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NICKOLAS J. MICHELACAKIS

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Let $\mathscr{AB}^n$ be the class of torsion-free, discrete groups that contain a normal, at most n-step, nilpotent subgroup of finite index. We give sufficient conditions for the fundamental group of a fibration $F \to T \to B$, with base B an infra-nilmanifold, to belong to $\mathscr{AB}^n$. Manifolds of this kind may, for example, appear as thin ends of nonpositively curved manifolds. We prove that if, in addition, we require that T be Kähler, then T possesses a flat Riemannian metric and the fundamental group $\pi_1(T)$ is necessarily a Bieberbach group. Further, we prove that a torsion-free, virtually polycyclic group that can be realised as the fundamental group of a compact, Kähler $\mathrm{K}(\pi, 1)$-manifold is necessarily Bieberbach.

1. Introduction

Torsion-free, discrete, cocompact subgroups of the group of affine motions of $\mathbb{R}^n$ were first studied by Bieberbach in 1912, and more recently by Charlap; they are called *Bieberbach groups*. They correspond precisely to the fundamental groups of compact manifolds endowed with a flat Riemannian metric [Charlap 1965], and such manifolds are finitely covered by flat tori [Bieberbach 1911].

L. Auslander [1960] and Lee and Raymond [1985] turned their attention to *almost-Bieberbach groups*, that is, torsion-free, discrete, cocompact subgroups of $G \rtimes C$, with C a maximal, compact subgroup of $\operatorname{Aut} G$ for G a simply connected, nilpotent Lie group. They succeeded in generalising much of Bieberbach's work. Malcev's equivalence [1949] shows that torsion-free, finitely generated, nilpotent groups correspond precisely to the fundamental groups of *nilmanifolds*, that is, compact manifolds of the form $M = G/N$, where G is a simply connected, nilpotent Lie group, and N a discrete subgroup. Theorem 3.2 shows that almost-Bieberbach groups correspond to *infra-nilmanifolds*, compact manifolds of the form G/Γ with G as above and Γ a discrete subgroup of $G \rtimes C$, where

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C is a maximal compact subgroup of $\operatorname{Aut} G$. We denote by $\mathcal{AB}^n$ the class of almost-Bieberbach groups whose maximal, normal, nilpotent subgroup is at most n-step nilpotent. We shall say that a group Γ admits an n-step almost-Bieberbach structure if and only if $\Gamma \in \mathcal{AB}^n$ and its maximal normal nilpotent subgroup is n-step nilpotent.

(We know from [Gromov 1981] and [Wolf 1968] that, among finitely generated groups, virtually nilpotent groups are precisely those groups that have polynomial growth. For details and precise definitions, see those works or [Tits 1981].)

We employ algebraic methods to study closed manifolds that fibre over infra-nilmanifolds. If $F \to T \to B$ is such a fibration, where F, T and B are all acyclic, the long homotopy exact sequence reduces to a group extension of the form

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(T) \longrightarrow \pi_1(B) \longrightarrow 1.$$

Manifolds of this type appear as thin ends of geometrically finite hyperbolic manifolds, which are an interesting subclass of nonpositively curved manifolds. More specifically, Apanasov and Xie [1997] proved that if $\Gamma \subset \mathcal{H}_n \rtimes U(n-1)$ is a torsion-free discrete group acting on the Heisenberg group $\mathcal{H}_n := \mathbb{C}^{n-1} \times \mathbb{R}$, the orbit space $\mathcal{H}_n / \Gamma$ is a Heisenberg manifold of zero Euler characteristic and a vector bundle over a compact manifold. Further, this compact manifold is finitely covered by a nilmanifold which is either a torus or a torus bundle over a torus. This generalises earlier results on almost flat manifolds concerning lattices in $\mathcal{H}_n \rtimes U(n-1)$ [Gromov 1978; Buser and Karcher 1981].

As mentioned above, groups in $\mathcal{AB}^n$ correspond to infra-nilmanifolds. In Section 2 we study extensions of the form

$$1 \longrightarrow G \longrightarrow \Gamma \longrightarrow K \longrightarrow 1,$$

with $K \in \mathcal{AB}^n$, to provide sufficient conditions under which Γ belongs to $\mathcal{AB}^n$. In particular, Proposition 2.2 guarantees the existence of an almost-Bieberbach structure on Γ provided G is a normal subgroup of Γ in a precise way. Proposition 2.4 does the same provided G lies in $\mathcal{AB}^n$ and the action of K on G respects some suitable minimal conditions.

In Section 3 we use the Johnson–Rees characterisation of fundamental groups of flat, Kähler [Johnson and Rees 1991], and projective [Johnson 1990] manifolds, and apply the Benson–Gordon theorem [1988] for the existence of a Kähler structure on a compact nilmanifold to show, in Theorem 3.3, that the existence of a Kähler structure on a special fibration as above implies the existence of a flat Riemann metric on T. In particular, in $\mathcal{AB}^n$, the classes of fundamental groups of Kähler and projective manifolds coincide, as shown in Corollary 3.4. Further, as a consequence of the Lefschetz hyperplane theorem and Bertini's theorem, this is a subclass of the class of fundamental groups of compact, closed, nonsingular projective surfaces.

Finally, in Section 4, we use a structure theorem concerning virtually polycyclic groups, proved in [Dekimpe and Igodt 1994], together with the results in [Arapura and Nori 1999], to prove, in Theorem 4.2, that a torsion-free, virtually polycyclic group can be realised as the fundamental group of a K(π, 1)-compact, Kähler manifold if and only if it is Bieberbach of a special kind, namely, its operator homomorphism is essentially complex.

2. Group extensions

A group N is said to be *nilpotent* if its upper central series

$$1 = N_0 \lhd N_1 = Z(N) \lhd N_2 \lhd \cdots,$$

defined by $N_{i+1}/N_i = Z(N/N_i)$, is finite. If n is the smallest integer such that $N_n = N$, then N is said to be *n-step nilpotent*. We shall say that a finitely generated, torsion-free group Γ admits an (n-step) *almost-Bieberbach* group structure if it can be written as an extension of a finitely generated, (n-step) nilpotent group N by a finite group Φ. Notice that, given such a torsion-free, finitely generated, nilpotent group, its quotients N_{i+1}/N_i are of a special form, namely $N_{i+1}/N_i \cong \mathbb{Z}^{i_j}$.

Lemma 2.1. *Let Γ fit in an extension*

$$0 \longrightarrow \mathbb{Z}^m \longrightarrow \Gamma \xrightarrow{p} G \longrightarrow 1,$$

where the torsion-free group G has an n-step nilpotent, normal subgroup N of finite index and $\mathbb{Z}^m$ a trivial N-module. Then $\Gamma \in \mathcal{AB}^{n+1}$.

Proof. Let G be defined by the extension

$$1 \longrightarrow N \longrightarrow G \longrightarrow \Phi \longrightarrow 1,$$

with N n-step nilpotent, Φ finite and $\phi : \Phi \to \operatorname{Out} N$ the operator homomorphism. Consider $\bar{\Gamma} := p^{-1}(N)$. Then the extension

$$0 \longrightarrow \mathbb{Z}^m \longrightarrow \bar{\Gamma} \xrightarrow{p} N \longrightarrow 1 \tag{2–1}$$

is central, which implies that $\bar{\Gamma}$ is at most $(n+1)$-step nilpotent. The proof is completed by the observation that $\bar{\Gamma} = p^{-1}(N) \lhd \Gamma$ and $\Gamma/\bar{\Gamma} \cong (\Gamma/\mathbb{Z}^m)/(\bar{\Gamma}/\mathbb{Z}^m) \cong G/N \cong \Phi$. Notice that Γ is torsion-free since so are $\mathbb{Z}^m$ and G. □

We now turn our attention to the fibre of the fibration $F \to T \to B$ to prove the following:

Proposition 2.2. *Let* Γ *be a torsion-free extension*

$$1 \longrightarrow K \longrightarrow \Gamma \xrightarrow{p} G \longrightarrow 1 \tag{2–2}$$

of a finitely generated group K *by a group* G *admitting an* n*-step nilpotent, almost-Bieberbach structure such that* $\operatorname{Im} c$ *is finite, where* $c : \Gamma \to \operatorname{Aut} K$ *denotes the conjugation map. Then* $\Gamma \in \mathscr{AB}^{n+1}$.

Proof. Since G admits an n-step almost-Bieberbach structure there is a short exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow \Phi \longrightarrow 1$$

where N is n-step nilpotent, Φ is finite, and $\phi : \Phi \to \operatorname{Out} N$ is the operator homomorphism. Let $\hat{\Gamma} := p^{-1}(N)$. Then $\hat{\Gamma}$ fits in a short exact sequence

$$1 \longrightarrow K \longrightarrow \hat{\Gamma} \xrightarrow{p} N \longrightarrow 1,$$

where we denote by $\bar{c}$ the restriction of the conjugation map $c : \Gamma \to \operatorname{Aut} K$ to $\hat{\Gamma}$. Let $\overline{\Gamma} := \operatorname{Ker} \bar{c}$, which is nonempty since Γ is infinite. Then the extension

$$1 \longrightarrow \overline{\Gamma} \cap K \longrightarrow \overline{\Gamma} \longrightarrow p(\overline{\Gamma}) \longrightarrow 1$$

is central, with $p(\overline{\Gamma}) \lhd N$, and therefore itself nilpotent. This means that $\overline{\Gamma} \cap K$ is a finitely generated, torsion-free, abelian group and $\overline{\Gamma}$ is at most $(n+1)$-step nilpotent. The proof is completed by observing that the normal subgroup $\overline{\Gamma}$ of $\hat{\Gamma}$ has finite index in $\hat{\Gamma}$, since $\operatorname{Im} \bar{c}$ is finite. □

The group $\operatorname{Aut} K$, for K a Bieberbach group, is not necessarily finite. For an example, see [Charlap 1986, p. 219]. It does, then, make sense to check what happens if the fibre admits an n-step almost-Bieberbach structure. But first:

Proposition 2.3. *Let* Γ *be a torsion-free extension*

$$1 \longrightarrow K \longrightarrow \Gamma \xrightarrow{p} \mathbb{Z}^n \longrightarrow 0$$

of a Bieberbach group K *by a free abelian group of rank* n*, such that* $\mathbb{Z}^m \subseteq Z(\Gamma)$*, where* $\mathbb{Z}^m$ *is the translation subgroup of* K *and* $Z(\Gamma)$ *the center of* Γ*. Then* $\Gamma \in \mathscr{AB}^2$.

Proof. First observe that $\mathbb{Z}^m \lhd \Gamma$, since $\mathbb{Z}^m \leq Z(\Gamma)$. We therefore have a short exact sequence

$$1 \longrightarrow K/\mathbb{Z}^m \longrightarrow \Gamma/\mathbb{Z}^m \xrightarrow{p} \mathbb{Z}^n \longrightarrow 0 \tag{2–3}$$

where $K/\mathbb{Z}^m$ is isomorphic to F, the finite holonomy group of K. We distinguish two cases:

(i) Assume that (2–3) is a central extension. Then choose $Q := (\mathbb{Z}^{n-1} \times k\mathbb{Z}) \lhd \mathbb{Z}^n$ of index $k = |F|$, with $|F|$ the exponent of F. Let $\Gamma' := p^{-1}(Q)$. Then Γ' fits in a short exact sequence

$$1 \longrightarrow F \longrightarrow \Gamma' \xrightarrow{\ p\ } Q \longrightarrow 0$$

that splits as a direct product. By construction, $\Gamma' = (F \times Q) \lhd \Gamma/\mathbb{Z}^m$ is of finite index. Let $q : \Gamma \to \Gamma/\mathbb{Z}^m$ be the identification map. The free abelian group Q imbeds as a normal subgroup of $\Gamma' \lhd \Gamma/\mathbb{Z}^m$ and so also, because $\operatorname{Aut}(F \times Q) = \operatorname{Aut} F \times \operatorname{Aut} Q$, as a normal subgroup of $\Gamma/\mathbb{Z}^m$. Let $\tilde{\Gamma} := q^{-1}(Q)$ and $\hat{\Gamma} := q^{-1}(\Gamma')$; then $Q \cong \tilde{\Gamma}/\mathbb{Z}^m$ and $F \times Q \cong \hat{\Gamma}/\mathbb{Z}^m$. Since $Q \le \mathbb{Z}^n$, it acts on $\mathbb{Z}^m$ in the same way as $\mathbb{Z}^n$, namely trivially. So $\tilde{\Gamma}$ is 2-step nilpotent normal in Γ. One can further check that its index $|\Gamma/\tilde{\Gamma}|$ in Γ is finite, because $|\Gamma/\tilde{\Gamma}| = |\Gamma/\hat{\Gamma}| \cdot |\hat{\Gamma}/\tilde{\Gamma}| = |\mathbb{Z}^n/Q| \cdot |F|$. This completes the proof in this case.

(ii) Assume that the sequence (2–3) is not central, and let $c : \Gamma/\mathbb{Z}^m \to \operatorname{Aut} F$ be the conjugation map. Since F is finite and $\Gamma/\mathbb{Z}^m$ infinite, the kernel of c is nontrivial. Let $\bar{\Gamma} := \operatorname{Ker} c \lhd \Gamma/\mathbb{Z}^m$, let $\bar{F} := F \cap \bar{\Gamma}$, and let $\overline{Q} := p(\bar{\Gamma})$. Then the extension

$$1 \longrightarrow \bar{F} \longrightarrow \bar{\Gamma} \longrightarrow \overline{Q} \longrightarrow 0$$

with $\overline{Q} \lhd \mathbb{Z}^n$ (so that $\overline{Q} \cong \mathbb{Z}^\rho$ for some $\rho \le n$) belongs to the previous case. The result now follows, since $\bar{\Gamma}$ has finite index in Γ. □

Proposition 2.4. *Let Γ be a torsion-free extension*

$$1 \longrightarrow K \longrightarrow \Gamma \xrightarrow{\ p\ } G \longrightarrow 1, \tag{2–4}$$

where K and G admit m-step and n-step almost-Bieberbach structures, respectively. If $Z(L/L_i) \subseteq Z(\Gamma/L_i)$, where $\{L_i\}_i$ is the upper central series of an m-step nilpotent, normal subgroup L of finite index in K, then $\Gamma \in \mathscr{AB}^{n+m}$.

Proof. We first check inductively that $L_i \lhd \Gamma$. This is clear for $i = 1$. Assume it is true for i and let $q_i : L \to L/L_i$ be the identification map, where

$$L_{i+1} = q^{-1}\big(Z(L/L_i)\big),$$

so that $L_{i+1}/L_i \cong Z(L/L_i)$. Then $L_{i+1}/L_i \lhd Z(\Gamma/L_i)$ and $L_{i+1} \lhd \Gamma$. The rest of the proof also follows by induction, first on m and then on n. The group G is of the form

$$1 \longrightarrow N \longrightarrow G \longrightarrow \Phi \longrightarrow 1,$$

where N is n-step nilpotent and Φ finite. By letting $\hat{\Gamma} := p^{-1}(N)$ we get a sequence

$$1 \longrightarrow K \longrightarrow \hat{\Gamma} \longrightarrow N \longrightarrow 1.$$

The case $m = n = 1$ follows from Proposition 2.3. Assuming the theorem is true for some m and $n = 1$, we shall show it is true for $m+1$ and $n = 1$. If K is of the form

$$1 \longrightarrow L \longrightarrow K \longrightarrow F \longrightarrow 1,$$

where L is m-step nilpotent and F finite, consider $L_1 = Z(L)$. The conditions of the theorem ensure that $L_1 = Z(L) \cong \mathbb{Z}^\rho \lhd \Gamma$ for some positive integer ρ. This gives a short exact sequence

$$1 \longrightarrow K/\mathbb{Z}^\rho \longrightarrow \hat{\Gamma}/\mathbb{Z}^\rho \longrightarrow \mathbb{Z}^\nu \longrightarrow 0,$$

with $\nu > 0$. Then $\{L_i/L_1\}_i$ is the upper central series of L/L_1 and $\hat{\Gamma}/\mathbb{Z}^\rho$ admits an $(n+1)$-step almost-Bieberbach structure by the induction hypothesis. $\hat{\Gamma}$ fits into a central short exact sequence

$$0 \longrightarrow \mathbb{Z}^\rho \longrightarrow \hat{\Gamma} \longrightarrow \hat{\Gamma}/\mathbb{Z}^\rho \longrightarrow 1.$$

Lemma 2.1 now applies to prove that $\hat{\Gamma}$, and therefore Γ, admit an $(n+2)$-step almost-Bieberbach structure. Now assume the theorem is true for all m and n up to a certain value. We complete the proof by showing it holds for all m and $n+1$. If $\{N_i\}_i$ is the upper central series of some $(n+1)$-step nilpotent N, define $\bar{\Gamma} := p^{-1}(N_n)$. Then $\bar{\Gamma}$ fits in

$$1 \longrightarrow K \longrightarrow \bar{\Gamma} \longrightarrow N_n \longrightarrow 1$$

and admits an $(n+m)$-step almost-Bieberbach structure. Also there is a positive integer μ such that the sequence

$$1 \longrightarrow \bar{\Gamma} \longrightarrow \hat{\Gamma} \longrightarrow \bar{\Gamma}/\hat{\Gamma} \cong \mathbb{Z}^\mu \longrightarrow 0$$

is exact. The induction argument on the fibre implies that $\hat{\Gamma} \in \mathcal{AB}^{n+m+1}$, and so $\Gamma \in \mathcal{AB}^{n+m+1}$ too. □

3. Almost-Bieberbach groups and Kähler structures

Let $\operatorname{Aut} G$ denote the group of automorphisms of a simply connected Lie group G. We shall be concerned with discrete subgroups Γ of $\operatorname{Aut} G$ that act properly discontinuously on G.

A group Γ is said to be *crystallographic* if it is a cocompact, discrete subgroup of $\mathbb{R}^n \rtimes O(n) \subset \mathrm{Aff}(\mathbb{R}^n)$, where $O(n)$ is the maximal compact subgroup of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{Aff}(\mathbb{R}^n)$ is the group of Euclidean motions of $\mathbb{R}^n$. It is a *Bieberbach* crystallographic group if it is torsion-free as well. Bieberbach groups are precisely the fundamental groups of compact, complete Riemannian manifolds that are flat (locally isometric to Euclidean space), as first proved in [Bieberbach 1911]. An alternative characterisation of flat Riemannian manifolds is that in such manifolds, transition maps can be extended to elements of $\mathbb{R}^n \rtimes O(n)$. Charlap [1965] classified these manifolds, up to connection-preserving diffeomorphisms, by associating to a manifold M a short exact sequence

$$1 \longrightarrow \Lambda \longrightarrow G \longrightarrow \Phi \longrightarrow 1$$

in which the holonomy group Φ of M is finite and $\Lambda \cong \mathbb{Z}^n$ is the translation subgroup of $\Gamma \cong \pi_1(M)$, a torsion-free, discrete, cocompact subgroup of $\mathbb{R}^n \rtimes O(n) \subset \mathrm{Aff}(\mathbb{R}^n)$.

More generally, if G is a simply connected, nilpotent Lie group, we consider a maximal compact subgroup $C \subseteq \operatorname{Aut} G$. A cocompact, discrete subgroup Γ of $G \rtimes C$ is called an *almost-crystallographic* group, and if torsion-free it is called *almost-Bieberbach*. The quotient G/Γ is called an *infra-nilmanifold*, and if $\Gamma \subseteq G$ it is a *nilmanifold*.

Most of Bieberbach's work has been generalised to the nilpotent case in [Auslander 1960] and [Lee and Raymond 1985]:

Theorem 3.1 (Auslander). *Let $\Gamma \subseteq G \rtimes \operatorname{Aut} G$ be an almost-crystallographic group, where G is a connected, simply connected, nilpotent Lie group. Then $(\Gamma \cap G) \lhd \Gamma$ is a cocompact lattice in G, and $\Gamma/(\Gamma \cap G)$ is finite.*

Parts of the statement of the following theorem can already be found in [Lee and Raymond 1985]. We simplify the proof.

Theorem 3.2. *Γ is almost-crystallographic if and only if it is of the form*

$$1 \longrightarrow N \longrightarrow \Gamma \longrightarrow \Phi \longrightarrow 1,$$

with N finitely generated, torsion-free, maximal nilpotent, and Φ finite.

Proof. If $\Gamma \subseteq G \rtimes \operatorname{Aut} G$ is an almost-Bieberbach group, Theorem 3.1 says that $N = \Gamma \cap G$ is a maximal nilpotent, normal subgroup of Γ of finite index, and finitely generated because it is a discrete subgroup of the nilpotent group G. To prove the converse, given an extension like the one in the statement of the theorem, with abstract kernel $\phi : \Phi \to \operatorname{Out} N$, consider the extension of the Malcev completion $\mathcal{N}$ of N,

$$1 \longrightarrow \mathcal{N} \longrightarrow S(\Gamma) \longrightarrow \Phi \longrightarrow 1,$$

with abstract kernel $\psi : \Phi \xrightarrow{\phi} \operatorname{Out} N \to \operatorname{Out} \mathcal{N}$. The claim is that there is exactly one extension of $\mathcal{N}$ by Φ, namely $\mathcal{N} \hookrightarrow \mathcal{N} \rtimes_{\hat{\psi}} \Phi$, where $\hat{\psi} : \Phi \to \operatorname{Aut} \mathcal{N}$ is a lifting morphism of ψ.

Since $Z(\mathcal{N})$ is a vector space and Φ is finite, $H^3(\Phi, Z(\mathcal{N}))$ vanishes and by [Mac Lane 1963, Theorem 8.7] the abstract kernel $[\Phi, \mathcal{N}, \psi]$ has an extension. Furthermore, [Mac Lane 1963, Theorem 8.8] says that this extension is unique because the set $H^2(\Phi, Z(\mathcal{N}))$ parametrizing all congruence classes of such extensions is null, for the same reason. So we know that there is precisely one extension $\mathcal{N} \hookrightarrow \Gamma \twoheadrightarrow \Phi$. If we can further show that ψ has a lifting morphism $\hat{\psi} : \Phi \to \operatorname{Aut}\mathcal{N}$, then $\Gamma \cong \mathcal{N} \rtimes_{\hat{\psi}} \Phi$. To this end, we apply induction on the nilpotency class of $\mathcal{N}$. If $\mathcal{N}$ is 1-step nilpotent then $\mathcal{N} \cong Z(\mathcal{N})$ and the result is obvious. If $\mathcal{N}$ is c-step nilpotent, consider the inverse image under the natural projection $q : \operatorname{Aut}\mathcal{N} \to \operatorname{Out}\mathcal{N}$ of the finite group $\psi(\Phi)$. This gives birth to a short exact sequence $\operatorname{Inn}\mathcal{N} \hookrightarrow q^{-1}(\psi(\Phi)) \twoheadrightarrow \psi(\Phi)$ with $\operatorname{Inn}\mathcal{N} \cong \mathcal{N}/Z(\mathcal{N})$ fulfilling the induction hypothesis. We can thus find a splitting morphism $s : \psi(\Psi) \to q^{-1}(\psi(\Phi)) < \operatorname{Aut}\mathcal{N}$. But now, $s \circ \psi$ is the lifting we were looking for, completing the proof. We thus have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & \Gamma & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & {\scriptstyle\imath}\big\downarrow & & {\scriptstyle\jmath}\big\downarrow & & \big\| & & \\ 1 & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{N}\rtimes\Phi & \longrightarrow & \Phi & \longrightarrow & 1 \end{array}$$

The map $\jmath$, with $\jmath(n, g) = (\imath(n), g)$, embeds Γ as a discrete, cocompact subgroup of the disconnected Lie group $S(\Gamma)$, proving the theorem. □

Given a short exact sequence $\mathbb{Z}^{2n} \hookrightarrow \Gamma \twoheadrightarrow \Phi$ with operator homomorphism $\phi : \Phi \to \operatorname{Aut}\mathbb{Z}^{2n}$, we say ϕ is *essentially complex* if there is a *complex structure* for the Φ-module $\mathbb{Z}^{2n} \otimes \mathbb{R}$, that is, a map $t \in \operatorname{End}_{\mathbb{R}[\Phi]}(\mathbb{Z}^{2n} \otimes \mathbb{R})$ such that $t^2 = -1$. In other words, $\phi : \Phi \to \operatorname{Aut}\mathbb{Z}^{2n}$ is essentially complex if $\operatorname{Im}\phi \subseteq \mathrm{GL}_{\mathbb{C}}\big((\mathbb{Z}^{2n} \otimes \mathbb{R})^t\big)$, with

$$\mathrm{GL}_{\mathbb{C}}\big((\mathbb{Z}^{2n} \otimes \mathbb{R})^t\big) := \{m \in \mathrm{GL}_{\mathbb{R}}(\mathbb{Z}^{2n} \otimes \mathbb{R}) \ \text{ such that } \ mt = tm\}.$$

Theorem 3.3. *Let Γ be the torsion-free extension*

$$1 \longrightarrow N \longrightarrow \Gamma \longrightarrow \Phi \longrightarrow 1,$$

where N is a torsion-free, finitely generated maximal nilpotent group and Φ is a finite group. Then there is a compact Kähler $K(\Gamma, 1)$-manifold M if and only if $N \cong \mathbb{Z}^{2n}$ and the operator homomorphism $\phi : \Phi \to \operatorname{Aut} N$ is essentially complex.

Proof. By Theorem 3.2 there is a connected, simply connected Lie group G such that Γ is a torsion-free, discrete, cocompact subgroup of $G \rtimes \operatorname{Aut} G$. Since M is a $\mathrm{K}(\Gamma, 1)$-manifold, its universal covering is homeomorphically equivalent to G and $M \cong G/\Gamma$. The hypotheses on N say that G contains $N \hookrightarrow \Gamma$ as a discrete cocompact subgroup. Then $\hat{M} \cong G/N$ is a compact $\mathrm{K}(N, 1)$-nilmanifold that covers M in a finite, unramified way. Because the Kähler condition is local, the fact that M admits a Kähler structure implies that $\hat{M}$ also admits a Kähler structure. The Benson–Gordon theorem says that this can happen only if $N \cong \mathbb{Z}^{2n}$, forcing the finite cover $\hat{M}$ of M to be holomorphically equivalent to the complex torus $\mathbb{C}^n/\mathbb{Z}^{2n}$. The converse is settled by [Johnson and Rees 1991, Theorem 3.1]. □

Let $\mathcal{B}_{\mathcal{K}}$ be the class of groups that can be realised as fundamental groups of compact, Kähler manifolds whose underlying Riemannian structure is flat, and $\mathcal{B}_{\mathcal{P}} \subseteq \mathcal{B}_{\mathcal{K}}$ the subclass consisting of groups that can be realised as fundamental groups of complex projective varieties. Let $\mathcal{AB}_{\mathcal{K}}$ denote the class of groups that can be realised as fundamental groups of compact nilmanifolds; that is, compact manifolds of the form G/Γ, where G is a simply connected, nilpotent Lie group and Γ a discrete subgroup admitting a Kähler structure, and let $\mathcal{AB}_{\mathcal{P}} \subseteq \mathcal{AB}_{\mathcal{K}}$ be the subclass consisting of groups that can be realised as fundamental groups of complex projective nilvarieties.

Corollary 3.4. (1) $\mathcal{AB}_{\mathcal{K}} \equiv \mathcal{B}_{\mathcal{K}} \equiv \mathcal{B}_{\mathcal{P}} \equiv \mathcal{AB}_{\mathcal{P}}$.

(2) *Every group in $\mathcal{AB}_{\mathcal{K}}$ is the fundamental group of a smooth, compact, complex algebraic surface.*

Proof. (1) The first equality follows directly from Theorem 3.3 and [Johnson and Rees 1991, Theorem 3.1]. The second is [Johnson 1990, Corollary 4.3], while the third stems from the first two together with the inclusion $\mathcal{AB}_{\mathcal{P}} \subseteq \mathcal{AB}_{\mathcal{K}}$.

(2) If M is a smooth projective manifold, then by Bertini's theorem there is a smooth hyperplane section $M_{(n-1)}$. By the Lefschetz hyperplane theorem [Milnor 1963], $\pi_l(M, M_{(n-1)}) = 0$ for $l < n$, so M and $M_{(n-1)}$ have isomorphic fundamental groups if $n \geq 3$. □

We now combine Proposition 2.2, Proposition 2.4 and Theorem 3.3:

Theorem 3.5. *If the Kähler manifold T is the total space of a fibration $F \to T \to B$ over an infra-nilmanifold B with aspherical fibre F and if the short exact sequence*

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(T) \longrightarrow \pi_1(B) \longrightarrow 1$$

of their respective fundamental groups satisfies the conditions of either Proposition 2.2 or Proposition 2.4, then T admits a flat Riemannian metric.

4. Virtually polycyclic groups and Kähler rigidity

An *affinely flat* manifold is an n-manifold endowed with an atlas whose transition maps can be extended to elements of $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$. A torsion-free group Γ is *virtually polycyclic* if it has a subgroup Γ_0 of finite index which is polycyclic, that is, one that admits a finite composition series $\Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_n = 1$ such that $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}$ for all i. The number n is an invariant, called the *rank* of Γ. Groups in $\mathcal{AB}^n$ are obviously virtually polycyclic. Auslander [1964] has conjectured that the fundamental group of a compact, complete, affinely flat manifold has to be virtually polycyclic. Milnor [1977] has shown that torsion-free, virtually polycyclic groups can be realised as fundamental groups of complete affinely flat manifolds. On the other hand, Johnson [1976] has proved that torsion-free, virtually polycyclic groups can be realised as fundamental groups of compact $K(\pi, 1)$-manifolds. However, contrary to the Bieberbach case, Benoist [1992] has given an example of a 10-step nilpotent group of rank 11, proving that it is not always possible to do both!

If Γ is a virtually polycyclic group, the *Fitting group* of Γ, denoted $\mathrm{Fitt}(\Gamma)$, is the unique maximal normal subgroup of Γ. The *closure* $\overline{\mathrm{Fitt}(\Gamma)}$ of the Fitting group of a group Γ is the maximal normal subgroup of Γ containing $\mathrm{Fitt}(\Gamma)$ as a normal subgroup of finite index. The basic property of $\overline{\mathrm{Fitt}(\Gamma)}$ is that it leaves the quotient $\Gamma/\overline{\mathrm{Fitt}(\Gamma)}$ with no finite, normal subgroup in it — in other words, *almost-torsion-free*. In [Dekimpe and Igodt 1994] it is proved that if Γ is a finitely generated virtually nilpotent group then Γ is almost-torsion-free if and only if $\overline{\mathrm{Fitt}(\Gamma)}$ is almost-crystallographic.

If N is a torsion-free, finitely generated, c-step nilpotent group, then to any extension

$$N \hookrightarrow \Gamma \overset{p}{\twoheadrightarrow} Q$$

with abstract kernel $\psi : Q \to \mathrm{Out}\, N$ we can inductively associate c morphisms $\psi_i : Q \to \mathrm{Aut}(N_{i+1}/N_i)$, where $N_{i+1}/N_i = Z(N/N_i)$. Now if $q \in \Gamma$ is such that $p(q)$ has finite order in Q, and $\langle q, N\rangle$ is nilpotent, then $p(q) \in \bigcap_1^c \mathrm{Ker}\, \psi_i$. Conversely, if $q \in \Gamma$ is such that $p(q) \in \bigcap_1^c \mathrm{Ker}\, \psi_i$, then $\langle q, N\rangle$ is nilpotent in Γ.

We shall use the following lemma, which is half of [Dekimpe and Igodt 1994, Theorem 2.2]. For completeness, we write a proof here.

Lemma 4.1. *Let Γ be a virtually polycyclic group. If Γ is almost-torsion-free,* $\mathrm{Fitt}(\Gamma)$ *is torsion-free maximal nilpotent in* Γ.

Proof. Since Γ is polycyclic-by-finite, $\mathrm{Fitt}(\Gamma)$ is finitely generated nilpotent. Therefore its torsion set is a finite characteristic subgroup of $\mathrm{Fitt}(\Gamma)$, and thus normal in Γ, and hence trivial since Γ is almost torsion-free. So, Γ fits in an extension

$$1 \longrightarrow \mathrm{Fitt}(\Gamma) \longrightarrow \Gamma \overset{p}{\longrightarrow} Q \longrightarrow 1 \tag{4–1}$$

with $\mathrm{Fitt}(\Gamma)$ torsion-free and Q abelian-by-finite, say $A \hookrightarrow Q \overset{j}{\twoheadrightarrow} F$. Now let $q \in \Gamma$ be such that $N := \langle q, \mathrm{Fitt}(\Gamma)\rangle$ is nilpotent, and look at $p(N)$. If $p(N) \cap A \neq \{1\}$ then $p^{-1}(p(N) \cap A)$ is normal in Γ since $(p(N) \cap A) \lhd A$ is nilpotent as a subgroup of N. Thus, $p^{-1}(p(N) \cap A) \subseteq \mathrm{Fitt}(\Gamma)$ and $p(N) \cap A = \{1\}$, a contradiction. We deduce that $p(N) \cong j(p(N)) \subseteq F$, and hence that $p(q)$ is of finite order in Q. The discussion preceding the theorem shows that $p(q) \in \bigcap_1^c(\psi_i) \cap p(N)$, where ψ_i are the morphisms associated with (4–1), which is a finite group since

F is finite; therefore $q \in \overline{\mathrm{Fitt}(\Gamma)}$. But since Γ is almost torsion-free, $\overline{\mathrm{Fitt}(\Gamma)}$ is almost crystallographic and $\mathrm{Fitt}(\overline{\mathrm{Fitt}(\Gamma)}) = \mathrm{Fitt}(\Gamma)$ is maximal nilpotent in $\overline{\mathrm{Fitt}(\Gamma)}$, implying $q \in \mathrm{Fitt}(\Gamma)$, a contradiction. □

Theorem 4.2. *Let Γ be a torsion-free, virtually polycyclic group. Then Γ can be realised as the fundamental group of a $K(\pi, 1)$ compact, Kähler manifold if and only if Γ is Bieberbach with essentially complex operator homomorphism.*

Proof. The converse is the second half of Theorem 3.3. For the direct statement, observe that since Γ is torsion-free, it is almost-torsion-free. Thus, by Lemma 4.1, $\mathrm{Fitt}(\Gamma)$ is torsion-free maximal nilpotent in Γ, and Γ fits in a short exact sequence of the form

$$1 \longrightarrow \mathrm{Fitt}(\Gamma) \longrightarrow \Gamma \xrightarrow{p} Q \longrightarrow 1,$$

where Q is abelian-by-finite. Since Γ is Kähler, by [Arapura and Nori 1999], there exists a nilpotent subgroup $\Delta \subseteq \Gamma$ of finite index. But Δ is necessarily contained in $\mathrm{Fitt}(\Gamma)$, so Q is finite, and Theorem 3.3 completes the proof with $N = \mathrm{Fitt}(\Gamma)$. □

Provided that the Auslander conjecture is true, Theorem 4.2 would immediately imply:

Conjecture 4.3. *If a Kähler manifold T is the total space of a fibration $F \to T \to B$ where both the base B and the fibre F are infra-nilmanifolds, then T admits a Riemann flat structure.*

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NICKOLAS J. MICHELACAKIS
MATHEMATICS AND STATISTICS DEPARTMENT
UNIVERSITY OF CYPRUS
P.O. BOX 20537
NICOSIA 1678
CYPRUS

njm@ucy.ac.cy