k, and $\boldsymbol{e}(\ell)$ is the disjoint union of the values of $\boldsymbol{i}(\ell)$ and $\boldsymbol{j}(\ell)$. Note that since $\boldsymbol{i}(\ell)$ must be increasing, it is determined by $\boldsymbol{e}(\ell)$ and $\boldsymbol{j}(\ell)$. For any splitting of $\boldsymbol{e}$ into such a sequence pair $(\boldsymbol{i}, \boldsymbol{j})$ we set $\boldsymbol{\Omega}_{\boldsymbol{e},\boldsymbol{j}} = \{\ell \in \boldsymbol{\Omega}_{\boldsymbol{e}} \mid \boldsymbol{j}(\ell) = \boldsymbol{j}\}$. These sets are also algebraic and G-invariant, and we refer to the collection of nonempty $\boldsymbol{\Omega}_{\boldsymbol{e},\boldsymbol{j}}$ as the *fine stratification* of $\mathfrak{g}^*$. There is an ordering on the fine stratification for which the minimal layer is a Zariski open subset of the minimal coarse layer.

(3) Now fix a layer $\boldsymbol{\Omega}_{\boldsymbol{e},\boldsymbol{j}}$ in the fine stratification. For each $\ell \in \boldsymbol{\Omega}_{\boldsymbol{e},\boldsymbol{j}}$, set

$$\varphi(\ell) = \{j \in \boldsymbol{e} \mid \mathfrak{s}_{j'}^\ell \cap \ker(\gamma_j) = \mathfrak{s}_{j''}^\ell \cap \ker(\gamma_j)\}.$$

The index set $\varphi(\ell)$ identifies those directions j in $\boldsymbol{e}$ where the coadjoint action of G dilates by its character μ_j. If $j \in \varphi$, then $j-1 \in I$, and $j \in I$ if and only if μ_j is real. It is easily seen that $\varphi(\ell)$ is contained in the values of $\boldsymbol{i}$, and there are examples where $\varphi(\ell)$ is not constant on the fine layer. For each $j \in \boldsymbol{i}$, there is a rational function $q_j : \mathfrak{g}^* \to \mathbb{C}$ such that q_j is relatively invariant with multiplier μ_j^{-1}, and such that for $\ell \in \boldsymbol{\Omega}_{\boldsymbol{e},\boldsymbol{j}}$, one has $j \in \varphi$ if and only if $q_j(\ell) \neq 0$. So for each subset φ of the values of $\boldsymbol{i}$, the set $\boldsymbol{\Omega}_{\boldsymbol{e},\boldsymbol{j},\varphi} = \{\ell \in \boldsymbol{\Omega}_{\boldsymbol{e},\boldsymbol{j}} \mid \varphi(\ell) = \varphi\}$ is an algebraic subset of $\boldsymbol{\Omega}_{\boldsymbol{e},\boldsymbol{j}}$. We refer to this further refinement of the fine stratification as the *ultrafine stratification* of $\mathfrak{g}^*$. The ultrafine stratification also has an ordering for which the minimal layer is a Zariski open subset of the minimal fine layer.

(4) Now fix an ultrafine layer $\boldsymbol{\Omega} = \boldsymbol{\Omega}_{\boldsymbol{e},\boldsymbol{j},\varphi}$ and let $\imath = \{j \in \boldsymbol{e} - \varphi \mid j \notin I \text{ and } j+1 \notin \boldsymbol{e}\}$. Let V_0 be the span of those Z_j^* for which either $j \notin \boldsymbol{e}$ or $j \in \varphi \cup \imath$. Then for each $i \in \imath$, there is a rational function $p_i : \mathfrak{g}^* \to \mathbb{C}$ such that the set

$$\Sigma = \Big\{\ell \in \boldsymbol{\Omega} \cap V_0 \mid p_i(\ell) = 0 \text{ for every } i \in \imath, \text{ and } |q_j(\ell)| = 1 \text{ for every } j \in \varphi\Big\}$$

is a topological cross-section for the orbits in $\boldsymbol{\Omega}$.

1.2. *Parametrizing an orbit.* Take $\ell \in \mathfrak{g}^*$ and write $\boldsymbol{e}(\ell) = \{e_1 < e_2 < \cdots < e_{2d}\}$. Then, for each $j \in \boldsymbol{e}$, one can select $X_j \in \mathfrak{g} \cap (\mathfrak{s}_{j''} - \mathfrak{s}_{j'})$ so that

$$(t_1, t_2, \ldots, t_{2d}) \to \exp(t_1 X_{e_1}) \exp(t_2 X_{e_2}) \cdots \exp(t_{2d} X_{e_{2d}})\ell$$

is an analytic diffeomorphism $Q(t, \ell, X_{e_1}, X_{e_1}, \ldots, X_{e_{2d}})$ of $\mathbb{R}^{2d}$ with the coadjoint orbit of ℓ. The starting point for the constructions of [Currey 1992] is a procedure for selecting the X_j, in terms of the elements ℓ belonging to a fine layer $\boldsymbol{\Omega}_{\boldsymbol{e},\boldsymbol{j}}$, so that the resulting map $Q(t, \ell)$ is analytic in ℓ and has a manageable and somewhat explicit form. The relevant result is [Currey 1992, Lemma 1.3]; the following lemma is a restatement of the important aspects of this result in a somewhat simplified form. We then include a description of the procedure by which this result is proved in [Currey 1992]. Finally, we show how this result is used to define the orbit parametrization, and we observe that a slight modification of the selection procedure in [Currey 1992] obtains a parametrization that is simpler in some cases.

Lemma 1.2.1 [Currey 1992, Lemma 1.3]. *Let $\mathfrak{g}$ be an exponential solvable Lie algebra over $\mathbb{R}$, and choose a good basis for $\mathfrak{s} = \mathfrak{g}_c$. Let $\boldsymbol{\Omega}_{\boldsymbol{e},\boldsymbol{j}}$ be a fine layer. Then there is a cover $F = \{O_t\}$ of $\boldsymbol{\Omega}_{\boldsymbol{e},\boldsymbol{j}}$ by finitely many Zariski open sets, and for each $O \in F$ and $1 \leq k \leq d$, there are analytic functions $X_k : O \to \mathfrak{g}$, $Y_k : O \to \mathfrak{g}$, and $\phi_k : O \to \mathbb{S}^1$ with the following properties.*

(i) $\ell[X_j(\ell), X_k(\ell)] = \ell[Y_j(\ell), Y_k(\ell)] = 0$ *for* $1 \leq j, k \leq d$.

(ii) $\ell[X_j(\ell), Y_k(\ell)] = 0$ *if and only if* $j \neq k$, *for* $1 \leq j, k \leq d$.

(iii) *For each* k, *the functions* $\ell \to \phi_k(\ell)X_k(\ell)$ *and* $\ell \to \phi_k(\ell)Y_k(\ell)$ *extend to rational functions from* $\Omega_{\boldsymbol{e},j}$ *into* $\mathfrak{s}$ *and are independent of* O.

(iv) *For each* $1 \le k \le d$, *set*

$$\mathfrak{m}_k(\ell) = \mathbb{C}\text{-span}\,\{\phi_1(\ell)Y_1(\ell), \phi_2(\ell)Y_2(\ell), \dots, \phi_k(\ell)Y_k(\ell), \phi_1(\ell)X_1(\ell), \phi_2(\ell)X_2(\ell), \dots, \phi_k(\ell)X_k(\ell)\},$$

so that $\mathfrak{s} = \mathfrak{m}_k(\ell) \oplus \mathfrak{m}_k(\ell)^\ell$ *for each* $\ell \in \Omega$. *For* $Z \in \mathfrak{s}$ *and* $\ell \in \Omega$, *let* $\rho_k(Z,\ell)$ *be the projection of* Z *into* $\mathfrak{m}_k(\ell)^\ell$ *parallel to* $\mathfrak{m}_k(\ell)$, *with* $\rho_0(Z,\ell) = Z$. *Then* $X_k(\ell)$ *and* $Y_k(\ell)$ *are in the image of* $\rho_{k-1}(\,\cdot\,,\ell)$, *and the function* ρ_k *is defined recursively by the formula*

$$\rho_k(Z,\ell) = \rho_{k-1}(Z,\ell) - \frac{\ell[\rho_{k-1}(Z,\ell), X_k(\ell)]}{\ell[Y_k(\ell), X_k(\ell)]}\, Y_k(\ell) - \frac{\ell[\rho_{k-1}(Z,\ell), Y_k(\ell)]}{\ell[X_k(\ell), Y_k(\ell)]}\, X_k(\ell). \tag{1.2.1}$$

(v) *For each* $\ell \in \Omega$, $\rho_k(\mathfrak{s}_j,\ell) \subset \mathfrak{s}_{j''}$ *for* $1 \le j \le n$ *and* $0 \le k \le d$, *and* $X_k(\ell) \in \mathfrak{s}_{j_k''}$, $Y_k(\ell) \in \mathfrak{s}_{i_k''}$.

(vi) *For* $1 \le k \le d$, $X_k(\ell)$ *has the form*

$$X_k(\ell) = \Re\big(\ell[\rho_{k-1}(\bar{Z}_{j_k},\ell), Y_k(\ell)]\rho_{k-1}(Z_{j_k},\ell)\big).$$

Remark 1.2.2. In the construction of [Currey 1992, Lemma 1.3], one actually has

$$X_k(\ell) = a(\ell)\, \Re\big(\ell[\rho_{k-1}(\bar{Z}_{j_k},\ell), Y_k(\ell)]\rho_{k-1}(Z_{j_k},\ell)\big),$$

where $a(\ell)$ is a real-valued analytic function on O. Formula (vi) above represents a simplification of the procedure there.

For the purposes of this paper it will be necessary to analyze the preceding objects in some detail, so we recall how these objects are defined. Let $1 \le k \le d$. If $k > 1$, assume that a Zariski open subset O of $\Omega_{\boldsymbol{e},j}$ has been selected, and that $Y_1, Y_2, \dots Y_{k-1}, X_1, X_2, \dots, X_{k-1}$ have been defined so as to satisfy (i)–(vi) above, so that we have the map ρ_{k-1}. If $k = 1$, set $O = \Omega_{\boldsymbol{e},j}$ and $\rho_0(Z,\ell) = Z$ for $Z \in \mathfrak{s}$ and $\ell \in \mathfrak{g}^*$. We then proceed to select a Zariski open subset of O and to construct Y_k and X_k. We consider several cases. In each of them $X_k(\ell)$ is defined essentially as in Lemma 1.2.1(vi) above, although in Cases 3 and 5, Remark 1.2.2 applies. In those cases we justify the remark.

Case 0. $i_k \in I$ and $i_k - 1 \in I$. Here $Z_{i_k} \in \mathfrak{g}$, and we set

$$Y_k(\ell) = \rho_{k-1}(Z_{i_k},\ell).$$

The rest of the cases are those for which $Z_{i_k} \ne \bar{Z}_{i_k}$.

Case 1. $i_k \notin I$ and $i_k + 1 \notin \boldsymbol{e}$. Here one finds that the complex numbers

$$\beta_{1,k}(\ell) = \ell[\rho_{k-1}(Z_{j_k}, \ell), \Re Z_{i_k}] \text{ and } \beta_{2,k}(\ell) = \ell[\rho_{k-1}(Z_{j_k}, \ell), \Im Z_{i_k}],$$

satisfy $\Im\big(\beta_{1,k}(\ell)\overline{\beta_{2,k}(\ell)}\big) = 0$. Write $O = O_1 \cup O_2$, where $O_t = \{\ell \in O \mid \beta_{t,k}(\ell) \neq 0\}$. For $\ell \in O_t$, set

$$\phi_{t,k}(\ell) = \frac{\beta_{t,k}(\ell)}{|\beta_{t,k}(\ell)|},$$

and

$$Y_{t,k}(\ell) = \phi_{t,k}(\ell)^{-1}\big(\beta_{1,k}(\ell)\rho_{k-1}(\Re Z_{i_k}, \ell) + \beta_{2,k}(\ell)\rho_{k-1}(\Im Z_{i_k}, \ell)\big), \quad t = 1, 2.$$

Case 2. $i_k - 1 = j_r \notin I$. Here we set

$$Y_k(\ell) = \rho_{k-1}(\tilde{X}_r(\ell), \ell)$$

where

$$\tilde{X}_r(\ell) = \Im\big(\ell[\rho_{r-1}(\bar{Z}_{j_r}, \ell), Y_r(\ell)]\rho_{r-1}(Z_{j_r}, \ell)\big).$$

Case 3. $i_k \notin I$ and $i_k + 1 = j_k$. Here $Y_k(\ell) = \rho_{k-1}(\Im Z_{i_k}, \ell)$ and in the proof of [Currey 1992, Lemma 1.3], $X_k(\ell) = \rho_{k-1}(\Re Z_{i_k}, \ell)$. Note that

$$\begin{aligned}&\Re\big(\ell[\rho_{k-1}(\bar{Z}_{j_k}, \ell), Y_k(\ell)]\rho_{k-1}(Z_{j_k}, \ell)\big)\\ &= \ell[\rho_{k-1}(\Re Z_{i_k}, \ell), \rho_{k-1}(\Im Z_{i_k}, \ell)]\ \rho_{k-1}(\Re Z_{i_k}, \ell)\end{aligned}$$

so that Remark 1.2.2 holds.

Case 4. $i_k \notin I$, $i_k + 1 = i_{k+1}$. This case splits into two subcases.

Case 4a. $Z_{j_{k+1}} = \bar{Z}_{j_k}$. Here $Y_k(\ell) = \rho_{k-1}(\Re Z_{i_k}, \ell)$.

Case 4b. $Z_{j_{k+1}} \neq \bar{Z}_{j_k}$. This case is just like Case 1: the functions $\beta_{t,1}(\ell)$, and the sets O_t, $t = 1, 2$ are defined exactly the same way, as is $Y_{t,k}(\ell)$, for $\ell \in O_t$, $t = 1, 2$.

Case 5. $i_k - 1 = i_{k-1} \notin I$. Again there are two subcases.

Case 5a. $Z_{j_{k-1}} = \bar{Z}_{j_k}$. Set $r = k - 1$ and note that Case 4 holds for r. We have $Y_k(\ell) = \rho_r(\Im Z_{i_r}, \ell)$, and in the proof of [Currey 1992, Lemma 1.3], $X_k(\ell)$ is defined as $X_k(\ell) = \rho_{k-1}(\tilde{X}_r(\ell), \ell)$ where

$$\tilde{X}_r(\ell) = \Im\big(\ell[\rho_{r-1}(\bar{Z}_{j_r}, \ell), Y_r(\ell)]\rho_{r-1}(Z_{j_r}, \ell)\big).$$

We claim that Remark 1.2.2 holds in this case also. Set

$$\beta_r(\ell) = \ell[\rho_{r-1}(Z_{j_r}, \ell), Y_r(\ell)];$$

then

$$\rho_{r-1}(Z_{j_r}, \ell) = \frac{\beta_r(\ell)}{|\beta_r(\ell)|^2}\big(X_r(\ell) + i\tilde{X}_r(\ell)\big)$$

and

$$\rho_{r-1}(Z_{j_k},\ell)=\overline{\rho_{r-1}(Z_{j_r},\ell)}=\frac{\overline{\beta_r(\ell)}}{|\beta_r(\ell)|^2}\big(X_r(\ell)-i\tilde{X}_r(\ell)\big).$$

Now

$$\begin{aligned}\rho_r(Z_{j_k},\ell)&=\rho_{r-1}(Z_{j_k},\ell)-\frac{\ell[\rho_{r-1}(Z_{j_k},\ell),X_r(\ell)]}{\ell[Y_r(\ell),X_r(\ell)]}Y_r(\ell)\\&\qquad-\frac{\ell[\rho_{r-1}(Z_{j_k},\ell),Y_r(\ell)]}{\ell[X_r(\ell),Y_r(\ell)]}X_r(\ell)\\&=\frac{\overline{\beta_r(\ell)}}{|\beta_r(\ell)|^2}\big(X_r(\ell)-i\tilde{X}_r(\ell)\big)-\frac{-i\overline{\beta_r(\ell)}}{|\beta_r(\ell)|^2}\,\frac{\ell[\tilde{X}_r(\ell),X_r(\ell)]}{\ell[Y_r(\ell),X_r(\ell)]}Y_r(\ell)\\&\qquad-\frac{\overline{\beta_r(\ell)}}{|\beta_r(\ell)|^2}\,X_r(\ell)\\&=\frac{-i\overline{\beta_r(\ell)}}{|\beta_r(\ell)|^2}\bigg(\tilde{X}_r(\ell)-\frac{\ell[\tilde{X}_r(\ell),X_r(\ell)]}{\ell[Y_r(\ell),X_r(\ell)]}Y_r(\ell)\bigg),\end{aligned}$$

and so because $\ell[Y_k(\ell),Y_r(\ell)]=0$, we get

$$\begin{aligned}&\Re\big(\ell[\rho_r(\bar{Z}_{j_k},\ell),Y_k(\ell)]\rho_r(Z_{j_k},\ell)\big)\\&\quad=\Re\bigg(\frac{i\beta_r(\ell)}{|\beta_r(\ell)|^2}\,\ell[\tilde{X}_r(\ell),Y_k(\ell)]\,\rho_r(Z_{j_k},\ell)\bigg)\\&\quad=\ell[\tilde{X}_r(\ell),Y_k(\ell)]\,\Re\bigg(\frac{i\beta_r(\ell)}{|\beta_r(\ell)|^2}\bigg(\frac{-i\overline{\beta_r(\ell)}}{|\beta_r(\ell)|^2}\bigg(\tilde{X}_r(\ell)-\frac{\ell[\tilde{X}_r(\ell),X_r(\ell)]}{\ell[Y_r(\ell),X_r(\ell)]}Y_r(\ell)\bigg)\bigg)\bigg)\\&\quad=\frac{\ell[\tilde{X}_r(\ell),Y_k(\ell)]}{|\beta_r(\ell)|^2}\bigg(\tilde{X}_r(\ell)-\frac{\ell[\tilde{X}_r(\ell),X_r(\ell)]}{\ell[Y_r(\ell),X_r(\ell)]}Y_r(\ell)\bigg)\\&\quad=\frac{\ell[\tilde{X}_r(\ell),Y_k(\ell)]}{|\beta_r(\ell)|^2}\,\rho_r(\tilde{X}_r(\ell),\ell).\end{aligned}$$

This proves the claim.

Case 5b. $Z_{j_{k-1}}\neq\bar{Z}_{j_k}$. Again we set $r=k-1$. Then Case 4b holds for r and we have

$$Y_k(\ell)=\beta_{1,r}(\ell)\rho_{r-1}(\Im Z_{i_r},\ell)-\beta_{2,r}(\ell)\rho_{r-1}(\Re Z_{i_r},\ell).$$

(In this subcase $X_k(\ell)$ is defined in [Currey 1992, Lemma 1.3] exactly as Lemma 1.2.1(vi) above.)

Write $\boldsymbol{e}=\{e_1<e_2<\cdots<e_{2d}\}$ and fix $O\in F$. In [Currey 1992, Proposition 1.5], the objects $X_k(\ell)$ and $Y_k(\ell)$ are used to define analytic functions $r_a:O\to\mathfrak{g}$

for the purpose of parametrizing the orbit of each $\ell \in O$ in a manageable way. The definition given there is

$$r_a(\ell) = \begin{cases} \dfrac{X_k(\ell)}{\left|\ell[Z_{e_a}, X_k(\ell)]\right|} & \text{if } e_a = i_k, \\ \dfrac{Y_k(\ell)}{\left|\ell[Z_{e_a}, Y_k(\ell)]\right|} & \text{if } e_a = j_k. \end{cases}$$

Suppose that $j = e_a \in \boldsymbol{e}$ with $j-1 \in I$. If also $j \in I$, then

$$\mathrm{ad}^* r_a(\ell)\ell = \zeta_a(\ell) Z_j^* \mod \mathfrak{s}_j^\perp,$$

where $\zeta_a(\ell) = \pm 1$ (and is constant on O). If $j \notin I$, then

$$\mathrm{ad}^* r_a(\ell)\ell = \zeta_a(\ell) Z_j^* + \overline{\zeta_a(\ell)} Z_{j+1}^* \mod \mathfrak{s}_{j+1}^\perp,$$

where $\zeta_a(\ell)$ is a complex number of modulus one. (Recall $Z_{j+1} = \bar{Z}_j$ in this case.) If also $j+1 = e_{a+1} \in \boldsymbol{e}$, then similarly

$$\mathrm{ad}^* r_{a+1}(\ell)\ell = \zeta_{a+1}(\ell) Z_j^* + \overline{\zeta_{a+1}(\ell)} Z_{j+1}^* \mod \mathfrak{s}_{j+1}^\perp$$

Note that $\zeta_a(\ell) = \ell[Z_j, r_a(\ell)]$ (and $\zeta_{a+1}(\ell) = \ell[Z_j, r_{a+1}(\ell)]$ if $j+1 \in \boldsymbol{e}$), so that $\ell \to \zeta_a(\ell)$ (and $\ell \to \zeta_{a+1}(\ell)$) are analytic functions on O.

It is shown in [Currey 1992, Proposition 1.5] that if $j \notin I$ and both $j = e_a$ and $j+1$ belong to $\boldsymbol{e}$, then for each ℓ the complex numbers $\zeta_a(\ell)$ and $\zeta_{a+1}(\ell)$ are linearly independent over $\mathbb{R}$. It will simplify a subsequent computation if we can show that in fact they are orthogonal, that is, that

$$\Re\big(\zeta_a(\ell)\overline{\zeta_{a+1}(\ell)}\big) = 0.$$

To do this it is necessary to alter (slightly) the definition of $r_a(\ell)$ in one particular case: suppose that $e_a = i_k$ and that Case 4a holds for k. In other words, suppose that $e_a = i_k \notin I$, that $i_k + 1 = i_{k+1}$, and that $Z_{j_{k+1}} = \bar{Z}_{j_k}$. Then I claim that we could have defined the $X_k(\ell)$, $X_{k+1}(\ell)$, $Y_k(\ell)$ and $Y_{k+1}(\ell)$ as follows. Set

$$X_k'(\ell) = \rho_{k-1}(\Re Z_{j_k}, \ell),$$

and then set

$$Y_k'(\ell) = \Re\big(\ell[\rho_{k-1}(\bar{Z}_{i_k}, \ell), X_k'(\ell)]\, \rho_{k-1}(Z_{i_k}, \ell)\big)$$

and

$$Y_{k+1}'(\ell) = \Im\big(\ell[\rho_{k-1}(\bar{Z}_{i_k}, \ell), X_k'(\ell)]\, \rho_{k-1}(Z_{i_k}, \ell)\big).$$

Note that $\ell[Y'_{k+1}(\ell), X'_k(\ell)] = \ell[Y'_{k+1}(\ell), Y'_k(\ell)] = 0$. Hence if we set

$$\begin{aligned} X'_{k+1}(\ell) &= \rho'_k(\Im Z_{j_k}, \ell) \\ &= \rho_{k-1}(\Im Z_{j_k}, \ell) - \frac{\ell[\rho_{k-1}(\Im Z_{j_k}, \ell), Y'_k(\ell)]}{\ell[X'_k(\ell), Y'_k(\ell)]} X'_k(\ell) \\ &\qquad\qquad - \frac{\ell[\rho_{k-1}(\Im Z_{j_k}, \ell), X'_k(\ell)]}{\ell[Y'_k(\ell), X'_k(\ell)]} Y'_k(\ell), \end{aligned}$$

then

$$\ell[X'_{k+1}(\ell), X'_k(\ell)] = \ell[X'_{k+1}(\ell), Y'_k(\ell)] = 0.$$

By virtue of our assumptions for this case, $\ell[X'_{k+1}(\ell), Y'_{k+1}(\ell)]$ does not vanish. This proves the claim. Now for this case, with $e_a = i_k$ and $e_{a+1} = i_{k+1} = i''_k$, we set

$$r_a(\ell) = \frac{X'_k(\ell)}{\left|\ell[Z_{e_a}, X'_k(\ell)]\right|} \quad \text{and} \quad r_{a+1}(\ell) = \frac{X'_{k+1}(\ell)}{\left|\ell[Z_{e_a}, X'_{k+1}(\ell)]\right|}.$$

We emphasize here that this is merely an alteration of the definitions of $r_a(\ell)$ and $r_{a+1}(\ell)$ in this case. In particular the definition of $\rho_k(\,\cdot\,, \ell)$ is not changed. The advantage of this alteration is that it allows for the following result, which is used in the proof of Proposition 1.4.1 (see also Proposition 2.1.1).

In the remainder of this paper we shall refer to Case 0 above as Case (1.2.0), Case 1 as Case (1.2.1), and so on.

Lemma 1.2.3. *Let O be a covering set in F for the fine layer $\Omega_{\boldsymbol{e},\boldsymbol{j}}$. Suppose that $j \notin I$, and that both j and $j+1$ belong to $\boldsymbol{e}$. Write $j = e_a$. Then for each $\ell \in O$ the complex numbers $\zeta_a(\ell) = \ell[Z_j, r_a(\ell)]$ and $\zeta_{a+1}(\ell) = \ell[Z_j, r_{a+1}(\ell)]$ are orthogonal.*

Proof. It suffices to show that in each of the above cases where $j \notin I$ and j and $j+1$ both belong to $\boldsymbol{e}$, one has $U(\ell)$ and $\tilde{U}(\ell)$ belonging to $\mathfrak{g}$ such that $\ell[U(\ell), r_{a+1}(\ell)] = \ell[\tilde{U}(\ell), r_a(\ell)] = 0$, and such that

$$Z_j = \alpha(\ell)\, U(\ell) + \tilde{\alpha}(\ell)\, \tilde{U}(\ell) \mod \mathfrak{s}_{j-1},$$

where $\alpha(\ell)$ and $\tilde{\alpha}(\ell)$ are orthogonal complex numbers.

First suppose that $\{j, j+1\}$ includes a term of the index sequence $\boldsymbol{j}$. Thus either $j = j_r$ and $j+1 = i_k$, or $\{j, j+1\} = \{j_r, j_k\}$, with $r < k$ in both cases. We set

$$U(\ell) = X_r(\ell) = \Re\big(\ell[\rho_{r-1}(\bar{Z}_j, \ell), Y_r(\ell)]\, \rho_{r-1}(Z_j, \ell)\big)$$

and

$$\tilde{U}(\ell) = \Im\big(\ell[\rho_{r-1}(\bar{Z}_{j_r}, \ell), Y_r(\ell)]\, \rho_{r-1}(Z_{j_r}, \ell)\big).$$

Then

$$Z_j = \frac{\beta_r(\ell)}{|\beta_r(\ell)|^2} \big(U(\ell) + i\tilde{U}(\ell)\big) \mod \mathfrak{s}_{j-1},$$

where

$$\beta_r(\ell) = \ell[\rho_{r-1}(Z_j, \ell), Y_r(\ell)].$$

It follows immediately that

$$Z_j = \alpha(\ell)U(\ell) + \tilde{\alpha}(\ell)\tilde{U}(\ell) \mod \mathfrak{s}_{j-1},$$

where $\alpha(\ell)$ and $\tilde{\alpha}(\ell)$ are orthogonal. If $j+1 = i_k$, an examination of Case (1.2.2) shows that $\tilde{U}(\ell) = Y_k(\ell) \mod \mathfrak{s}_{j-1}$, while if $\{j, j+1\} = \{j_r, j_k\}$, a computation exactly as in Case (1.2.5a) shows that $\tilde{U}(\ell)$ is a real multiple of $X_k(\ell)$. Hence in either case we have $\ell[U(\ell), r_{a+1}(\ell)] = \ell[\tilde{U}(\ell), r_a(\ell)] = 0$.

Secondly, suppose that $j = i_k$ and $j+1 = i_{k+1}$. If $Z_{j_{k+1}} = \bar{Z}_{j_k}$, we set $U(\ell) = Y'_k(\ell)$ and $\tilde{U}(\ell) = Y'_{k+1}(\ell)$. From the definitions of $Y'_k(\ell)$ and $Y'_{k+1}(\ell)$ we have

$$Z_j = \frac{\beta_k(\ell)}{|\beta_k(\ell)|^2}\big(U(\ell) + i\tilde{U}(\ell)\big) \mod \mathfrak{s}_{j-1},$$

where now

$$\beta_k(\ell) = \ell[\rho_{k-1}(Z_j, \ell), X'_k(\ell)].$$

and (with the alternate definitions of r_a and r_{a+1}) we find that $\ell[U(\ell), r_{a+1}(\ell)] = \ell[\tilde{U}(\ell), r_a(\ell)] = 0$. Finally, if $j = i_k$, $j+1 = i_{k+1}$, and $Z_{j_{k+1}} \neq \bar{Z}_{j_k}$, then we set $U(\ell) = Y_k(\ell)$ and $\tilde{U}(\ell) = Y_{k+1}(\ell)$. From the definitions of Y_k and Y_{k+1} in this case we have

$$Z_j = \frac{\beta_{1,k}(\ell) + i\beta_{2,k}(\ell)}{\beta_{1,k}(\ell)^2 + \beta_{2,k}(\ell)^2}\big(U(\ell) + i\tilde{U}(\ell)\big) \mod \mathfrak{s}_{j-1}.$$

As in the previous cases we find that U and $\tilde{U}$ satisfy the desired conditions. This completes the proof. □

For $\ell \in O$ and $t \in \mathbb{R}$, set $g_a(t, \ell) = \exp(t r_a(\ell))$ and set

$$g^a(t, \ell) = g_1(t_1, \ell)g_2(t_2, \ell)\cdots g_a(t_a, \ell) \quad \text{for } t \in \mathbb{R}^{2d},$$

with $g(t, \ell) = g^{2d}(t, \ell)$. Then, for each $\ell \in O$,

$$Q(t, \ell) = g(t, \ell)\ell = \sum_{j=1}^{n} Q_j(t, \ell)Z_j^*$$

defines a diffeomorphism of $\mathbb{R}^{2d}$ onto the coadjoint orbit of ℓ. Note that for each $1 \le j \le n$, if $1 \le b \le 2d$ is defined by $e_b \le j'' < e_{b+1}$, then $Q_j(t, \ell) = \big(g^b(t, \ell)\ell\big)_j$, so that $Q_j(\cdot, \ell)$ depends only upon $t_1, t_2, \ldots, t_b$. Note also that if $j \notin I$, then $Q_{j+1} = \bar{Q}_j$.

1.3. ***A closer look at parametrization.*** The form of the functions $Q_j(t, \ell)$ as functions of $t \in \mathbb{R}^{2d}$ is well-known. We wish to closely examine these functions not just as functions of t, but as functions of $\ell_1, \ell_2, \ldots, \ell_n$ as well. We assume that we have fixed a layer $\Omega_{\boldsymbol{e},j}$ belonging to the fine stratification, with all associated objects as described in the preceding section. We begin with some observations that follow immediately from the results of [Currey 1992].

Remark 1.3.1. The definition of ρ_k implies that $\rho_k(\rho_r(Z,\ell),\ell) = \rho_k(Z,\ell)$ for $0 \le r \le k,\ Z \in \mathfrak{s},\ \ell \in \Omega$.

Remark 1.3.2. Because $X_k(\ell)$ and $Y_k(\ell)$ are in the image of $\rho_{k-1}(\,\cdot\,,\ell)$, we have

$$\ell([V, Y_k(\ell)]) = \ell([\rho_{k-1}(V,\ell), Y_k(\ell)]),$$
$$\ell([V, X_k(\ell)]) = \ell([\rho_{k-1}(V,\ell), X_k(\ell)]),$$

for any $V \in \mathfrak{s}$, by the definition of $\rho_{k-1}(\,\cdot\,,\ell)$. Formula (1.2.1) can be simplified accordingly.

Remark 1.3.3. Fix $1 \le k \le d$ and let $Z \in \mathfrak{s}$. Then $\rho_{k-1}(Z,\ell)$ belongs to $\mathfrak{s}^{\ell}_{i_k{}'}$.

Lemma 1.3.4. *Fix a covering set $O \in F$, let Y_k and X_k be the functions described in Lemma 1.2.1 and let $1 \le k \le d$.*

(i) *One has*

$$X_k(\ell) = a_{1,k}(\ell)\,\rho_{k-1}(\Re Z_{j_k},\ell) + a_{2,k}(\ell)\,\rho_{k-1}(\Im Z_{j_k},\ell),$$
$$Y_k(\ell) = b_{1,k}(\ell)\,\rho_{k-1}(\Re Z_{i_k},\ell) + b_{2,k}(\ell)\,\rho_{k-1}(\Im Z_{i_k},\ell)$$

where $a_{1,k}(\ell)$, $a_{2,k}(\ell)$, $b_{1,k}(\ell)$, and $b_{2,k}(\ell)$ all depend only upon $\ell_1, \ldots, \ell_{i_k}$. Moreover, if Case (1.2.4a) holds for k, the above statement also holds for the functions X'_k, Y'_k, X'_{k+1}, and Y'_{k+1}.

(ii) *Fix j such that $1 \le j \le n$, and let $Z \in \mathfrak{s}_{j''}$, $V \in \mathfrak{s}$. Then $\ell \to \ell[Z, \rho_k(V,\ell)]$ depends only on $\ell_1, \ell_2, \ldots, \ell_j$.*

Proof. We proceed by induction on k; suppose that $k = 1$. Note that $\mathfrak{s}_{i_1-1} = \bar{\mathfrak{s}}_{i_1-1}$. An examination of the construction of $Y_1(\ell)$ and $X_1(\ell)$ in [Currey 1992, Proof of Lemma 1.3], and outlined in the various cases of Section 1.2, shows that (i) is true. In fact, the functions $a_{1,1}(\ell)$, $a_{2,1}(\ell)$, $b_{1,1}(\ell)$, and $b_{2,1}(\ell)$ depend only upon the expressions

$$\ell[\Re Z_{j_1}, \Re Z_{i_1}], \quad \ell[\Re Z_{j_1}, \Im Z_{i_1}], \quad \ell[\Im Z_{j_1}, \Re Z_{i_1}], \quad \ell[\Im Z_{j_1}, \Im Z_{i_1}].$$

We now turn to the statement (ii) when $k = 1$. Observe first that, having verified (i) for $k = 1$, and referring to Lemma 1.2.1(v), we see that the function

$$\ell \to \ell[X_1(\ell), Y_1(\ell)]$$

depends only on $\ell_1, \ell_2, \ldots, \ell_{i_1}$. Now consider the function $\ell \to \ell[Z, \rho_1(V, \ell)]$, where $Z \in s_{j''}$ and V is any element of $\mathfrak{s}$. We have

$$\ell[Z, \rho_1(V, \ell)] = \ell[Z, V] - \frac{\ell[V, X_1(\ell)]}{\ell[Y_1(\ell), X_1(\ell)]}\,\ell[Z, Y_1(\ell)] - \frac{\ell[V, Y_1(\ell)]}{\ell[X_1(\ell), Y_1(\ell)]}\,\ell[Z, X_1(\ell)].$$

If $j \le i_1'$, then $\ell[Z, Y_1(\ell)]$ and $\ell[Z, X_1(\ell)]$ are both zero, whence $\ell[Z, \rho_1(V, \ell)] = \ell[Z, V]$ and the conclusion follows. If $j > i_1'$ but $j \le j_1'$, then $\ell[Z, Y_1(\ell)] = 0$, so

$$\ell[Z, \rho_1(V, \ell)] = \ell[Z, V] - \frac{\ell[V, Y_1(\ell)]}{\ell[X_1(\ell), Y_1(\ell)]}\,\ell[Z, X_1(\ell)].$$

Again using parts (i) and (v) of Lemma 1.2.1, we have that

$$\ell \to \ell[V, Y_1(\ell)] \;\text{ and }\; \ell \to \ell[Z, X_1(\ell)]$$

depend only on $\ell_1, \ell_2, \ldots, \ell_j$, and the result follows. Finally, if $j > j_1'$, using (i) and Lemma 1.2.1 in a similar way, we find that each factor in each term of the above depends only on $\ell_1, \ell_2, \ldots, \ell_j$. This completes the case $k = 1$.

Now suppose that $k > 1$ and that (i) and (ii) hold for all $1 \le r \le k-1$. We note that the induction hypothesis (together with the properties of the functions $\rho_r(\,\cdot\,, \ell)$) implies that for each $1 \le r \le k-1$ and $1 \le s \le k$, the function

$$\ell \to \ell[\rho_r(\Re Z_{j_s}, \ell), \rho_r(\Re Z_{i_s}, \ell)] = \ell[\rho_r(\Re Z_{j_s}, \ell), \Re Z_{i_s}]$$

depends only upon $\ell_1, \ldots, \ell_{i_k}$. (Recall here that $i_s \le i_k$.) Similarly, the expressions

$$\begin{aligned}(1.3.1)\quad & \ell[\rho_r(\Re Z_{j_s}, \ell), \rho_r(\Im Z_{i_s}, \ell)], \\ & \ell[\rho_r(\Im Z_{j_s}, \ell), \rho_r(\Re Z_{i_s}, \ell)] \quad\text{and}\quad \ell[\rho_r(\Im Z_{j_s}, \ell), \rho_r(\Im Z_{i_s}, \ell)]\end{aligned}$$

depend only upon $\ell_1, \ldots, \ell_{i_k}$.

To see that (i) holds for k, we begin by observing that if (i) is true for $Y_k(\ell)$, it is true for $X_k(\ell)$ as well, by virtue of the formula

$$\begin{aligned}X_k(\ell) &= \Re\big(\ell[\rho_{k-1}(\bar{Z}_{j_k}, \ell), Y_k(\ell)]\,\rho_{k-1}(Z_{j_k}, \ell)\big) \\ &= \ell[\rho_{k-1}(\Re Z_{j_k}, \ell), Y_k(\ell)]\,\rho_{k-1}(\Re Z_{j_k}, \ell) \\ &\quad + \ell[\rho_{k-1}(\Im Z_{j_k}, \ell), Y_k(\ell)]\,\rho_{k-1}(\Im Z_{j_k}, \ell).\end{aligned}$$

As for $Y_k(\ell)$, we examine each of the five cases outlined in Section 1.2 for the formulae by which $Y_k(\ell)$ is defined. In Case (1.2.0), $b_{1,k}(\ell) = 1$ and $b_{2,k}(\ell) = i$, while in Case (1.2.1), $b_{1,k}(\ell) = \phi_{t,k}(\ell)^{-1}\beta_{1,k}(\ell)$ and $b_{2,k}(\ell) = \phi_{t,k}(\ell)^{-1}\beta_{2,k}(\ell)$ are easily seen to depend upon the expressions (1.3.1), with $r = k-1$. Suppose that we are in Case (1.2.2), which means that we have $r < k$ such that $j_r = i_k - 1 \notin I$ and $Z_{i_k} = \bar{Z}_{j_r}$. The formula for $Y_k(\ell)$ in this case is

$$\begin{aligned}&\rho_{k-1}\big(\Im\big(\ell[\rho_{r-1}(\bar{Z}_{j_r},\ell),Y_r(\ell)]\rho_{r-1}(Z_{j_r},\ell)\big),\ell\big)\\ &\quad=\ell[\rho_{r-1}(\Re Z_{j_r},\ell),Y_r(\ell)]\rho_{k-1}(\Im Z_{j_r},\ell)-\ell[\rho_{r-1}(\Im Z_{j_r},\ell),Y_r(\ell)]\rho_{k-1}(\Re Z_{j_r},\ell)\\ &\quad=-\ell[\rho_{r-1}(\Im Z_{j_r},\ell),Y_r(\ell)]\rho_{k-1}(\Re Z_{i_k},\ell)-\ell[\rho_{r-1}(\Re Z_{j_r},\ell),Y_r(\ell)]\rho_{k-1}(\Im Z_{i_k},\ell),\end{aligned}$$

where we have used Remark 1.3.1. So $b_{1,k}(\ell) = -\ell[\rho_{r-1}(\Im Z_{i_k}, \ell), Y_r(\ell)]$ and $b_{2,k}(\ell) = -\ell[\rho_{r-1}(\Re Z_{i_k}, \ell), Y_r(\ell)]$ are seen to depend only upon the expressions (1.3.1). Cases (1.2.3), (1.2.4a), and (1.2.5a), are trivial: $b_{t,k}(\ell) = 0$ or ± 1, and Cases (1.2.4b) and (1.2.5b) are similar to Cases (1.2.1) and (1.2.2), respectively. Finally, in Case (1.2.4a), the definitions of $X'_k(\ell)$, $X'_{k+1}(\ell)$, $Y'_k(\ell)$, and $Y'_{k+1}(\ell)$ resemble those for $X_k(\ell)$, $X_{k+1}(\ell)$, $Y_k(\ell)$, and $Y_{k+1}(\ell)$, except with the letters X and Y interchanged, and we leave it to the reader to check that they also satisfy (i). This completes the induction step for statement (i).

Turning to the statement (ii), we argue as we did for $k = 1$. We observe using (i) and Lemma 1.2.1(v) that the function

$$\ell \to \ell[X_k(\ell), Y_k(\ell)]$$

depends entirely upon the expressions (1.3.1) with $r = k-1$, and hence only upon $\ell_1, \ell_2, \dots, \ell_{i_k}$. Let $Z \in \mathfrak{s}_{j''}$ and let V be any element of $\mathfrak{s}$. From the simplified form of (1.2.1) (Remark 1.3.2), we have

$$\begin{aligned}\ell[Z, \rho_k(V, \ell)] = \ell[Z, \rho_{k-1}(V, \ell)] &- \frac{\ell[V, X_k(\ell)]}{\ell[Y_k(\ell), X_k(\ell)]}\,\ell[Z, Y_k(\ell)]\\ &- \frac{\ell[V, Y_k(\ell)]}{\ell[X_k(\ell), Y_k(\ell)]}\,\ell[Z, X_k(\ell)].\end{aligned}$$

If $j \le i'_k$, then $\ell[Z, Y_k(\ell)]$ and $\ell[Z, X_k(\ell)]$ are both zero, whence $\ell[Z, \rho_k(V, \ell)] = \ell[Z, \rho_{k-1}(V, \ell)]$ and the conclusion follows by induction. If $j > i'_k$ but $j \le j'_k$, then $\ell[Z, Y_k(\ell)] = 0$, so

$$\ell[Z, \rho_k(V, \ell)] = \ell[Z, \rho_{k-1}(V, \ell)] - \frac{\ell[V, Y_k(\ell)]}{\ell[X_k(\ell), Y_k(\ell)]}\,\ell[Z, X_k(\ell)].$$

Again using (i) and Lemma 1.2.1(v), we see that

$$\ell \to \ell[V, Y_k(\ell)] \quad \text{and} \quad \ell \to \ell[Z, X_k(\ell)]$$

depend only on $\ell_1, \ell_2, \dots, \ell_j$, and the result follows. Finally if $j > j'_k$, using (i) and Lemma 1.2.1 in a similar way, we find that each factor in each term of the above depends only on $\ell_1, \ell_2, \dots, \ell_j$. This completes the induction step for part (ii). □

Lemma 1.3.5. *Assume given*:

(a) *an index* j, $1 \le j \le n$ *such that* $j-1 \in I$;

(b) *indices* $1 \le k_1, k_2, \ldots, k_p \le d$ *and* $1 \le e_{a_1} \le e_{a_2} \le \cdots \le e_{a_p} \le j''$ *such that* e_{a_s} *is equal to one of* i_{k_s} *or* j_{k_s}, $1 \le s \le p$;

(c) *for each* $1 \le s \le p$, *an element* $V_s \in \mathfrak{s}$ *such that* $\rho_{k_s-1}(V_s, \ell)$ *belongs to* $\mathfrak{s}^{\ell}_{e'_{a_s}}$ *for every* $\ell \in \Omega$;

(d) *an element* $Z \in \mathfrak{s}_{j''}$.

Then the function

$$\ell \to \ell\big[[\cdots[[Z, \rho_{k_1-1}(V_1, \ell)], \rho_{k_2-1}(V_2, \ell)], \cdots], \rho_{k_p-1}(V_p, \ell)\big]$$

depends only on $\ell_1, \ell_2, \ldots, \ell_j$.

Proof. We proceed by induction on $N = \sum_{s=1}^{p} k_s$; if $N = 1$ then $p = 1$ and $k_1 = 1$, and the result is obvious. Assume that $N > 1$. It is clear that we may assume that $k_1 > 1$, and by Lemma 1.3.4, we may assume that $p > 1$. Note also that $Y_{k_1-1}(\ell) \in \mathfrak{s}_{i''_{k_1-1}} \subset \mathfrak{s}_{e'_{a_s}}$ for all $2 \le s \le p$. By the assumption about the elements V_s we have

$$\ell\big[[\cdots[[Z, Y_{k_1-1}(\ell)], \rho_{k_2-1}(V_2, \ell)], \cdots], \rho_{k_p-1}(V_p, \ell)\big] = 0,$$

and hence, for each $\ell \in \Omega$,

$$\begin{aligned} \ell\big[[\cdots[[Z, \rho_{k_1-1}(V_1, \ell)], \rho_{k_2-1}(V_2, \ell)], \cdots], \rho_{k_p-1}(V_p, \ell)\big] \\ = \ell\big[[\cdots[[Z, \rho_{k_1-2}(V_1, \ell)], \rho_{k_2-1}(V_2, \ell)], \cdots], \rho_{k_p-1}(V_p, \ell)\big] \\ - b(\ell)\,\ell\big[[\cdots[[Z, X_{k_1-1}(\ell)], \rho_{k_2-1}(V_2, \ell)], \cdots], \rho_{k_p-1}(V_p, \ell)\big], \end{aligned} \tag{1.3.2}$$

where

$$b(\ell) = \frac{\ell[\rho_{k_1-2}(V_1, \ell), Y_{k_1-1}(\ell)]}{\ell[X_{k_1-1}(\ell), Y_{k_1-1}(\ell)]}.$$

Now the data

$$1 \le k_1 - 1, k_2, \ldots, k_p \le d, \quad i_{k_1-1} < e_{a_2} < \cdots < e_{a_p}, \quad V_1, V_2, \ldots, V_p$$

satisfy the conditions of the lemma since $\rho_{k_1-2}(V_1, \ell)$ belongs to $\mathfrak{s}^{\ell}_{i'_{k_1-1}}$. Hence by induction the first term of the right-hand side above, namely,

$$\ell\big[[\cdots[[Z, \rho_{k_1-2}(V_1, \ell)], \rho_{k_2-1}(V_2, \ell)], \cdots], \rho_{k_p-1}(V_p, \ell)\big]$$

depends only on $\ell_1, \ell_2, \ldots, \ell_j$.

As for the second term of (1.3.2), we apply the formulas Lemma 1.3.4(i):

$$\begin{aligned} Y_{k_1-1}(\ell) &= b_1(\ell)\rho_{k_1-2}(\Re Z_{i_{k_1-1}}, \ell) + b_2(\ell)\rho_{k_1-2}(\Im Z_{i_{k_1-1}}, \ell), \\ X_{k_1-1}(\ell) &= a_1(\ell)\rho_{k_1-2}(\Re Z_{j_{k_1-1}}, \ell) + a_2(\ell)\rho_{k_1-2}(\Im Z_{j_{k_1-1}}, \ell), \end{aligned}$$

with $a_1(\ell), a_2(\ell), b_1(\ell)$ and $b_2(\ell)$ depending only upon $\ell_1, \ell_2, \dots, \ell_{i_{k_1-1}}$. From this and Lemma 1.3.4(ii) it follows that $b(\ell)$ depends only upon $\ell_1, \ell_2, \dots, \ell_{i_{k_1-1}}$.

Moreover, we observe that the data

$$1 \le k_1 - 1, k_2, \dots, k_p \le d, \quad i_{k_1-1} < e_{a_2} < \dots < e_{a_p}, \quad \Re Z_{j_{k_1-1}}, V_2, \dots, V_p$$

satisfy the conditions for this lemma and so, by induction,

$$\ell\big[[\cdots[[Z, \rho_{k_1-2}(\Re Z_{j_{k_1-1}}, \ell)], \rho_{k_2-1}(V_2, \ell)], \cdots], \rho_{k_p-1}(V_p, \ell)\big]$$

depends only upon $\ell_1, \ell_2, \dots, \ell_j$. Similarly,

$$\ell\big[[\cdots[[Z, \rho_{k_1-2}(\Im Z_{j_{k_1-1}}, \ell)], \rho_{k_2-1}(V_2, \ell)], \cdots], \rho_{k_p-1}(V_p, \ell)\big]$$

depends only upon $\ell_1, \ell_2, \dots, \ell_j$. We conclude that the second term of (1.3.2) depends only upon $\ell_1, \ell_2, \dots, \ell_j$. This completes the proof. □

Proposition 1.3.6. *Fix $O \in F$, and for each $\ell \in O$, let $Q(t, \ell) = g(t, \ell)\ell$ be defined as above. Then for each $1 \le j \le n$ and for each $t \in \mathbb{R}^{2d}$, the function $\ell \to Q_j(t, \ell)$ depends only on $\ell_1, \ell_2, \dots, \ell_j$.*

Proof. Fix $1 \le j \le n$; we may assume that $j - 1 \in I$. Set $a = \max\{1 \le b \le 2d \mid e_b \le j''\}$. Note that $r_b(\ell) \in \mathfrak{s}^{\ell}_{j''}$ for $b > a$, and hence $\exp(t_b r_b(\ell))\ell Z_j = \ell_j$ then. Now fix $t \in \mathbb{R}^{2d}$. Let $q \in \{0, 1, 2, \dots\}^a$ be a multi-index. With the conventions $t^q = t_1^{q_1} t_2^{q_2} \cdots t_a^{q_a}$ and $q! = q_1! q_2! \cdots q_a!$, we have

$$\begin{aligned} Q_j(t, \ell) &= g(t, \ell)\ell Z_j \\ &= \exp(t_1 r_1(\ell)) \dots \exp(t_a r_a(\ell))\ell Z_j = \sum_{q \in \{0,1,2,\dots\}^a} w_j(q, t, \ell), \end{aligned}$$

where

$$w_j(q, t, \ell) = \frac{t^q}{q!} (\mathrm{ad}^* r_1(\ell))^{q_1} (\mathrm{ad}^* r_2(\ell))^{q_2} \cdots (\mathrm{ad}^* r_a(\ell))^{q_a} \ell Z_j. \tag{1.3.3}$$

It remains to show that for each $t \in \mathbb{R}^{2d}$ and each multi-index q, the function $\ell \to w_j(q, t, \ell)$ depends only on $\ell_1, \ell_2, \dots, \ell_j$. Fix a multi-index q and write

$$(e_1, e_1, \dots, e_1, e_2, \dots, e_2, \dots, e_a, \dots e_a) = (e_{a_1}, e_{a_2}, \dots, e_{a_p}),$$

where on the left-hand side e_b is listed q_b times, for $1 \le b \le a$. For each $1 \le s \le p$, let $1 \le k_s \le d$ be such that $e_{a_s} \in \{i_{k_s}, j_{k_s}\}$. Note that $i_{k_s} \le j''$ holds for $1 \le s \le p$. Writing

$$Y_k(\ell) = b_{1,k}(\ell)\, \rho_{k-1}(\Re Z_{i_k}, \ell) + b_{2,k}(\ell)\, \rho_{k-1}(\Im Z_{i_k}, \ell)$$

as in Lemma 1.3.4, the functions b_{1,k_s} and b_{2,k_s}, for each $1 \le s \le p$, depend only on $\ell_1, \ell_2, \dots, \ell_j$. Similarly for the functions a_{1,k_s} and a_{2,k_s} that appear in the formula for $X_{k_s}(\ell)$. Also by Lemma 1.3.4, the functions $\ell[Z_j, X_{k_s}(\ell)]$ and $\ell[Z_j, Y_{k_s}(\ell)]$

depend only upon $\ell_1, \ell_2, \ldots, \ell_j$. (If Case (1.2.4a) holds for k_s, replace $X_{k_s}(\ell)$ by $X'_{k_s}(\ell)$ and the same statements hold.) Substituting the formula for $r_b(\ell)$ into (1.3.3) we obtain a function $A(\ell) = A(\ell_1, \ell_2, \ldots, \ell_j)$ such that

$$w_j(q_1, q_2, \ldots, q_a, t, \ell) \\ = \frac{t^q}{q!} A(\ell)\, \ell\big[[\cdots[[Z, \rho_{k_1-1}(V_1, \ell)], \rho_{k_2-1}(V_2, \ell)], \cdots], \rho_{k_p-1}(V_p, \ell)\big],$$

where V_s is one of $\Re Z_{j_{k_s}}, \Im Z_{j_{k_s}}$ if $e_{a_s} = i_{k_s}$, and one of $\Re Z_{i_{k_s}}, \Im Z_{i_{k_s}}$ if $e_{a_s} = j_{k_s}$. The factor

$$\ell\big[[\cdots[[Z, \rho_{k_1-1}(V_1, \ell)], \rho_{k_2-1}(V_2, \ell)], \cdots], \rho_{k_p-1}(V_p, \ell)\big]$$

appearing in the preceding satisfies the hypothesis of Lemma 1.3.5, and hence depends only upon $\ell_1, \ell_2, \ldots, \ell_j$. This completes the proof. □

Lemma 1.3.7. *Let $1 \le j \le n$ be an index with $j-1 \in I$, and let i_k be a term of the index sequence $\boldsymbol{i} = \{i_1 < i_2 < \cdots < i_d\}$ with $i_k < j$. Then for each $V \in \mathfrak{s}$ and for $0 \le r \le k$, the function $\ell \to \gamma_j(\rho_r(V, \ell))$ depends only upon $\ell_1, \ell_2, \ldots, \ell_{j-1}$.*

Proof. We proceed by induction on r: if $r = 0$, the result is obvious. Suppose that $r > 0$, and assume that the result holds for $r-1$. We have

$$\gamma_j(\rho_r(V, \ell)) = \gamma_j(\rho_{r-1}(V, \ell)) - \frac{\ell[\rho_{r-1}(V, \ell), X_r(\ell)]}{\ell[Y_r(\ell), X_r(\ell)]}\gamma_j(Y_r(\ell)) \\ - \frac{\ell[\rho_{r-1}(V, \ell), Y_r(\ell)]}{\ell[X_r(\ell), Y_r(\ell)]}\gamma_j(X_r(\ell)).$$

Note that $i_r \le i_k < j$; hence $\gamma_j(Y_r(\ell)) = 0$. If also $j_r \le j''$ then $\gamma_j(X_r(\ell)) = 0$, so $\gamma_j(\rho_r(V, \ell)) = \gamma_j(\rho_{r-1}(V, \ell))$, and the induction step is complete. Suppose that $j_r > j''$. It remains to check that each of the expressions

$$\ell[\rho_{r-1}(V, \ell), Y_r(\ell)], \quad \ell[X_r(\ell), Y_r(\ell)] \quad \text{and} \quad \gamma_j(X_r(\ell))$$

depend only upon $\ell_1, \ell_2, \ldots, \ell_{j-1}$. Using formulas Lemma 1.3.4(i) for $X_r(\ell)$ and $Y_r(\ell)$, the fact that $i_r < j$, and Lemma 1.3.4(ii), we see that both $\ell[\rho_{r-1}(V, \ell), Y_r(\ell)]$ and $\ell[X_r(\ell), Y_r(\ell)]$ depend only upon $\ell_1, \ell_2, \ldots, \ell_{j-1}$. As for $\gamma_j(X_r(\ell))$, we apply the formula for $X_r(\ell)$ again:

$$\gamma_j(X_r(\ell)) = a_{1,r}(\ell)\gamma_j(\rho_{r-1}(\Re Z_{j_r}, \ell)) + a_{2,r}(\ell)\gamma_j(\rho_{r-1}(\Im Z_{j_r}, \ell))$$

where $a_{1,r}(\ell)$ and $a_{2,r}(\ell)$ depend only on $\ell_1, \ell_2, \ldots, \ell_{i_r}$. By the induction assumption, $\gamma_j(\rho_{r-1}(\Re Z_{j_r}, \ell))$ and $\gamma_j(\rho_{r-1}(\Im Z_{j_r}, \ell))$ depend only on $\ell_1, \ell_2, \ldots, \ell_{j-1}$. This completes the induction step and the proof. □

We now recall the procedure of substitution [Currey 1992, Proposition 2.6] by which $Q(t, \ell)$ is simplified to obtain a map $P(z, \ell)$. Let $\Omega \subset \Omega_{\boldsymbol{e}, \boldsymbol{j}}$ be a layer belonging to the ultrafine stratification. Given any covering set $O \in F$, then for each $\ell \in O$, we make substitutions

$$z_1 = \xi_1(t,\ell), z_2 = \xi_2(t,\ell), \ldots, z_{2d} = \xi_{2d}(t,\ell),\ t \in R^{2d},\ \ell \in \Omega \cap O,$$

that result in a simplification of the expressions $Q_{e_a}(t,\ell)$, for $1 \le a \le 2d$. If $j = e_a \notin \varphi$ and $e''_a \in \boldsymbol{e}$, then $z_a = Q_j(t,\ell)$ (this is always the situation in the nilpotent case.) If $j = e_a \notin \varphi$ and $e''_a \notin \boldsymbol{e}$ (that is, $j \in \imath$), then $z_a = c_j(t,\ell)\,\Re\big(c_j(t,\ell)^{-1} Q_j(t,\ell)\big)$, where

$$c_j(t,\ell) = \operatorname{sign}\big(\mu_j(g^{a-1}(t,\ell)\big)\,\zeta_a(\ell).$$

(Here $\operatorname{sign} w = w/|w|$ for a nonzero complex number w.) If $j = e_a \in \varphi$, then $z_a = \mu_j(g^a(t,\ell))\,q_j(\ell)^{-1}$, where

$$q_j(\ell) = \frac{\gamma_j(r_a(\ell))}{\zeta_a(\ell)}$$

is a nonvanishing, μ_j^{-1}-relatively invariant rational function on Ω; see [Currey 1992, Proposition 1.8, Corollary 2.2, and the definition of Ω on p. 256]. Solving for t_a in terms of $z_1, z_2, \ldots, z_a$ and ℓ, we obtain inverse maps $\Phi_1(z,\ell)$, $\Phi_2(z,\ell)$, $\ldots$, $\Phi_{2d}(z,\ell)$ as described in [Currey 1992, proof of Proposition 2.6, p. 261], so that

$$Q(\Phi(z,\ell),\ell) = P(z,\ell) = \sum_{j=1}^{n} P_j(z,\ell)\,Z_j^*.$$

For each $\ell \in \Omega$ there is a submanifold $T(\ell)$ of $\mathbb{C}^{2d}$, depending only on the orbit of ℓ, such that $P(\,\cdot\,,\ell)$ is an analytic bijection of $T(\ell)$ with the coadjoint orbit of ℓ. The functions $P_j(z,\ell)$, for $1 \le j \le n$, satisfy

(i) $P_j(z, s\ell) = P_j(z,\ell)$ for $s \in G$;

(ii) $P_j(z,\ell) = 0 \mod (z_1, z_2, \ldots, z_a)$, where $e_a \le j < e_{a+1}$;

(iii) $P_{e_a}(z,\ell) = z_a \mod (z_1, z_2, \ldots, z_{a-1})$, with $P_{e_a}(z,\ell) \equiv z_a$ unless $e_a \in \imath \cup \varphi$. (In the nilpotent case, $\imath \cup \varphi = \varnothing$.)

The function $P(z,\ell)$ is defined on the entire ultrafine layer Ω, independently of the covering set O, and is a precise generalization of the map of [Pukánszky 1967, Lemma 4].

Finally, one has an analytic map $z : \Omega \to \mathbb{C}^m$ with $z(s\ell) = z(\ell)$ and $z(\ell) \in T(\ell)$ such that $P^* : \ell \to P(z(\ell),\ell)$ maps Ω onto an orbital cross-section Σ. The map $z(\ell) = (z_1(\ell), z_2(\ell), \ldots, z_{2d}(\ell))$ is defined as follows. If $e_a \notin \varphi$, set $z_a(\ell) = 0$. Suppose that $j = e_a \in \varphi$. Assume that if $b < a$, then $z_b(\ell)$ is defined, and set

$$g^{a-1}(\ell) = g^{a-1}\big(\Phi_1(z_1(\ell),\ell), \ldots, \Phi_{a-1}(z_1(\ell), \ldots, z_{a-1}(\ell),\ell),\ell\big).$$

Then

$$z(\ell)=\frac{\mu_j(g^{a-1}(\ell))}{q_j(\ell)}\left|\frac{q_j(\ell)}{\mu_j(g^{a-1}(\ell))}\right|^{1+i\alpha_j},$$

where $1+i\alpha_j=\mu_j/\Re\mu_j$. Set

$$\theta_j(\ell)=\ell_j-\frac{1}{q_j(\ell)},\quad \ell\in\Omega.$$

It is also shown in [Currey 1992, Lemma 2.1] that the function $\theta_j(\ell)$ depends only upon $\ell_1,\ell_2,\ldots,\ell_{j-1}$. It follows from this, from the definition of the substitutions $z_a=\xi_a(t,\ell)$ [Currey 1992, p. 263], and from Proposition 1.3.6 and Lemma 1.3.7 that for each $1\le a\le 2d$, both $\xi_a(t,\ell)$ and $\Phi_a(z,\ell)$ depend only on $\ell_1,\ell_2,\ldots,\ell_{e_a}$. Thus the following is immediate.

Corollary 1.3.8. *For each $1\le j\le n$ and for z fixed, $P_j(z,\cdot)$ and P_j^* depend only upon $\ell_1,\ell_2,\ldots,\ell_j$.*

We now proceed with more technical results aimed at a better understanding of the structure of Ω as a bundle over the cross-section Σ. If $j=e_a\in\boldsymbol{e}$ but $j\notin\imath\cup\varphi$, we already know that $P_j(z,\ell)=z_a$. What is needed is a better understanding of the functions $Q_j(t,\ell)$, and hence the functions $P_j(z,\ell)$, in the cases where $j\in\imath\cup\varphi$. This will be our present focus.

Lemma 1.3.9. *Let $1\le j\le n$ be an index with $j\notin I$, $j\in\boldsymbol{e}$, and $j+1\notin\boldsymbol{e}$. Then, for any $\ell\in\Omega_{\boldsymbol{e},j}$,*

(i) $\mathfrak{s}^{\ell}_{j''}\subset\ker(\gamma_j)$, *and*

(ii) *if $j=j_k$, then $\mathfrak{s}^{\ell}_{i_k'}\subset\ker(\gamma_j)$.*

Proof. Let $1\le k\le d$ with $j\in\{i_k,j_k\}$, and fix $\ell\in\Omega$. From the definition of i_k and j_k, we have $Y(\ell)\in\mathfrak{h}_{k-1}(\ell)\cap\mathfrak{s}_{i_k}$ and $X(\ell)\in\mathfrak{h}_{k-1}(\ell)\cap\mathfrak{s}_{j_k}$ so that $X(\ell)=Z_j \mod \mathfrak{s}_{j-1}$, $Y(\ell)=Z_{i_k}\mod\mathfrak{s}_{i_k-1}$, and $\ell[X(\ell),Y(\ell)]\neq 0$. Moreover, we have $Z(\ell)\in\mathfrak{s}(\ell)$, such that $\mathfrak{s}_{j''}=\mathfrak{s}_{j'}+\mathbb{C}\text{-span}\{Z_j,Z(\ell)\}$.

To prove part (i), assume that $j=i_k$. If $V\in\mathfrak{s}^{\ell}_{j''}$,

$$\ell[V,[X(\ell),Y(\ell)]]=0\quad\text{and}\quad \ell[Z(\ell),[X(\ell),V]]=0.$$

By the Jacobi identity it follows that

$$\ell[X(\ell),[V,Z(\ell)]]=0.$$

Since $j\notin I$, this can only happen if $\gamma_j(V)=0$. If $j=j_k$, the proof is the same, with $Y(\ell)$ and $X(\ell)$ reversing roles. This proves part (i).

Now to prove (ii), assume that $j=j_k$ and let $V\in\mathfrak{s}^{\ell}_{i_k'}$. Then $V\in\mathfrak{h}_{k-1}(\ell)$ and so

$$[V,X(\ell)]=\gamma_j(V)\ Z_j+W,\qquad [V,Y(\ell)]=\gamma_{i_k}(V)\ Z_{i_k}+U,$$

with $W \in \mathfrak{h}_{k-1}(\ell) \cap \mathfrak{s}_{j-1}$ and $U \in \mathfrak{h}_{k-1}(\ell) \cap \mathfrak{s}_{i_k-1}$. Hence, from the Jacobi identity, we get

$$0 = \ell[V, [X(\ell), Y(\ell)]] = (\gamma_j(V) + \gamma_{i_k}(V))\ell[X(\ell), Y(\ell)],$$

so that $\gamma_j(V) = -\gamma_{i_k}(V)$. (Since $\gamma_j(V)$ is not real, it follows that $i_k{}'' - i_k{}' = 2$.)

Now referring to the cases described in Section 1.2, the proof here branches into several cases:

(a) Case (1.2.0) or Case (1.2.1) holds for k: We have $[X(\ell), Y(\ell)] \in \mathfrak{s}_{i_k'}$. Since $V \in \mathfrak{s}^{\ell}_{i_k'}$, we may repeat the same argument given for part (i) verbatim.

(b) Case (1.2.2) holds for k: Here $i_k - 1 = j_r$ with $r \le k-1$ and we have $X_r(\ell) \in \mathfrak{s}_{i_k} \cap \mathfrak{g}$ and $\tilde{X}_r(\ell) \in \mathfrak{s}_{i_k} \cap \mathfrak{g}$ such that $\mathfrak{s}_{i_k} = \mathfrak{s}_{i_k-2} +$ span$\{X_r(\ell), \tilde{X}_r(\ell)\}$ and such that $X_r(\ell) \notin \mathfrak{h}_r(\ell)$, $\tilde{X}_r(\ell) \in \mathfrak{h}_r(\ell)$. Now $V \in \mathfrak{h}_r(\ell)$, so $[V, \mathfrak{h}_r(\ell)] \subset \mathfrak{h}_r(\ell)$. But if $\gamma_{i_k}(V) \ne 0$, then $[V, \tilde{X}_r(\ell)] = a\tilde{X}_r(\ell) + bX_r(\ell) + W$, where $W \in \mathfrak{s}_{i_k-2}$ and $b \ne 0$. This would mean that $\mathfrak{s}_{j_r''} =$ span$\{[V, \tilde{X}_r(\ell)], \tilde{X}_r(\ell)\} + \mathfrak{s}_{j_r'} \subset \mathfrak{h}_r(\ell) + \mathfrak{s}_{j_r'}$, which contradicts the definition of $j_r = \min\{1 \le j \le n \mid \mathfrak{h}_{r-1}(\ell) \cap \mathfrak{s}_j \not\subset \mathfrak{h}_r(\ell)\}$.

(c) Case (1.2.4) holds for k: We have $X_k(\ell)$ and $\tilde{X}_k(\ell)$ belonging to $\mathfrak{s}_{j''} \cap \mathfrak{g}$ with $\mathfrak{s}_{j''} =$ span$\{X_k(\ell), \tilde{X}_k(\ell)\} + \mathfrak{s}_{j'}$, $\ell[X_k(\ell), Y_k(\ell)] \ne 0$, and $\ell[\tilde{X}_k(\ell), Y_k(\ell)] = 0$. This means that $X_k(\ell) \notin \mathfrak{h}_k(\ell)$, and $\tilde{X}_k(\ell) \in \mathfrak{h}_{k+1}(\ell)$, the latter because, by virtue of our assumption that $j+1 \notin \boldsymbol{e}$, we have $j_{k+1} > j+1$. Now $\gamma_j(V) = 0$ if and only if $\gamma_j(\rho_{k+1}(V, \ell)) = 0$, and $\rho_{k+1}(V, \ell)$ belongs to $\mathfrak{h}_{k+1}(\ell)$. Hence if $\gamma_j(\rho_{k+1}(V, \ell)) \ne 0$, then $[\rho_{k+1}(V, \ell), \tilde{X}_k(\ell)] = a\tilde{X}_k(\ell) + bX_k(\ell) \mod \mathfrak{s}_{j'}$, where $b \ne 0$. This would imply that $\mathfrak{s}_j \subset \mathfrak{h}_k(\ell) + \mathfrak{s}_{j'}$, contradicting the definition of $j_k = j$.

(d) Case (1.2.5) holds for k: This case is similar to (c), and we omit the details. □

Lemma 1.3.10. *Suppose given j with $1 \le j \le n$ and $j-1 \in I$, and k $1 \le k \le d$ with $i_k < j$. Assume further that if $j = j_r$ for some $r < k$, then $j \notin I$ and $j+1 \notin \boldsymbol{e}$. Then, for $0 \le r \le k-1$ and for each $V \in \mathfrak{s}$, the function $\ell \to \ell[Z_j, \rho_r(V, \ell)]$ defined on $\Omega_{\boldsymbol{e},j}$ is of the form*

$$\ell[Z_j, \rho_r(V, \ell)] = \gamma_j(\rho_r(V, \ell))\ell_j + u(\ell),$$

where $u(\ell)$ depends only upon $\ell_1, \ell_2, \dots, \ell_{j-1}$.

Proof. We proceed by induction on r, the result being clear for $r = 0$. Assume that $r > 0$ and that the result holds for $r-1$. This means in particular that we may assume that

$$\ell[Z_j, \rho_{r-1}(V, \ell)] = \gamma_j(\rho_{r-1}(V, \ell))\ell_j + u_0(\ell_1, \ell_2, \dots, \ell_{j-1}).$$

By our hypothesis and the properties of sequence pairs we have $i_r < j$, and also $j+1 \ne j_r$ if $j \notin I$. We therefore have three cases: $j'' < j_r$, $j = j_r$, and $j \ge j_r''$.

Case 1: $j'' < j_r$. Here $\ell[Z_j, Y_r(\ell)] = \ell[\rho_{r-1}(Z_j, \ell), Y_r(\ell)] = 0$, so

$$\ell[Z_j, \rho_r(V, \ell)] = \ell[Z_j, \rho_{r-1}(V, \ell)] - c(\ell)\,\ell[Z_j, X_r(\ell)].$$

where

$$c(\ell) = \frac{\ell[\rho_{r-1}(V, \ell), Y_r(\ell)]}{\ell[X_r(\ell), Y_r(\ell)]}.$$

By Lemma 1.3.4, we have

$$\begin{aligned} Y_r(\ell) &= b_{r,1}(\ell)\rho_{r-1}(\Re Z_{i_r}, \ell) + b_{r,2}(\ell)\rho_{r-1}(\Im Z_{i_r}, \ell), \\ X_r(\ell) &= a_{r,1}(\ell)\rho_{r-1}(\Re Z_{j_r}, \ell) + a_{r,2}(\ell)\rho_{r-1}(\Im Z_{j_r}, \ell), \end{aligned}$$

where $a_{r,1}(\ell), a_{r,2}(\ell), b_{r,1}(\ell)$, and $b_{r,2}(\ell)$ depend only upon $\ell_1, \ell_2, \ldots, \ell_{i_r}$. It follows from these formulas and the induction hypothesis that $c(\ell)$ depends only upon $\ell_1, \ell_2, \ldots, \ell_{j-1}$. Also by induction we have

$$\begin{aligned} \ell[Z_j, \rho_{r-1}(\Re Z_{j_r}, \ell)] &= \gamma_j(\rho_{r-1}(\Re Z_{j_r}, \ell))\ell_j + v_1(\ell_1, \ell_2, \ldots, \ell_{j-1}), \\ \ell[Z_j, \rho_{r-1}(\Im Z_{j_r}, \ell)] &= \gamma_j(\rho_{r-1}(\Im Z_{j_r}, \ell))\ell_j + v_2(\ell_1, \ell_2, \ldots, \ell_{j-1}), \end{aligned}$$

and it follows that we have $u_1(\ell_1, \ell_2, \ldots, \ell_{j-1})$ such that

$$\ell[Z_j, X_r(\ell)] = \gamma_j(X_r(\ell))\ell_j + u_1(\ell_1, \ell_2, \ldots, \ell_{j-1}).$$

Hence

$$\begin{aligned} \ell[Z_j, \rho_r(V, \ell)] &= \gamma_j(\rho_{r-1}(V, \ell))\ell_j + u_0(\ell) - c(\ell)\left(\gamma_j(X_r(\ell))\ell_j + u_1(\ell)\right) \\ &= \gamma_j(\rho_r(V, \ell))\ell_j + u(\ell), \end{aligned}$$

where $u(\ell) = u_0(\ell) - c(\ell)u_1(\ell)$ depends only upon $\ell_1, \ell_2, \ldots, \ell_{j-1}$.

Case 2: $j = j_r$. Here $j \notin I$ and $j + 1 \notin \boldsymbol{e}$. By Remark 1.3.3 and Lemma 1.3.9, for each $\ell \in \Omega$, the image of $\rho_{r-1}(\cdot, \ell)$ is contained in $\ker(\gamma_j)$. This, combined with the induction hypothesis, implies that for any $V \in \mathfrak{s}$, the expression $\ell[Z_j, \rho_{r-1}(V, \ell)]$ depends only upon $\ell_1, \ell_2, \ldots, \ell_{j-1}$. Combining this with Lemma 1.3.4, we find that the expressions

$$c(\ell) = \frac{\ell[\rho_{r-1}(V, \ell), Y_r(\ell)]}{\ell[X_r(\ell), Y_r(\ell)]}, \qquad d(\ell) = \frac{\ell[\rho_{r-1}(V, \ell), X_r(\ell)]}{\ell[Y_r(\ell), X_r(\ell)]}$$

depend only upon $\ell_1, \ell_2, \ldots, \ell_{j-1}$, and hence that

$$\ell[Z_j, \rho_r(V, \ell)] = \ell[Z_j, \rho_{r-1}(V, \ell)] - c(\ell)\,\ell[Z_j, X_r(\ell)] - d(\ell)\ell[Z_j, Y_r(\ell)]$$

depends only upon $\ell_1, \ell_2, \ldots, \ell_{j-1}$ also. Since $\rho_r(V, \ell) \in \ker(\gamma_j)$, we are done with this case.

Case 3: $j \geq j_r''$. This is similar to Case 1, with an additional term that is handled in a way precisely analogous to the arguments in Case 1. We omit the details. □

Lemma 1.3.11. *Assume given*:

(a) *an index j with $1 \le j \le n$ and $j-1 \in I$, and such that, if $j = j_r$, then $j \notin I$ and $j+1 \notin \boldsymbol{e}$*;

(b) *indices $1 \le k_1, k_2, \dots, k_p \le d$ and $1 \le e_{a_1} \le e_{a_2} \le \cdots \le e_{a_p} \le j$ such that e_{a_s} is equal to one of i_{k_s} or j_{k_s}, for $1 \le s \le p$*;

(c) *for each $1 \le s \le p$, an element $V_s \in \mathfrak{s}$ such that for every $\ell \in \Omega_{\boldsymbol{e},\boldsymbol{j}}$, $\rho_{k_s-1}(V_s, \ell)$ belongs to $\mathfrak{s}^{\ell}_{e'_{a_s}}$.*

Then, for each $\ell \in \Omega_{\boldsymbol{e},\boldsymbol{j}}$,

$$\ell\big[[\cdots[[Z_j, \rho_{k_1-1}(V_1,\ell)], \rho_{k_2-1}(V_2,\ell)], \cdots], \rho_{k_p-1}(V_p,\ell)\big] = \prod_{s=1}^{p} \gamma_j(\rho_{k_s-1}(V_s,\ell))\,\ell_j + y(\ell),$$

where $y(\ell)$ depends only upon $\ell_1, \ell_2, \dots, \ell_{j-1}$.

Proof. Set $r_s = k_s - 1$, for $1 \le s \le p$. As in Lemma 1.3.5, we proceed by induction on $N = \sum_{s=1}^{p} k_s$, and by Lemma 1.3.10, we may assume that $p > 1$.

Suppose first that $r_1 = 0$. Writing $[Z_j, V_1] = \gamma_j(V_1) Z_j + W$ with $W \in \mathfrak{s}_{j-1}$, we apply induction to

$$\ell\big[[\dots[Z_j, \rho_{r_2}(V_2,\ell)], \dots], \rho_{r_p}(V_p,\ell)\big]$$

and Lemma 1.3.5 to

$$y_1(\ell) = \ell\big[[\dots[W, \rho_{r_2}(V_2,\ell)], \dots], \rho_{r_p}(V_p,\ell)\big],$$

obtaining that

$$\begin{aligned}
&\ell\big[[\cdots[[Z_j, V_1], \rho_{r_2}(V_2,\ell)], \cdots], \rho_{r_p}(V_p,\ell)\big] \\
&= \gamma_j(V_1)\left(\prod_{s=2}^{p} \gamma_j(\rho_{r_s}(V_s,\ell))\,\ell_j + y_0(\ell)\right) + \ell\big[[\dots[W, \rho_{r_2}(V_2,\ell)], \dots], \rho_{r_p}(V_p,\ell)\big] \\
&= \gamma_j(V_1)\prod_{s=2}^{p} \gamma_j(\rho_{r_s}(V_s,\ell))\,\ell_j + \gamma_j(V_1)\ y_0(\ell) + y_1(\ell).
\end{aligned}$$

Now suppose that $r_1 > 0$. From our assumption about the indices and the elements V_s, we have $\ell\big[[\dots[Y_{r_1}(\ell), \rho_{r_2}(V_2,\ell)], \dots], \rho_{r_p}(V_p,\ell)\big] = 0$. Thus

$$\begin{aligned}
\text{(1.3.4)}\quad &\ell\big[[\cdots[[Z_j, \rho_{r_1}(V_1,\ell)], \rho_{r_2}(V_2,\ell)], \cdots], \rho_{r_p}(V_p,\ell)\big] \\
&= \ell\big[[\cdots[[Z_j, \rho_{r_1-1}(V_1,\ell)], \rho_{r_2}(V_2,\ell)], \cdots], \rho_{r_p}(V_p,\ell)\big] \\
&\quad - c(\ell)\,\ell\big[[\cdots[[Z_j, X_{r_1}(\ell)], \rho_{r_2}(V_2,\ell)], \cdots], \rho_{r_p}(V_p,\ell)\big],
\end{aligned}$$

where

$$c(\ell) = \frac{\ell[\rho_{r_1-1}(V_1,\ell), Y_{r_1}(\ell)]}{\ell[X_{r_1}(\ell), Y_{r_1}(\ell)]}.$$

We now proceed in much the same way as in the proof of Lemma 1.3.5. Looking at the first term of the right-hand side of 1.3, we observe that the data

$$1 \le k_1 - 1, k_2, \dots, k_p \le d, \quad i_{k_1-1} < e_{a_2} < \cdots < e_{a_p}, \quad V_1, \dots, V_p$$

satisfy the hypothesis of this lemma, so by induction,

$$\ell\big[[\cdots[[Z_j, \rho_{r_1-1}(V_1,\ell)], \rho_{r_2}(V_2,\ell)], \cdots], \rho_{r_p}(V_p,\ell)\big] \\ = \gamma_j(\rho_{r_1-1}(V_1,\ell)) \prod_{s=2}^{p} \gamma_j(\rho_{r_s}(V_s,\ell))\ell_j + y_0(\ell),$$

where $y_0(\ell)$ depends only upon $\ell_1, \ell_2, \dots, \ell_{j-1}$.

Turning to the second term, we apply formulas Lemma 1.3.4(i) to conclude that $c(\ell)$ depends only upon $\ell_1, \ell_2, \dots, \ell_{i_{k_1-1}}$. We then observe that the data

$$1 \le k_1 - 1, k_2, \dots, k_p \le d, \quad i_{k_1-1} < e_{a_2} < \cdots < e_{a_p}, \quad \Re Z_{j_{k_1-1}}, V_2, \dots, V_p$$

satisfy the conditions for this lemma, and so, by induction,

$$\ell\big[[\cdots[[Z_j, \rho_{r_1-1}(\Re Z_{j_{k_1-1}},\ell)], \rho_{r_2}(V_2,\ell)], \cdots], \rho_{r_p}(V_p,\ell)\big] \\ = \gamma_j(\rho_{r_1-1}(\Re Z_{j_{r_1}},\ell)) \prod_{s=2}^{p} \gamma_j(\rho_{r_s}(V_s,\ell))\ell_j + y_1(\ell),$$

where $y_1(\ell)$ depends only upon $\ell_1, \ell_2, \dots, \ell_{j-1}$. An entirely similar formula holds involving $\Im Z_{j_{k_1-1}}$ instead of $\Re Z_{j_{k_1-1}}$ and a remainder $y_2(\ell)$ depending only upon $\ell_1, \ell_2, \dots, \ell_j$. Using the formula for $X_{r_1}(\ell)$ from Lemma 1.3.4, we can substitute the preceding into equation 1.3 to get

$$\begin{aligned} &\ell\big[[\cdots[[Z_j, \rho_{k_1-1}(V_1,\ell)], \rho_{k_2-1}(V_2,\ell)], \cdots], \rho_{k_p-1}(V_p,\ell)\big] \\ &\quad = \gamma_j(\rho_{r_1-1}(V_1,\ell)) \prod_{s=2}^{p} \gamma_j(\rho_{r_s}(V_s,\ell))\ell_j + y_0(\ell) \\ &\qquad - c(\ell)a_1(\ell)\Big(\gamma_j(\rho_{r_1-1}(\Re Z_{j_{r_1}},\ell)) \prod_{s=2}^{p} \gamma_j(\rho_{r_s}(V_s,\ell))\ell_j + y_1(\ell)\Big) \\ &\qquad - c(\ell)a_2(\ell)\Big(\gamma_j(\rho_{r_1-1}(\Im Z_{j_{r_1}},\ell)) \prod_{s=2}^{p} \gamma_j(\rho_{r_s}(V_s,\ell))\ell_j + y_2(\ell)\Big) \end{aligned}$$

$$= \gamma_j(\rho_{r_1-1}(V_1,\ell)) \prod_{s=2}^{p} \gamma_j(\rho_{r_s}(V_s,\ell))\,\ell_j + y_0(\ell)$$

$$-c(\ell)\gamma_j(X_{r_1}(\ell)) \prod_{s=2}^{p} \gamma_j(\rho_{r_s}(V_s,\ell))\,\ell_j - c(\ell)a_1(\ell)y_1(\ell) - c(\ell)a_2(\ell)y_2(\ell)$$

$$= \gamma_j(\rho_{r_1}(V_1,\ell)) \prod_{s=2}^{p} \gamma_j(\rho_{r_s}(V_s,\ell))\,\ell_j + y(\ell),$$

where $y(\ell) = y_0(\ell) - c(\ell)a_1(\ell)y_1(\ell) - c(\ell)a_2(\ell)y_2(\ell)$ depends only upon ℓ_1, ℓ_2, $\dots$, ℓ_{j-1}. This completes the proof. □

We now examine the functions Q_j, $1 \le j \le n$, in light of the preceding results. Observe that Lemma 1.3.11 applies to every index j belonging to $\imath \cup \varphi$, and recall that it is these indices that primarily concern us at present.

Fix a covering set $O \in F$. Choose $1 \le j \le n$ such that $j-1 \in I$, set

$$a = \min\{1 \le b \le 2d \mid e_b \ge j\},$$

and define $Q_j^\circ(t,\ell) = g^{a-1}(t,\ell)\ell\, Z_j$. We begin by computing $Q_j^\circ(t,\ell)$.

Lemma 1.3.12. *We have*

$$Q_j^\circ(t,\ell) = \mu_j(g^{a-1}(t,\ell))\,\ell_j + Y_j^\circ(t,\ell),$$

where $Y_j^\circ(0,\ell) \equiv 0$ *for every* $\ell \in O$. *Moreover,* $Y_j^\circ(t,\ell)$ *depends only upon* $\ell_1,\dots,\ell_{j-1}$, *unless* $j \in \boldsymbol{j}$ *and* $j'' \in \boldsymbol{e}$.

Proof. We compute in much the same way as Proposition 1.3.6, with the added information of subsequent lemmas. If $q = q_1, q_2, \dots, q_{a-1} \in \{0,1,2,\dots\}^{a-1}$ is a multi-index, we have

$$Q_j^\circ(t,\ell) = g^{a-1}(t,\ell)\ell\, Z_j = \sum_{q\in\{0,1,2,\dots\}^{a-1}} w_j(q,t,\ell),$$

where

$$w_j(q,t,\ell) = \frac{t^q}{q!}\big(\operatorname{ad}^* r_1(\ell)^{q_1}\ \operatorname{ad}^* r_2(\ell)^{q_2} \cdots \operatorname{ad}^* r_{a-1}(\ell)^{q_{a-1}}\,\ell\big) Z_j.$$

Fix a multi-index $q \ne (0,0,\dots,0)$ and write

$$(e_1,e_1,\dots,e_1,e_2,\dots,e_2,\dots,e_{a-1},\dots,e_{a-1}) = (e_{a_1},e_{a_2},\dots,e_{a_p}),$$

where on the left-hand side each index e_b is listed q_b times, for $1 \le b \le a-1$. For each $1 \le s \le p$, let $1 \le k_s \le d$ be such that $e_{a_s} \in \{i_{k_s}, j_{k_s}\}$. If $e_{a_s} = j_{k_s}$, then

$$\begin{aligned} r_{a_s}(\ell) &= \frac{Y_{k_s}(\ell)}{\left|\ell[Z_{e_{a_s}}, Y_{k_s}(\ell)]\right|} \\ &= \frac{b_{1,k_s}(\ell)}{\left|\ell[Z_{e_{a_s}}, Y_{k_s}(\ell)]\right|}\, \rho_{k_s-1}(\Re Z_{i_{k_s}}, \ell) + \frac{b_{2,k_s}(\ell)}{\left|\ell[Z_{e_{a_s}}, Y_{k_s}(\ell)]\right|}\, \rho_{k_s-1}(\Im Z_{i_{k_s}}, \ell). \end{aligned}$$

Similarly, if $e_{a_s} = i_{k_s}$,

$$r_{a_s}(\ell) = \frac{a_{1,k_s}(\ell)}{\left|\ell[Z_{e_{a_s}}, X_{k_s}(\ell)]\right|}\, \rho_{k_s-1}(\Re Z_{j_{k_s}}, \ell) + \frac{a_{2,k_s}(\ell)}{\left|\ell[Z_{e_{a_s}}, X_{k_s}(\ell)]\right|}\, \rho_{k_s-1}(\Im Z_{j_{k_s}}, \ell).$$

Substituting these expressions into the formula for $w_j(q, t, \ell)$ above we get, for $q \neq (0, 0, \ldots, 0)$,

$$\begin{aligned} w_j(q, t, \ell) = \frac{t^q}{q!} \sum_{\substack{c_s=1,2 \\ 1\le s\le p}} A(q, c_1, \ldots, c_p, \ell) \\ \cdot \left(\mathrm{ad}^*\rho_{k_1-1}(V_{c_1}, \ell)\ \mathrm{ad}^*\rho_{k_2-1}(V_{c_2}, \ell) \times \mathrm{ad}^*\rho_{k_p-1}(V_{c_p}, \ell)\ell\right) Z_j, \end{aligned}$$

where

$$V_{c_s} = \begin{cases} \Re Z_{j_{k_s}} & \text{if } e_{a_s} = i_{k_s} \text{ and } c_s = 1, \\ \Im Z_{j_{k_s}} & \text{if } e_{a_s} = i_{k_s} \text{ and } c_s = 2, \\ \Re Z_{i_{k_s}} & \text{if } e_{a_s} = j_{k_s} \text{ and } c_s = 1, \\ \Im Z_{i_{k_s}} & \text{if } e_{a_s} = j_{k_s} \text{ and } c_s = 2, \end{cases}$$

and where $A(q, c, \ell)$ is the product of the corresponding coefficients. Specifically,

$$A(q, c, \ell) = \prod_{s=1}^{p} A_s(q, c, \ell),$$

where

$$A_s(q, c_1, \ldots, c_p, \ell) = \begin{cases} \dfrac{a_{1,k_s}(\ell)}{\left|\ell[Z_{e_{a_s}}, X_{k_s}(\ell)]\right|} & \text{if } e_{a_s} = i_{k_s} \text{ and } c_s = 1, \\ \dfrac{a_{2,k_s}(\ell)}{\left|\ell[Z_{e_{a_s}}, X_{k_s}(\ell)]\right|} & \text{if } e_{a_s} = i_{k_s} \text{ and } c_s = 2, \\ \dfrac{b_{1,k_s}(\ell)}{\left|\ell[Z_{e_{a_s}}, Y_{k_s}(\ell)]\right|} & \text{if } e_{a_s} = j_{k_s} \text{ and } c_s = 1, \\ \dfrac{b_{2,k_s}(\ell)}{\left|\ell[Z_{e_{a_s}}, Y_{k_s}(\ell)]\right|} & \text{if } e_{a_s} = j_{k_s} \text{ and } c_s = 2. \end{cases}$$

By Lemma 1.3.4, $A(q, c, \ell)$ depends only upon $\ell_1, \ell_2, \ldots, \ell_{j-1}$. Turning next to the expression

$$\left(\mathrm{ad}^*\rho_{k_1-1}(V_{c_1}, \ell)\ \mathrm{ad}^*\rho_{k_2-1}(V_{c_2}, \ell) \cdots \mathrm{ad}^*\rho_{k_p-1}(V_{c_p}, \ell)\ell\right)Z_j,$$

we see that it can be written as

$$\prod_{s=1}^{p} \gamma_j(\rho_{k_s-1}(V_{c_s}, \ell))\ell_j + y_j(q, c, \ell).$$

We may apply Lemma 1.3.11 to this expression unless $j \in \boldsymbol{j}$ and $j'' \in \boldsymbol{e}$: for each multi-index $q \neq (0, 0, \ldots, 0)$ and c, $y_j(q, c, \ell)$ depends only upon $\ell_1, \ell_2, \ldots, \ell_{j-1}$. We obtain

$$w_j(q, t, \ell) = \frac{t^q}{q!} \sum_{\substack{c_s=1,2 \\ 1\le s\le p}} A(q, c, \ell) \left(\prod_{s=1}^{p} \gamma_j(\rho_{k_s-1}(V_{c_s}, \ell))\ell_j + y_j(q, c, \ell) \right),$$

and finally,

$$\begin{aligned} Q_j^\circ(t, \ell) &= \sum_q w_j(q, t, \ell) \\ &= \sum_q \frac{t^q}{q!} \sum_{\substack{c_s=1,2 \\ 1\le s\le p}} A(q, c, \ell) \left(\prod_{s=1}^{p} \gamma_j(\rho_{k_s-1}(V_{c_s}, \ell))\ell_j + y_j(q, c, \ell) \right) \\ &= \sum_q \frac{t^q}{q!} \sum_{\substack{c_s=1,2 \\ 1\le s\le p}} A(q, c, \ell) \prod_{s=1}^{p} \gamma_j(\rho_{k_s-1}(V_{c_s}, \ell))\ell_j \\ &\qquad\qquad + \sum_{q\neq(0,0,\ldots 0)} \frac{t^q}{q!} \sum_{\substack{c_s=1,2 \\ 1\le s\le p}} A(q, c, \ell) y_j(q, c, \ell) \\ &= \left(\sum_q \frac{t^q}{q!} \prod_{s=1}^{p} \gamma_j(r_{a_s}(\ell)) \right) \ell_j + \sum_{q\neq(0,0,\ldots 0)} \frac{t^q}{q!} Y_j^\circ(q, \ell) \\ &= \mu_j(g_1(t_1, \ell)g_2(t_2, \ell) \cdots g_{a-1}(t_{a-1}, \ell))\ell_j + Y_j^\circ(t, \ell), \end{aligned}$$

where $Y_j^\circ(t, \ell)$ satisfies the conditions of the lemma. This completes the proof. □

Note that $\mu_j(g_b(t, \ell)) = \exp(t_b\gamma_j(r_b(\ell))$ for $1 \le b \le a-1$, and from Lemmas 1.3.4 and 1.3.7, the function $\ell \to \gamma_j(r_b(\ell))$ depends only upon $\ell_1, \ldots, \ell_{j-1}$. Hence the function $\ell \to \mu_j(g^{a-1}(t, \ell)) = \mu_j(g_1(t, \ell)) \cdots \mu_j(g_{a-1}(t, \ell))$ depends only upon $\ell_1, \ldots, \ell_{j-1}$.

We now use this to describe $Q_j(t, \ell)$, for $1 \le j \le n$. Fix an index j such that $j-1 \in I$. The value of $\dim(\mathfrak{s}_{j'}^{\ell}/\mathfrak{s}_{j''}^{\ell})$ is constant on Ω (equal to 0, 1, or 2) and we denote it by d_j. If $d_j = 0$, that is, $j \notin \boldsymbol{e}$, then $Q_j^\circ(t,\ell) = Q_j(t,\ell)$. Suppose that $d_j = 1$; then $j = e_a \in \boldsymbol{e}$ and

$$Q(t,\ell) = Q^\circ(t, g_a(t_a,\ell)\ell).$$

Now $g_a(t_a,\ell)\ell\,|_{\mathfrak{s}_{j-1}} = \ell\,|_{s_{j-1}}$ and

$$(g_a(t_a,\ell)\ell)_j = \ell_j + t_a\,F(t_a\gamma_j(r_a(\ell)))\,\zeta_a(\ell),$$

where $F : \mathbb{R} \to \mathbb{R}$ is the real analytic function

$$F(x) = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots .$$

Recall the rational, relatively-invariant function

$$q_j(\ell) = \frac{\gamma_j(r_a(\ell))}{\zeta_a(\ell)}.$$

If $j \notin \varphi$, then $q_j(\ell) = 0$ for all $\ell \in \boldsymbol{\Omega}$ and one computes from the above that $(g_a(t_a,\ell)\ell)_j = \ell_j + t_a\zeta_a(\ell)$. If $j \in \varphi$, then q_j is nonvanishing on $\boldsymbol{\Omega}$, and

$$(g_a(t_a,\ell)\ell)_j = e^{t_a\gamma_j(r_a(\ell))}\,q_j(\ell)^{-1} + \theta_j(\ell),$$

where, as before, $\theta_j(\ell) = \ell_j - q_j(\ell)^{-1}$. Suppose that $d_j = 2$. Then both j and $j+1 = j''$ belong to $\boldsymbol{e}$, and

$$Q(t,\ell) = Q^\circ\big(t, g_a(t_a,\ell)g_{a+1}(t_{a+1},\ell)\ell\big).$$

We have

$$(g_a(t_a,\ell)g_{a+1}(t_{a+1},\ell)\ell)\,|_{\mathfrak{s}_{j-1}} = \ell\,|_{s_{j-1}}$$

and, because $\mathfrak{g}$ is exponential, $j, j+1 \notin \varphi$. It follows that

$$\big(g_a(t_a,\ell)g_{a+1}(t_{a+1},\ell)\ell\big)_j = \ell_j + t_a\zeta_a(\ell) + t_{a+1}\zeta_{a+1}(\ell).$$

Proposition 1.3.13. *Fix a covering set $O \in F$, and let $1 \le j \le n$ such that $j-1 \in I$. Then the function $Q_j(t,\ell)$ has the form*

$$\begin{cases} \mu_j(g^{a-1}(t,\ell))\ell_j + Y_j(t,\ell) & \text{if } j \notin \boldsymbol{e},\\ \mu_j(g^{a-1}(t,\ell))\,(\ell_j + t_a\zeta_a(\ell)) + Y_j(t,\ell) & \text{if } d_j = 1 \text{ and } j \notin \varphi,\\ \mu_j(g^{a-1}(t,\ell))\,(e^{t_a\gamma_j(r_a(\ell))}q_j(\ell)^{-1}) + Y_j(t,\ell) & \text{if } d_j = 1 \text{ and } j \in \varphi,\\ \mu_j(g^{a-1}(t,\ell))\,(\ell_j + t_a\zeta_a(\ell) + t_{a+1}\zeta_{a+1}(\ell)) + Y_j(t,\ell) & \text{if } d_j = 2. \end{cases}$$

Moreover, if $j \notin \boldsymbol{e}$, or if $j \in \imath \cup \varphi$, then $Y_j(t,\ell)$ depends only upon $\ell_1, \ell_2, \ldots, \ell_{j-1}$.

Proof. If $j \notin \boldsymbol{e}$ or if $\gamma_j(r_a(\ell)) = 0$, the formula holds with $Y_j = Y_j^\circ$, whereas if $j \in \varphi$, then $Q_j(t, \ell)$ has the indicated form with

$$Y_j(t, \ell) = \mu_j(g^{a-1}(t, \ell))\,\theta_j(\ell) + Y_j^\circ(t, \ell). \qquad \square$$

Combining these formulas with the substitutions of [Currey 1992, p. 263], we obtain the following, which will be useful in the sequel. Recall the definition of $g^{a-1}(\ell)$ given before Corollary 1.3.8; note that $g^{a-1}(\ell)$ depends only upon $\ell_1, \ell_2, \dots, \ell_{j-1}$. Recall that $\operatorname{sign} w = w/|w|$ for a nonzero complex number w.

Corollary 1.3.14. *Let Ω be an ultrafine layer with $P^* : \Omega \to \Sigma$ the natural projection onto its cross-section Σ. For each $1 \le j \le n$, there is a function $Y_j^*(\ell)$ depending only upon $\ell_1, \ell_2, \dots, \ell_{j-1}$ such that P^* is given by*

$$P_j^*(\ell) = \begin{cases} \mu_j(g^{a-1}(\ell))\ell_j + Y_j^*(\ell) & \text{if } j \notin \boldsymbol{e}, \\ 0 & \text{if } j \in \boldsymbol{e} \text{ but } j \notin \iota \cup \varphi, \\ \zeta_a(\ell) \operatorname{sign}(\mu_j(g^{a-1}(\ell)))i & \\ \qquad \cdot \Im\big(|\mu_j(g^{a-1}(\ell))|\zeta_a(\ell)^{-1}\ell_j + Y_j^*(\ell)\big) & \text{if } j \in \iota, \\ \dfrac{\mu_j(g^{a-1}(\ell))}{q_j(\ell)} \left|\dfrac{q_j(\ell)}{\mu_j(g^{a-1}(\ell))}\right|^{1+i\alpha_j} + Y_j^*(\ell) & \text{if } j \in \varphi. \end{cases}$$

1.4. *The local trivializations.* Let $\Omega \subset \Omega_{\boldsymbol{e},j}$ be an ultrafine layer with cross-section Σ and with the covering F of Lemma 1.2.1. Let F^* be the covering of Σ defined by $F^* = \{E = \Sigma \cap O \mid O \in F\}$. For each $E \in F^*$, set

$$\Omega_E = \bigcup \{\mathbb{O} \in \Omega/G \mid \mathbb{O} \cap E \neq \varnothing\} = (P^*)^{-1}(E).$$

It is evident that

$$(t, \lambda) \to Q(t, \lambda)$$

defines a diffeomorphism of Ω_E with $\mathbb{R}^{2d} \times E$. In this way we see that Q furnishes us with local trivializations of Ω/G, with fiber $\mathbb{R}^{2d}$. The local trivialization $\tilde{P}$ referred to above represents a simplification of the map Q, obtained by changing the fiber. Let $W = W_1 \times W_2 \times \cdots \times W_{2d}$ be the subset of $\mathbb{R}^{2d}$ defined by $W_a = \mathbb{R}$ if $e_a \notin \varphi$ and $W_a = (0, +\infty)$ if $e_a \in \varphi$. The description of Ω as a bundle over Σ with fiber W is given in [Currey 1992, Theorem 2.8]. We make this description more explicit here: we describe how the local trivialization can be obtained by a method of substitution, in a way that is analogous to the construction of the Pukánszky map $P(z, \ell)$.

Proposition 1.4.1. *Let W be the subset of $\mathbb{R}^{2d}$ defined as above. Let $O \in F$ be a covering set for the ultrafine layer Ω, let $E = O \cap \Sigma$, and let $\tilde{P} : W \times E \to \Omega_E$ be the local trivialization map for which $P^*(\tilde{P}(w, \lambda)) = \lambda$ for all $\lambda \in E$, as described in*

[Currey 1992, Theorem 2.8]. *Then there is an analytic function* $\psi : W \times \Sigma \to \mathbb{R}^{2d}$ *such that*

$$\tilde{P}(w,\lambda) = Q(\psi(w,\lambda),\lambda) \quad \textit{for } w \in W,\ \lambda \in \Sigma.$$

Set $g^a(w,\lambda) = g^a(\psi(w,\lambda),\lambda)$ *for* $1 \le a \le 2d$. *For* $1 \le a \le 2d$, *write* $j = e_a$ *and assume* $j-1 \in I$. *Then* ψ_a *satisfies the following.*

(a) *For each* $w \in W$, *the function* $\psi_a(w,\lambda)$ *depends only on* $\lambda_1, \lambda_2, \dots, \lambda_j$.

(b) *For each* $\lambda \in E$, *if* $d_j = 1$, *then*

$$\psi_a(w,\lambda) = \begin{cases} \zeta_a(\lambda)^{-1}\mu_j(g^{a-1}(w,\lambda))^{-1} w_a & \text{mod } (w_1,\dots,w_{a-1}) \quad \textit{if } j \notin \imath \cup \varphi, \\ \left|\mu_j(g^{a-1}(w,\lambda))\right|^{-1} w_a & \text{mod } (w_1,\dots,w_{a-1}) \quad \textit{if } j \in \imath, \\ \Re\left(\gamma_j(r_a(\lambda))\right)^{-1} \log w_a & \text{mod } (w_1,\dots,w_{a-1}) \quad \textit{if } j \in \varphi. \end{cases}$$

If $d_j = 2$, *then*

$$\begin{bmatrix} \psi_a(w,\lambda) \\ \psi_{a+1}(w,\lambda) \end{bmatrix} = A(w,\lambda) \begin{bmatrix} w_a \\ w_{a+1} \end{bmatrix} \quad \text{mod } (w_1,\dots,w_{a-1}),$$

where

$$|\det A(w,\lambda)| = \left|\mu_j(g^{a-1}(w,\lambda))^{-2}\right| = \left|\mu_j(g^{a-1}(w,\lambda))^{-1}\mu_{j+1}(g^{a-1}(w,\lambda))^{-1}\right|.$$

Proof. It is the inverse mapping $\Theta : (P^*)^{-1}(E) \to W \times E$ of $\tilde{P}$ that is described in [Currey 1992, Theorem 2.8]: Θ has the form $\Theta(\ell) = (w(\ell), P^*(\ell))$ where $w(\ell)$ is as follows. For $1 \le a \le 2d$ such that $j = e_a$ and $j-1 \in I$, if $d_j = 1$ then

$$w_a(\ell) = \begin{cases} \ell_j & \text{if } j \notin \imath \cup \varphi, \\ \Re\big(\text{sign}(\mu_j(s))^{-1}\zeta_a(\lambda)^{-1}\ell_j\big) & \text{if } j \in \imath, \\ |q_j(\ell)|^{-1} & \text{if } j \in \varphi. \end{cases}$$

Here $s \in G$ satisfies $s\lambda = \ell$. If $d_j = 2$, then

$$w_a(\ell) = \Re(\ell_j),\ w_{a+1}(\ell) = \Im(\ell_j).$$

The map ψ can therefore be obtained by the substitutions $t_a = \psi_a(w,\lambda)$, $1 \le a \le 2d$, as follows. First, let $j = e_1$. If $d_j = 1$ and $j \notin \imath \cup \varphi$, since $\mathfrak{g}$ is exponential and by virtue of condition (ii) of our chosen basis (page 101), we have $j \in I$. Setting $w_1 = Q_j(t_1,\lambda)$, then $\psi_1(w_1,\lambda)$ is obtained by solving for t_1 in terms of w_1 and λ. If $j \in \varphi$, then $w_1 = |q_j(g_1(t_1,\lambda)\lambda)|^{-1} = |\mu_j(g_1(t_1,\lambda))|$ (recall that $|q_j(\lambda)| = 1$). The desired formula for $\psi_1(w_1,\lambda)$ is again obtained by solving for t_1. If $d_j = 2$, setting $w_1 = \Re\big(Q_j(t_1,\lambda)\big)$ and $w_2 = \Im\big(Q_j(t_1,\lambda)\big)$, we get

$$w_1 + i w_2 = t_1\zeta_1(\lambda) + t_2\zeta_2(\lambda) + \lambda_j.$$

Hence
$$\begin{bmatrix}\psi_1(w,\lambda)\\ \psi_2(w,\lambda)\end{bmatrix} = Z(\lambda)^{-1}\begin{bmatrix} w_1-\Re(\lambda_j)\\ w_2-\Im(\lambda_j)\end{bmatrix},$$
where
$$Z(\lambda)=\begin{bmatrix}\Re(\zeta_1(\lambda)) & \Re(\zeta_2(\lambda))\\ \Im(\zeta_1(\lambda)) & \Im(\zeta_2(\lambda))\end{bmatrix}.$$
By Lemma 1.2.3, $|\det Z\,|=1$. This finishes the case $j=e_1$.

Suppose that $1<a\le 2d$ and that we have defined $\psi_1(w,\lambda)$, $\psi_2(w,\lambda)$, $\dots$, $\psi_{a-1}(w,\lambda)$, each of which satisfy conditions (a) and (b) of the proposition. For $j=e_a$, if $d_j=1$ and $j\notin\iota\cup\varphi$, let $w_a=Q_j(t_1,t_2,\dots,t_a,\lambda)$ and solve for t_a, while at the same time substituting $t_b=\psi_b(w,\lambda)$, $1\le b\le a-1$. Thus $\psi_a(w_1,w_2,\dots,w_a,\lambda)$ is obtained. If $j\in\iota$, set
$$\begin{aligned} w_a &= \Re\big(\operatorname{sign}(\mu_j(g^{a-1}(t,\lambda)))^{-1}\zeta_a(\lambda)^{-1}Q_j(t,\lambda)\big)\\ &= |\mu_j(g^{a-1}(t,\lambda))|t_a+\Re\big(|\mu_j(g^{a-1}(t,\lambda))|\zeta_a(\lambda)^{-1}\lambda_j\\ &\qquad\qquad + \operatorname{sign}(\mu_j(g^{a-1}(t,\lambda)))^{-1}\zeta_a(\lambda)^{-1}Y_j(t,\lambda)\big).\end{aligned}$$
It is evident that, solving for t_a and substituting $t_b=\psi_b(w,\lambda)$ for $1\le b\le a-1$, the desired form for $\psi_a(w_1,w_2,\dots,w_a,\lambda)$ is obtained. If $j\in\varphi$, one gets
$$w_a=|q_j(g(t,\lambda)\lambda)|^{-1}=|\mu_j(g^a(t,\lambda))|=|\mu_j(g^{a-1}(t,\lambda))|\,e^{t_a\Re\gamma_j(r_a(\lambda))},$$
from which $\psi_a(w,\lambda)$ is obtained by solving for t_a. Suppose that $d_j=2$. Making the substitution we get
$$w_a+iw_{a+1}=\mu_j(g^{a-1}(t,\lambda))\big(t_a\zeta_a(\lambda)+t_{a+1}\zeta_{a+1}(\lambda)+\lambda_j\big)+Y_j(t,\lambda),$$
and substituting $t_b=\psi_b(w,\lambda)$ for $1\le b\le a-1$,
$$t_a\zeta_a(\lambda)+t_{a+1}\zeta_{a+1}(\lambda)=\mu_j(g^{a-1}(w,\lambda)^{-1})\big(w_a+iw_{a+1}-Y_j(w,\lambda)\big)-\lambda_j.$$
Setting $\psi_a(w,\lambda)=t_a$ and $\psi_{a+1}(w,\lambda)=t_{a+1}$, we get
$$\begin{bmatrix}\psi_a(w,\lambda)\\ \psi_{a+1}(w,\lambda)\end{bmatrix}=Z(w,\lambda)^{-1}\left(M(w,\lambda)^{-1}\begin{bmatrix}w_a-\Re(Y_j(w,\lambda))\\ w_{a+1}-\Im(Y_j(w,\lambda))\end{bmatrix}-\begin{bmatrix}\Re\lambda_j\\ \Im\lambda_j\end{bmatrix}\right),$$
where
$$Z(w,\lambda)=\begin{bmatrix}\Re(\zeta_a(\lambda)) & \Re(\zeta_{a+1}(\lambda))\\ \Im(\zeta_a(\lambda)) & \Im(\zeta_{a+1}(\lambda))\end{bmatrix}$$
and
$$M(w,\lambda)=\begin{bmatrix}\Re(\mu_j(g^{a-1}(w,\lambda))) & -\Im(\mu_j(g^{a-1}(w,\lambda)))\\ \Im(\mu_j(g^{a-1}(w,\lambda))) & \Re(\mu_j(g^{a-1}(w,\lambda)))\end{bmatrix}.$$
Again by Lemma 1.2.3, $|\det Z(w,\lambda)|=1$, and also, as desired,
$$\det M(w,\lambda)=\big|\mu_j(g^{a-1}(w,\lambda))\big|^2. \qquad \square$$

Making the substitutions indicated above yields the following description of $\tilde{P}$. We use the notation $w^{a-1} = w_1, \dots, w_{a-1}$.

Proposition 1.4.2. *Let $1 \le j \le n$ be such that $j - 1 \in I$. Let $1 \le a \le 2d$ be defined by $e_{a-1} < j \le e_a$. There is an analytic function $Y_j(w, \ell)$, depending only upon $w_1, w_2, \dots, w_{a-1}$ and $\lambda_1, \lambda_2, \dots, \lambda_{j-1}$, such that $\tilde{P}_j(w, \lambda)$ has the following form, according to the cases below.*

(i) *$j \notin \boldsymbol{e}$. If $j < e_1$, then $\tilde{P}_j(w, \lambda) = \lambda_j$. If $j > e_1$, then*

$$\tilde{P}_j(w, \lambda) = \mu_j(g^{a-1}(w, \lambda))\, \lambda_j + Y_j(w, \lambda).$$

(ii) *$d_j = 1$.*

- *If $j \notin \imath \cup \varphi$, then $\tilde{P}_j(w, \lambda) = w_a$.*
- *If $j \in \imath$, then, with $c_j(w^{a-1}, \lambda) = \operatorname{sign}\big(\mu_j(g^{a-1}(w, \lambda))\big)\, \zeta_a(\lambda)$,*

$$\tilde{P}_j(w, \lambda) = c_j(w^{a-1}, \lambda)\, \big(w_a + i\Im\big(c_j(w^{a-1}, \lambda)^{-1} \mu_j(g^{a-1}(w, \lambda))\, \lambda_j\big)\big) + Y_j(w, \lambda).$$

- *If $j \in \varphi$, then*

$$\tilde{P}_j(w, \lambda) = \frac{\mu_j(g^{a-1}(w, \lambda))}{\big|\mu_j(g^{a-1}(w, \lambda))\big|^{1+i\alpha_j}}\, w_a^{1+i\alpha_j} q_j(\lambda)^{-1} + Y_j(w, \lambda),$$

where $1 + i\alpha_j = \gamma_j / \Re(\gamma_j)$.

(iii) *$d_j = 2$. Then $\tilde{P}_j(w, \lambda) = w_a + i w_{a+1}$.*

2. The Plancherel Measure

2.1. ***Computation of the canonical measure on an orbit.*** We now proceed to apply the results of Section 1 to harmonic analysis on an exponential Lie group G. Let $\mathfrak{s}$ be the complexification of $\mathfrak{g}$ and assume that we have chosen a basis $\{Z_1, Z_2, \dots, Z_n\}$ for $\mathfrak{s}$ satisfying conditions (i)–(iii) of page 101. We retain all other notations from Section 1 as well. We begin by computing the canonical measure on any coadjoint orbit [Pukánszky 1968] in terms of the data from Proposition 1.4.2. Set $\mu_{\boldsymbol{e}} = \prod_{j \in \boldsymbol{e}} \mu_j$.

Proposition 2.1.1. *Let Ω be an ultrafine layer with cross-section Σ. Fix $\lambda \in \Sigma$, let $\mathcal{O}_\lambda$ be the coadjoint orbit through λ and let β_λ be the canonical measure on $\mathcal{O}_\lambda$. Choose any covering set $E \in F^*$ that contains λ and let $\tilde{P} : W \times E \to \Omega_E$ be the local trivialization of Proposition 1.4.2. For any nonnegative Borel measurable function f on $\mathcal{O}_\lambda$, one has*

$$\int_{\mathcal{O}_\lambda} f \, d\beta_\lambda = \frac{c}{|P_{\boldsymbol{e}}(\lambda)|} \int_W f(\tilde{P}(w, \lambda))\, \big|\mu_{\boldsymbol{e}}(g(w, \lambda))\big|^{-1} dw,$$

where $c = (2\pi)^d \prod_{j \in \varphi} |1 + i\alpha_j|$.

Proof. From [Pedersen 1984, Lemma 2.1.3] and the definitions above, we have

$$\int_{\mathcal{O}_\lambda} f\, d\beta_\lambda = \frac{(2\pi)^d}{|P_{\boldsymbol{e}}(\lambda)|}\int_{\mathbb{R}^{2d}} f(g(t,\lambda)\lambda)\prod_{a<b}\bigl|\mu_{e_a}(\exp(t_b r_b(\lambda)))\bigr|^{-1}\,dt$$
$$= \frac{(2\pi)^d}{|P_{\boldsymbol{e}}(\lambda)|}\int_{W} f(\tilde{P}(w,\lambda))\prod_{a<b}\bigl|\mu_{e_a}(\exp(\psi_b(w,\lambda) r_b(\lambda)))\bigr|^{-1}|J_\psi(w,\lambda)|\,dw.$$

It remains to compute $\prod_{a<b}\bigl|\mu_{e_a}(\exp(\psi_b(w,\lambda) r_b(\lambda)))\bigr|^{-1}|J_\psi(w,\lambda)|$, and for this we refer to the description of the functions $\psi_a(w,\lambda)$ given in Proposition 1.4.1. If $j=e_a\in\boldsymbol{e}-\varphi$ and $d_j=1$, we have

$$\left|\frac{\partial\psi_a}{\partial w_b}\right| = \begin{cases}\bigl|\mu_j(g^{a-1}(w,\lambda))\bigr|^{-1} & \text{if } b=a,\\ 0 & \text{if } b>a.\end{cases}$$

If $j=e_a\in\boldsymbol{e}-\varphi$ with $j-1\in I$ and $d_j=2$, then for all $b>a+1$ we have $\partial\psi_a/\partial w_b=\partial\psi_{a+1}/\partial w_b=0$ and

$$\left|\det\begin{pmatrix}\partial\psi_a/\partial w_a & \partial\psi_a/\partial w_{a+1}\\ \partial\psi_{a+1}/\partial w_a & \partial\psi_{a+1}/\partial w_{a+1}\end{pmatrix}\right| = |\det A(w,\ell)|$$
$$= \bigl|\mu_j(g^{a-1}(w,\lambda))\bigr|^{-1}\bigl|\mu_{j+1}(g^{a-1}(w,\lambda))\bigr|^{-1}.$$

On the other hand, if $j=e_a\in\varphi$, then

$$\frac{\partial\psi_a}{\partial w_b} = \begin{cases}\bigl|\mu_j(g^{a}(w,\lambda))\bigr|^{-1}\bigl(\Re(\gamma_j(r_a(\lambda)))\bigr)^{-1} & \text{if } b=a,\\ 0 & \text{if } b>a.\end{cases}$$

Now, by [Currey 1992, Proposition 1.8], $j=e_a\in\boldsymbol{e}-\varphi$ implies $\gamma_j(r_a(\lambda))=0$, hence $\mu_j(g^{a-1}(w,\lambda))=\mu_j(g^a(w,\lambda))$. It follows that

$$|J_\psi(w,\lambda)| = \prod_{j\in\boldsymbol{e}-\varphi}\bigl|\mu_j(g^{a-1}(w,\lambda))\bigr|^{-1}\prod_{j\in\varphi}\bigl|\mu_j(g^{a}(w,\lambda))\bigr|^{-1}|\Re(\gamma_j(r_a(\lambda)))|^{-1}$$
$$= \prod_{j\in\boldsymbol{e}}\bigl|\mu_j(g^{a}(w,\lambda))\bigr|^{-1}\prod_{j\in\varphi}\bigl|\Re(\gamma_j(r_a(\lambda)))\bigr|^{-1}.$$

Finally we combine this with the fact that $\bigl|\Re(\gamma_j(r_a(\lambda)))\bigr|^{-1}=|1+i\alpha_j|$ to get

$$\prod_{a<b}\bigl|\mu_{e_a}(\exp(\psi_b(w,\lambda) r_b(\lambda)))\bigr|^{-1}|J_\psi(w,\lambda)| = \Bigl(\prod_{j\in\varphi}|1+i\alpha_j|\Bigr)|\mu_{\boldsymbol{e}}(g(w,\lambda))|^{-1}.$$

This completes the proof. □

2.2. *A natural measure on the cross-section.* As indicated in Section 1.1 and proved in [Currey 1992, Proposition 1.9], the ultrafine stratification has a total ordering for which the minimal layer is Zariski open and consists of coadjoint orbits of maximal dimension. Let Ω denote this minimal "generic" layer, with Σ its cross-section. In [Currey 1992, Theorem 2.9] it is shown that Σ is a submanifold of $\mathfrak{g}^*$; in this section we describe it in more detail.

Let $\mathbb{K}_j = \mathbb{R}$ if $j'' - j' = 1$ and $\mathbb{K}_j = \mathbb{C}$ if $j'' - j' = 2$. For each $1 \le j \le n$, let $\Sigma_j = \pi_j(\Sigma) = \{(\lambda_1, \lambda_2, \dots, \lambda_j) \mid \lambda \in \Sigma\}$. To obtain a picture of the cross-section Σ we describe Σ_j in terms of Σ_{j-1} and a subset of $\mathbb{K}_j$, for each j such that $j-1 \in I$.

Fix $1 \le j \le n$, $j-1 \in I$. For each $(\lambda_1, \lambda_2, \dots, \lambda_{j-1}) \in \Sigma_{j-1}$, set

$$L_j(\lambda_1, \lambda_2, \dots, \lambda_{j-1}) = \begin{cases} \mathbb{K}_j & \text{if } j \notin \boldsymbol{e}, \\ \{0\} & \text{if } j \in \boldsymbol{e} \text{ but } j \notin \imath \cup \varphi, \\ \mathbb{R} i \zeta_a(\lambda_1, \lambda_2, \dots, \lambda_{j-1}) & \text{if } j = e_a \in \imath, \\ \mathbb{S}^{j''-j'-1} + \theta_j(\lambda_1, \lambda_2, \dots, \lambda_{j-1}) & \text{if } j \in \varphi. \end{cases}$$

Proposition 2.2.1. *Let $\Sigma = P^*(\Omega)$ be the orbital cross-section in Ω. Fix $1 \le j \le n$ with $j-1 \in I$. For each $(\lambda_1, \lambda_2, \dots, \lambda_{j-1}) \in \Sigma_{j-1}$ let $U_j(\lambda_1, \lambda_2, \dots, \lambda_{j-1})$ be the subset of $\mathbb{K}_j$ defined by*

$$\Sigma_j = \big\{(\lambda_1, \lambda_2, \dots, \lambda_j) \mid (\lambda_1, \lambda_2, \dots, \lambda_{j-1}) \in \Sigma_{j-1} \\ \text{and } \lambda_j \in U_j(\lambda_1, \lambda_2, \dots, \lambda_{j-1})\big\}.$$

Then the set $U_j(\lambda_1, \lambda_2, \dots, \lambda_{j-1})$ is a dense open subset of $L_j(\lambda_1, \lambda_2, \dots, \lambda_{j-1})$.

Proof. Fix $1 \le j \le n$ with $j-1 \in I$, and for each $(\lambda_1, \lambda_2, \dots, \lambda_{j-1}) \in \Sigma_{j-1}$, let $W(\lambda_1, \lambda_2, \dots, \lambda_{j-1}) = \{(\lambda_1, \lambda_2, \dots, \lambda_{j-1}, x) \mid x \in K_j\}$. By Corollary 1.3.8, P_j^* can be regarded as a function on $\pi_j(\Omega)$, and it is clear that

$$U_j(\lambda_1, \lambda_2, \dots, \lambda_{j-1}) = P_j^*(W(\lambda_1, \lambda_2, \dots, \lambda_{j-1}) \cap \pi_j(\Omega)).$$

With $(\lambda_1, \lambda_2, \dots, \lambda_{j-1}) \in \Sigma_{j-1}$ fixed, let $h_j : \mathbb{K}_j \to L_j(\lambda_1, \lambda_2, \dots, \lambda_{j-1})$ be the map defined by

$$h_j(x) = \begin{cases} x & \text{if } j \notin \boldsymbol{e}, \\ 0 & \text{if } j \in \boldsymbol{e} \text{ but } j \notin \imath \cup \varphi, \\ i\zeta_j(\lambda_1,\lambda_2,\dots,\lambda_{j-1}) \Im\big(\zeta_j(\lambda_1,\lambda_2,\dots,\lambda_{j-1})^{-1} x\big) & \text{if } j \in \imath, \\ \dfrac{x - \theta_j(\lambda_1, \lambda_2, \dots, \lambda_{j-1})}{|x - \theta_j(\lambda_1, \lambda_2, \dots, \lambda_{j-1})|^{1+i\alpha_j}} \\ \qquad\qquad + \theta_j(\lambda_1, \lambda_2, \dots, \lambda_{j-1}) & \text{if } j \in \varphi. \end{cases}$$

It is easily seen that h_j is a continuous, open mapping; we claim that

$$P_j^*(\ell) = h_j(\ell_j) \quad \text{for each } \ell \in W(\lambda_1, \lambda_2, \ldots, \lambda_{j-1}) \cap \pi_j(\Omega).$$

To see this, observe first that for $\ell \in W(\lambda_1, \lambda_2, \ldots, \lambda_{j-1}) \cap \pi_j(\Omega)$,

$$\pi_{j-1}(\ell) = \pi_{j-1}(P^*(\ell)) = \pi_{j-1}(Q(\Phi_1(z(\ell), \ell), \ldots, \Phi_{a-1}(z(\ell), \ell), \ell)),$$

where $a = \min\{1 \le b \le 2d \mid j \le e_b\}$. This shows that $\Phi_1(z(\ell), \ell) = \cdots = \Phi_{a-1}(z(\ell), \ell) = 0$, and hence $g^{a-1}(\ell) = e$. Furthermore, $Y_j^*(\ell) = 0$ unless $j \in \varphi$, whence $Y_j^*(\ell) = \theta_j(\ell)$.

With this in mind we apply Corollary 1.3.14: if $j \notin \boldsymbol{e}$, then $P_j^*(\ell) = \ell_j$, while if $j \in \boldsymbol{e} - \imath \cup \varphi$, there is nothing to prove. If $j \in \imath$, then

$$P_j^*(\ell) = i\zeta_j(\lambda_1, \lambda_2, \ldots, \lambda_{j-1})\Im\big(\zeta_j(\lambda_1, \lambda_2, \ldots, \lambda_{j-1})^{-1}\ell_j\big),$$

while if $j \in \varphi$, then, recalling that $q_j(\ell)^{-1} = \ell_j - \theta_j(\lambda_1, \lambda_2, \ldots, \lambda_{j-1})$, the claim follows in this case as well.

Now, since Ω is dense and open in $\mathfrak{g}^*$, the intersection $W(\lambda_1, \lambda_2, \ldots, \lambda_{j-1}) \cap \pi_j(\Omega)$ is dense and open in $W(\lambda_1, \lambda_2, \ldots, \lambda_{j-1})$, and we have a dense open subset $V(\lambda_1, \lambda_2, \ldots, \lambda_{j-1})$ of $\mathbb{K}_j$ such that

$$U(\lambda_1, \lambda_2, \ldots, \lambda_{j-1}) = h_j(V(\lambda_1, \lambda_2, \ldots, \lambda_{j-1})).$$

Since h_j is a continuous and open mapping, the proof is complete. □

The picture of the cross-section thus obtained is therefore as a line bundle over circles. More specifically, for $j \in \boldsymbol{e}^c \cup \imath \cup \varphi$ such that $j - 1 \in I$, let S_j be defined by

$$S_j = \begin{cases} \mathbb{K}_j & \text{if } j \notin \boldsymbol{e}, \\ \mathbb{R} & \text{if } j \in \imath, \\ \mathbb{S}^0 = \{\pm 1\} & \text{if } j \in \varphi \text{ and } j'' - j' = 1, \\ \mathbb{S}^1 & \text{if } j \in \varphi \text{ and } j'' - j' = 2. \end{cases}$$

Recall that Σ is covered by the sets $E = \Sigma \cap O$, where $O \in F$, and that we denoted this covering by F^*. Fix $E \in F^*$ and define $\sigma = \sigma_E : E \to S = \prod_{j-1 \in I} S_j$ by

$$\sigma_j(\lambda) = \begin{cases} \lambda_j & \text{if } j \notin \boldsymbol{e}, \\ \Im(\zeta_a(\lambda)^{-1}\lambda_j) & \text{if } j = e_a \in \imath, \\ q_j(\lambda)^{-1} & \text{if } j \in \varphi \end{cases}$$

Corollary 2.2.2. *The mapping σ_E is a diffeomorphism between E and a dense, open subset of S.*

Proof. Clearly σ has rank $n-2d$, hence its image is an open submanifold of S. We claim that it is also dense in S. Let $s \in S$, and assume that we have a sequence $s(n)$ in $\sigma(\Sigma)$ such that for some $j \in \boldsymbol{e}^c \cup \imath \cup \varphi$, we have $s_i(n) \to s_i$ for all $i \in \boldsymbol{e}^c \cup \imath \cup \varphi$ with $i < j$. Let $\lambda(n) = \sigma^{-1}(s(n))$. If $j \notin \boldsymbol{e}$, by density of $U_j(\lambda_1(n), \ldots, \lambda_{j-1}(n))$ for each n, we can choose $s_j(n) \in U_j(\lambda_1(n), \ldots, \lambda_{j-1}(n))$ such that $s_j(n) \to s_j$. Similarly, if $j \in \imath$, we can choose $s_j(n) \in \overline{i\zeta_j(\lambda)} U_j(\lambda_1(n), \ldots, \lambda_{j-1}(n)) \subset \mathbb{R}$ such that $s_j(n) \to s_j$, and if $j \in \varphi$, we can choose $s_j(n) \in U_j(\lambda_1(n), \ldots, \lambda_{j-1}(n)) - \theta_j(\lambda)$ such that $s_j(n) \to s_j$. □

Now let m be Lebesgue measure on S, and define the Borel measure μ on Σ by $\mu(A) = m(\sigma_E(A \cap E))$. We claim that the measure μ is independent of the choice of the covering set $E \in F^*$.

Note first that, by the constructions of [Currey 1992] (see, for example, remarks preceding 2.4 of that reference), if O_1 and O_2 are any two elements of F and $\zeta_{a,1}$ and $\zeta_{a,2}$ are the functions on O_1 and O_2 (respectively) with values in S^1 associated with the index e_a and as defined on page 107, then $\zeta_{a,1}(\ell) = \pm\zeta_{a,2}(\ell)$ for each $\ell \in O_1 \cap O_2$. Now let E_1 and E_2 be any two elements of F^*; the preceding observation shows that if A is a Borel subset of Σ, then

$$m(\sigma_{E_1}(A \cap E_1 \cap E_2)) = m(\sigma_{E_2}(A \cap E_1 \cap E_2)).$$

Let p be a polynomial function on $\mathfrak{g}^*$ such that $E_2 = \{\lambda \in \Sigma \mid p(\lambda) \neq 0\}$. Then

$$\sigma_{E_1}(A \cap E_1 \cap E_2^c) = \sigma_{E_1}(A \cap E_1) \cap \{s \in \sigma_{E_1}(E_1) \mid p(\sigma_{E_1}^{-1}(s)) = 0\},$$

and hence

$$m(\sigma_{E_1}(A \cap E_1)) = m(\sigma_{E_1}(A \cap E_1 \cap E_2)).$$

Applying the same argument with E_1, E_2 reversed, we conclude that

$$m(\sigma_{E_1}(A\cap E_1)) = m(\sigma_{E_1}(A\cap E_1\cap E_2)) = m(\sigma_{E_2}(A\cap E_1\cap E_2)) = m(\sigma_{E_2}(A\cap E_2)).$$

Thus the claim is verified. We shall use the simplified notation $d\mu(\lambda) = d\lambda$.

Lemma 2.2.3. *Let $1 \leq j \leq n$ such that $j-1 \in I$ and $j \notin \boldsymbol{e}$. Let $0 \leq k \leq d$, and let $V \in \mathfrak{g}$.*

(i) *The function $\ell \to \gamma_j(\rho_k(V, \ell))$ on Ω depends only upon $\ell_1, \ell_2, \ldots, \ell_{j-1}$.*

(ii) *There is a function $\upsilon(\ell)$ on Ω, depending only upon $\ell_1, \ell_2, \ldots, \ell_{j-1}$, such that*

$$\ell[Z_j, \rho_k(V, \ell)] = \gamma_j(\rho_k(V, \ell))\ell_j + \upsilon(\ell) \quad \textit{for each } \ell \in \Omega.$$

The proof of this lemma is quite similar to that of Lemmas 1.3.7 and 1.3.10 (see also [Currey 1991, Lemma 2.3]) and is therefore omitted here.

By [Duflo and Raïs 1976, Lemme 5.2.2], the stabilizer algebra $\mathfrak{g}(\ell)$ is abelian for each $\ell \in \Omega$. Since the roots of the action of $\mathfrak{g}(\ell)$ on $\mathfrak{g}/\mathfrak{g}(\ell)$ are already of the form $\pm\nu_1, \cdots \pm\nu_d$, it follows that $G(\ell)$ is contained in the kernel of Δ. This allows the following. Fix $\mathbb{O} \in \mathfrak{g}^*/G$ with parameter $\lambda \in \Sigma$, let β_λ denote the canonical measure on $\mathbb{O}$, and let $\tilde{\beta}_\lambda$ denote the corresponding measure on $G/G(\lambda)$. Given any positive, Δ^{-1} relatively invariant function ψ on $\mathfrak{g}^*$, we have

$$\psi(\lambda) \int_{\mathbb{O}} f(\ell)\psi(\ell)^{-1} d\beta_\lambda(\ell) = \int_{G/G(\lambda)} f(a\lambda)\Delta(a) d\tilde{\beta}_\lambda(\overline{a}).$$

Hence we have defined a relatively invariant measure on $\mathbb{O}$ independent of the choice of ψ. In particular, the relatively invariant Borel measure ω_λ on $\mathbb{O}$ given by

$$\int_{\mathbb{O}} f\, d\omega_\lambda = r_\psi(\lambda) \int_{\mathbb{O}} f(\ell)\psi(\ell)^{-1} d\beta_\lambda(\ell),$$

where

$$r_\psi(\lambda) = \frac{|P_{\boldsymbol{e}}(\lambda)|\; \psi(\lambda)}{(2\pi)^d \prod_{j\in\varphi} |1+i\alpha_j|},$$

is independent of the choice of ψ. Choose any covering set $E \in F^*$ that contains λ and let $\tilde{P} : W \times E \to \Omega_E$ be the local trivialization of Proposition 1.4.2. Then Proposition 2.1.1 yields

$$\begin{aligned}
\int_{\mathbb{O}} f\, d\omega_\lambda &= r_\psi(\lambda) \int_{\mathbb{O}} f(\ell)\psi(\ell)^{-1} d\beta_\lambda(\ell) \\
&= \psi(\lambda) \int_W f(\tilde{P}(w,\lambda))\, \psi(\tilde{P}(w,\lambda))^{-1} \left|\mu_{\boldsymbol{e}}(g(w,\lambda))\right|^{-1} dw \\
&= \psi(\lambda) \int_W f(\tilde{P}(w,\lambda))\, \psi(g(w,\lambda)\lambda)^{-1} \left|\mu_{\boldsymbol{e}}(g(w,\lambda))\right|^{-1} dw \\
&= \int_W f(\tilde{P}(w,\lambda))\, \Delta(g(w,\lambda)) \left|\mu_{\boldsymbol{e}}(g(w,\lambda))\right|^{-1} dw \\
&= \int_W f(\tilde{P}(w,\lambda)) \prod_{j\notin \boldsymbol{e}} \left|\mu_j(g(w,\lambda))\right| dw.
\end{aligned}$$

We sum up these observations:

Proposition 2.2.4. *Let $\mathbb{O}$ be a coadjoint orbit in Ω with parameter $\lambda \in \Sigma$, and let ω_λ be the relatively invariant measure defined by*

$$\int_{\mathbb{O}} f\, d\omega_\lambda = \frac{|P_{\boldsymbol{e}}(\lambda)|}{(2\pi)^d \prod_{j\in\varphi} |1+i\alpha_j|} \int_{G/G_\lambda} f(a\lambda)\, \Delta(a)\, d\tilde{\beta}_\lambda(\overline{a}).$$

Choose any covering set $E \in F^*$ that contains λ and let $\tilde{P} : W \times E \to \Omega_E$ be the local trivialization of Proposition 1.4.2. Then for any nonnegative Borel measurable *function f on $\mathbb{O}$, we have*

$$\int_{\mathbb{O}} f \, d\omega_\lambda = \int_W f(\tilde{P}(w, \lambda)) \prod_{j \notin \boldsymbol{e}} \left|\mu_j(g(w, \lambda))\right| dw.$$

We are now ready for the main result of this paper.

Theorem 2.2.5. *For any nonnegative measurable function h on $\mathfrak{g}^*$,*

$$\int_{\mathfrak{g}^*} h(\ell) \, d\ell = \int_\Sigma \int_{\mathbb{O}_\lambda} h(\ell) \, d\omega_\lambda(\ell) \, d\lambda.$$

Proof. Fix $E \in F^*$ and set $S_E = \sigma_E(E) \subset S$. Let $\tilde{P}$ be the associated trivialization of Ω_E and write $\tilde{P}(w, s) = \tilde{P}(w, \sigma^{-1}(s))$ and $g(w, s) = g(w, \sigma^{-1}(s))$, for $w \in W$, $s \in S_E$. It is enough to show that

$$\int_{\Omega_E} h(\ell) \, d\ell = \int_{S_E} \int_W h(\tilde{P}(w, s)) \prod_{j \notin \boldsymbol{e}} \left|\mu_j(g(w, s))\right| dw \, dm(s).$$

A straightforward computation, based upon the formulas of Proposition 1.4.2 and Corollary 2.2.2, shows that $\tilde{P}_j(w, s)$ is given as follows. Assume that $j - 1 \in I$, and define the index $a = a(j)$ as before. If $j \notin \boldsymbol{e}$, then

$$\tilde{P}_j(w, s) = \mu_j(g^{a-1}(w, s)) s_j + Y_j(w, s),$$

where $g^{a-1}(w, s)$ and $Y(w, s)$ depend only upon $w_1, \ldots, w_{a-1}$ and the s_i, for $i < j$ with $i \in \boldsymbol{e}^c \cup \imath \cup \varphi$. If $j = e_a \in \boldsymbol{e}$ but $j \notin \imath \cup \varphi$, then

$$\tilde{P}_j(w, s) = \begin{cases} w_a & \text{if } j'' - j' = 1, \\ w_a + i w_{a+1} & \text{if } j'' - j' = 2. \end{cases}$$

If $j \in \imath$, then

$$\tilde{P}_j(w, s) = c_j(w, s)\big(w_a + i \left|\mu_j(g^{a-1}(w, s))\right| s_j + i \Im(Y_j(w, s))\big),$$

while if $j \in \varphi$, then

$$\tilde{P}_j(w, s) = \frac{\mu_j(g^{a-1}(w, s))}{|\mu_j(g^{a-1}(w, s))|^{1+i\alpha_j}} \, w_a^{1+i\alpha_j} \, s_j + Y_j(w, s).$$

Here again $g^{a-1}(w, s)$ and $Y_j(w, s)$ depend only upon $w_1, \ldots, w_{a-1}$ and the s_i, for $i < j$ with $i \in \boldsymbol{e}^c \cup \imath \cup \varphi$. Given this explicit description of $\tilde{P}$, it follows from the change of variables theorem in calculus that

$$\int_{\Omega_E} h(\ell) \, d\ell = \int_{S_E} \int_W h(\tilde{P}(w, s)) J(w, s) \, dw \, dm(s),$$

where

$$J(w,s)=\prod_{\substack{j-1\in I\\ j\notin \boldsymbol{e}}}\left|\mu_j(g^{a(j)-1}(w,s))\right|^{j''-j'}\prod_{j\in\imath}\left|\mu_j(g^{a(j)-1}(w,s))\right|\prod_{\substack{j=e_a\in\varphi\\ j''-j'=2}} w_a.$$

It remains for us to simplify the expression $J(w,s)$. By Lemma 2.2.3,

$$\prod_{\substack{j-1\in I\\ j\notin \boldsymbol{e}}}\left|\mu_j(g^{a(j)-1}(w,s))\right|^{j''-j'}=\prod_{\substack{j\notin \boldsymbol{e}\\ d_j=0}}\left|\mu_j(g(w,s))\right|.$$

By Lemma 1.3.9(i) and the fact that $|\mu_j|=|\mu_{j''}|$, we have

$$\prod_{j\in\imath}\left|\mu_j(g^{a(j)-1}(w,s))\right|=\prod_{\substack{j\notin \boldsymbol{e}\\ j-1\in\imath}}\left|\mu_j(g(w,s))\right|.$$

Finally, if $j=e_a\in\varphi$, we have observed in the proof of Proposition 1.4.2 that $w_a=\left|\mu_j(g^a(w,s))\right|$ and by Lemma 1.3.9, $\mu_j(g^a(w,s))=\mu_j(g(w,s))$. Again using the fact that $|\mu_j|=|\mu_{j''}|$,

$$\prod_{\substack{j=e_a\in\varphi\\ j''-j'=2}} w_a=\prod_{\substack{j\notin \boldsymbol{e}\\ j-1\notin I\\ j-1\in\varphi}}\left|\mu_j(g(w,s))\right|.$$

Hence

$$J(w,s)=\prod_{j\notin \boldsymbol{e}}\left|\mu_j(g(w,s))\right|.$$

This completes the proof. □

We now show how this gives a natural and explicit computation of the Plancherel measure. For each $\lambda\in\Sigma$, let $\mathfrak{b}(\lambda)=\mathfrak{h}_d(\lambda)\cap\mathfrak{g}$ with $B(\lambda)=\exp(\mathfrak{b}(\lambda))$ and let $\pi_\lambda=\operatorname{ind}_{B(\lambda)}^G(\chi_\lambda)$ be the representation induced from the character χ_λ of $B(\lambda)$ with differential $i\lambda$. As is well-known, π_λ is irreducible, and it is clear that $\{\pi_\lambda,\mathcal{H}_\lambda\}_{\lambda\in\Sigma}$ is a measurable field of irreducible representations. From the construction of $\mathfrak{h}_d(\lambda)$ and the fact that $G(\lambda)\subset\ker\Delta$, it follows that $B(\lambda)$ is contained in $\ker\Delta$. Thus we can define the positive, self-adjoint operator D_λ on (a dense subset of) $\mathcal{H}_\lambda$ by $D_\lambda f(a)=\Delta(a)f(a)$.

Now let ψ be any positive Borel function on $\mathfrak{g}^*$ satisfying $\psi(a\ell)=\Delta(a)^{-1}\psi(\ell)$ for $\ell\in\mathfrak{g}^*$, $a\in G$. For each $\lambda\in\Sigma$, let $A_{\psi,\lambda}$ be the densely defined operator on $\mathcal{H}_\lambda$ defined by $A_{\psi,\lambda}f(a)=\psi(a\lambda)^{1/2}f(a)$. Let m_ψ be the measure on $\mathfrak{g}^*/G$ given by

$$\int_{\mathfrak{g}^*}h(\ell)\psi(\ell)d\ell=\int_{\mathfrak{g}^*/G}\int_{\mathcal{O}}h(\ell)d\beta_{\mathcal{O}}(\ell)dm_\psi(\mathcal{O}).$$

As is shown in [Duflo and Raïs 1976], the Plancherel measure is $A_{\psi,\lambda}^{-2} dm_\psi(\mathbb{O}_\lambda)$. But it is clear that $\psi(\lambda)A_{\psi,\lambda}^{-2} = D_\lambda$ and from Proposition 2.2.4 and Theorem 2.2.5, an easy calculation shows that $dm_\psi(\mathbb{O}_\lambda) = r_\psi(\lambda) d\lambda$. Hence

$$A_{\psi,\lambda}^{-2} dm_\psi(\mathbb{O}_\lambda) = K_\lambda \, d\lambda,$$

where

$$K_\lambda = \frac{|P_{\boldsymbol{e}}(\lambda)|}{(2\pi)^{n+d} \prod_{j\in\varphi} |1+i\alpha_j|} D_\lambda.$$

We sum up:

Corollary 2.2.6. *Let G be an exponential solvable Lie group and fix a good basis for the complexified Lie algebra $\mathfrak{s} = \mathfrak{g}_c$. Then there is an algorithm for constructing, in a unique and natural way,*

(i) *an explicit cross-section Σ for almost all orbits in $\mathfrak{g}^*/G$,*

(ii) *a Lebesgue measure $d\lambda$ on Σ,*

(iii) *a measurable field $\{\pi_\lambda, \mathcal{H}_\lambda\}$ of irreducible representations (associated with the parameters λ via the Kirillov–Bernat correspondence) and a measurable field $\{K_\lambda\}_{\lambda\in\Sigma}$ of positive, self-adjoint, semi-invariant operators acting in $\mathcal{H}_\lambda$, such that*

$$\phi(e) = \int_\Sigma \mathrm{Tr}\big(K_\lambda^{1/2} \pi_\lambda(\phi) K_\lambda^{1/2}\big)\, d\lambda$$

for any smooth function ϕ on G having compact support.

For each $\lambda \in \Sigma$, one has

$$K_\lambda = \frac{|P_{\boldsymbol{e}}(\lambda)|}{(2\pi)^{n+d} \prod_{j\in\varphi} |1+i\alpha_j|} D_\lambda,$$

where D_λ is the multiplication operator determined by Δ on $\mathcal{H}_\lambda$.

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