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ORTHOGONAL FUNCTIONS IN  $H^\infty$

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We construct examples of  $H^\infty$  functions  $f$  on the unit disk such that the push-forward of Lebesgue measure on the circle is a radially symmetric measure  $\mu_f$  in the plane, and we characterize which symmetric measures can occur in this way. Such functions have the property that  $\{f^n\}$  is orthogonal in  $H^2$ , and provide counterexamples to a conjecture of W. Rudin, independently disproved by Carl Sundberg. Among the consequences is that there is an  $f$  in the unit ball of  $H^\infty$  such that the corresponding composition operator maps the Bergman space isometrically into a closed subspace of the Hardy space.

## 1. Introduction

Let  $H^\infty$  denote the algebra of bounded holomorphic functions on the unit disk  $\mathbb{D}$ , let  $\mathcal{U}$  be the closed unit ball of  $H^\infty$  and let  $\mathcal{U}_0 = \{f \in \mathcal{U} : f(0) = 0\}$ . If  $f \in H^\infty$  then it has radial boundary values (which we also call  $f$ ) almost everywhere on the unit circle  $\mathbb{T}$ . We say that  $f$  is *orthogonal* if the sequence of powers  $\{f^n : n = 0, 1, \dots\}$  is orthogonal, that is, if

$$\int_{\mathbb{T}} f^n \bar{f}^m d\theta = 0$$

whenever  $n \neq m$ . In this paper we will characterize orthogonal functions in  $H^\infty$  in terms of the Borel probability measure  $\mu_f(E) = |f^{-1}(E)|$ , where  $|\cdot|$  denotes Lebesgue measure on  $\mathbb{T}$ , normalized to have mass 1. We will also determine exactly which measures arise in this way. We say a measure is *radial* if  $\mu(E) = \mu(e^{i\theta}E)$  for  $-\infty < \theta < \infty$  and every measurable set  $E$ . We will prove:

**Theorem 1.1.** *If  $f \in \mathcal{U}_0$  then  $\{f^n : n = 0, 1, \dots\}$  is an orthogonal sequence if and only if  $\mu_f$  is a radial probability measure supported in the closed unit disk and*

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satisfying

$$(1-1) \quad \int_{|z| \leq 1} \log \frac{1}{|z|} d\mu_f(z) < \infty.$$

Moreover, given any measure  $\mu$  satisfying these conditions there exists  $f \in \mathcal{U}_0$  such that  $\mu = \mu_f$ .

The result is motivated by the observation that if  $f$  is an inner function (that is,  $f \in H^\infty$  and  $|f| = 1$  almost everywhere on  $\mathbb{T}$ ) with  $f(0) = 0$  then  $\mu_f$  is normalized Lebesgue measure on  $\mathbb{T}$  (Lemma 2.3) and  $f$  is orthogonal since, if  $m > n$ ,

$$\int_{\mathbb{T}} f^n \bar{f}^m d\theta = \int_{\mathbb{T}} f^{n-m} d\theta = 2\pi f^{n-m}(0) = 0.$$

At a 1988 MSRI conference Walter Rudin asked if the converse is true, that is, are multiples of inner functions the only orthogonal bounded holomorphic functions on the disk? In other words, is normalized Lebesgue measure on the circle the only radial measure which can occur as a  $\mu_f$ ? Our characterization shows that many other symmetric measures can occur and hence provide counterexamples to Rudin's "orthogonality conjecture". The conjecture was independently disproved by Carl Sundberg [2003].

The simplest example of a measure satisfying Theorem 1.1 (other than Lebesgue measure on a circle) is to take  $\mu$  to be Lebesgue measure on the union of two circles  $\{z : |z| = \frac{1}{2}\} \cup \{z : |z| = 1\}$ , normalized to give each mass  $\frac{1}{2}$ . The corresponding function  $f$  is orthogonal by the theorem, but is clearly not inner since  $|f| = \frac{1}{2}$  on a subset of  $\mathbb{T}$  of positive measure.

A more interesting example of a radial measure satisfying (1-1) is normalized area measure on the disk. Thus there is an  $f \in \mathcal{U}_0$  such that  $\mu_f$  is normalized area measure. We will show (Lemma 6.1) that for any holomorphic  $g$  on the disk, and  $f \in \mathcal{U}_0$  orthogonal,

$$(1-2) \quad \|g \circ f\|_{H^p}^p = \int_{\mathbb{D}} |g|^p d\mu_f + \mu_f(\mathbb{T}) \|g\|_{H^p}^p,$$

and hence:

**Corollary 1.2.** *There is an  $f \in \mathcal{U}_0$  such that for any analytic  $g$  on  $\mathbb{D}$ ,  $g$  is in the Bergman space  $A^p$ , if and only if  $g \circ f$  is in the Hardy space  $H^p$ , and the norms are equal.*

Thus the subspace  $M_f$  spanned by the powers of  $f$  in  $H^2$  is isomorphic to the Bergman space, and multiplication by  $f$  on  $M_f$  is isomorphic to multiplication by  $z$  on the Bergman space. Since both spaces are Hilbert spaces, of course one is isomorphic to a subspace of the other, but it is perhaps a little surprising that this isomorphism can be accomplished with a composition operator. Similar statements

can be made for Bergman spaces with respect to radial weights  $w \, dx \, dy = d\mu$  of finite mass which satisfy (1–1).

More generally, it would be interesting to know for which pair of spaces  $X, Y$ , of analytic functions on  $\mathbb{D}$ , there is an  $f \in \mathcal{U}_0$  such that  $g \in X$  if and only if  $g \circ f \in Y$ , and to characterize such  $f$ 's when they exist. The latter problem is interesting even when  $X = Y$  (for example, see [Cima and Hansen 1990]). In Corollary 6.3 we characterize orthogonal functions with this property when  $X = Y = H^p$  (it is true if and only if  $\mu_f(\mathbb{T}) > 0$ ). In particular, all inner functions have this property (as claimed in [Cima and Hansen 1990]).

Paul Bourdon has pointed out that (1–2) implies that orthogonal functions  $f$  where  $\mu_f(\mathbb{T}) > 0$  give examples of composition operators with closed range. See [Cima et al. 1974/75] and [Zorboska 1994] for characterizations of such functions.

The radial symmetry of a “Rudin counterexample” has also been noted by Paul Bourdon [1997a]. He showed that  $f$  is orthogonal if and only if the Nevanlinna counting function,

$$N_f(w) = \sum_{f(z)=w} \log \frac{1}{|z|}$$

is almost everywhere constant on each circle centered on the origin. He also showed that the answer to Rudin’s question is “yes” if  $f$  is univalent, and that if  $f$  is orthogonal, the closure of the range of  $f$  is a disk (since the range of  $f$  equals the set where  $N_f$  is positive). The Nevanlinna function  $N_f$  is related to  $\mu_f$  by the formula

$$N_f(w) = \log \frac{1}{|w|} - \int \log \frac{1}{|z - w|} d\mu_f(z)$$

(except possibly on a set of logarithmic capacity zero). This is due to W. Rudin [1967] but we shall give a proof for completeness (Lemma 3.1).

**Corollary 1.3.** *If  $f \in \mathcal{U}_0$  is nonconstant and orthogonal then  $N_f(w) = N(|w|)$  for all  $w$  outside an exceptional set of zero logarithmic capacity, where*

$$N(r) = \int_r^1 \frac{1 - \mu(t)}{t} dt$$

*for some increasing function  $\mu$  on  $[0, 1]$  such that  $\mu(0) = 0$  and  $\mu(1) = 1$ , and  $\int_0^1 \mu(t) dt/t < \infty$  (in fact,  $\mu(r) = \mu_f(D(0, r))$ ). Moreover, for every such  $N$  there is an  $f \in \mathcal{U}_0$  such that  $N_f(w) = N(|w|)$  except possibly on a set of logarithmic capacity zero.*

The first part of this is due to Paul Bourdon [1997b]. The condition on  $N$  in the previous result has many equivalent formulations; for example, it holds if and only if  $M(r) = N(e^r)$  on  $(-\infty, 0]$  is concave up, has  $M(0) = 0$  and  $\sup_{r < 0} M(r) + r < \infty$ , or if  $N(|z|)$  is subharmonic on  $\mathbb{D} \setminus \{0\}$  and  $N(|z|) + \log |z|$  is bounded above.

The behavior of the composition operator  $C_f : g \rightarrow g \circ f$  can often be expressed in terms of  $N_f$ , for example, see [Shapiro 1987; Smith 1996; Smith and Yang 1998]. The result above provides radial examples with any desired rate of decay faster than  $1 - r$  as  $r \rightarrow 1$ .

If  $f$  is orthogonal, then  $f(0) = 2\pi \int f d\theta = 0$ , so  $f$  cannot be an outer function. However, our construction can be modified to give:

**Corollary 1.4.** *There is an orthogonal  $f$  such that  $f(z)/z$  is a nonconstant outer function.*

Thus, not only are there orthogonal functions which are not inner, there are examples with only the most trivial possible inner factor. I do not know whether there is an example where  $f(z)/z$  is bounded away from zero on  $\mathbb{D}$  or which symmetric measures  $\mu$  are of the form  $\mu_f$  with  $f(z)/z$  outer.

One can also construct examples with other properties. For example,  $f \in \mathcal{U}_0$  is said to be in the hyperbolic little Bloch class  $\mathcal{B}_0^h$  if

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)|f'(z)|}{1 - |f(z)|^2} = 0.$$

(This is contained in the usual little Bloch space, where only the numerator is required to go to zero.) We will show (Lemma 5.2) that if  $g$  is inner and  $f \in H^\infty$  then  $\mu_{f \circ g} = \mu_f$ . Thus taking  $g$  to be an inner function in the hyperbolic little Bloch class (which exists by a result of Wayne Smith [1998] and independently of Aleksandrov, Anderson and Nicolau [Aleksandrov et al. 1999]; also see [Cantón 1998]), we can deduce:

**Corollary 1.5.** *Any of the measures in Theorem 1.1 is  $\mu_f$  for some  $f \in \mathcal{B}_0^h$ .*

Cima, Korenblum and Stessin [Cima et al. 1993] also identified symmetric properties of orthogonal functions and showed the answer to Rudin’s question is “yes” if  $f$  is Hölder of order  $\alpha > \frac{1}{2}$  on  $\mathbb{T}$ . I do not know if there exists any (noninner) orthogonal function which is continuous up to the boundary, but expect that it might be possible to build one by modifying the construction in this paper. If there is a continuous orthogonal function, it would be very interesting to know if the result of Cima, Korenblum and Stessin is sharp, and if not, what the best modulus of continuity for such a function could be. What other natural conditions on an orthogonal function imply that it is actually inner?

The remaining sections are organized as follows:

**Section 2:** We describe some elementary properties of  $\mu_f$  and prove it is radial if and only if  $f$  is orthogonal.

**Section 3:** We prove Corollary 1.3 (given Theorem 1.1).

**Section 4:** We prove some results concerning the convergence of  $\mu_f$ .

**Section 5:** We prove [Corollary 1.5](#) (given [Theorem 1.1](#)).

**Section 6:** We prove [Corollary 1.2](#) (given [Theorem 1.1](#)).

**Section 7:** We construct a symmetric  $\mu_f$  which is supported on two circles.

**Section 8:** We construct all examples supported in  $\{\frac{1}{2} \leq |z| \leq 1\}$ .

**Section 9:** We complete the proof of [Theorem 1.1](#).

**Section 10:** We prove [Corollary 1.4](#).

## 2. Elementary properties of $\mu_f$

We begin by recalling a few simple facts about analytic functions  $f$  and their corresponding measures  $\mu_f$ . Many of these are well known but we include them for the convenience of the reader.

**Lemma 2.1.** *If  $f \in H^\infty$  then  $\mu_f$  satisfies*

$$\int \log \frac{1}{|z|} d\mu_f(z) < \infty.$$

*Proof.* If  $f$  has a zero of order  $n$  at the origin, then  $g(z) = f(z)/z^n$  is holomorphic on the unit disk and  $|g| = |f|$  on  $\mathbb{T}$ , hence  $\mu_g(A) = \mu_f(A)$  for any annulus  $A = \{z : r_1 \leq |z| \leq r_2\}$ . Thus

$$\int \varphi(z) d\mu_f(z) = \int \varphi(z) d\mu_g(z)$$

for any radial function  $\varphi$ . Using Fatou's lemma and the fact that  $\log |g(z)|^{-1}$  is superharmonic on the disk (see [[Garnett 1981](#), page 35]), we deduce

$$\begin{aligned} \int \log \frac{1}{|z|} d\mu_f(z) &= \int \log \frac{1}{|z|} d\mu_g(z) = \frac{1}{2\pi} \int \log |g(e^{i\theta})|^{-1} d\theta \\ &= \frac{1}{2\pi} \int \lim_{r \rightarrow 1} \log |g(re^{i\theta})|^{-1} d\theta \\ &\leq \frac{1}{2\pi} \lim_{r \rightarrow 1} \int \log |g(re^{i\theta})|^{-1} d\theta \leq \log |g(0)|^{-1} < \infty. \quad \square \end{aligned}$$

A similar estimate is true for other points, for example,

$$\int \log \frac{1}{|z-a|} d\mu_f(z) < \infty.$$

In particular, this implies the well-known fact that the set where  $f$  has radial limit  $a$  must have measure zero.

Given an arc  $I \subset \mathbb{T}$  we define the *Carleson box* with *base*  $I$  to be

$$Q = Q_I = \{z \in \mathbb{D} : z/|z| \in I, 1 - |z| \leq |I|\}.$$

A positive measure  $\mu$  is a *Carleson measure* if there exists a  $C < \infty$  such that  $\mu(Q_I) \leq C|I|$ , for every arc  $I \in \mathbb{D}$ .

**Lemma 2.2.** *If  $f \in \mathcal{U}_0$  then  $\mu_f$  is a Carleson measure with constant independent of  $f$ .*

*Proof.* Define  $\varphi(z) = \omega(z, Q, \mathbb{D} \setminus Q)$  for  $z \in \mathbb{D} \setminus Q$  and  $\varphi(z) = 1$  for  $z \in Q$ . It is easy to see that  $\omega(z, I, \mathbb{D}) \geq M^{-1} > 0$  for every  $z \in \partial Q \cap \mathbb{D}$  and some  $M < \infty$  (independent of  $I$  and  $z \in \partial Q$ ), so the maximal principle implies

$$\varphi(0) \leq M\omega(0, I, \mathbb{D}) \leq M|I|.$$

Let  $f_r(z) = f(rz)$ . Note that  $\lim_{r \rightarrow 1} \varphi(f(rx)) = \varphi(f(x))$  for almost every  $x \in \mathbb{T}$ , because  $\varphi$  is continuous on the closed disk except at two points, and the set where  $f$  has a radial limit equal to one of these has measure zero (by the remark following [Lemma 2.1](#)). So by the Lebesgue dominated convergence theorem,

$$(2-1) \quad \mu_f(Q) \leq \int \varphi d\mu_f = \frac{1}{2\pi} \int \varphi \circ f d\theta = \lim_{r \rightarrow 1} \frac{1}{2\pi} \int \varphi \circ f_r d\theta.$$

Since  $\varphi$  is superharmonic on  $\mathbb{D}$ , it follows that  $\varphi \circ f$  is too, so the right-hand side of (2-1) is at most  $\varphi(f(0)) = \varphi(0) \leq M|I|$ .  $\square$

If  $f(0) \neq 0$  then  $\mu_f$  is still a Carleson measure, but with norm depending on  $|f(0)|$ .

One can think of the previous lemma as a weak version of the Littlewood subordination principle: that if  $f$  is an analytic self-map of the disk then  $g \in H^p$  implies  $g \circ f \in H^p$  (with smaller or equal norm). Formally, this implies that if  $f(0) = 0$ , then

$$\int |g|^p d\mu_f \leq \frac{1}{2\pi} \int_{\mathbb{T}} |g \circ f|^p d\theta = \|g \circ f\|_{H^p}^p \leq \|g\|_{H^p}^p$$

for every  $g \in H^p$ . This implies that  $d\mu_f$  is a Carleson measure with norm independent of  $f$  (see, for example, [\[Garnett 1981, Theorem I.5.6\]](#)).

The following result appears in many places (for example, [\[Löwner 1923; Nordgren 1968, Lemma 1; Rudin 1980, page 405; Tsuji 1959, Theorem VIII.30\]](#)) and is sometimes called “Löwner’s lemma”. See [\[Fernández et al. 1996\]](#) and its references for various generalizations.

**Lemma 2.3.** *If  $f$  is an inner function such that  $f(0) = 0$ , then  $\mu_f$  is normalized Lebesgue measure on the unit circle.*

*Proof.* It is enough to check that  $\mu_f(I) = |I|$  for arcs. Let  $I$  be an arc on the unit circle and let  $\varphi(z) = \omega(z, I, \mathbb{D})$ . Then  $\varphi \circ f$  is bounded and harmonic, and takes radial boundary values 1 and 0 almost everywhere (1 almost everywhere that  $f$  has

radial limit in  $I$ , and 0 almost everywhere that  $f$  has radial limit outside  $I$ ). Thus

$$|I| = \varphi(0) = \varphi(f(0)) = \frac{1}{2\pi} \int_{f^{-1}(I)} d\theta = \mu_f(I). \quad \square$$

As noted before, the following lemma is similar to results in [Bourdon 1997a] and [Cima et al. 1993].

**Lemma 2.4.** *Suppose  $f \in H^\infty$ . Then the measure  $\mu_f$  is radial if and only if  $\{f^n\}$  is orthogonal.*

*Proof.* If  $\mu_f$  is radial, it can be written so that

$$\int g(z) d\mu_f(z) = \int_0^{2\pi} \int_0^\infty g(re^{i\theta}) d\theta dv(r)$$

for every  $g \in C_c(\mathbb{R}^2)$ , the set of continuous functions of compact support defined on  $\mathbb{R}^2$ , and for some measure  $\nu$  on  $(0, \infty)$ . Thus

$$\int_{\mathbb{T}} f^n \bar{f}^m d\theta = \int_{\mathbb{C}} z^n \bar{z}^m d\mu_f(z) = \int_0^\infty \int_0^{2\pi} r^{n+m} e^{i(n-m)\theta} d\theta dv(r) = 0$$

if  $n \neq m$ , so  $f$  is orthogonal. Conversely, if  $f$  is orthogonal, then  $\mu_f$  satisfies

$$\int_{\mathbb{C}} z^n \bar{z}^m d\mu_f(z) = \int_0^{2\pi} \int_0^\infty r^{n+m} e^{i(n-m)\theta} d\mu_f(re^{i\theta}) = 0$$

for  $n \neq m$ . Thus

$$\int_{\mathbb{C}} P(z, \bar{z}) d\mu_f(z) = \int_{\mathbb{D}} \sum_n a_{n,n} r^{2n} d\mu_f(z)$$

for any polynomial  $P(z, \bar{z}) = \sum_{n,m} a_{n,m} z^n \bar{z}^m$  in  $z$  and  $\bar{z}$ , and hence

$$\int_{\mathbb{C}} P(\lambda z, \bar{\lambda} \bar{z}) d\mu_f(z) = \int_{\mathbb{C}} P(z, \bar{z}) d\mu_f(z)$$

for any  $|\lambda| = 1$ . Since polynomials in  $z$  and  $\bar{z}$  are dense in the continuous functions on the closed unit disk, we deduce that

$$\int_{\mathbb{D}} g(z) d\mu_f(z) = \int_{\mathbb{D}} g(\lambda z) d\mu_f(z)$$

for any  $g \in C_c(\mathbb{R}^2)$  and any  $|\lambda| = 1$ . This implies  $\mu_f$  is radial.  $\square$

The following lemma greatly simplifies the construction of the basic example, where  $\mu_f$  is supported on two circles. It says that if we can construct an example where  $\mu_f$  is radial on the smaller circle, then it automatically looks like Lebesgue measure on the larger one.



**Lemma 2.5.** *Suppose  $f$  lies in  $\mathcal{U}_0$ , and  $\mu_f$  is supported on the circles  $C_{1/2} \cup C_1 = \{|z| = \frac{1}{2}\} \cup \{|z| = 1\}$ . If  $\mu_f$  restricted to  $C_{1/2}$  is a multiple of Lebesgue 1-dimensional measure, then so is  $\mu_f$  restricted to  $C_1$ .*

*Proof.* Suppose  $u$  is any bounded harmonic function on  $\mathbb{D}$ . Then  $v(z) = u(f(z))$  is also bounded and harmonic on  $\mathbb{D}$  and  $u(0) = v(0)$ . Thus

$$\begin{aligned} u(0) = v(0) &= \frac{1}{2\pi} \int_0^{2\pi} u(f(e^{i\theta})) d\theta = \int u(z) d\mu_f(z) \\ &= \int_{C_{1/2}} u(z) d\mu_f(z) + \int_{C_1} u(z) d\mu_f(z) \\ &= \mu_f(C_{1/2})u(0) + \int_{C_1} u(z) d\mu_f(z). \end{aligned}$$

Hence  $\int_{C_1} u d\mu_f = \mu_f(C_1)u(0)$  for any bounded harmonic function  $u$  on  $\mathbb{D}$ . This easily implies that  $\mu_f$  restricted to  $C_1$  is a multiple of Lebesgue measure on  $C_1$ .  $\square$

The same proof gives the following generalization of [Lemma 2.5](#).

**Lemma 2.6.** *Suppose  $f \in \mathcal{U}_0$ . Then  $\mu_f$  restricted to the unit circle is of the form  $\frac{1}{2\pi}(1 - g(\theta)) d\theta$ , where  $g$  is the balayage of  $\mu_f$  onto the circle, that is,*

$$g(\theta) = \int_{\mathbb{D}} P_z(\theta) d\mu_f(z),$$

where  $P_z(\theta)$  is the Poisson kernel for  $\mathbb{D}$  with respect to the point  $z$ .

### 3. The Nevanlinna counting function

For  $f \in H^\infty$ , the Nevanlinna counting function is defined to be

$$N_f(w) = \sum_{f(z)=w} \log \frac{1}{|z|}.$$

If  $f \in \mathcal{U}_0$  then  $N_f(w) \leq \log |w|^{-1}$ . Clearly this is just the Green's function for the Riemann surface associated to  $f$  (projected to the plane by summing over sheets). Since  $\mu_f$  is the projection of harmonic measure for the Riemann surface, the following is analogous to the standard result for Green's functions of planar domains. Let  $\Delta = \partial_x^2 + \partial_y^2$  denote the Laplacian and let  $\delta_0$  be the Dirac mass at the origin.

**Lemma 3.1** [[Rudin 1967](#)]. *If  $f \in \mathcal{U}_0$  then  $\Delta N_f = -\delta_0 + \mu_f$  in the sense of distributions, and*

$$(3-1) \quad N_f(w) = \log \frac{1}{|w|} - \int \log \frac{1}{|z - w|} d\mu_f(z)$$

for all  $w$ , except possibly for an exceptional set  $E$  of logarithmic capacity zero where “ $<$ ” holds.

The exceptional set is required. For example, if  $f$  is the universal covering map of  $\mathbb{D}$  minus a compact set  $E$  of zero logarithmic capacity,  $f$  is an inner function,  $\mu_f$  is normalized Lebesgue measure on the circle and  $N_f(z) = \chi_{\mathbb{D} \setminus E}(z) \log |z|^{-1}$ .

*Proof.* For  $0 < r < 1$ , let  $f_r(z) = f(rz)$  and let  $\gamma_r = f_r(\mathbb{T})$ . If we choose  $r$  so that  $f'$  never vanishes on the circle of radius  $r$ , then  $\gamma_r$  is a smooth curve and it is easy to check using Green’s theorem that  $\Delta N_{f_r} = -\delta_0 + \mu_{f_r}$ . To see that (3–1) holds for  $\mu_{f_r}$ , note that both sides of the equation have the same distributional Laplacian, so they differ by a harmonic function.  $N_{f_r}$  vanishes outside the unit disk by definition, and the right side of (3–1) vanishes there because  $\mu_{f_r}$  evaluates harmonic functions at 0. Hence the difference between the left and right sides is the constant zero function.

For any smooth  $\varphi$  with compact support,

$$\int N_{f_r} \Delta \varphi \, dx \, dy = -\varphi(0) + \int \varphi \, d\mu_{f_r}.$$

We shall see later that  $\mu_{f_r}$  weakly converges to  $\mu_f$  (Corollary 4.4), and clearly  $N_{f_r} \nearrow N_f$  as  $r \nearrow 1$ . Thus taking  $r \rightarrow 1$  and applying the monotone convergence theorem we get

$$\int N_f \Delta \varphi \, dx \, dy = -\varphi(0) + \int \varphi \, d\mu_f.$$

This proves the first claim of the lemma. Next we verify (3–1).

We already know that if we replace  $f$  by  $f_r$  then we have equality in (3–1) for all  $z$  and as  $r \rightarrow 1$ , and we know  $N_{f_r}(z) \nearrow N_f(z)$  for all  $z$ . Thus the question reduces to whether

$$(3-2) \quad U_r(w) \rightarrow U_1(w) \text{ as } r \rightarrow 1$$

for all  $w$  except a set  $E$  of logarithmic capacity zero, where

$$U_r(w) = \int \log \frac{1}{|z - w|} \, d\mu_{f_r}(z).$$

Note that  $U_r$  is decreasing in  $r$ , by the superharmonicity of  $\log |f|^{-1}$ , and that  $U_1$  is bounded below by  $-\log 2$ , since  $|z - w| < 2$  for points in the unit disk.

To prove that (3–2) holds, we follow the proof of Frostman’s theorem (see [Garnett 1981, Theorem II.6.4], for example). Suppose  $\sigma$  is a measure such that  $V(z) = \int \log |z - w|^{-1} \, d\sigma(w)$  is bounded. It suffices to show  $\sigma(E) = 0$ . By Fatou’s lemma

$$\lim_{r \rightarrow 1} \int \log \frac{1}{|z - w|} \, d\mu_{f_r}(z) \geq \int \log \frac{1}{|z - w|} \, d\mu_f(z),$$

so  $\lim_{r \rightarrow 1} U_r(w) \geq U_1(w)$  for all  $w$ . On the other hand, by Fatou's lemma, Fubini's theorem and the Lebesgue dominated convergence theorem,

$$\begin{aligned} \int_E \lim_{r \rightarrow 1} U_r(w) d\sigma(w) &\leq \lim_{r \rightarrow 1} \int_E U_r(w) d\sigma(w) \\ &= \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} V(f(re^{i\theta})) d\theta = \frac{1}{2\pi} \int_0^{2\pi} V(f(e^{i\theta})) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_E \log \frac{1}{|f(e^{i\theta}) - w|} d\sigma(w) d\theta = \int_E U_1(z) d\sigma(w). \end{aligned}$$

Thus we must have  $\lim_{r \rightarrow 1} U_r(w) = U_1(w)$  except on a set of zero  $\sigma$  measure.  $\square$

**Lemma 3.1** clearly implies that  $\mu_f$  is radial if and only if  $N_f$  is (except for the exceptional set). Thus we see that  $\{f^n\}$  is an orthogonal sequence if and only if  $\mu_f$  is radial, if and only if  $N_f$  is radial, except on a set of logarithmic capacity zero. This gives an alternate approach to the results of Bourdon [1997a].

We can also compute exactly which radial functions can occur as  $N_f$  for some  $f \in \mathcal{U}_0$ . Note that

$$\frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{|re^{i\theta} - w|} d\theta = \begin{cases} \log(1/|w|) & \text{if } r \leq |w|, \\ \log(1/r) & \text{if } r \geq |w|. \end{cases}$$

Thus if  $\mu_f$  is radial and we set  $\mu(r) = \mu_f(D(0, r))$ , then

$$N_f(w) = \log \frac{1}{|w|} - \int \log \frac{1}{|z - w|} d\mu_f(z) = \int_{|w|}^1 \frac{1 - \mu(r)}{r} dr.$$

Moreover, the integral condition

$$\int_{\mathbb{D}} \log \frac{1}{|z|} d\mu < \infty$$

becomes

$$\int_0^1 \mu(r) \frac{dr}{r} < \infty.$$

Thus **Theorem 1.1** implies the following corollary.

**Corollary 3.2.** *Suppose  $N(r) = \int_r^1 (1 - \mu(t)) dt/t$  for some increasing function  $\mu$  such that  $\int_0^1 \mu(r) dr/r < \infty$ , with  $\mu(0) = 0$  and  $\mu(1) = 1$ . Then there is an  $f \in \mathcal{U}_0$  such that  $N_f(z) = N(|z|)$  except on a set of zero logarithmic capacity.*

For example, if  $\mu_f$  is normalized area measure on the unit disk then  $\mu(r) = r^2$  and  $N_f(z) = \log 1/r - (1 - r) \approx (1 - r)^2$  as  $r \rightarrow 1$ .

#### 4. Weak\* convergence of $\mu_f$

We will obtain the functions  $f$  in [Theorem 1.1](#) by a “cut and paste” construction of the corresponding Riemann surface. What this means is that we shall build a sequence of nested Riemann surfaces  $R_0 \subset R_1 \subset R_2 \subset \cdots \subset \bigcup R_n = R$  by identifying subdomains of the unit disk along common boundary arcs. The projection of  $R$  into the unit disk is a bounded holomorphic function on  $R$ , and hence  $R$  must be hyperbolic, that is, its universal covering space is the unit disk  $\mathbb{D}$ . The desired map will be the covering map  $f : \mathbb{D} \rightarrow R$  followed by the projection into the disk and the corresponding measure  $\mu_f$  is simply the harmonic measure for the surface  $R$ , projected into the plane. In fact, we shall abuse notation and consider the covering map  $f : \mathbb{D} \rightarrow R$  as actually mapping into the complex numbers (that is, we identify the covering map and this map followed by the projection into the plane). By a similar abuse we shall think of harmonic measure on  $R$  and the corresponding projected measure  $\mu_f$  as the same. Similarly, we will fix a point in  $R_0$  which projects to 0 and call it 0 as well. All our covering maps will be chosen to map 0 in the disk to 0 on the surface. See [\[Bishop 1993\]](#) and [\[Stephenson 1988\]](#), where a similar procedure has been used in different problems.

The main point we must be careful about is to show that the harmonic measure for  $R$  is the limit of the measures for  $R_n$ . To see that there might be a problem in general, consider what can happen when the surfaces are not nested. For example,  $R_n$  is the unit disk minus the points  $\{z_k = \frac{1}{2} \exp(i2\pi k 2^{-n}) : k = 1, \dots, 2^n\}$ . Then the universal covering map  $f_n : \mathbb{D} \rightarrow R_n$  is an inner function (the isolated boundary points do not have any harmonic measure, so all the measure lives on the part of the boundary above the unit circle) and hence  $\mu_{f_n}$  is Lebesgue measure on the unit circle. However, one can show (with some work) that  $f_n(z) \rightarrow \frac{1}{2}z$  uniformly on compact sets of  $\mathbb{D}$ , so that  $\mu_f$  is Lebesgue measure on the circle of radius  $\frac{1}{2}$ . However, if the Riemann surfaces are nested by (increasing) inclusion, then we will show the corresponding measures converge weak\*, that is,

$$\lim_{n \rightarrow \infty} \int g d\mu_n = \int g d\mu$$

for any  $g \in C_c(\mathbb{R}^2)$ .

**Lemma 4.1.** *Suppose  $\epsilon > 0$  and  $D(0, \epsilon) = R_0 \subset R_1 \subset \cdots$  are obtained by identifying subdomains of the unit disk along boundary arcs. Let  $R = \bigcup_{n=1}^\infty R_n$ . Choose covering maps  $f_n : \mathbb{D} \rightarrow R_n$  and  $f : \mathbb{D} \rightarrow R$  so that  $f_n(0) = f(0) = 0$ . Then  $\mu_{f_n}$  converges weak\* to  $\mu_f$  on the closed unit disk.*

The easiest way to see this is using Brownian motion; we shall first sketch such a proof and then give a more classical proof without using Brownian motion.

Let  $\mathcal{W}$  be the Wiener space of continuous paths in  $\mathbb{C}$  starting at the origin. If  $R$  is a Riemann surface constructed as above then we can think of the paths as taking values in  $R$  and for each path  $w \in \mathcal{W}$ , we define the stopping time  $t_w$  as the first time  $t$  such that  $w(t) \notin R$ . Then  $w \rightarrow t_w$  is measurable and the harmonic measure for  $R$  is simply the push-forward of Wiener measure on  $\mathcal{W}$  under the map given by  $w \rightarrow w(t_w)$ . Given a sequence of nested surfaces  $R_0 \subset R_1 \subset \dots$  as in the lemma, we get a corresponding sequence of maps  $g_n : \mathcal{W} \rightarrow \mathbb{C}$ . Moreover, if  $R = \bigcup_n R_n$  and  $g : \mathcal{W} \rightarrow \mathbb{C}$  is the corresponding map, then  $g(w) = \lim_n g_n(w)$ ; this is because the inclusions imply that for any continuous path in the plane, the first time it leaves  $R$  is the limit of the first time it left  $R_n$ . Thus for any bounded, continuous function  $\varphi$  on the plane,  $\varphi(g_n(w)) \rightarrow \varphi(g(w))$  for all  $w$ , so the Lebesgue dominated convergence theorem implies that

$$\int_{\mathcal{W}} \varphi(g(w)) dw = \lim_{n \rightarrow \infty} \int_{\mathcal{W}} \varphi(g_n(w)) dw,$$

which is the desired weak\* convergence.

The sketch above is simple and explains why the result is true, but uses the existence of Wiener measure and deep connections between it and harmonic measure. It therefore seems desirable to provide a second proof which uses only function theory. Moreover, we will need some corollaries of the following classical proof for our applications to composition operators.

Let  $\{R_n\}$ ,  $R$ ,  $\{f_n\}$  and  $f$  be as in the lemma and let  $\Omega_n = f^{-1}(R_n) \subset \mathbb{D}$ . Then  $\Omega_0 \subset \Omega_1 \subset \dots$  and  $\bigcup_n \Omega_n = \mathbb{D}$ . Let  $\omega_n$  be the harmonic measure for  $\Omega_n$  with respect to the origin and let  $\varphi$  be any continuous function on the plane. We want to show that

$$\lim_{n \rightarrow \infty} \int \varphi(f(z)) d\omega_n(z) = \int_{\mathbb{T}} \varphi(f(e^{i\theta})) d\theta/2\pi.$$

We start by proving the much easier fact that  $\omega_n$  converges weak\* to normalized Lebesgue measure on the circle. (Since  $f$  need not be continuous up to the boundary,  $\varphi \circ f$  need not be continuous either, so weak\* convergence of  $\omega_n$  is not, by itself, enough to prove weak\* convergence of  $\mu_{f_n}$ .)

**Lemma 4.2.** *If  $\{0\} \in \Omega_0 \subset \Omega_1 \subset \dots$  is a sequence of subdomains such that  $\bigcup_n \Omega_n = \mathbb{D}$ , and  $\omega_n = \omega(0, \cdot, \Omega_n)$  is the corresponding harmonic measure with respect to the origin, then  $\{\omega_n\}$  converges weak\* to (normalized) Lebesgue measure on  $\mathbb{T}$ . Moreover, the measures  $\omega_n$  are all Carleson with a uniform constant.*

*Proof.* The Carleson condition follows from [Lemma 2.2](#) applied to the covering map onto  $\Omega_n$ , so we need only prove weak\* convergence. Since  $\bigcap_n (\mathbb{D} \setminus \Omega_n) = \mathbb{T}$ , there is a sequence  $\{r_n\} \nearrow 1$  such that  $D_n = \{z : |z| < r_n\} \subset \Omega_n \subset \mathbb{D}$ . Suppose that  $I \subset \mathbb{T}$  is an open arc and let  $Q = \{z \in \mathbb{D} : z/|z| \in I, 1 - |z| \leq |I|\}$  be the corresponding

Carleson square. To show  $\omega_n$  converges weak\* to normalized Lebesgue measure, it is clearly enough to show that  $\omega_n(Q) \rightarrow |I|$ .

Let  $U_n = D_n \cup Q$  and  $V_n = \mathbb{D} \setminus (Q \setminus D_n)$ . Then  $\omega(0, I, U_n) \rightarrow |I|$ . To see this, first note that  $\omega(0, I, U_n) \leq |I|$  follows immediately from the maximum principle applied to  $\omega(z, I, U_n)$  on  $U_n$ . For the other direction, suppose that  $J \subset I$  is a proper subinterval and note that

$$\omega(0, I, U_n) \geq \omega(0, J, U_n) = |J| - \int_{\partial U_n \setminus I} \int_J P_z(\theta) d\theta d\omega(0, \cdot, U_n),$$

and that  $\int_J P_z(\theta) d\theta \rightarrow 0$  as  $n \rightarrow \infty$  for  $z \in \partial U_n \setminus I$ . Thus the Lebesgue dominated convergence theorem implies  $\liminf \omega(0, I, U_n) \geq |J|$ . Since this holds for any proper subinterval  $J$ , we see that  $\omega(0, I, U_n) \rightarrow |I|$  as desired. A similar argument shows that  $\omega(0, Q \cap V_n, V_n) \rightarrow |I|$  as  $n \rightarrow \infty$ .

Thus, by the monotonicity of harmonic measure,

$$\omega_n(Q) \geq \omega(0, \partial\Omega_n \cap Q, U_n) \geq \omega(0, I, U_n) \rightarrow |I|,$$

and so  $\liminf_n \omega_n(Q) \geq |I|$ . On the other hand,

$$\omega_n(Q) \leq \omega(0, Q \cap \partial V_n, V_n) \rightarrow |I|,$$

which implies that  $\omega_n(Q) \rightarrow |I|$ . This proves the lemma.  $\square$

**Lemma 4.3.** *Suppose  $g$  is a bounded, continuous function on  $\mathbb{D}$  which has nontangential limit  $g(x)$  almost everywhere on  $\mathbb{T}$ , and that  $\nu_n$  is a sequence of probability measures on  $\overline{\mathbb{D}}$  which converge weak\* to (normalized) Lebesgue measure on the circle and which are all Carleson measures with a uniform constant. Then*

$$\lim_{n \rightarrow \infty} \int_{\overline{\mathbb{D}}} g(z) d\nu_n(z) = \frac{1}{2\pi} \int_{\mathbb{T}} g(e^{i\theta}) d\theta.$$

*Proof.* We may assume that  $\|g\|_\infty = 1$ . Fix some  $\epsilon > 0$ . Since  $g$  has nontangential limits almost everywhere, given almost any  $x \in \mathbb{T}$  there is a  $\delta(x) > 0$  such that if  $I$  is any interval containing  $x$  with length less than  $\delta(x)$ , then  $g$  is within  $\epsilon/2$  of  $g(x)$  on the top half of the corresponding Carleson box  $Q$ . Fix a complex number  $a$ , and a  $\delta > 0$ , and assume that  $E_a = \{x \in \mathbb{T} : |g(x) - a| \leq \epsilon/2, \delta(x) > \delta\}$  has positive Lebesgue measure. Using the Lebesgue density theorem choose a dyadic interval  $I$  of length less than  $\delta$  so that  $|I \cap E_a| \geq (1 - \epsilon)|I|$  and let  $Q_k$  be the collection of maximal dyadic subsquares with bases  $\{I_k\} \subset I$  such that  $|g(z) - a| > \epsilon$  for some  $z$  in the top half of  $Q_k$ . Let  $Q$  be the Carleson square with base  $I$  and let  $W = Q \setminus \bigcup Q_k$ . Then  $g$  is within  $\epsilon$  of a constant on  $W$ , and  $|\partial W \cap I| \geq (1 - \epsilon)|I|$ .

We claim that for these domains  $\lim_n v_n(W) = |\partial W \cap \mathbb{T}|$ . To prove this, we use the weak\* convergence of  $\{v_n\}$  to deduce

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \int_W dv_n &= \lim_{n \rightarrow \infty} \left( v_n(Q) - \sum_k v_n(Q_k) \right) \\
 &= |I| - \lim_{n \rightarrow \infty} \sum_k v_n(Q_k) \\
 &= |I| - \sum_k \lim_{n \rightarrow \infty} v_n(Q_k) \\
 &= |I| - \sum_k |I_k| \\
 &= |\partial W \cap \mathbb{T}|,
 \end{aligned}$$

where we used the Lebesgue dominated convergence theorem on the sequence space  $\ell^1$  to interchange the limit and the infinite sum (our assumption that the measures are uniformly Carleson implies that  $v_n(Q_k) \leq C|I_k|$ , independent of  $n$ ; this gives the  $\ell^1$  upper bound).

Moreover, the intervals  $I$  with these properties form a Vitali cover of  $\mathbb{T}$  (see, for example, [Wheeden and Zygmund 1977, Section 7.3]), so we can form a disjoint cover of almost every point of  $\mathbb{T}$  using such intervals. Thus we can construct a finite number of disjoint domains  $W_j = Q_j \setminus \bigcup_k Q_k^j$ , where

- (1)  $Q_j$  is a Carleson square with base  $I_j$  and  $|\partial W_j \cap I_j| \geq (1 - \epsilon)|I_j|$ ,
- (2)  $g$  is within  $\epsilon$  of a constant  $c_j$  on each  $W_j$ ,
- (3)  $\sum_j |\partial W_j \cap \mathbb{T}| \geq 1 - \epsilon$ .

Let  $W = \bigcup_j W_j$  be this finite union. The weak\* convergence of  $\{v_n\}$  implies that

$$\limsup_{n \rightarrow \infty} v_n(\mathbb{D} \setminus W) \leq \epsilon,$$

and so if  $\|g\|_\infty \leq 1$ ,

$$\begin{aligned}
 \left| \lim_{n \rightarrow \infty} \int g dv_n - \int_{\mathbb{T}} g d\theta / 2\pi \right| &\leq \lim_{n \rightarrow \infty} \left| \int_W g dv_n - \frac{1}{2\pi} \int_{\partial W \cap \mathbb{T}} g d\theta \right| \\
 &\quad + \int_{\mathbb{D} \setminus W} |g| dv_n + \frac{1}{2\pi} \int_{\mathbb{T} \setminus \partial W} |g| d\theta \\
 &\leq C\epsilon \sum_j |\partial W_j \cap \mathbb{T}| + 2|\mathbb{T} \setminus \bigcup \partial W_j| \\
 &\leq C\epsilon.
 \end{aligned}$$

Letting  $\epsilon \rightarrow 0$  proves [Lemma 4.3](#) and thus completes our function-theoretic proof of [Lemma 4.1](#). □

A very special (and easier) case of [Lemma 4.3](#) is:

**Corollary 4.4.** *Suppose  $f \in \mathcal{O}_{\mathbb{U}_0}$  and let  $f_r(z) = f(rz)$  for  $r < 1$ . Then  $\mu_{f_r}$  converges weak\* to  $\mu_f$  as  $r \rightarrow 1$ .*

**Corollary 4.5.** *If  $f$  is inner and  $f(0) = 0$  then  $\mu_{f_r}$  converges weak\* to normalized Lebesgue measure on  $\mathbb{T}$ .*

## 5. A change of variables

The following result was suggested by Paul Bourdon and simplifies certain arguments from an earlier version of the paper.

**Lemma 5.1.** *Suppose  $g$  is a positive, continuous function on  $\mathbb{D}$  and has nontangential boundary values almost everywhere on  $\mathbb{T}$ . Then, for any  $f \in \mathcal{O}_{\mathbb{U}}$ ,*

$$\int g(z) d\mu_f(z) = \frac{1}{2\pi} \int_0^{2\pi} g(f(e^{i\theta})) d\theta.$$

The integral on the left requires some interpretation since  $g$  is not necessarily continuous on the support of  $\mu_f$ . On the interior of the disk,  $g$  is continuous and positive so the integral is well defined (possibly infinite). On the circle,  $\mu_f$  is absolutely continuous with respect to Lebesgue measure and the boundary values of  $g$  are Borel, so the integral on the circle is also well defined.

*Proof.* Using the monotone convergence theorem we can reduce to the case when  $g$  is bounded (just truncate and let the truncation tend to  $\infty$ ). So assume  $g$  is bounded by  $M$ . For any  $\epsilon > 0$  we can easily construct a sawtooth region  $W$  so that  $|\mathbb{T} \cap \partial W| > 1 - \epsilon$  and  $g$  extends continuously to the closure of  $W$ . Thus we can write  $g = (g - h) + h$  where  $h$  is continuous, bounded by  $M$  and  $g - h$  is zero on  $W$ . The lemma is true for continuous functions by the definition of  $\mu_f$ , and

$$\int (g - h) d\mu_f \leq 2M\mu_f(\overline{\mathbb{D}} \setminus W) \leq 2MC\epsilon,$$

since  $\mu_f$  is Carleson with a uniform constant. Similarly

$$\int (g - h) \circ f(e^{i\theta}) d\theta \leq 2MC\epsilon,$$

so taking  $\epsilon \rightarrow 0$  proves the lemma. □

The following lemma is now immediate.

**Lemma 5.2.** *If  $g \in H^\infty$  and  $f$  is inner with  $f(0) = 0$  then  $\mu_g = \mu_{g \circ f}$ .*



The hyperbolic little Bloch space,  $\mathcal{B}_0^h$ , is defined to be the space of those holomorphic maps  $f \in \mathcal{U}$  such that

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)|f'(z)|}{1 - |f(z)|^2} = 0,$$

and is contained in the usual little Bloch space,  $\mathcal{B}_0$ . Schwarz's inequality implies the left side is bounded by 1 for any analytic self-map of the disk, and from this it is easy to verify that  $g$  and  $f$  are both holomorphic self-maps of the disk, and  $f$  is hyperbolic little Bloch then so is  $g \circ f$ . It is far from obvious that there is an inner function in the hyperbolic little Bloch space, but they do exist (see [Aleksandrov et al. 1999; Cantón 1998; Smith 1998]). This and Lemma 5.2 thus imply:

**Corollary 5.3.** *If  $g \in \mathcal{U}$ , then there is an  $f \in \mathcal{B}_0^h$  such that  $\mu_f = \mu_g$ .*

Recall that the Hardy space,  $H^p$ , is the set of holomorphic functions  $g$  such that

$$\|g\|_{H^p} = \lim_{r \rightarrow 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p d\theta \right)^{1/p} < \infty.$$

Such a function has radial boundary values almost everywhere on  $\mathbb{T}$ , which we also denote by  $g$ . If we know  $g \in H^p$  for  $p > 1$ , then the radial maximal function of  $g$  is in  $L^p$  and so on can use the dominated convergence theorem to deduce that

$$\|g\|_{H^p} = \frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^p d\theta.$$

In general, however, the right-hand side might be finite but  $g$  might not be in  $H^p$  (there exist nonzero holomorphic functions on the disk that have radial value zero almost everywhere, and hence are not in  $H^p$ ). If  $f \in \mathcal{U}$  then  $\mu_f$  restricted to  $\mathbb{T}$  is absolutely continuous with respect to Lebesgue measure, so  $\int_{\mathbb{D}} |g|^p d\mu_f$  makes sense.

As another application of Lemma 5.1 we can show

**Lemma 5.4.** *Suppose  $g \in H^p$  on the unit disk and  $f \in \mathcal{U}_0$ . Then for any  $0 < p < \infty$ ,*

$$\|g \circ f\|_{H^p}^p = \lim_{r \rightarrow 1} \int_{\mathbb{D}} |g|^p d\mu_{f_r} = \int_{\mathbb{D}} |g|^p d\mu_f.$$

*Proof.* The first equality is the definition of the  $H^p$  norm, so we only have to prove the second. If  $g \in H^p$  and  $f \in \mathcal{U}_0$  then by a result of Ryff [1966],  $g \circ f \in H^p$  with smaller or equal norm. Thus  $|g|^p$  is positive, continuous function on the disk which has nontangential boundary values almost everywhere, so Lemma 5.1 shows that

$$\int |g(z)|^p d\mu_f = \frac{1}{2\pi} \int_0^{2\pi} |g(f(e^{i\theta}))|^p d\theta,$$

and since we already know  $g \circ f \in H^p$ , we can deduce that the right-hand side equals  $\|g \circ f\|_{H^p}$ .  $\square$

## 6. Mapping the Bergman space into the Hardy space

For our applications to composition operators, we need a version of [Lemma 5.4](#) that works without the assumption that  $g \in H^p$ . The proof given above doesn't work in general because if  $g$  is not in  $H^p$  we can't say that  $\|g\|_{H^p} = \int_0^{2\pi} |g|^p d\theta/2\pi$ . In fact, we will not even assume  $g$  has boundary values on the circle, so this integral is not necessarily defined.

**Lemma 6.1.** *Suppose  $g$  is holomorphic on the open unit disk,  $f \in \mathcal{U}_0$  and  $\mu_f$  is radial. Then, for any  $0 < p < \infty$ ,*

$$(6-1) \quad \|g \circ f\|_{H^p}^p = \lim_{r \rightarrow 1} \int_{\mathbb{D}} |g|^p d\mu_{f_r} = \int_{\mathbb{D}} |g|^p d\mu_f + \mu_f(\mathbb{T}) \|g\|_{H^p}^p.$$

*Proof.* Let  $g_s(z) = g(sz)$  for  $0 < s < 1$ . First, we want to show that, for any  $0 < p < \infty$ ,

$$(6-2) \quad \lim_{s \rightarrow 1} \int_{\mathbb{D}} |g(sz)|^p d\mu_f = \int_{\mathbb{D}} |g(z)|^p d\mu_f + \mu_f(\mathbb{T}) \|g\|_{H^p}^p,$$

Since  $g$  is holomorphic,  $|g|^p$  is subharmonic for  $0 < p < \infty$  (see, for example, [[Garnett 1981](#), page 35]) and hence  $m(r) = \frac{1}{2\pi} \int |g(re^{i\theta})|^p d\theta$ , is defined on  $[0, 1]$  and is an increasing function of  $r$  [[Garnett 1981](#), Corollary I.6.6]. Therefore we can extend it to be defined at  $r = 1$  by  $\|g\|_{H^p}^p = m(1) = \lim_{r \rightarrow 1} m(r)$ . Thus  $m_s(r) \equiv m(sr)$  increases to  $m(r)$  as  $s \rightarrow 1$  for all  $r \in [0, 1]$ . Let  $\nu$  be the measure on  $[0, 1]$  defined by  $\nu(E) = \mu_f(\{z : |z| \in E\})$ . Since  $\mu_f$  is radial we have

$$\int_{\mathbb{D}} \varphi d\mu_f = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \varphi(re^{i\theta}) d\theta d\nu(r).$$

Thus by the monotone convergence theorem,

$$\lim_{s \rightarrow 1} \int_{\mathbb{D}} |g_s|^p d\mu_f = \lim_{s \rightarrow 1} \int m_s(r) d\nu = \int_{[0,1]} m(r) d\nu = \int_{\mathbb{D}} |g|^p d\mu_f + \mu_f(\mathbb{T}) m(1).$$

This is (6-2).

We will break the proof of (6-1) into three cases.

**Case 1:**  $\int_{\mathbb{D}} |g|^p d\mu_f = \infty$ .

For any  $M > 0$  choose  $0 < t < 1$  so that  $\int_{|z| < t} |g|^p d\mu_f > 2M$  and write  $|g|^p = g_1 + g_2$  where  $g_1$  and  $g_2$  are nonnegative,  $g_1 = |g|^p$  on  $|z| < t$ , and  $g_1$  is continuous and compactly supported in  $\mathbb{D}$ . Then

$$\int |g|^p d\mu_{f_r} \geq \int g_1 d\mu_{f_r} > \frac{1}{2} \int g_1 d\mu_f \geq M$$

if  $r$  is close enough to 1. Thus  $\int |g|^p d\mu_{f_r} \rightarrow \infty = \int |g|^p d\mu_f$ .

**Case 2:**  $\int_{\mathbb{D}} |g|^p d\mu_f < \infty$  and  $\mu_f(\mathbb{T}) = 0$ .

Since  $\mu_{f_r}$  converges weak\* to  $\mu_f$ ,

$$\lim_{r \rightarrow 1} \int |g_s|^p d\mu_{f_r} = \int |g_s|^p d\mu_f$$

for any fixed  $s < 1$ . Since  $g_s(f(z))$  is holomorphic on the open disk,  $|g_s(f(z))|^p$  is subharmonic. Thus  $\int |g_s|^p d\mu_{f_r}$  is increasing in  $r$ , and hence

$$\int |g_s|^p d\mu_{f_r} \leq \int |g_s|^p d\mu_f.$$

Now take  $s \rightarrow 1$ . For  $r$  fixed,  $\mu_{f_r}$  is compactly supported in the disk, so  $|g_s|^p$  is uniformly bounded on its support and hence the left-hand side converges to  $\int |g|^p d\mu_{f_r}$ . Condition (6–2) implies the right-hand side converges to  $\int |g|^p d\mu_f$ . Thus

$$\int |g|^p d\mu_{f_r} \leq \int |g|^p d\mu_f$$

for all  $r < 1$ .

Fix  $\epsilon > 0$  and choose  $0 < t < 1$  so that  $\int_{t < |z| < 1} |g|^p d\mu_f < \epsilon$ . Write  $|g|^p = g_1 + g_2$  as in Case 1. Thus  $\int g_2 d\mu_f < \epsilon$ . Also, if  $r$  is close enough to 1 then, by weak\* convergence,

$$\left| \int g_1 d\mu_f - \int g_1 d\mu_{f_r} \right| < \epsilon.$$

Thus

$$\int g_2 d\mu_{f_r} \leq \left| \int g_1 d\mu_f - \int g_1 d\mu_{f_r} \right| + \int g_2 d\mu_f \leq 2\epsilon.$$

Hence

$$\begin{aligned} \left| \int |g|^p d\mu_f - \int |g|^p d\mu_{f_r} \right| &\leq \int g_2 d\mu_{f_r} + \left| \int g_1 d\mu_f - \int g_1 d\mu_{f_r} \right| + \int g_2 d\mu_f \\ &\leq 4\epsilon, \end{aligned}$$

if  $r$  is close enough to 1.

**Case 3:**  $\int_{\mathbb{D}} |g|^p d\mu_f < \infty$  and  $\mu_f(\mathbb{T}) > 0$ .

If  $\lim_{r \rightarrow 1} \int |g|^p d\mu_{f_r} = \infty$  then by the subharmonicity of  $|g \circ f|^p$  we see that  $\int |g|^p d\mu_f = \infty$ , so (6–1) holds. Thus we may assume that  $\lim_{r \rightarrow 1} \int |g|^p d\mu_{f_r} < \infty$ , that is, we may assume that  $g \circ f \in H^p$ , and hence that  $|g(f(z))|^p$  has a harmonic majorant  $u$  on  $\mathbb{D}$  (see [Garnett 1981, Lemma II.1.1]).

First we show that  $g \in H^p$ . For  $0 < r < 1$  let  $D_r = D(0, r)$ . Let  $\Omega_r$  be the component of  $f^{-1}(D_r)$  which contains the origin, and let  $\omega_r$  be the harmonic measure on  $\Omega_r$  with respect to the origin. Let  $\nu_r$  be the push-forward of  $\omega_r$  under

the map  $f$ . Then clearly  $\nu_r$  is supported on  $\overline{D}_r$  and  $\nu_r(E) \leq \mu_f(E)$  for any  $E \subset D_r$ . By [Lemma 2.6](#),  $\nu_r$  on  $C_r = \partial D_r$  must be  $\frac{1}{2\pi} d\theta$  minus the balayage of  $\nu_r$  restricted to  $D_r$ . Since  $\nu_r \leq \mu_f$ , this means that  $\nu_r$  on  $C_r$  is at least  $\frac{1}{2\pi} d\theta$  minus the balayage of  $\mu_f$  restricted to  $D_r$ . Since  $\mu_f$  is radial, its balayage onto  $C_r$  is also radial, that is, equal to  $\frac{1}{2\pi} \mu_f(D_r) d\theta \leq \frac{1}{2\pi} (1 - \mu_f(\mathbb{T})) d\theta$ . Thus  $\nu_r \geq \frac{1}{2\pi} \mu_f(\mathbb{T}) d\theta$  on  $C_r$ . Hence, for any  $g$  holomorphic on  $\mathbb{D}$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p d\theta \leq \frac{1}{\mu_f(\mathbb{T})} \int |g|^p d\nu_r = \frac{1}{\mu_f(\mathbb{T})} \int |g \circ f|^p d\omega_r.$$

Thus, if  $u$  is a harmonic majorant of  $|g \circ f|^p$  on  $\mathbb{D}$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p d\theta \leq \frac{1}{\mu_f(\mathbb{T})} \int u d\omega_r = \frac{u(0)}{\mu_f(\mathbb{T})} < \infty.$$

In other words,  $g \in H^p$  and thus (6–1) follows from [Lemma 5.4](#). □

Recall that the Bergman space  $A^p$  is defined as the set of holomorphic functions  $g$  on the disk  $\mathbb{D}$  such that

$$\|g\|_{A^p} = \left( \frac{1}{\pi} \int_{\mathbb{D}} |g|^p dx dy \right)^{1/p} < \infty.$$

**Corollary 6.2.** *If  $f \in H^\infty$  such that  $d\mu_f = \frac{1}{\pi} \chi_{\mathbb{D}} dx dy$ , then any function  $g$ , analytic on the disk, is in the Bergman space if and only if  $g \circ f$  is in the Hardy space, and  $\|g\|_{A^p} = \|g \circ f\|_{H^p}$ , that is, the composition operator  $C_f : A^p \rightarrow H^p$  is an isometry.*

*Proof.* Using [Lemma 6.1](#) we see that

$$\begin{aligned} \|g \circ f\|_{H^p} &= \lim_{r \rightarrow 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |g(f(re^{i\theta}))|^p d\theta \right)^{1/p} \\ &= \lim_{r \rightarrow 1} \left( \int |g|^p d\mu_{f_r} \right)^{1/p} = \left( \int |g|^p d\mu_f \right)^{1/p} = \|g\|_{A^p}. \end{aligned} \quad \square$$

This corollary may seem a little surprising, since functions in  $H^p$  have nontangential limits almost everywhere, whereas those in  $A^p$  need not, but since  $f$  has almost all of its boundary values in the interior of the disk, this is not a contradiction. Of course, it still remains to show (see [Section 9](#)) that there is an  $f \in H^\infty$  such that  $\mu_f$  is area measure.

[Corollary 6.2](#) obviously holds for any weighted Bergman space where the weight is a radial measure of finite mass satisfying the integral condition (1–1) in [Theorem 1.1](#). If instead of an isometry, we merely want  $\|g\|_{A_p} \simeq \|g \circ f\|_{H^2}$  we could take a much bigger class of functions  $f$ , for example,  $\mu_f = w dx dy$  for some weight  $w$  which is bounded above and below on an annulus  $\{r < |z| < 1\}$ . Constructing such examples only needs the techniques of [Section 8](#), not the full proof of [Theorem 1.1](#).

Similarly, by appropriate choices of  $\mu_f$  one can construct composition operators on  $H^p$  which satisfy conditions like

$$\|C_f(g)\|_{H^p}^p = \frac{1}{2}\|g\|_{H^p}^p + \frac{1}{2}\|g\|_{A^p}^p \quad \text{or} \quad \|C_f(g)\|_{H^p}^p = \frac{1}{2}\|g\|_{H^p}^p + \frac{1}{2}\|g_{1/2}\|_{H^p}^p.$$

In [Cima and Hansen 1990], a function  $f$  is said to have property  $(*)$  relative to  $H^p$  if  $g \circ f \in H^p$  implies that  $g \in H^p$ , for any holomorphic  $g$  on  $\mathbb{D}$ . Paul Bourdon has pointed out that for general  $f \in \mathcal{U}$ , the condition  $\mu_f(\mathbb{T}) = 0$  implies condition  $(*)$ , which implies  $N_f(z) = o(1 - |z|)$  which, by J. Shapiro's theorem [1987], implies that  $C_f$  is compact and hence does not have a bounded right inverse. Since  $f$  is nonconstant,  $C_f$  is 1-to-1 and so does not have closed range (this is a consequence of the open mapping theorem, for example [Rudin 1973, Corollary 2.12c]). Thus  $C_f$  does not have property  $(*)$ , since any function in  $\overline{C_f(H^p)} \setminus C_f(H^p)$  is an  $H^p$  function without an  $H^p$  preimage. Lemma 6.1 clearly implies the following corollary.

**Corollary 6.3.** *If  $f \in \mathcal{U}_0$  is orthogonal, then  $f$  has property  $(*)$  relative to  $H^p$  if and only if  $\mu_f(\mathbb{T}) > 0$ .*

*Proof.* If  $\mu_f(\mathbb{T}) > 0$  then the argument in Case 3 of the proof of Lemma 6.1 shows that  $g \circ f \in H^p$  implies  $g \in H^p$ . Thus  $f$  has property  $(*)$  with respect to  $H^p$ .  $\square$

A special case of Corollary 6.3 is when  $\mu_f(\mathbb{T}) = 1$ , that is, all inner functions have property  $(*)$ . It would be very interesting to have a similar characterization of property  $(*)$  for general functions in  $\mathcal{U}_0$ .

## 7. An example of $\mu_f$ supported on two circles

In this section we will construct an  $f \in H^\infty$  so that  $\mu_f$  is supported on the union of two circles  $C_{1/2}$  and  $C_1$  (where  $C_r = \{z : |z| = r\}$ ) and is a multiple of Lebesgue measure on each. This example suffices to disprove Rudin's orthogonality conjecture, and introduces the estimates and techniques needed for the general case of Theorem 1.1. In the next section we will show that any radial probability measure supported in  $\{\frac{1}{2} \leq |z| \leq 1\}$  can occur as a  $\mu_f$ , and in Section 9 we will do the general case of measures supported on  $\mathbb{D}$ .

Based on Lemmas 4.1 and 2.5, it suffices to build an increasing sequence of Riemann surfaces  $\{R_n\}$  so that the corresponding maps  $\{f_n\}$  satisfy  $f_n(0) = 0$ , that  $\mu_{f_n}$  is supported on the two circles  $C_{1/2} \cup C_1$ , and that  $\mu_{f_n}$  restricted to  $C_{1/2}$  is of the form  $\frac{1}{2\pi} g_n(\theta) d\theta$ , where  $g_n$  converges uniformly to a positive constant.

We start by taking  $f_0(z) = \frac{1}{2}z$ , that is,  $f_0$  is the (trivial) Riemann mapping from  $\mathbb{D}$  to the disk  $R_0 = \{|z| < 1/2\}$ . The corresponding measure  $\mu_0 = \mu_{f_0}$  is normalized Lebesgue measure on the circle  $C_{1/2}$ , that is,  $\frac{1}{2\pi} g_0(\theta) d\theta$  where  $g_0(\theta) = 1$ .

Now we describe the idea of the construction of  $R_1$  (we will give the details later). First we replace  $R_0$  with a slightly smaller disk,  $S_1$ . We divide the boundary

of  $S_1$  into a large number of alternating intervals which we call type  $I$  and type  $J$ . Along each type  $I$  interval we attach a copy of a certain Riemann surface with boundary over  $C_{1/2}$  (attaching different copies to different intervals) and along each type  $J$  interval we attach copies of certain surfaces with boundary over  $C_1$ . This gives the surface  $R_1$ . With appropriate choices of the parameters involved we can show that, with high probability, the Brownian paths which first hit  $\partial S_1$  at a type  $I$  interval go on to hit the part of  $\partial R_1$  over  $C_{1/2}$  and the paths which hit the  $J$  intervals go on to hit  $\partial R_1$  over  $C_1$ . Thus we have “rerouted” a certain fraction of the harmonic measure on  $C_{1/2}$  out to  $C_1$ . By choosing various parameters correctly, we can make the harmonic measure over  $C_{1/2}$  in  $R_1$  be close to any multiple of Lebesgue measure we want (as long as the total mass is less than 1). The resulting measure may not be radial but, by iterating the construction with variable size barriers, we can make harmonic measure as close to a multiple of Lebesgue measure as we wish, obtaining a radial measure in the limit.

Now we give the construction of  $R_1$  in more detail. Choose  $\delta_1$  very small and let  $S_1 = D(0, r_1)$ , where  $r_1 = \frac{1}{2} - \delta_1$ . Obviously harmonic measure on  $S_1$  is just normalized Lebesgue measure on its boundary. Choose a large integer  $m_1$  and points  $\{z_j : j = 1, \dots, m_1\}$  equally spaced on the circle  $C_{r_1}$ . Choose a continuous function  $0 < \eta(x) < 1$  on  $C_{r_1}$ , let  $I_j$  be an arc of  $\partial S_1$  of angle measure  $\eta(z_j)2\pi/m_1$  centered at  $z_j$ , and let  $\{J_j\}$  be the complementary arcs. For the first step of the construction we can take  $\eta(x) = \eta_1$  to be a constant for simplicity, but in later steps we will have to use nonconstant  $\eta$ 's.

Fix some  $0 < \tau_1 < 1$  and, for each arc of the form  $I_j$  with endpoints  $\{p, q\}$ , choose a countable collection of points  $E = \{w_k^j\} \subset I_j$ , accumulating only at the endpoints of  $I_j$ , so that for any  $z \in I_j$

$$(7-1) \quad \text{dist}(z, E) \leq \tau_1 \text{dist}(z, \{p, q\}).$$

Let the components of  $I_j \setminus E$  be denoted  $\{I_k^j\}$ . For each  $I_k^j$ , consider the (infinitely connected) planar domain  $\mathbb{D} \setminus E$  and the universal cover of the domain. Take a copy of the arc  $I_k^j$  in the universal cover; it is on the boundary of a simply connected domain  $D$  in the universal cover which covers  $D(0, \frac{1}{2})$ . The arc cuts the universal cover into two components and we let  $R_k^j$  denote the component which does not contain  $D$ . For each interval  $I_k^j$ , we attach a copy of  $R_k^j$  to  $S_1$  along the arc  $I_k^j$ .

For the intervals  $\{J_j\}$  we follow the same procedure, defining a set  $E \subset J_j$  and sub intervals  $\{J_k^j\}$ , but replacing  $D(0, \frac{1}{2})$  with  $D(0, 1)$ . That is, we attach a component of the universal cover of  $D(0, 1) \setminus E$ , cut along  $J_k^j$ . Doing this for all  $j$  and  $k$  gives the surface  $R_1$ . The harmonic measure for  $R_1$  is now supported on  $C_{1/2} \cup C_1$ , (the rest of the ideal boundary covers a countable set, so has zero measure) so we only need to check that it is still close to radial on  $C_{1/2}$ .

Now we want to discuss the two main estimates for describing the harmonic measure of  $R_1$ . The first says that a continuous convolution of the Poisson kernel is well approximated by a discrete version if the sample points are sufficiently close together. The second says that the harmonic measure of  $I$  intervals is small when viewed from a  $J$  interval, and vice versa.

Suppose  $D(0, r)$  is a disk and  $g$  is a continuous function on a smaller circle  $C_s$ ,  $s < r$ . The balayage of  $g$  onto the circle  $C_r$  is

$$Bg(\theta) = \int_0^{2\pi} g(se^{it}) P_{se^{it}}(\theta) dt,$$

where  $P_z(\theta)$  is the Poisson kernel for  $D(0, r)$  with respect to the point  $z$ .

**Lemma 7.1.** *With the intervals  $\{I_j\}$  defined as above, and  $F = \bigcup_j I_j$ , for any continuous  $0 < g < 1$  on the circle  $C_s$*

$$B(g\chi_F)(\theta) = \int_F g(se^{it}) P_{se^{it}}(\theta) dt \rightarrow B(g\eta)(\theta),$$

uniformly as  $m_1 \rightarrow \infty$ .

*Proof.* Let  $K_j$  be the interval on  $C_s$ , centered at  $z_j$ , of angle measure  $2\pi/m_1$  (choose them to be half-open, so that they form a disjoint cover of the circle). Define piecewise constant functions  $a(x)$  and  $b(x)$  on  $C_{r_1}$  by

$$a(x) = \sum_j \chi_{K_j}(x) \eta(z_j), \quad b(x, \theta) = \sum_j \chi_{K_j}(x) g(z_j) P_{z_j}(\theta),$$

and let

$$A(m_1) = \|\eta(z) - a(z)\|_\infty, \quad B(m_1) = \|g(x) P_x(\theta) - b(x, \theta)\|_\infty.$$

It is clear that, by uniform continuity, both quantities tend to zero as  $m_1 \rightarrow \infty$ . Thus by using the fact that  $\chi_F(x) - a(x)$  has mean value zero on each interval  $K_j$  where  $b(x, \theta)$  is constant in  $x$  we get

$$\begin{aligned} & |B(g\chi_F)(\theta) - B(g\eta)(\theta)| \\ &= \left| \int_0^{2\pi} (g(se^{it}) P_{se^{it}}(\theta) - b(se^{it}, \theta) + b(se^{it}, \theta)) (\chi_F(se^{it}) - \eta(se^{it})) dt \right| \\ &\leq B(m_1) \int_0^{2\pi} |(\chi_F - \eta)(se^{it})| dt + \int_0^{2\pi} b(se^{it}, \theta) |a(se^{it}) - \eta(se^{it})| dt \\ &\leq 2\pi B(m_1) + A(m_1) \max |b|. \end{aligned}$$

This clearly tends to zero as  $m_1 \rightarrow \infty$ , as desired.  $\square$

Now for the second estimate. We want to show that the harmonic measure of  $C_{1/2}$  is much larger than that of  $C_1$  with respect to a point  $z \in I_k^j$ .

**Lemma 7.2.** *Suppose that  $z \in I_k^j$ , and suppose that  $\gamma$  is a circular arc in  $S_1$  with endpoints in the corresponding set  $E$  such that  $\text{dist}(\gamma, z) \simeq \text{dist}(z, \{p, q\})$  (with constants independent of  $\tau_1$ ), and which separates  $z$  from all the  $J$ -intervals. Let  $\Omega$  be the component of  $R_1 \setminus \gamma$  which contains  $z$ . Then  $\omega(z, \gamma, \Omega) \rightarrow 0$  as  $\tau_1$  does.*

*Proof.* Standard estimates of hyperbolic metric imply that  $\gamma$  is within a bounded hyperbolic distance of a geodesic in  $R_1$ , and that the hyperbolic distance from  $\gamma$  to  $z$  is at least  $C \log \tau_1^{-1}$ . Lifted to the disk, this implies the harmonic measure of  $\gamma$  with respect to  $z$  is  $\leq \exp(C \log \tau_1) \leq \tau_1^\alpha$ , for some  $\alpha > 0$ , as desired. Obviously, the same estimate holds if we reverse the rôles of the  $I$  and  $J$  intervals.  $\square$

The previous result has a simple explanation in terms of Brownian motion. Consider a Brownian motion on the Riemann surface started at  $z$  and run until it either hits  $\gamma$  or leaves  $R_1$ . The path will only hit  $\gamma$  if it stays on the correct sheet of  $R_1$ , but this is extremely unlikely because it will cross the arc  $I_j$  many times and each time it has a certain chance (which is large if  $\tau$  is small) of becoming “tangled” and ending up on the wrong sheet.

We can now show that the harmonic measure of  $R_1$  on the circle  $C_{1/2}$  can be taken as close to a multiple of Lebesgue measure as we wish (depending on our choices of  $m_1$ ,  $\tau_1$  and  $\eta$ ). The harmonic measure of  $R_1$  on the circle  $C_{1/2}$  will be the balayage of the harmonic measure of  $S_1$  restricted to the  $I$  intervals, with an error bounded by  $C\tau_1^\alpha$ . The harmonic measure is (normalized) angle measure restricted to the  $I$ -intervals. Thus if  $m_1$  is large enough, the harmonic measure on  $C_{1/2}$  will be of the form  $\frac{1}{2\pi} g_1(x) d\theta$ , with  $g_1$  as close to a constant as we wish. Take  $\frac{1}{2} + \frac{1}{10} \leq g_1(x) \leq \frac{1}{2} + \frac{3}{10}$ , to be concrete.

Now suppose we have constructed  $R_{n-1}$ . To construct  $R_n$ , we follow the method above. We start passing to a subsurface  $S_n \subset R_{n-1}$  where the boundary circles over  $C_{1/2}$  are replaced by boundaries over  $C_{1/2-\delta_n}$ . The parameter  $\delta_n$  is chosen so small that every component of  $R_{n-1} \setminus S_n$  is a regular cover of the annulus  $\{\frac{1}{2} - \delta_n < |z| < \frac{1}{2}\}$  (which will be possible by the construction of  $R_{n-1}$ ) and so that harmonic measure  $\mu_{S_n}$  on  $S_n$  is very close to harmonic measure on  $R_{n-1}$ , say

$$(7-2) \quad \left| \int \varphi d(\mu_{S_n} - \mu_{R_{n-1}}) \right| \leq 2^{-n}$$

for every smooth  $\varphi$  with gradient bounded by  $n$ .

As before we choose  $m_n$  equally spaced points  $\{z_j^n\}$  on  $C_n = C_{\frac{1}{2}-\delta_n}$  and define intervals  $\{I_j^n\}$  of  $C_n$ , centered at these points, of angle measure  $2\pi\eta_n(z_j^n)/m_n$ , where

$$\eta_n(x) = \left( \frac{1}{2} + \frac{2}{10^n} \right) / g_{n-1}(x).$$

The complementary intervals are denoted  $\{J_j^n\}$ . We choose a very small  $\tau_n$  and sets  $E$  in each interval which satisfies (7-1) with  $\tau_n$ . We then attach copies  $D(0, \frac{1}{2}) \setminus E$



to the copies of the  $I$  intervals in  $\partial S_n$  and copies of  $D(0, 1) \setminus E$  to the  $J$  intervals. Then if we choose  $\delta_n$  and  $\tau_n$  small enough and  $m_n$  large enough, we can get the harmonic measure of  $R_n$  over  $C_{1/2}$  to be  $g_n(x) d\theta/2\pi$  with  $g_n$  as close to  $g_{n-1}\eta_n$  as we wish, say

$$\frac{1}{2} + \frac{1}{10^n} \leq g_n \leq \frac{1}{2} + \frac{3}{10^n}.$$

Continuing in this way we can clearly construct a sequence  $\{R_n\}$  of Riemann surfaces so that the harmonic measures over  $C_{1/2}$  converge to a multiple of Lebesgue measure. This almost finishes the proof, except that the surfaces  $\{R_n\}$  are not nested by inclusion. However, the subsurfaces  $\{S_n\}$  constructed as part of the induction are nested and their union is also  $R$ . Hence their harmonic measures converge to that of  $R$ . By (7-2), the weak\* limit for the measures on  $\{S_n\}$  and  $\{R_n\}$  must be the same, so we are done.

The same proof shows that we can build an  $f \in H^\infty$  so that  $\mu_f|_{C_{1/2}} = \frac{1}{2\pi} g d\theta$  for any continuous  $g$  with  $0 \leq g < 1$  (or any  $g$  which is the decreasing limit of such functions). Similarly, the circle can be replaced by any smooth curve  $\gamma$ , and  $g$  by a continuous function such that  $g ds \leq d\omega(0, \cdot, \mathbb{D} \setminus \gamma)$ .

The construction in this section clearly generalizes as follows.

**Lemma 7.3.** *Suppose  $R$  is a Riemann surface built by attaching subdomains of  $\mathbb{D}$  along boundary arcs. Let  $\Pi$  denote the corresponding projection of  $R$  into the plane. Suppose  $\Pi(\partial R)$  hits  $C_r$  and there is a  $\delta > 0$  such that every component of  $\Pi^{-1}(C_r)$  in  $\partial R$  is the boundary of a domain in  $R$  which is a regular cover of the annulus  $\{r - \delta < |z| < r\}$  (or  $\{r < |z| < r + \delta\}$ ). Suppose the harmonic measure of  $R$  over  $C_r$  projects to a measure of the form  $\frac{1}{2\pi} g d\theta$  on  $C_r$ , where  $0 < g < 1$ . Choose  $s < r$  (or  $s > r$ ) very close to  $r$ . Suppose we are given  $N$  functions  $\{\eta_k\}$  such that  $0 < \eta_k < 1$ . Choose a large integer  $m$  and choose  $mN$  equally spaced points  $\{z_i\}$  on  $C_s$ . Let  $I_j^k$  be the interval of length  $2\pi\eta_k(z_{k+jN})/mN$  centered at  $z_{k+jN}$ . Let  $J_i$  denote the components of  $C_s \setminus \bigcup_{j,k} I_j^k$ . Choose a small  $\tau$  and choose sets  $E$  satisfying (7-1) in every interval. For  $k = 0, \dots, N$ , choose  $s_k < s < r_k$ . For each arc in  $\partial R$  projecting to  $I_j^k$  attach a copy of  $A_k \setminus E = \{s_k < |z| < r_k\} \setminus E$ . To each arc projecting to a  $J_i$  attach a copy of  $A_0 \setminus E = \{s_0 < |z| < r_0\} \setminus E$ . If  $s$  is close enough to  $r$ , if  $m$  is large enough and if  $\tau$  is small enough, then the projected harmonic measure of the new surface  $S$  on  $\partial S \setminus R$  is as close to  $\sum_k B_k(\eta_k g)$  as we wish, where  $B_k$  denotes balayage from  $C_s$  onto  $\partial A_k$ .*

For the proof of [Theorem 1.1](#), we can always take  $s_k = 0$ , that is, we can attach disks instead of annuli. Only for the proof of [Corollary 1.4](#) will we have to attach proper annuli.

## 8. Theorem 1.1 on an annulus

In this section we will show that any radial probability measure  $\mu$  supported in the annulus  $\{z : \frac{1}{2} \leq |z| \leq 1\}$  is of the form  $\mu_f$  for some  $f \in \mathcal{U}_0$ , and in the next section we will extend this to the general case.

First some notation. For  $0 < r < s < 1$  let  $A(r, s) = \{z : r \leq |z| < s\}$ . When  $s = 1$ , we let  $A(r, 1) = \{z : r \leq |z| \leq 1\}$ . For  $0 < r < 1$ , let  $\mu(r) = \mu(A(0, r))$ . Let  $r^0 = \frac{1}{2}$ , let  $r_0^1 = \frac{1}{2}$ , let  $r_1^1 = \frac{3}{4}$  and, more generally, let  $r_k^n = \frac{1}{2} + k2^{-n-1}$  for  $k = 0, \dots, 2^n - 1$ . Let  $\mu_k^n = \mu(A(r_k^n, r_{k+1}^n))$ , and let  $C_k^n = C_{r_k^n}$ .

By rescaling, we may assume that  $\mathbb{T} \subset \text{supp}(\mu) \subset \overline{\mathbb{D}}$  and hence that  $\mu_{2^n-1}^n$  is positive for all  $n$ .

We will construct a sequence  $R_0 \subset R_1 \subset \dots$  of Riemann surfaces, such that the corresponding measure  $\mu_n$  is supported on the union of  $2^n$  circles,  $\bigcup_{k=0}^{2^n-1} C_k^n$ . On  $C_k^n$  the measure  $\mu_n$  will have the form  $\frac{1}{2\pi} g_k^n d\theta$  where

$$(8-1) \quad \mu_k^n < g_k^n \leq \mu_k^n + \epsilon_n$$

for  $k = 0, \dots, 2^n - 2$  and any  $\epsilon_n > 0$  we choose, and for  $k = 2^n - 1$  we have

$$(8-2) \quad \mu_{2^{n+1}-2}^{n+1} < g_k^n \leq \mu_{2^n-1}^n.$$

Recall that since  $\mu_n$  is a probability measure, if it gives too much mass to the first  $2^n - 1$  annuli, then it must give too little to the last one. It is obvious that such measures  $\{\mu_n\}$  converge weak\* to  $\mu$ , so by the argument at the end of the previous section, the  $\mu_f$  corresponding to the limiting surface  $R = \bigcup_n R_n$  must equal  $\mu$ .

Thus it only remains to construct the surfaces. As in the previous section we start with  $R_0 = D(0, \frac{1}{2})$ . To construct  $R_1$ , we will proceed exactly as in the previous section, except that instead of redirecting harmonic measure to the unit circle, we send it to the circle  $C_{3/4}$ . The estimates are all the same so we can obtain a surface  $R_1$  such that the corresponding  $\mu_1$  is supported on  $C_{1/2} \cup C_{3/4}$  and is of the form  $\frac{1}{2\pi} g_0^1 d\theta$  on  $C_0^1$  and  $\frac{1}{2\pi} g_1^1 d\theta$  on  $C_1^1$  where

$$\mu_0^1 < g_0^1 < \mu_0^1 + \epsilon_1 \quad \text{and} \quad \mu_2^2 < g_1^1 < \mu_1^1,$$

for any  $\epsilon_1 > 0$  we choose.

To construct  $R_{n+1}$  for  $n \geq 1$ , we just make one small change. The mass on the outermost circle  $C_{2^n-1}^n$  is redistributed to itself,  $C_{2^n-1}^n = C_{2^{n+1}-2}^{n+1}$ , and to the outermost circle of the next stage,  $C_{2^{n+1}-1}^{n+1}$ . The mass of any other circle  $C_j^n$  is redistributed to three circles; itself,  $\tilde{C}_j^n = C_{2j}^{n+1}$ , the next circle out in the next generation,  $C_{2j+1}^{n+1}$  and the outermost circle of the next generation,  $C_{2^{n+1}-1}^{n+1}$ .

To do this we let  $\tilde{C}_j^n$  be the circle of radius  $r_j^n - \delta_n$ , where  $\delta_n < 2^{-n-10}$  is chosen so small that the harmonic measure on  $S_n$  (the subsurface of  $R_n$  bounded by the lifts of  $\tilde{C}_j^n$  which contain 0 and hence contain  $S_{n-1}$ ) is as close as we wish to harmonic

measure on  $R_{n-1}$ , that is, it satisfies (7–2). We now just apply the construction of Lemma 7.3, with  $N = 2$ ,  $s_0 = s_1 = s_2 = 0$ ,  $r_0 = r_{2j}^{n+1}$ ,  $r_1 = r_{2j+1}^{n+1}$  and  $r_2 = r_{2^{n+1}-1}^{n+1}$ . More precisely, suppose that we have two continuous functions  $\eta_1$  and  $\eta_2$  defined on  $\tilde{C}_j^n$ , such that  $\eta_1 + \eta_2 < 2$ , together with  $m_n$  equidistributed points  $\{z_j\}$  on  $\partial S_n$ , and choose intervals centered at these points. However, instead of having two types of intervals, we will have three:  $\{I_j\}$  of angle measure  $2\pi\eta_1(\theta)/m_n$  centered at  $z_j$  for  $j$  even,  $\{K_j\}$  of angle measure  $2\pi\eta_2(\theta)/m_n$  centered at  $z_j$  for  $j$  odd, and the remaining intervals  $\{J_j\}$ . We choose a very small  $\tau_n$  and a countable set  $E$  in each interval which satisfies (7–1). Then along type  $I$  intervals we attach a copy of the universal cover of  $D(0, r_j^n) \setminus E$ , along the type  $K$  intervals we attach the universal cover of  $D(0, r_{2j+1}^{n+1}) \setminus E$ , and along the type  $J$  intervals we attach that of  $D(0, r_{2^{n+1}-1}^{n+1}) \setminus E$ . Then if we take  $m_n$  large enough and  $\delta_n$  and  $\tau_n$  small enough, the harmonic measure of the surface  $R_{n+1}$  over  $C_j^n$  will be as close to the balayage of  $\eta_1 g_j^n$  onto  $C_j^n$  as we wish and the harmonic measure over  $C_{2j+1}^{n+1}$  will be as close to the balayage of  $\eta_2 g_j^n$  onto that circle as we wish, independent of what changes we make at circles other than  $C_j^n$ .

Now do a similar construction around each circle  $C_j^n$ , for  $j = 0, \dots, 2^n - 2$ . At the outermost circle  $C_{2^n-1}^n$ , we redirect the measure to only two circles: itself and the outermost circle of the next generation,  $C_{2^{n+1}-1}^{n+1}$ . By construction, condition (8–1) holds with any constant  $\epsilon_n$  we want. Then by Lemma 2.6,  $\mu_n$  on the outermost circle must be normalized Lebesgue measure minus the balayage of the measures on the inner circles. Since these measures have total mass as close to, but larger than,

$$\mu\left(A\left(\frac{1}{2}, r_{2^n-1}^n\right)\right) = \sum_{j=0}^{2^n-2} \mu_j^n$$

as we wish, the mass of the outermost circle is as close to, but smaller than,  $\mu\left(A\left(r_{2^n-1}^n, 1\right)\right) = \mu_{2^n-1}^n$ . Moreover, since the measures on the inner circles are as close to radial as we wish, so is their balayage onto the outermost circle and hence so is  $\mu_f$  restricted to the outermost circle (this condition defines our choice of  $\epsilon_n$ ). This gives condition (8–2). The proof is completed by taking limits just as before.

## 9. Theorem 1.1 on the whole disk

To complete the proof of Theorem 1.1 we need to show how to obtain any measure satisfying (1–1). As in the last section we can assume  $\mathbb{T} \subset \text{supp}(\mu) \subset \overline{\mathbb{D}}$ . We can also simplify the situation slightly by observing that it is enough to assume that most of the mass of  $\mu$  lives away from the origin, that is,

$$(9-1) \quad \int \log \frac{1}{|z|} d\mu \leq \delta.$$

This is because for  $f \in H^\infty$  the measure  $\mu_{f^d}$  is the push-forward under  $z \rightarrow z^d$  of the measure  $\mu_f$  and so

$$\int \log \frac{1}{|z|} d\mu_f = \frac{1}{d} \int \log \frac{1}{|z|} d\mu_{f^d}.$$

By taking  $d$  large we can make the right-hand side as small as we wish. Thus for any  $\mu$  on the disk satisfying (1-1), it suffices to construct an  $f$  corresponding to the pull-back of  $\mu$  under  $z^d$ , that is, it suffices to consider only measures satisfying (9-1) for any  $\delta > 0$  we choose.

Start by taking  $R_0 = D(0, \frac{1}{4})$ . Let  $r_n = 2^{-n}$  for  $n = 0, 1, 2, \dots$  and let  $\mu_n = \mu(A(r_n, r_{n-1}))$ . Then

$$(9-2) \quad \sum_{n \geq 2} (n-1)(\log 2)\mu_n \leq \int \log \frac{1}{|z|} d\mu \leq \delta,$$

so

$$(9-3) \quad \mu_n \leq \frac{\delta}{(\log 2)(n-1)} \leq \frac{\delta'}{n},$$

where  $\delta'$  is as small as we wish.

We need two simple facts about harmonic measure on an annulus.

**Lemma 9.1.** *Suppose  $A = \{z : s < |z| < r\}$  and  $s < t < r$ . Then  $\omega(z, C_s, A) = u_{s,r}(z) = (\log |z| - \log r)/(\log s - \log r)$  for any  $z$  with  $|z| = t$ .*

*Proof.* This is immediate since the given function is harmonic in  $A$ , equals 1 on  $C_s$  and equals 0 on  $C_r$ .  $\square$

**Lemma 9.2.** *Suppose  $s, t, r$  and  $A$  are as in Lemma 9.1. Then if  $t \geq 2s$ , there is an  $M < \infty$ , independent of  $s, t$  and  $r$ , such that for  $|z| = t$ ,  $\omega(z, \cdot, A)$  restricted to  $C_s$  has the form  $\frac{1}{2\pi} g d\theta$  and  $g$  satisfies  $\max_{C_s} g \leq M \min_{C_s} g$ .*

*Proof.* Recall that harmonic measure on  $\partial A$  is the normal derivative of Green's function  $G$  with pole at  $z$ . Let  $t' = \frac{2}{3}t > s$ . By Harnack's inequality there is an  $M$  such that  $\max_{C_{t'}} G \leq M \min_{C_{t'}} G$ , and hence there is a constant  $C$  such that

$$C(1 - u_{s,t'}) \leq G \leq MC(1 - u_{s,t'}),$$

on  $\{s < |z| < t'\}$ . Since the normal derivative of  $u_{s,t'}$  is constant on  $C_s$  (since  $u$  is radial), this implies the normal derivative of  $G$  on  $C_s$  is trapped between two constants  $A$  and  $MA$ , as desired.  $\square$

Consider the annulus  $A_n = \{z : 2^{-n} < |z| < 2^{-1}, n = 3, 4, \dots\}$  and a point  $z$  such that  $|z| = \frac{1}{3}$ . The two previous results imply that there is a constant  $B$  such that harmonic measure for  $A$  on the circle  $C_{2^{-n}}$  is of the form  $\frac{1}{2\pi} g d\theta$  where  $g \geq B/n$  for  $n \geq 3$ . By (9-2) we can assume  $\mu$  is chosen so that  $\sum_n n\mu_n \leq (2B)^{-1}$ .

Thus  $\sum_n Bn\mu_n \leq \frac{1}{2}$ , and hence it is possible to choose a collection of disjoint, adjacent intervals  $\{I_n : n = 2, 3, 4, \dots\}$  on  $C_{1/4}$ , of angle measure  $4\pi n\mu_n/B$ . In each interval  $I_n$  choose a countable set  $E_n$  satisfying the “thickness” condition (7–1) with some  $\tau_n$ , and attach to  $I_n$  a copy of the universal cover of  $A_{n+1} \setminus E_n$ . The resulting Riemann surface has harmonic measures supported over the union of circles  $\bigcup_n C_{2^{-n}}$  for  $n = 1, 3, 4, 5, \dots$  and, moreover, if we choose  $\tau_n \rightarrow 0$  quickly enough, the harmonic measure of the circles corresponding to  $n = 3, 4, 5, \dots$  is of the form  $\frac{1}{2\pi} g_n d\theta$  with  $g_n > \mu_{n-1}$ , but might not be close to radial.

For each such circle  $C_{2^{-n}}$ , choose  $I$  and  $J$  intervals in the usual way and attach copies of  $D(0, \frac{1}{2}) \setminus E$  and  $D(0, 2^{-n}) \setminus E$  respectively. As we have seen before, we can choose  $\eta$ ,  $m$  and  $\tau$  so that the harmonic measure  $\frac{1}{2\pi} g_n d\theta$  on  $C_{2^{-n}}$  is as close to (but larger than)  $\mu_n$  as we wish. Using Lemma 2.6, the harmonic measure of  $C_{1/2}$  will be as close to (but less than)  $\mu_1$  as we wish and, in particular, it is larger than  $\mu(\{\frac{1}{2} \leq |z| < \frac{3}{4}\})$  (this is where we use the assumption that  $\mathbb{T}$  is in the support of  $\mu$ ).

The rest of the proof is now the same as the previous section. On each annulus we redistribute the harmonic measure from the circle into the annulus, sending any “extra” measure to the outermost circle,  $C_{1-2^{-n}}$ . In the limit, we obtain the desired measure  $\mu$ .

## 10. An example which is almost an outer function

In this section we will construct an orthogonal function  $f$  whose only inner factor is the required zero at 0, that is,  $f(z)/z$  is outer. We will construct  $f$  so that 0 is the only zero of  $f$ ; thus  $f(z)/z$  has no Blaschke factor. In order to prove it has no singular inner factor, recall that if  $f(z)/z = gh$  with  $g$  outer and  $h$  a nontrivial singular inner function, then

$$\log |f|^{-1} = \log |g|^{-1} + \log |h|^{-1},$$

and that the first term on the right is the Poisson integral of its boundary values on  $\mathbb{T}$ , but that the second term is the Poisson integral of a singular measure on  $\mathbb{T}$  and has boundary value zero almost everywhere on  $\mathbb{T}$ . Let

$$H_\epsilon = \{z \in \mathbb{D} : |h(z)| < \epsilon\} \quad \text{and} \quad F_\epsilon = \{z \in \mathbb{D} : |f(z)| < \epsilon\}.$$

Since  $\log |h(0)|^{-1} = \log(1/\epsilon)\omega(0, H_\epsilon, \mathbb{D} \setminus H_\epsilon)$ , we deduce that

$$\omega(0, H_\epsilon, \mathbb{D} \setminus H_\epsilon) \geq C/\log(1/\epsilon),$$

where  $C = \log |h(0)|^{-1}$  and consequently, since  $H_\epsilon \subset F_\epsilon$ ,

$$(10-1) \quad \omega(0, F_\epsilon, \mathbb{D} \setminus F_\epsilon) \geq C/\log(1/\epsilon).$$

We will construct  $R$  so that the harmonic measure of  $\{z \in R \setminus D(0, \frac{1}{2}) : |z| \leq 2^{-n}\}$  has harmonic measure (in  $R$ , with respect to 0) less than  $\lambda^n$  for some  $\lambda < 1$ . This contradicts (10–1), so the covering map has no singular inner factor.

Since we have already seen several constructions of this type in great detail, I will only sketch the construction. Start with  $R_0 = D(0, \frac{1}{2})$ . Divide  $C_{1/2}$  into a finite collection of intervals  $\{I_n\}$  and in each choose a set  $E$  satisfying (7–1). Along each interval attach a copy of  $\{\frac{1}{4} < |z| < 1\} \setminus E$ . This gives  $R_2$ .

Lemmas 9.1 and 9.2 imply that harmonic measure of  $R_2$  over  $C_{1/4}$  is of the form  $\frac{1}{2\pi} g d\theta$  where the max of  $g$  is bounded by a universal constant times the minimum. Thus there is a constant  $c < \min(g)$  and a  $\lambda < 1$  such that

$$\int (g - c) d\theta \leq \lambda \int g d\theta.$$

In other words, we can truncate  $g$  to be a constant and still retain a fixed fraction of the harmonic measure.

Now do the standard construction of  $I$  and  $J$  intervals on  $C_{\frac{1}{4}+\delta}$ , attaching copies of  $\{\frac{1}{8} < |z| < 1\}$  and  $\{\frac{1}{4} < |z| < 1\}$  respectively, so that the new harmonic measure on  $C_{1/4}$  is very close to radial (say within  $\epsilon_1$  of constant) and has mass at least  $(1 - \lambda)$  times the previous mass.

At the next stage we do the construction near both circles  $C_{1/4}$  and  $C_{1/8}$ . At  $C_{1/8}$  we repeat the process of the previous paragraph, making the harmonic measure above  $C_{1/8}$  as close to radial as we wish, while retaining at least  $(1 - \lambda)$  of the total mass, transferring the excess to  $C_1$  and  $C_{1/16}$ . On  $C_{1/4}$  we only make the measure within  $\epsilon_2$  of constant (while losing at most  $\epsilon_1$  of the mass), the excess being transferred to  $C_{1/8}$  and  $C_1$ .

We now iterate the process in the obvious way. At stage  $n$  we have a surface  $R_n$  which only covers the origin once, and such that the harmonic measure is supported on the circles  $\{C_{2^{-k}}\}$ , with the  $k$ -th circle getting mass at most  $\lambda^k$ . Thus the same is true for the limiting measure  $\mu$ , and hence the harmonic measure of the set  $\{z \in R \setminus R_0 : |z| < 2^{-n}\}$  has harmonic measure less than  $C\lambda^n$  in  $R$ . This proves that  $f(z)/z$  is outer.

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## References

- [Aleksandrov et al. 1999] A. B. Aleksandrov, J. M. Anderson, and A. Nicolau, “[Inner functions, Bloch spaces and symmetric measures](#)”, *Proceedings London Math. Soc.* (3) **79**:2 (1999), 318–352. [MR 2000g:46029](#) [Zbl 01463593](#)
- [Bishop 1993] C. J. Bishop, “An indestructible Blaschke product in the little Bloch space”, *Publ. Mat.* **37**:1 (1993), 95–109. [MR 94j:30032](#) [Zbl 0810.30024](#)
- [Bourdon 1997a] P. S. Bourdon, “[Rudin’s orthogonality problem and the Nevanlinna counting function](#)”, *Proc. Amer. Math. Soc.* **125**:4 (1997), 1187–1192. [MR 98b:30034](#) [Zbl 0866.30028](#)
- [Bourdon 1997b] P. S. Bourdon, “Rudin’s orthogonality problem and the Nevanlinna counting function, II”, 1997. Unpublished notes.
- [Cantón 1998] A. Cantón, “Singular measures and the little Bloch space”, *Publ. Mat.* **42**:1 (1998), 211–222. [MR 99g:30046](#) [Zbl 0916.30032](#)
- [Cima and Hansen 1990] J. A. Cima and L. J. Hansen, “[Space-preserving composition operators](#)”, *Michigan Math. J.* **37**:2 (1990), 227–234. [MR 91m:47042](#) [Zbl 0715.30028](#)
- [Cima et al. 1974/75] J. A. Cima, J. Thomson, and W. Wogen, “On some properties of composition operators”, *Indiana Univ. Math. J.* **24** (1974/75), 215–220. [MR 50 #2979](#) [Zbl 0276.47038](#)
- [Cima et al. 1993] J. A. Cima, B. Korenblum, and M. Stessin, “Composition isometries and Rudin’s problems”, preprint, 1993.
- [Fernández et al. 1996] J. L. Fernández, D. Pestana, and J. M. Rodríguez, “[Distortion of boundary sets under inner functions, II](#)”, *Pacific J. Math.* **172**:1 (1996), 49–81. [MR 97b:30035](#) [Zbl 0847.32005](#)
- [Garnett 1981] J. B. Garnett, *Bounded analytic functions*, Pure and Applied Mathematics **96**, Academic Press Inc., New York, 1981. [MR 83g:30037](#) [Zbl 0469.30024](#)
- [Löwner 1923] K. Löwner, “Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I”, *Math. Ann.* **89** (1923), 103–121. [JFM 49.0714.01](#)
- [Nordgren 1968] E. A. Nordgren, “Composition operators”, *Canad. J. Math.* **20** (1968), 442–449. [MR 36 #6961](#) [Zbl 0161.34703](#)
- [Rudin 1967] W. Rudin, “A generalization of a theorem of Frostman”, *Math. Scand* **21** (1967), 136–143 (1968). [MR 38 #3463](#) [Zbl 0185.33301](#)
- [Rudin 1973] W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973. [MR 51 #1315](#) [Zbl 0185.33301](#)
- [Rudin 1980] W. Rudin, *Function theory in the unit ball of  $\mathbb{C}^n$* , Grundlehren der Mathematischen Wissenschaften **241**, Springer, New York, 1980. [MR 82i:32002](#) [Zbl 0495.32001](#)
- [Ryff 1966] J. V. Ryff, “[Subordinate  \$H^p\$  functions](#)”, *Duke Math. J.* **33** (1966), 347–354. [MR 33 #289](#) [Zbl 0148.30205](#)
- [Shapiro 1987] J. H. Shapiro, “[The essential norm of a composition operator](#)”, *Ann. of Math.* (2) **125**:2 (1987), 375–404. [MR 88c:47058](#) [Zbl 0642.47027](#)
- [Smith 1996] W. Smith, “[Composition operators between Bergman and Hardy spaces](#)”, *Trans. Amer. Math. Soc.* **348**:6 (1996), 2331–2348. [MR 96i:47056](#) [Zbl 0857.47020](#)
- [Smith 1998] W. Smith, “[Inner functions in the hyperbolic little Bloch class](#)”, *Michigan Math. J.* **45**:1 (1998), 103–114. [MR 2000e:30070](#) [Zbl 0976.30018](#)
- [Smith and Yang 1998] W. Smith and L. Yang, “[Composition operators that improve integrability on weighted Bergman spaces](#)”, *Proc. Amer. Math. Soc.* **126**:2 (1998), 411–420. [MR 98d:47070](#) [Zbl 0892.47031](#)

- [Stephenson 1988] K. Stephenson, “Construction of an inner function in the little Bloch space”, *Trans. Amer. Math. Soc.* **308**:2 (1988), 713–720. [MR 89k:30031](#) [Zbl 0654.30024](#)
- [Sundberg 2003] C. Sundberg, “Measures induced by analytic functions and a problem of Walter Rudin”, *J. Amer. Math. Soc.* **16**:1 (2003), 69–90. [MR 2003i:30049](#) [Zbl 1012.30022](#)
- [Tsuji 1959] M. Tsuji, *Potential theory in modern function theory*, Maruzen Co. Ltd., Tokyo, 1959. [MR 22 #5712](#) [Zbl 0322.30001](#)
- [Wheeden and Zygmund 1977] R. L. Wheeden and A. Zygmund, *Measure and integral*, Marcel Dekker Inc., New York, 1977. [MR 58 #11295](#) [Zbl 0362.26004](#)
- [Zorboska 1994] N. Zorboska, “Composition operators with closed range”, *Trans. Amer. Math. Soc.* **344**:2 (1994), 791–801. [MR 94k:47050](#) [Zbl 0813.47037](#)

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