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Let k be a finite field, a global field, or a local non-archimedean field, and let H_1 and H_2 be split, connected, semisimple algebraic groups over k. We prove that if H_1 and H_2 share the same set of maximal k-tori, up to k-isomorphism, then the Weyl groups $W(H_1)$ and $W(H_2)$ are isomorphic, and hence the algebraic groups modulo their centers are isomorphic except for a switch of a certain number of factors of type B_n and C_n.

(Due to a recent result of Philippe Gille, this result also holds for fields which admit arbitrary cyclic extensions.)

1. Introduction

Let H be a connected, semisimple algebraic group over a field k. It is natural to ask to what extent the group H is determined by the k-isomorphism classes of maximal k-tori contained in it. We study this question over finite fields, global fields and local non-archimedean fields, and prove the following theorem.

Theorem 1.1 (Theorem 4.1). *Let k be a finite field, a global field or a local non-archimedean field, and let H_1 and H_2 be split, connected, semisimple algebraic groups over k. Suppose that for every maximal k-torus $T_1 \subset H_1$ there exists a maximal k-torus $T_2 \subset H_2$ such that the tori T_1 and T_2 are k-isomorphic, and vice versa. Then the Weyl groups $W(H_1)$ and $W(H_2)$ are isomorphic.*

Moreover, if we write $W(H_1)$ and $W(H_2)$ as a direct product of Weyl groups of simple algebraic groups, $W(H_1) = \prod_{\Lambda_1} W_{1,\alpha}$, and $W(H_2) = \prod_{\Lambda_2} W_{2,\beta}$, then there exists a bijection $i : \Lambda_1 \to \Lambda_2$ such that $W_{1,\alpha}$ is isomorphic to $W_{2,i(\alpha)}$ for every $\alpha \in \Lambda_1$.

Since a split simple algebraic group with trivial center is determined by its Weyl group, except for the groups of the type B_n and C_n, we have following theorem.

Theorem 1.2. *Let k be as in Theorem 1.1, and let H_1 and H_2 be split, connected, semisimple algebraic groups over k, with trivial center. Write H_1 and H_2 as direct products of simple groups: $H_1 = \prod_{\Lambda_1} H_{1,\alpha}$, and $H_2 = \prod_{\Lambda_2} H_{2,\beta}$. If H_1 and H_2 satisfy the condition given in Theorem 1.1, then there is a bijection $i : \Lambda_1 \to \Lambda_2$*

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such that $H_{1,\alpha}$ is isomorphic to $H_{2,i(\alpha)}$, except for the case where $H_{1,\alpha}$ is a simple group of type B_n or C_n, in which case $H_{2,i(\alpha)}$ could be of type C_n or B_n.

From the explicit description of maximal k-tori in $\mathrm{SO}(2n+1)$ and $\mathrm{Sp}(2n)$ (see for instance [Kariyama 1989, Proposition 2]) one finds that these groups contain the same set of k-isomorphism classes of maximal k-tori.

We show by an example that the existence of split tori in the groups H_1 and H_2 is necessary. Note that if k is $\mathbb{Q}_p$, then the Brauer group of k is $\mathbb{Q}/\mathbb{Z}$. Consider the central division algebras of degree five, D_1 and D_2, corresponding to $1/5$ and $2/5$ in $\mathbb{Q}/\mathbb{Z}$ respectively, and let

$$H_1 = \mathrm{SL}_1(D_1) \quad \text{and} \quad H_2 = \mathrm{SL}_1(D_2).$$

The maximal tori in H_1 and H_2 correspond, respectively, to the maximal commutative subfields in D_1 and D_2. But over $\mathbb{Q}_p$ every division algebra of a fixed degree contains every field extension of that degree (see [Pierce 1982, Proposition 17.10 and Corollary 13.3]), so H_1 and H_2 share the same set of maximal tori over k. But they are not isomorphic, since it is known that $\mathrm{SL}_1(D) \cong \mathrm{SL}_1(D')$ if and only if $D \cong D'$ or $D \cong (D')^{\mathrm{op}}$ [Knus et al. 1998, 26.11].

This paper is arranged as follows. The description of the k-conjugacy classes of maximal k-tori in an algebraic group H defined over k can be given in terms of the Galois cohomology of the normalizer in H of a fixed maximal torus. Similarly, the k-isomorphism classes of n-dimensional tori defined over k can be described in terms of n-dimensional integral representations of the Galois group of $\bar{k}$ (the algebraic closure of k) over k. Using these two descriptions, in Section 2 we obtain a Galois cohomological description for the k-isomorphism classes of maximal k-tori in H. Since we are dealing with groups that are split over the base field k, the Galois action on the Weyl groups is trivial. This enables us to prove, in Section 4, that if split, connected, semisimple algebraic groups H_1 and H_2 of rank n share the same set of maximal k-tori up to k-isomorphism, then the Weyl groups $W(H_1)$ and $W(H_2)$, considered as subgroups of $\mathrm{GL}_n(\mathbb{Z})$, share the same set of elements up to conjugacy in $\mathrm{GL}_n(\mathbb{Z})$.

This then is the main question to be answered: if the Weyl groups of two split, connected, semisimple algebraic groups, W_1 and W_2, embedded in $\mathrm{GL}_n(\mathbb{Z})$ in the natural way, i.e., by their action on the character group of a fixed split maximal torus, have the property that every element of W_1 is $\mathrm{GL}_n(\mathbb{Z})$-conjugate to one in W_2 and vice versa, are the Weyl groups isomorphic? Much of the work in this paper seeks to prove this statement by using elaborate information available about the conjugacy classes in Weyl groups of simple algebraic groups together with their standard representations in $\mathrm{GL}_n(\mathbb{Z})$. Our analysis finally depends on the knowledge of characteristic polynomials of elements in the Weyl groups considered as subgroups of $\mathrm{GL}_n(\mathbb{Z})$. This information is summarized in Section 3. Using it we prove the main theorem in Section 4.

We emphasize that if we were proving Theorems 1.1 and 1.2 for simple algebraic groups, our proofs would be relatively very simple. However, for semisimple groups, we have to make a somewhat complicated inductive argument on the maximal rank among the simple factors of the semisimple groups H_1 and H_2.

We use the term "simple Weyl group of rank r" for the Weyl group of a simple algebraic group of rank r. Any Weyl group is a product of simple Weyl groups in a unique way (up to permutation). We say that two Weyl groups are *isomorphic* if and only if the simple factors and their multiplicities are the same.

The question studied in this paper seems relevant for the study of Mumford–Tate groups over number fields. The author was informed, after the completion of the paper, that Theorem 1.1 over a finite field is implicit in the work of Larsen and Pink [1992]. We would like to mention that although much of the paper could be said to be implicitly contained in [Larsen and Pink 1992], the theorems we state (and prove) are not explicitly stated there, and our proofs are also different.

2. Galois cohomological lemmas

We begin by fixing notation. Let k denote an arbitrary field and let $G(\bar{k}/k)$ be the Galois group of $\bar{k}$ (the algebraic closure of k) over k. Let H denote a split, connected, semisimple algebraic group defined over k and let T_0 be a fixed split maximal torus in H, of dimension n. Let W be the Weyl group of H with respect to T_0. Then we have an exact sequence of algebraic groups defined over k,

$$0 \longrightarrow T_0 \longrightarrow N(T_0) \longrightarrow W \longrightarrow 1$$

where $N(T_0)$ denotes the normalizer of T_0 in H.

The above exact sequence gives us a map $\psi : H^1(k, N(T_0)) \to H^1(k, W)$. It is well known that a certain subset of $H^1(k, N(T_0))$ classifies k-conjugacy classes of maximal k-tori in H. For the sake of completeness, we formulate this as a lemma in the case of split, connected, semisimple groups.

Lemma 2.1. *Let H be a split, connected, semisimple algebraic group defined over a field k and let T_0 be a fixed split maximal torus in H. The natural embedding $N(T_0) \hookrightarrow H$ induces a map $\Psi : H^1(k, N(T_0)) \to H^1(k, H)$. The set of k-conjugacy classes of maximal tori in H are in one-one correspondence with the subset of $H^1(k, N(T_0))$ which is mapped to the neutral element in $H^1(k, H)$ by the map Ψ.*

Proof. Let T be a maximal k-torus in H and let L be a splitting field of T, that is, assume that the torus T splits as a product of $\mathbb{G}_m$s over L. We assume that the field L is Galois over k. By the uniqueness of maximal split tori up to conjugacy, there exists an element $a \in H(L)$ such that $aT_0a^{-1} = T$. Then for any $\sigma \in G(L/k)$, we have $\sigma(a)T_0\,\sigma(a)^{-1} = T$, as both T_0 and T are defined over k. This implies that

$$\big(a^{-1}\sigma(a)\big)T_0\big(a^{-1}\sigma(a)\big)^{-1} = T_0.$$

Therefore $a^{-1}\sigma(a) \in N(T_0)$. This enables us to define a map $G(L/k) \to N(T_0)$ which sends σ to $a^{-1}\sigma(a)$, and by composing this map with the natural map $G(\bar{k}/k) \to G(L/k)$, we get a map $\phi_a : G(\bar{k}/k) \to N(T_0)$. We check that

$$\phi_a(\sigma\tau) = \phi_a(\sigma)\sigma\big(\phi_a(\tau)\big)$$

for all $\sigma, \tau \in G(\bar{k}/k)$, and hence that ϕ_a is a 1-cocycle. If $b \in H(L)$ is another element such that $bT_0\,b^{-1} = T$, we see that

$$\phi_a(\sigma) = (b^{-1}a)^{-1}\phi_b(\sigma)\sigma(b^{-1}a).$$

Thus the element $[\phi_a] \in H^1(k, N(T_0))$ is determined by the maximal torus T, and so we denote it by $\phi(T)$. It is clear that $\phi(T)$ is determined by the k-conjugacy class of T. Moreover, if $\phi(T) = \phi(S)$ for two maximal tori T and S in H, then one can check that these two tori are conjugate over k. Indeed, if $T = aT_0\,a^{-1}$ and $S = bT_0\,b^{-1}$ for $a, b \in H(\bar{k})$ then for any $\sigma \in G(\bar{k}/k)$,

$$a^{-1}\sigma(a) = c^{-1}\big(b^{-1}\sigma(b)\big)\sigma(c)$$

for some $c \in N(T_0)$. Then $\sigma(bca^{-1}) = bca^{-1}$ for all $\sigma \in G(\bar{k}/k)$, and hence $bca^{-1} \in H(k)$ and $(bca^{-1})T(bca^{-1})^{-1} = S$. Further, it is clear that the image of ϕ in $H^1(k, N(T_0))$ is mapped to the neutral element in $H^1(k, H)$ by Ψ.

Moreover, if $\phi_1 : G(\bar{k}/k) \to N(T_0)$ is a 1-cocycle such that $\Psi(\phi_1)$ is neutral in $H^1(k, H)$, then $\phi_1(\sigma) = a^{-1}\sigma(a)$ for some $a \in H(\bar{k})$. Then the cohomology class $[\phi_1] \in H^1(k, N(T_0))$ corresponds to the maximal torus $S_1 = aT_0a^{-1}$ in H. Since $a^{-1}\sigma(a) = \phi_1(\sigma) \in N(T_0)$, the torus S_1 is invariant under the Galois action, and so we conclude that it is defined over k. Thus the image of ϕ is the inverse image of the neutral element in $H^1(k, H)$ under the map Ψ. This is the complete description of the k-conjugacy classes of maximal k-tori in the group H.

Finally, we observe that the detailed proof we have given above amounts to looking at the exact sequence $1 \to N(T_0) \to H \to H/N(T_0) \to 1$ which gives an exact sequence

$$H/N(T_0)(k) \longrightarrow H^1(k, N(T_0)) \longrightarrow H^1(k, H)$$

of pointed sets. Therefore $H/N(T_0)(k)$, which is the variety of conjugacy classes of k-tori in H, is identified with the elements of $H^1(k, N(T_0))$ which become trivial in $H^1(k, H)$. □

We also recall the correspondence between k-isomorphism classes of n-dimensional k-tori and equivalence classes of n-dimensional integral representations of

$G(\bar{k}/k)$. Let $T_0 = \mathbb{G}_m^n$ be the split torus of dimension n, let T_1 be an n-dimensional torus defined over k, and let L_1 denote the splitting field of T_1. Since the torus T_1 is split over L_1, we have an L_1-isomorphism $f : T_0 \to T_1$. The Galois action on T_0 and T_1 gives us another isomorphism, $f^\sigma := \sigma f \sigma^{-1} : T_0 \to T_1$. Again one sees that the map $\varphi_f : G(\bar{k}/k) \to \mathrm{Aut}_{L_1}(T_0)$, given by $\sigma \mapsto f^{-1} f^\sigma$, is a 1-cocycle. Since the torus T_0 is already split over k, we have $\mathrm{Aut}_{L_1}(T_0) \cong \mathrm{Aut}_k(T_0)$, and hence the Galois group $G(\bar{k}/k)$ acts trivially on $\mathrm{Aut}_{L_1}(T_0)$, which is isomorphic to $\mathrm{GL}_n(\mathbb{Z})$. Therefore, φ_f is actually a homomorphism from the Galois group $G(\bar{k}/k)$ to $\mathrm{GL}_n(\mathbb{Z})$. This homomorphism gives an n-dimensional integral representation of the absolute Galois group, $G(\bar{k}/k)$. By changing the isomorphism f to any other L_1-isomorphism from T_0 to T_1, we get a conjugate of φ_f. Thus the element $[\varphi_f]$ in $H^1(k, \mathrm{GL}_n(\mathbb{Z}))$ is determined by T_1 and we denote it by $\varphi(T_1)$. Thus a k-isomorphism class of an n-dimensional torus gives us an equivalence class of n-dimensional integral representations of the Galois group, $G(\bar{k}/k)$. This correspondence is known to be bijective [Platonov and Rapinchuk 1994, 2.2].

Since the group H that we consider here is split over the base field k, the Weyl group W of H is defined over k, and $W(\bar{k}) = W(k)$. Therefore $G(\bar{k}/k)$ acts trivially on W, and hence $H^1(k, W)$ is the set of conjugacy classes of elements in $\mathrm{Hom}(G(\bar{k}/k), W)$. Since W acts faithfully on the character group of T_0, we can consider $W \hookrightarrow \mathrm{GL}_n(\mathbb{Z})$ and thus each element of $H^1(k, W)$ gives us an integral representation of the absolute Galois group. For a maximal torus T in H, we already have an n-dimensional integral representation of $G(\bar{k}/k)$, as described above. We prove that this representation is equivalent to a Galois representation given by an element of $H^1(k, W)$.

Lemma 2.2. *Let H be a split, connected, semisimple algebraic group defined over k. Fix a maximal split k-torus T_0 in H. Let T be a maximal k-torus in H, let $\phi(T) \in H^1(k, N(T_0))$ be the cohomology class corresponding to the k-conjugacy class of T in H, and let $\varphi(T) \in H^1(k, \mathrm{GL}_n(\mathbb{Z}))$ be the cohomology class corresponding to the k-isomorphism class of T. Then the integral representations given by $\varphi(T)$ and $i \circ \psi \circ \phi(T)$ are equivalent, where $\psi : H^1(k, N(T_0)) \to H^1(k, W)$ is induced by the natural map from $N(T_0)$ to W, and i is the natural map from $H^1(k, W)$ to $H^1(k, \mathrm{GL}_n(\mathbb{Z}))$.*

Proof. Let L be a splitting field of T, then an element $a \in H(L)$ such that $aT_0a^{-1} = T$ enables us to define a 1-cocycle $\phi_a : G(\bar{k}/k) \to N(T_0)$ given by $\phi_a(\sigma) = a^{-1}\sigma(a)$. The element $\phi(T) \in H^1(k, N(T_0))$ is precisely the class $[\phi_a]$.

Further, we treat conjugation by a as an L-isomorphism $f : T_0 \to T$, and then it can be checked that the map $f^\sigma := \sigma f \sigma^{-1}$ is precisely conjugation by $\sigma(a)$. The element $\varphi(T) \in H^1(k, \mathrm{GL}_n(\mathbb{Z}))$ is equal to $[\varphi_f]$, where $\varphi_f(\sigma) = f^{-1} f^\sigma$. Now, the map $\psi : N(T_0) \to W$ is the natural map taking an element $\alpha \in N(T_0)$ to

$\bar{\alpha} := \alpha \cdot T_0 \in W = N(T_0)/T_0$. Hence we have

$$\psi\big(\phi_a(\sigma)\big) = \overline{a^{-1}\sigma(a)} = f^{-1} f^{\sigma} = \varphi_f(\sigma).$$

Since the action of W on T_0 is given by conjugation, it is clear that the integral representation of the Galois group $G(\bar{k}/k)$, given by $\psi(\phi(T))$, is equivalent to the one given by $\varphi(T)$. □

Thus, a k-isomorphism class of a maximal torus in H gives an element in $H^1(k, W)$. We note here that not every subgroup of the Weyl group W may appear as a Galois group of some finite extension K/k. For instance, if k is a local field of characteristic zero it is known that the Galois group of any finite extension over k is a solvable group [Serre 1979, IV].

If we assume that the base field k is either a finite field or a local non-archimedean field, we have the following result.

Lemma 2.3. *Let k be a finite field or a local non-archimedean field and let H be a split, connected, semisimple algebraic group defined over k. Fix a split maximal torus T_0 in H and let W denote the Weyl group of H with respect to T_0. An element in $H^1(k, W)$ which corresponds to a homomorphism $\rho : G(\bar{k}/k) \to W$ with cyclic image, corresponds to a k-isomorphism class of a maximal torus in H under the mapping $\psi : H^1(k, N(T_0)) \to H^1(k, W)$.*

Proof. Consider the map $\Psi : H^1(k, N(T_0)) \to H^1(k, H)$ induced by the inclusion $N(T_0) \hookrightarrow H$. If we denote the neutral element in $H^1(k, H)$ by ι, then by Lemma 2.1 the set

$$X := \big\{ f \in H^1(k, N(T_0)) : \Psi(f) = \iota \big\}$$

is in one-one correspondence with the k-conjugacy classes of maximal k-tori in H. By Lemma 2.2, it is enough to show that $[\rho] \in \psi(X)$, where $\psi : H^1(k, N(T_0)) \to H^1(k, W)$ is induced by the natural map from $N(T_0)$ to W.

By Tits' theorem [1966, 4.6], there exists a subgroup $\overline{W}$ of $N(T_0)(k)$ such that the sequence

$$0 \longrightarrow \mu_2^n \longrightarrow \overline{W} \longrightarrow W \longrightarrow 1$$

is exact. Let N denote the image of ρ in W. We know that N is a cyclic subgroup of W. Let w be a generator of N and $\overline{w}$ be a lifting of w to $\overline{W}$. Since the base field k admits cyclic extensions of any given degree, there exists a map ρ_1 from $G(\bar{k}/k)$ to $\overline{W}$ whose image is the cyclic subgroup generated by $\overline{w}$. Since the Galois action on $\overline{W}$ is trivial, as $\overline{W}$ is a subgroup of $N(T_0)(k)$, the map ρ_1 could be treated as a 1-cocycle from $G(\bar{k}/k)$ to $N(T_0)$. Consider $[\rho_1]$ as an element in $H^1(k, N(T_0))$, then $\psi[\rho_1] = [\rho] \in H^1(k, W)$. We now consider two cases.

Case 1: k is a finite field.

By Lang's Theorem [1956, Corollary to Theorem 1], $H^1(k, H)$ is trivial and so the set X coincides with $H^1(k, N(T_0))$. Therefore the element $[\rho_1] \in H^1(k, N(T_0))$ corresponds to a k-conjugacy class of maximal k-torus in H. Then, by Lemma 2.2, $[\rho] = \psi[\rho_1]$ corresponds to a k-isomorphism class of maximal k-tori in H.

Case 2: k is a local non-archimedean field.

By [Platonov and Rapinchuk 1994, Proposition 2.10] there exists a semisimple, simply connected algebraic group $\widetilde{H}$, which is defined over k, together with a k-isogeny $\pi : \widetilde{H} \to H$. We have already fixed a split maximal torus T_0 in H; let $\widetilde{T}_0$ be the split maximal torus in $\widetilde{H}$ which gets mapped to T_0 by the covering map π. It can be seen that by restriction we get a surjective map $\pi : N(\widetilde{T}_0) \to N(T_0)$, where the normalizers are taken in appropriate groups. Moreover, the induced map $\pi_1 : \widetilde{W} \to W$ is an isomorphism.

We define the maps

$$\tilde{\psi} : H^1(k, N(\widetilde{T}_0)) \to H^1(k, \widetilde{W}) \quad \text{and} \quad \tilde{\Psi} : H^1(k, N(\widetilde{T}_0)) \to H^1(k, \widetilde{H})$$

in the same way as the maps ψ and Ψ are defined for the group H.

Consider the following diagram, where the horizontal arrows represent natural maps.

$$\begin{array}{ccccc} \widetilde{H} & \longleftarrow & N(\widetilde{T}_0) & \longrightarrow & \widetilde{W} \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ H & \longleftarrow & N(T_0) & \longrightarrow & W, \end{array}$$

It is clear that this diagram is commutative and hence so is the following one.

$$\begin{array}{ccccc} H^1(k, \widetilde{H}) & \xleftarrow{\tilde{\Psi}} & H^1(k, N(\widetilde{T}_0)) & \xrightarrow{\tilde{\psi}} & H^1(k, \widetilde{W}) \\ {\scriptstyle \pi^*}\downarrow & & {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle \pi_1^*} \\ H^1(k, H) & \xleftarrow[\Psi]{} & H^1(k, N(T_0)) & \xrightarrow[\psi]{} & H^1(k, W). \end{array}$$

Since π_1 is an isomorphism, the map π_1^* is a bijection. Now consider an element $[\rho] \in H^1(k, W)$ such that the image of the 1-cocycle ρ is a cyclic subgroup of W, and let $[\tilde{\rho}]$ be its inverse image in $H^1(k, \widetilde{W})$ under the bijection π_1^*. Using Tits' theorem [1966] as above, we lift $[\tilde{\rho}]$ to an element $[\tilde{\rho}_1]$ in $H^1(k, N(\widetilde{T}_0))$. Since $\widetilde{H}$ is simply connected and k is a non-archimedean local field, $H^1(k, \widetilde{H})$ is trivial [Bruhat and Tits 1967; Kneser 1965a, 1965b]. Therefore, $\tilde{\Psi}[\tilde{\rho}_1]$ is neutral in $H^1(k, \widetilde{H})$ and so is $\pi^*(\tilde{\Psi}[\tilde{\rho}_1])$ in $H^1(k, H)$. By commutativity of the diagram, we have that the element $[\rho] \in H^1(k, W)$ has a lift $\pi^*[\tilde{\rho}_1]$ in $H^1(k, N(T_0))$ such that $\Psi(\pi^*[\tilde{\rho}_1])$ is neutral in $H^1(k, H)$. Thus the element $[\rho]$ corresponds to a k-isomorphism class of a maximal torus in H. □

3. Characteristic polynomials

For a finite subgroup W of $\mathrm{GL}_n(\mathbb{Z})$, we define $\mathrm{ch}(W)$ to be the set of characteristic polynomials of elements of W, and $\mathrm{ch}^*(W)$ to be the set of irreducible factors of elements of $\mathrm{ch}(W)$. Since all the elements of W are of finite order, the irreducible factors (over $\mathbb{Q}$) of the characteristic polynomials are cyclotomic polynomials. We denote by ϕ_r the r-th cyclotomic polynomial, that is, the irreducible monic polynomial over $\mathbb{Z}$ satisfied by a primitive r-th root of unity. We define

$$\mathfrak{m}_i(W) = \max\left\{t : \phi_i^t \text{ divides } f \text{ for some } f \in \mathrm{ch}(W)\right\}$$

and

$$\mathfrak{m}'_i(W) = \min\left\{t : \phi_2^t \cdot \phi_i^{\mathfrak{m}_i(W)} \text{ divides } f \text{ for some } f \in \mathrm{ch}(W)\right\}.$$

For positive integers $i \neq j$, we define

$$\mathfrak{m}_{i,j}(W) = \max\left\{t + s : \phi_i^t \cdot \phi_j^s \text{ divides } f \text{ for some } f \in \mathrm{ch}(W)\right\}.$$

If U_1 is a subgroup of $\mathrm{GL}_n(\mathbb{Z})$ and U_2 is a subgroup of $\mathrm{GL}_m(\mathbb{Z})$, then $U_1 \times U_2$ can be treated as a subgroup of $\mathrm{GL}_{m+n}(\mathbb{Z})$. Then

$$\mathrm{ch}(U_1 \times U_2) = \left\{f_1 \cdot f_2 : f_1 \in \mathrm{ch}(U_1),\ f_2 \in \mathrm{ch}(U_2)\right\}.$$

Moreover, one can easily check that

$$\mathfrak{m}_i(U_1 \times U_2) = \mathfrak{m}_i(U_1) + \mathfrak{m}_i(U_2),$$
$$\mathfrak{m}'_i(U_1 \times U_2) = \mathfrak{m}'_i(U_1) + \mathfrak{m}'_i(U_2)$$

for all i, and

$$\mathfrak{m}_{i,j}(U_1 \times U_2) = \mathfrak{m}_{i,j}(U_1) + \mathfrak{m}_{i,j}(U_2)$$

for all i, j. A simple Weyl group W of rank n has a natural embedding in $\mathrm{GL}_n(\mathbb{Z})$. We obtain a description of the sets $\mathrm{ch}^*(W)$ with respect to this natural embedding. Here we use the following result due to T. A. Springer [1974, Theorem 3.4(i)] about the fundamental degrees of the Weyl group W. We recall that the degrees of the generators of the invariant algebra of the Weyl group are called as the fundamental degrees of the Weyl group.

Theorem 3.1 (Springer). *Let W be a complex reflection group with fundamental degrees $d_1, d_2, \dots, d_m$. An r-th root of unity occurs as an eigenvalue for some element of W if and only if r divides one of the fundamental degrees d_i of W.*

Equivalently, the irreducible polynomial ϕ_r is in $\mathrm{ch}^*(W)$ if and only if r divides one of the fundamental degrees d_i of the reflection group W.

Table 3.2 lists the fundamental degrees and the divisors of degrees for the simple Weyl groups (see [Humphreys 1990, 3.7]).

Type	Degrees	Divisors of degrees	
A_n	$2, 3, \ldots, n+1$	$1, 2, \ldots, n+1$	
B_n	$2, 4, \ldots, 2n$	$1, 2, \ldots, n, n+2, n+4, \ldots, 2n$	for n even
		$1, 2, \ldots, n, n+1, n+3, \ldots, 2n$	for n odd
D_n	$2, 4, \ldots, 2n-2, n$	$1, 2, \ldots, n, n+2, n+4, \ldots, 2n-2$	for n even
		$1, 2, \ldots, n, n+1, n+3, \ldots, 2n-2$	for n odd
G_2	2, 6	1, 2, 3, 6	
F_4	2, 6, 8, 12	1, 2, 3, 4, 6, 8, 12	
E_6	2, 5, 6, 8, 9, 12	1, 2, 3, 4, 5, 6, 8, 9, 12	
E_7	2, 6, 8, 10, 12, 14, 18	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18	
E_8	2, 8, 12, 14, 18, 20, 24, 30	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 24, 30	

Table 3.2. Fundamental degrees and divisors of the simple Weyl groups

Using Theorem 3.1 and Table 3.2, we can now easily compute the set $\mathrm{ch}^*(W)$ for any simple Weyl group W. We summarize them below.

$$\mathrm{ch}^*(W(A_n)) = \{\phi_1, \phi_2, \ldots, \phi_{n+1}\}$$
$$\mathrm{ch}^*(W(B_n)) = \{\phi_i, \phi_{2i} : i = 1, 2, \ldots, n\}$$
$$\mathrm{ch}^*(W(D_n)) = \{\phi_i, \phi_{2j} : i = 1, 2, \ldots, n, \;\; j = 1, 2 \ldots, n-1\}$$
$$\mathrm{ch}^*(W(G_2)) = \{\phi_1, \phi_2, \phi_3, \phi_6\}$$
$$\mathrm{ch}^*(W(F_4)) = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_6, \phi_8, \phi_{12}\}$$
$$\mathrm{ch}^*(W(E_6)) = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_8, \phi_9, \phi_{12}\}$$
$$\mathrm{ch}^*(W(E_7)) = \{\phi_1, \phi_2, \ldots, \phi_{10}, \phi_{12}, \phi_{14}, \phi_{18}\}$$
$$\mathrm{ch}^*(W(E_8)) = \{\phi_1, \phi_2, \ldots, \phi_{10}, \phi_{12}, \phi_{14}, \phi_{15}, \phi_{18}, \phi_{20}, \phi_{24}, \phi_{30}\}$$

4. Main result

In this section, k is either a finite field, a global field or a non-archimedean local field. We now restate the main result, Theorem 1.1.

Theorem 4.1. *Let H_1 and H_2 be split, connected, semisimple algebraic groups defined over k. Suppose that for every maximal k-torus $T_1 \subset H_1$ there exists a maximal k-torus $T_2 \subset H_2$ such that the torus T_2 is k-isomorphic to the torus T_1 and vice versa. Then, the Weyl groups $W(H_1)$ and $W(H_2)$ are isomorphic.*

Moreover, if we write $W(H_1)$ and $W(H_2)$ as a direct product of Weyl groups of simple algebraic groups, $W(H_1) = \prod_{\Lambda_1} W_{1,\alpha}$, and $W(H_2) = \prod_{\Lambda_2} W_{2,\beta}$, then there exists a bijection $i : \Lambda_1 \to \Lambda_2$ such that $W_{1,\alpha}$ is isomorphic to $W_{2,i(\alpha)}$ for every $\alpha \in \Lambda_1$.

The proof of this theorem occupies the rest of this section. Clearly the groups H_1 and H_2 are of the same rank, say n. Let W_1 and W_2 denote the Weyl groups of H_1 and H_2, respectively. We always treat W_1 and W_2 as subgroups of $\mathrm{GL}_n(\mathbb{Z})$. We first prove a lemma which transforms the information about k-isomorphism of maximal k-tori in the groups H_1 and H_2 into some information about the conjugacy classes of the elements of the corresponding Weyl groups W_1 and W_2.

Lemma 4.2. *Under the hypotheses of Theorem 4.1, for every element $w_1 \in W_1$, there exists an element $w_2 \in W_2$ such that w_2 is conjugate to w_1 in $\mathrm{GL}_n(\mathbb{Z})$ and vice versa.*

Proof. Let $w_1 \in W_1$ and let N_1 denote the subgroup of W_1 generated by w_1. Since the base field k admits any cyclic group as a Galois group, there is a map $\rho_1 : G(\bar{k}/k) \to W_1$ such that $\rho_1(G(\bar{k}/k)) = N_1$.

We first consider the case where k is a finite field or a local non-archimedean field. By Lemma 2.3, the element $[\rho_1] \in H^1(k, W_1)$ corresponds to a maximal k-torus in H_1, say T_1. By the hypothesis, there exists a torus $T_2 \subset H_2$ which is k-isomorphic to T_1. We know by Lemma 2.2 that there exists an integral Galois representation $\rho_2 : G(\bar{k}/k) \to \mathrm{GL}_n(\mathbb{Z})$ corresponding to the k-isomorphism class of T_2 which factors through W_2. Let $N_2 := \rho_2(G(\bar{k}/k)) \subseteq W_2$. Since T_1 and T_2 are k-isomorphic tori, the corresponding Galois representations, ρ_1 and ρ_2, are equivalent. This implies that there exists $g \in \mathrm{GL}_n(\mathbb{Z})$ such that $N_2 = gN_1g^{-1}$. Then $w_2 := gw_1g^{-1} \in N_2 \subseteq W_2$ is a conjugate of w_1 in $\mathrm{GL}_n(\mathbb{Z})$. We can start with an element $w_2 \in W_2$ and obtain its $\mathrm{GL}_n(\mathbb{Z})$-conjugate in W_1 in the same way.

Now we consider the case when k is a global field. Let v be a non-archimedean valuation of k and let k_v be the completion of k with respect to v. Clearly the groups H_1 and H_2 are defined over k_v. Let $T_{1,v}$ be a maximal k_v-torus in H_1. Then by Grothendieck's theorem [Borel and Springer 1968, 7.9, 7.11] and the weak approximation property [Platonov and Rapinchuk 1994, Proposition 7.3], there exists a k-torus in H, say T_1, such that $T_{1,v}$ is obtained from T_1 by the base change. By hypothesis, we have a k-torus T_2 in H_2 which is k-isomorphic to T_1. Then the torus $T_{2,v}$, obtained from T_2 by the base change, is k_v-isomorphic to $T_{1,v}$. Thus, every maximal k_v-torus in H_1 has a k_v-isomorphic torus in H_2. Similarly, we can show that every maximal k_v-torus in H_2 has a k_v-isomorphic torus in H_1. Then the proof follows by the previous case. □

Corollary 4.3. *Under the hypotheses of Theorem 4.1,* $\mathrm{ch}(W_1) = \mathrm{ch}(W_2)$ *and* $\mathrm{ch}^*(W_1) = \mathrm{ch}^*(W_2)$. *In particular,* $\mathfrak{m}_i(W_1) = \mathfrak{m}_i(W_2)$, $\mathfrak{m}'_i(W_1) = \mathfrak{m}'_i(W_2)$ *and* $\mathfrak{m}_{i,j}(W_1) = \mathfrak{m}_{i,j}(W_2)$ *for all i, j.*

Proof. Since the Weyl groups W_1 and W_2 share the same set of elements up to conjugacy in $\mathrm{GL}_n(\mathbb{Z})$, the sets $\mathrm{ch}(W_1)$ and $\mathrm{ch}(W_2)$ are the same, and hence the sets $\mathrm{ch}^*(W_1)$ and $\mathrm{ch}^*(W_2)$ are also the same. Further, for a fixed integer i, $\phi_i^{\mathfrak{m}_i(W_1)}$ divides an element $f_1 \in \mathrm{ch}(W_1)$. But since $\mathrm{ch}(W_1) = \mathrm{ch}(W_2)$, the polynomial $\phi_i^{\mathfrak{m}_i(W_1)}$ also divides an element $f_2 \in \mathrm{ch}(W_2)$. Therefore $\mathfrak{m}_i(W_1) \leq \mathfrak{m}_i(W_2)$. We obtain the inequality in the other direction in the same way and hence $\mathfrak{m}_i(W_1) = \mathfrak{m}_i(W_2)$. Similarly, we can prove that $\mathfrak{m}'_i(W_1) = \mathfrak{m}'_i(W_2)$, and also that, for integers $i \neq j$, the sets

$$\{(t_1, s_1) : \phi_i^{t_1} \cdot \phi_j^{s_1} \text{ divides some element } f_1 \in \mathrm{ch}(W_1)\},$$
$$\{(t_2, s_2) : \phi_i^{t_2} \cdot \phi_j^{s_2} \text{ divides some element } f_2 \in \mathrm{ch}(W_2)\}$$

are the same for $i = 1, 2$. It follows that $\mathfrak{m}_{i,j}(W_1) = \mathfrak{m}_{i,j}(W_2)$. □

We now prove the following result before going on to prove the main theorem.

Theorem 4.4. *Let H_1 and H_2 be split, connected, semisimple algebraic groups of rank n. Suppose that $\mathfrak{m}_i(W(H_1)) = \mathfrak{m}_i(W(H_2))$, that $\mathfrak{m}'_i(W(H_1)) = \mathfrak{m}'_i(W(H_2))$, and that $\mathfrak{m}_{i,j}(W(H_1)) = \mathfrak{m}_{i,j}(W(H_2))$ for all i, j. Let m be the maximum possible rank among the simple factors of H_1 and H_2. Let W'_1 and W'_2 denote the product of the Weyl groups of rank m simple factors of, respectively, H_1 and H_2. Then the groups W'_1 and W'_2 are isomorphic.*

Proof. We denote $W(H_1)$ by W_1 and $W(H_2)$ by W_2. We prove that if a simple Weyl group of rank m appears as a factor of W_1 with multiplicity p, then it appears as a factor of W_2, with the same multiplicity. We prove this lemma case by case, depending on the type of rank m simple factors of H_1 and H_2.

We prove this result by comparing the sets $\mathrm{ch}^*(W)$ for the simple Weyl groups of rank m. We observe from Table 3.2 that the maximal degree of the simple Weyl group of exceptional type, if any, is the largest among the maximal degrees of simple Weyl groups of rank m. The next largest maximal degree is that of $W(B_m)$, the next one is that of $W(D_m)$, and finally the Weyl group $W(A_m)$ has the smallest maximal degree. We use the relation between the elements of $\mathrm{ch}^*(W)$ and the degrees of the Weyl group W, given by Theorem 3.1. So, we begin the proof of the lemma with the case of exceptional groups of rank m, prove that it occurs with the same multiplicity for $i = 1, 2$. Then we prove the lemma for B_m, then for D_m and finally we prove the lemma for the group A_m.

Case 1: One of H_1 or H_2 contains a simple exceptional factor of rank m.

We first treat the case of the simple group E_8, that is, we assume that 8 is the maximum possible rank of the simple factors of the groups H_1 and H_2. We know that $\mathfrak{m}_{30}(W(E_8)) = 1$. Observe that ϕ_{30} is an irreducible polynomial of degree 8, and hence cannot occur in $\mathrm{ch}^*(W)$ for any simple Weyl group of rank at most 7. Moreover, from Theorem 3.1 and Table 3.2, it is clear that $\mathfrak{m}_{30}(W(A_8)) = \mathfrak{m}_{30}(W(B_8)) = \mathfrak{m}_{30}(W(D_8)) = 0$. Hence the multiplicity of E_8 in H_i is given by $\mathfrak{m}_{30}(W_i)$ which is the same for $i = 1, 2$.

Similarly for the simple algebraic group E_7, observe that $\mathfrak{m}_{18}(W(E_7)) = 1$ and $\mathfrak{m}_{18}(W) = 0$ for any simple Weyl group W of rank at most 7. Then the multiplicity of E_7 in H_i is given by $\mathfrak{m}_{18}(W_i)$ which is the same for $i = 1, 2$.

The case of E_6 is done by using $\mathfrak{m}_9$, since it is clear that $\mathfrak{m}_9(W) = 0$ for any simple Weyl group W of rank at most 6.

The cases of F_4 and G_2 are done similarly by using $\mathfrak{m}_{12}$ and $\mathfrak{m}_6$ respectively.

Case 2: One of H_1 or H_2 has B_m or C_m as a factor.

Since $W(B_m) \cong W(C_m)$, we treat the case of B_m only. By case 1, we can assume that the exceptional group of rank m, if any, occurs with the same multiplicities in both H_1 and H_2, and hence while counting the multiplicities $\mathfrak{m}_i$, $\mathfrak{m}'_i$ and $\mathfrak{m}_{i,j}$, we can (and will) ignore the exceptional groups of rank m.

Observe that $\mathfrak{m}_{2m}(W(B_m)) = 1$ and $\mathfrak{m}_{2m}(W) = 0$ for any other simple Weyl group W of classical type of rank at most m. However, it is possible that $\mathfrak{m}_{2m}(W) \neq 0$ for a simple Weyl group W of exceptional type of rank less than m. If $m \geq 16$ then this problem does not arise, therefore the multiplicity of B_m in H_i for $m \geq 16$ is given by $\mathfrak{m}_{2m}(W_i)$, which is the same for $i = 1, 2$. We do the cases of B_m for $m \leq 15$ separately.

For the group B_2, we observe that $\mathfrak{m}_4(W(B_2)) = 1$ and $\mathfrak{m}_4(W) = 0$ for any other simple Weyl group W of rank at most 2. Thus, the case of B_2 is done using $\mathfrak{m}_4(W_1) = \mathfrak{m}_4(W_2)$.

For the group B_3, we have $\mathfrak{m}_6(W(B_3)) = 1$, but then $\mathfrak{m}_6(W(G_2))$ is also 1. Observe that $\mathfrak{m}_4(W(B_3)) = 1$ and $\mathfrak{m}_4(W(G_2)) = 0$. We do this case by looking at the multiplicities of ϕ_4 and ϕ_6, so we do not worry about the simple Weyl groups W of rank at most 3 for which the multiplicities $\mathfrak{m}_4(W)$ and $\mathfrak{m}_6(W)$ are both zero. Now, let the multiplicities of B_3, G_2 and B_2 in the group H_i be, respectively, p_i, q_i and r_i, for $i = 1, 2$. Then, using $\mathfrak{m}_6(W_1) = \mathfrak{m}_6(W_2)$, we see that $p_1 + q_1 = p_2 + q_2$. Using $\mathfrak{m}_4$ we have $p_1 + r_1 = p_2 + r_2$ and using $\mathfrak{m}_{4,6}$ we see $p_1 + q_1 + r_1 = p_2 + q_2 + r_2$. Combining these equalities, we see that $p_1 = p_2$, that is, the group B_3 appears in both the groups H_1 and H_2 with the same multiplicity.

For the group B_4, we observe that $\mathfrak{m}_8(W(B_4)) = 1$. Since ϕ_8 has degree 4, it cannot occur in $\mathrm{ch}(W)$ for any simple Weyl group of rank at most 3 and $\mathfrak{m}_8(W(A_4)) = \mathfrak{m}_8(W(D_4)) = 0$. Since we are assuming by case 1 that the group F_4 occurs in both H_1 and H_2 with the same multiplicity, we are done in this case also.

For the group B_5, we have $\mathfrak{m}_{10}(W(B_5)) = 1$ and $\mathfrak{m}_{10}(W) = 0$ for any other simple Weyl group of classical type of rank at most 5. Since 5 does not divide the order of $W(G_2)$ or $W(F_4)$, it follows that $\mathfrak{m}_{10}(W(G_2)) = \mathfrak{m}_{10}(W(F_4)) = 0$ and so we are done.

The group B_6 is another group where the exceptional groups give problems. We have $\mathfrak{m}_{12}(W(B_6)) = 1$, but $\mathfrak{m}_{12}(W(F_4))$ is also 1. Observe that $\mathfrak{m}_{10}(W(B_6)) = 1$, but $\mathfrak{m}_{10}(W(F_4)) = 0$. Now, let the multiplicities of B_6, D_6, B_5 and F_4 in H_i be, respectively, p_i, q_i, r_i and s_i. Then $p_1 + s_1 = \mathfrak{m}_{12}(W_1) = \mathfrak{m}_{12}(W_2) = p_2 + s_2$. Similarly, comparing $\mathfrak{m}_{10}$, we see that

$$p_1 + q_1 + r_1 = p_2 + q_2 + r_2.$$

Then, we compare $\mathfrak{m}_{10,12}$ of the groups W_1 and W_2, to see that

$$p_1 + q_1 + r_1 + s_1 = p_2 + q_2 + r_2 + s_2.$$

Combining this equality with the one obtained by $\mathfrak{m}_{10}$, we get that $s_1 = s_2$ and hence $p_1 = p_2$. Thus the group B_6 occurs in both H_1 and H_2 with the same multiplicity.

We have $\mathfrak{m}_{14}(W(E_6)) = 0$, therefore the group B_7 is characterized by ϕ_{14} and hence it occurs in both H_1 and H_2 with the same multiplicity.

For the group B_8, we have $\mathfrak{m}_{16}(W(B_8)) = 1$. Since ϕ_{16} has degree 8, it cannot occur in $\mathrm{ch}^*(W)$ for any of the Weyl groups of G_2, F_4, E_6 or E_7. Thus, the group B_8 is characterized by ϕ_{16} and hence it occurs in both H_1 and H_2 with the same multiplicity.

The group B_9 has the property that $\mathfrak{m}_{18}(W(B_9)) = 1$. But $\mathfrak{m}_{18}(W(E_7)) = \mathfrak{m}_{18}(W(E_8)) = 1$, and so we conclude that the multiplicity of E_8 is the same for both W_1 and W_2 using $\mathfrak{m}_{30}$. Then we compare the multiplicities $\mathfrak{m}_{18}, \mathfrak{m}_{16}$ and $\mathfrak{m}_{16,18}$ to prove that the group B_9 occurs in both the groups H_1 and H_2 with the same multiplicity.

Now we examine the case B_{10}. Here $\mathfrak{m}_{20}(W(B_{10})) = 1$. Observe that $\mathfrak{m}_{20}(W) = 0$ for any other simple Weyl group W of rank at most 10, except E_8. Then the multiplicity of B_{10} in H_i is $\mathfrak{m}_{20}(W_i) - \mathfrak{m}_{30}(W_i)$ and hence it is the same for $i = 1, 2$.

The same method also works for B_{12}, that is, the multiplicity of B_{12} in H_i is $\mathfrak{m}_{24}(W_i) - \mathfrak{m}_{30}(W_i)$.

The multiplicities of B_{11}, B_{13} and B_{14} in H_i are given by $\mathfrak{m}_{22}(W_i)$, $\mathfrak{m}_{26}(W_i)$ and $\mathfrak{m}_{28}(W_i)$ and hence they are the same for $i = 1, 2$.

For B_{15}, we have $\mathfrak{m}_{30}(W(B_{15})) = \mathfrak{m}_{30}(W(E_8)) = 1$, and $\mathfrak{m}_{30}(W) = 0$ for any other simple Weyl group W of rank at most 15. Observe also that $\mathfrak{m}_{28}(W(B_{15})) = \mathfrak{m}_{28}(W(B_{14})) = 1$, and $\mathfrak{m}_{28}(W) = 0$ for any other simple Weyl group W of rank at most 15. Then by comparing $\mathfrak{m}_{30}$, $\mathfrak{m}_{28}$ and $\mathfrak{m}_{28,30}$ we get the desired result that B_{15} occurs in both H_1 and H_2 with the same multiplicity.

Case 3: One of H_1 or H_2 has D_m as a factor.

For this case, we assume that the exceptional group of rank m, if any, and the group B_m occur in both H_1 and H_2 with the same multiplicities.

We observe that $2m - 2$ is the largest integer r such that $\phi_r \in \mathrm{ch}^*(W(D_m))$, but $\mathfrak{m}_{2m-2}(W(B_{m-1})) = 1$. Hence we always have to compare the group D_m with the group B_{m-1}.

Let us assume that $m \geq 17$, so that $\phi_{2m-2} \notin \mathrm{ch}^*(W)$ for any simple Weyl group of exceptional type of rank less than m.

We know that $\mathfrak{m}_{2m-2}(W(D_m)) = \mathfrak{m}_{2m-2}(W(B_{m-1})) = 1$ and that $\mathfrak{m}_{2m-2}(W) = 0$ for any other simple Weyl group W of classical type of rank at most m. Further, $(X+1)(X^{m-1}+1)$ is the only element in $\mathrm{ch}(W(D_m))$ which has ϕ_{2m-2} as a factor. Similarly $X^{m-1}+1$ is the only element in $\mathrm{ch}(W(B_{m-1}))$ which has ϕ_{2m-2} as a factor. Observe that $\mathfrak{m}'_{2m-2}(W(D_m)) = \mathfrak{m}'_{2m-2}(W(B_{m-1})) + 1$ and $m'_{2m-2}(W) = 0$ for any other simple Weyl group W of rank at most m. Let p_i and q_i be, respectively, the multiplicities of the groups D_m and B_{m-1} in H_i, for $i = 1, 2$. Then by considering $\mathfrak{m}_{2m-2}$, we have $p_1 + q_1 = p_2 + q_2$. Further if m is even, then by considering $\mathfrak{m}'_{2m-2}$ we have $2p_1 + q_1 = 2p_2 + q_2$. These two equalities imply that $p_1 = p_2$. If m is odd then $\mathfrak{m}'_{2m-2}$ itself gives $p_1 = p_2$. Thus the group D_m appears in both H_1 and H_2 with the same multiplicity.

Now we consider the groups D_m, for $m \leq 16$.

For D_4, we have to consider the simple algebraic groups B_3 and G_2. Comparing the multiplicities $\mathfrak{m}_6$, $\mathfrak{m}_4$ and $\mathfrak{m}_{4,6}$ we see that G_2 occurs in both H_1 and H_2 with the same multiplicity, and then we proceed as above to prove that D_4 also occurs with the same multiplicity in both the groups H_1 and H_2

For the group D_5, we first prove that the multiplicity of F_4 is the same for both H_1 and H_2 using $\mathfrak{m}_{12}$ and then prove the required result by considering $\mathfrak{m}_5$, $\mathfrak{m}_8$ and $\mathfrak{m}_{5,8}$. While dealing with the case D_6, we observe that $\mathfrak{m}_{10}(W(G_2)) = \mathfrak{m}_{10}(W(F_4)) = 0$, and so this case follows by an argument similar to that for $m \geq 17$. The case D_7 is proved by considering $\mathfrak{m}_7$, $\mathfrak{m}_{12}$ and $\mathfrak{m}_{7,12}$. For D_8, we first prove that the group E_7 occurs in both H_1 and H_2 with the same multiplicity by considering $\mathfrak{m}_{18}$ and then proceed as above. For D_9, we prove that E_8 occurs in both H_1 and H_2 with the same multiplicity by considering $\mathfrak{m}_{30}$ and proceed as for $m \geq 17$. For D_{10}, we prove that E_8 appears in both H_1 and H_2 with the same multiplicity by considering $\mathfrak{m}_{30}$, and the same follows for E_7 by considering $\mathfrak{m}_{18}$, $\mathfrak{m}_{16}$ and $\mathfrak{m}_{16,18}$.

For the groups D_m, where $m \geq 11$, the only simple Weyl group W of exceptional type such that $\phi_{2m-2} \in \mathrm{ch}^*(W)$ is $W(E_8)$, but for D_m, with $m \leq 15$, we can assume that E_8 occurs in both H_1 and H_2 with the same multiplicity by considering $\mathfrak{m}_{30}$ and hence we are done. For the group D_{16}, we take care of E_8 by considering $\mathfrak{m}_{30}$, $\mathfrak{m}_{28}$ and $\mathfrak{m}_{28,30}$. Other arguments are similar to the case $m \geq 17$.

Case 4: One of H_1 or H_2 has A_m as a factor.

We now consider the case of simple algebraic group of type A_m. Here, as usual, we assume that all other simple algebraic groups of rank m occur with the same multiplicities in both H_1 and H_2.

If m is even, then $m+1$ is odd and hence $\mathfrak{m}_{m+1}(W) = 0$ for any simple Weyl group W of classical type of rank less than m. If $m \geq 30$, then we do not have to bother about the exceptional simple groups of rank less than m. If m is odd and $m \geq 31$, then ϕ_{m+1} occurs in $\mathrm{ch}^*(W(B_r))$ and $\mathrm{ch}^*(W(D_{r+1}))$ for $r \geq (m+1)/2$. Then we compare the multiplicities $\mathfrak{m}_m$, $\mathfrak{m}_{m+1}$ and $\mathfrak{m}_{m,m+1}$ and find that the group A_m occurs in H_1 and H_2 with the same multiplicity. We must therefore consider the cases $m \leq 29$ separately.

The cases A_1 and A_2 are easy since there are no exceptional groups of rank 1. For A_3 we use $\mathfrak{m}_3$, $\mathfrak{m}_4$ and $\mathfrak{m}_{3,4}$ to get the result, and the case A_4 follows similarly by using $\mathfrak{m}_5$. The group A_5 is more problematic, since neither $\mathfrak{m}_6(W(B_3))$ nor $\mathfrak{m}_6(W(G_2))$ nor $\mathfrak{m}_6(W(F_4))$ vanish, but this is solved by first proving that F_4 appears with the same multiplicity using $\mathfrak{m}_{12}$ and then using the multiplicities $\mathfrak{m}_5$, $\mathfrak{m}_6$ and $\mathfrak{m}_{5,6}$. The case A_6 is solved by using $\mathfrak{m}_7$, and for A_7 we use $\mathfrak{m}_7$, $\mathfrak{m}_8$ and $\mathfrak{m}_{7,8}$.

With A_8, we can first assume that the multiplicity of E_7 is the same for both H_1 and H_2 by using $\mathfrak{m}_{18}$, and then use $\mathfrak{m}_7$, $\mathfrak{m}_9$ and $\mathfrak{m}_{7,9}$ to get the result. For A_9 we can again get rid of E_7 and E_8 using the multiplicities $\mathfrak{m}_{18}$ and $\mathfrak{m}_{30}$. Then we are left with the groups B_5 and E_6, and so here we use $\mathfrak{m}_7$, $\mathfrak{m}_{10}$ and $\mathfrak{m}_{7,10}$ to get the result.

Further, we note that for even $m \in \{10, 12, 16, \ldots, 28\}$, we have $\mathfrak{m}_{m+1}(W) = 0$ for any simple Weyl group of rank less than m. Thus, the multiplicities of the groups A_m in H_i, for even $m \in \{10, 12, 16, \ldots, 28\}$, are characterized by considering $\mathfrak{m}_{m+1}(W_i)$ and are hence the same for $i = 1, 2$. The case A_{14} follows by using $\mathfrak{m}_{13}$, $\mathfrak{m}_{15}$ and $\mathfrak{m}_{13,15}$.

Thus, the only remaining cases are A_m where m is odd and $11 \leq m \leq 29$. We observe that for odd $m \in \{11, 13, 17, \ldots, 29\}$, the only simple Weyl group W of rank less than m, with $\mathfrak{m}_m(W) \neq 0$, is A_{m-1}. Moreover, $\mathfrak{m}_{m+1}(W(A_{m-1})) = 0$, so the cases of the groups A_m, for odd $m \in \{11, 13, 17, \ldots, 29\}$, are solved by considering $\mathfrak{m}_m$, $\mathfrak{m}_{m+1}$ and $\mathfrak{m}_{m,m+1}$.

The only remaining case is A_{15}, which can be solved by considering $\mathfrak{m}_{13}$, $\mathfrak{m}_{16}$ and $\mathfrak{m}_{13,16}$. □

We now prove the main theorem of this paper.

Proof of Theorem 4.1. Recall that W_1 and W_2 denote the Weyl groups of H_1 and H_2 respectively. Let m_0 be the maximum among the ranks of simple factors of the groups H_1 and H_2. It is clear from Corollary 4.3 that $\mathfrak{m}_i(W_1) = \mathfrak{m}_i(W_2)$, that $\mathfrak{m}'_i(W_1) = \mathfrak{m}'_i(W_2)$ and that $\mathfrak{m}_{i,j}(W_1) = \mathfrak{m}_{i,j}(W_2)$ for any i, j. Then we apply Theorem 4.4 to conclude that the products of rank m_0 simple factors in W_1 and W_2 are isomorphic.

Let m be a positive integer less than m_0. For $i = 1, 2$, let W'_i be the subgroup of W_i which is the product of the Weyl groups of simple factors of H_i of rank greater than m. We assume that the groups W'_1 and W'_2 are isomorphic and then we prove that the products of the Weyl groups of rank m simple factors of H_1 and H_2 are isomorphic. This will complete the proof of the theorem by an induction argument.

Let U_i be the subgroup of W_i such that $W_i = U_i \times W_i'$. Then, since $\mathfrak{m}_j(W_1') = \mathfrak{m}_j(W_2')$ and $\mathfrak{m}_j'(W_1') = \mathfrak{m}_j'(W_2')$, we have

$$\mathfrak{m}_j(U_1) = \mathfrak{m}_j(W_1) - \mathfrak{m}_j(W_1') = \mathfrak{m}_j(W_2) - \mathfrak{m}_j(W_2') = \mathfrak{m}_j(U_2),$$
$$\mathfrak{m}_j'(U_1) = \mathfrak{m}_j'(W_1) - \mathfrak{m}_j'(W_1') = \mathfrak{m}_j'(W_2) - \mathfrak{m}_j'(W_2') = \mathfrak{m}_j'(U_2)$$

and similarly

$$\mathfrak{m}_{i,j}(U_1) = \mathfrak{m}_{i,j}(U_2).$$

Now we use Theorem 4.4 to conclude that the subgroups of W_i which are products of the Weyl groups of simple factors of H_i of rank m are isomorphic, for $i = 1, 2$.

The proof of the theorem can now be completed by the downward induction on m. It also follows from the proof of Theorem 4.4, that the Weyl groups of simple factors of H_1 and H_2 are pairwise isomorphic. □

Remark 4.5. We remark here that the above proof is valid even if we assume that the Weyl groups $W(H_1)$ and $W(H_2)$ share the same set of elements up to conjugacy in $\mathrm{GL}_n(\mathbb{Q})$, not just in $\mathrm{GL}_n(\mathbb{Z})$. Thus Theorem 1.1 is true under the weaker assumption that the groups H_1 and H_2 share the same set of maximal k-tori up to k-isogeny, not just up to k-isomorphism.

We also remark that the above proof holds over the fields k which admit arbitrary cyclic extensions and which have cohomological dimension ≤ 1.

Remark 4.6. Philippe Gille [2004] has recently proved that the map ψ described in Lemma 2.2 is surjective for any quasisplit semisimple group H. Therefore our main result, Theorem 1.1, now holds for all fields k which admit cyclic extensions of arbitrary degree.

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References

[Borel and Springer 1968] A. Borel and T. A. Springer, "Rationality properties of linear algebraic groups, II", *Tôhoku Math. J.* (2) **20** (1968), 443–497. MR 39 #5576 Zbl 0211.53302

[Bruhat and Tits 1967] F. Bruhat and J. Tits, "Groupes algébriques simples sur un corps local", pp. 23–36 in *Proc. Conf. Local Fields* (Driebergen, 1966), edited by T. A. Springer, Springer, Berlin, 1967. MR 37 #6396 Zbl 0155.00204

[Gille 2004] P. Gille, "Type des tores maximaux des groupes semisimples", *J. Ramanujan Math. Soc.* **19**:3 (2004), 1–18.

[Humphreys 1990] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, Cambridge, 1990. MR 92h:20002 Zbl 0768.20016

[Kariyama 1989] K. Kariyama, "On conjugacy classes of maximal tori in classical groups", *J. Algebra* **125**:1 (1989), 133–149. MR 90j:20090 Zbl 0682.20031

[Kneser 1965a] M. Kneser, "Galois-Kohomologie halbeinfacher algebraischer Gruppen über $\mathfrak{p}$-adischen Körpern, I", *Math. Z.* **88** (1965), 40–47. MR 30 #4760 Zbl 0143.04702

[Kneser 1965b] M. Kneser, "Galois-Kohomologie halbeinfacher algebraischer Gruppen über $\mathfrak{p}$-adischen Körpern, II", *Math. Z.* **89** (1965), 250–272. MR 32 #5658 Zbl 0143.04702

[Knus et al. 1998] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, AMS Colloquium Publications **44**, American Mathematical Society, Providence, RI, 1998. MR 2000a:16031 Zbl 0955.16001

[Lang 1956] S. Lang, "Algebraic groups over finite fields", *Amer. J. Math.* **78** (1956), 555–563. MR 19,174a Zbl 0073.37901

[Larsen and Pink 1992] M. Larsen and R. Pink, "On l-independence of algebraic monodromy groups in compatible systems of representations", *Invent. Math.* **107**:3 (1992), 603–636. MR 93h:22031 Zbl 0778.11036

[Pierce 1982] R. S. Pierce, *Associative algebras*, Graduate Texts in Mathematics **88**, Springer, New York, 1982. MR 84c:16001 Zbl 0497.16001

[Platonov and Rapinchuk 1994] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, Pure Appl. Math. **139**, Academic Press, Boston, 1994. MR 95b:11039 Zbl 0841.20046

[Serre 1979] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics **67**, Springer, New York, 1979. MR 82e:12016 Zbl 0423.12016

[Springer 1974] T. A. Springer, "Regular elements of finite reflection groups", *Invent. Math.* **25** (1974), 159–198. MR 50 #7371 Zbl 0287.20043

[Tits 1966] J. Tits, "Normalisateurs de tores, I; Groupes de Coxeter étendus", *J. Algebra* **4** (1966), 96–116. MR 34 #5942 Zbl 0145.24703

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