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KNOT MUTATION: 4-GENUS OF KNOTS AND ALGEBRAIC CONCORDANCE

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Kearton observed that mutation can change the concordance class of a knot. A close examination of his example reveals that it is of 4-genus 1 and has a mutant of 4-genus 0. The first goal of this paper is to show by examples that for any pair of nonnegative integers *m* and *n* there is a knot of 4-genus *m* with a mutant of 4-genus *n*.

A second result is a crossing change formula for the algebraic concordance class of a knot, which is then applied to prove the invariance of the algebraic concordance class under mutation. We conclude with an application of crossing change formulas to give a short new proof of Long's theorem that strongly positive amphicheiral knots are algebraically slice.

1. Introduction

The main goal of this paper is to examine the effect of knot mutation on two concordance invariants of knots, the 4-ball genus and the algebraic concordance class. We completely describe the extent to which mutation can change the 4-genus, and show that the algebraic concordance class of a knot, as defined in [Levine 1969b], is invariant under mutation. In the course of our work we develop a crossing change formula for the algebraic concordance class of a knot. We apply such an approach to demonstrate that Long's theorem that strongly positive amphicheiral knots are algebraically slice is an immediate corollary of the Hartley–Kawauchi theorem that such knots have Alexander polynomials that are squares. Lastly, we show that the Hartley–Kawauchi theorem also follows from a similar crossing change approach.

Mutation and algebraic concordance. The construction of a mutant K^* of a knot K consists in removing a 3-ball B from S^3 that meets K in two proper arcs and gluing it back in via an involution τ of its boundary S, where τ is orientation-preserving and leaves the set $S \cap K$ invariant. This is among the subtlest constructions of knot theory in that it leaves a wide range of knot invariants unchanged [Adams 1989; Kawauchi 1994; 1996; Kirk 1989; Kirk and Klassen 1990; Meyerhoff and Ruberman 1990; Rong 1994; Ruberman 1987; 1999]. Most relevant to

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the work here is the statement of [Cooper and Lickorish 1999] that the Tristram–Levine signatures, σ_ω, are invariant under mutation, since, for ω a prime power root of unity, these provide the strongest classical bounds on the 4-genus [Murasugi 1965; Tristram 1969]: $\frac{1}{2}|\sigma_\omega(K)| \leq g_4(K)$. We will prove a more general result involving Levine's homomorphism [1969b] from the knot concordance group $\mathscr{C}$ to the algebraic concordance group $\mathscr{G}$:

Theorem 1.1. *Mutation does not change the image of a knot under Levine's homomorphism.*

One proof, given in Section 7, is entirely self-contained and gives a previously unnoticed crossing change formula for the algebraic concordance class of a knot. (As a side note, in Section 9 we use this crossing change formula to give a quick derivation of a result of Long that strongly positive amphicheiral knots are algebraically slice.) Section 8 present an alternate proof of Theorem 1.1; this argument is briefer, but depends on the detailed analysis of Seifert forms given in [Cooper and Lickorish 1999].

Mutation and the 4-genus of a knot. The 4-genus of a knot, $g_4(K)$, is the least genus of an embedded surface bounded by K in the 4-ball. This can be defined in either the smooth or topological locally flat category; the results of this paper apply in either. It is an especially challenging invariant to compute; there remain knots of low crossing number for which it is uncomputed, though the smooth category has advanced considerably in recent years, most notably with the solution of the Milnor conjecture giving the 4-genus of torus knots [Kronheimer and Mrowka 1993].

Almost nothing has been known concerning the interplay between mutation and the 4-genus. Basically the only success in this realm consists of Kearton's observation [1989] that an example of [Livingston 1983] yields an example for which mutation changes the concordance class of a knot. A close examination of that example shows that it has 4-genus 1, but it has a mutant of 4-genus 0. Further such examples have since been developed in [Kirk and Livingston 1999; 2001]. Our main result regarding the 4-genus is:

Theorem 1.2. *For every pair of nonnegative integers m and n, there is a knot K with mutant K^* satisfying $g_4(K) = m$ and $g_4(K^*) = n$.*

It should be noted that the original argument of [Livingston 1983] was based on [Gilmer 1983], in which it is now known an error appears. To correct for that, one must base the argument of [Livingston 1983] on a 3-fold branched cover rather than the 2-fold cover. We do this here.

Strongly positive amphicheiral knots. A knot K is called strongly positive amphicheiral if, when viewed as a knot in $\mathbb{R}^3$, it has a representative that is invariant under the map $\tau(x, y, z) = (-x, -y, -z)$ of $\mathbb{R}^3$. We consider two theorems:

Theorem 1.3 [Long 1984]. *A strongly positive amphicheiral knot is algebraically slice.*

Theorem 1.4 [Hartley and Kawauchi 1979]. *If K is strongly positive amphicheiral, the Alexander polynomial Δ_K is the square of a symmetric polynomial.*

In Section 9 we use crossing change formulas developed earlier to prove that Long's theorem is an immediate corollary of the Hartley–Kawauchi result. In Section 10 we use a crossing change argument to give a new proof of the Hartley–Kawauchi theorem.

2. Background on Casson–Gordon invariants

A key tool in the proof of Theorem 1.2 is the main theorem from [Gilmer 1982] bounding Casson–Gordon invariants in terms of the 4-genus of a knot. Here is a simplified description of that result, based on the statement of the theorem and later remarks in [Gilmer 1982].

Theorem 2.1 (Gilmer). *Let K be an algebraically slice knot such that $g_4(K) = g$ and let M_q be the q-fold branched cover of S^3 branched over K, with q a prime power. Let β denote the linking form on $H_1(M_q, \mathbb{Z})$. Then β can be written as a direct sum $\beta_1 \oplus \beta_2$ such that*

(1) *β_1 has a presentation of rank $2(q-1)g$, and*

(2) *β_2 has a metabolizer D such that, for any character χ of prime power order on $H_1(M_q, \mathbb{Z})$ given by linking with an element in D, one has*

$$|\sigma(K, \chi)| \leq 2qg.$$

Here $\sigma(K, \chi)$ is the Casson–Gordon invariant, originally denoted $\sigma_1\tau(K, \chi)$ in [Casson and Gordon 1986; Gilmer 1982]. We will need to know that D can be taken to be equivariant with respect to the deck transformation of M_q. Details concerning this and other points will be given below, as they arise.

In our applications the group $H_1(M_q, \mathbb{Z})$ will also be a vector space over a finite field, in which case a metabolizer for β_2 will be half-dimensional. Hence:

Corollary 2.2. *In Theorem 2.1, if $H_1(M_q, \mathbb{Z})$ is isomorphic to $H_1(M_q, \mathbb{Z}_p)$, a $\mathbb{Z}_p$-vector space, conclusion (1) can be restated as*

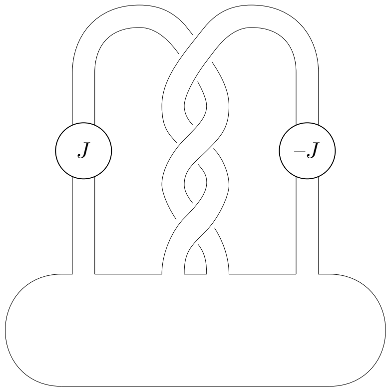
$$\dim \beta_1 \leq 2(q-1)g \tag{1}$$

and in (2) the metabolizer D satisfies

$$\dim D \geq \tfrac{1}{2}\big(\dim H_1(M_q, \mathbb{Z}_p) - 2(q-1)g\big).$$

3. The building blocks

The figure illustrates a knot K_J of genus 1. The bands in the surface are tied in knots J and $-J$, for a knot J to be determined later. The twisting of the bands is such that the Seifert matrix for K_J is $\left(\begin{smallmatrix}0&2\\1&0\end{smallmatrix}\right)$.



Here $-J$ denotes the concordance inverse of J, formed from J by reversing the orientations of S^3 and the knot. A diagram for $-J$ is constructed by reflecting a diagram for J through a vertical line on the page and reversing the orientation of the knot. For K_J, the knot in the right band is the reflection through a vertical line of the knot in the left band. In all examples here, J can be taken to be reversible, so the details of the orientation issues for J are not critical.

Knots related to this one have been carefully analyzed elsewhere, for example [Gilmer and Livingston 1992; Livingston 1983; 2001], and the details of the following results can be found there. Here are the relevant facts.

(1) If M_3 denotes the 3-fold branched cover of S^3 branched over K_J, then

$$H_1(M_3, \mathbb{Z}) = \mathbb{Z}_7 \oplus \mathbb{Z}_7.$$

(2) As a $\mathbb{Z}_7$-vector space, $H_1(M_3, \mathbb{Z})$ splits as the direct sum of a 2-eigenspace, spanned by a vector e_2, and a 4-eigenspace, spanned by a vector e_4, with respect to the linear transformation induced by the deck transformation.

(3) Linking with e_i induces a character $\chi_i : H_1(M_3, \mathbb{Z}) \to \mathbb{Z}_7$. Results of Litherland [1984] (see also [Gilmer 1993; Gilmer and Livingston 1992]) give

$$\sigma(K, \chi_2) = \sigma_{1/7}(J) + \sigma_{2/7}(J) + \sigma_{3/7}(J),$$
$$\sigma(K, \chi_4) = -\sigma_{1/7}(J) - \sigma_{2/7}(J) - \sigma_{3/7}(J),$$

where $\sigma_{a/b}$ denotes the classical Levine–Tristram signature, also written as σ_ω with $\omega = e^{(a/b)2\pi i}$. To simplify notation we set, for any knot J,

$$s_7(J) = \sigma_{1/7}(J) + \sigma_{2/7}(J) + \sigma_{3/7}(J).$$

There are knots for which s_7 is arbitrarily large, for instance connected sums of trefoil knots, which are reversible.

4. The Basic Examples

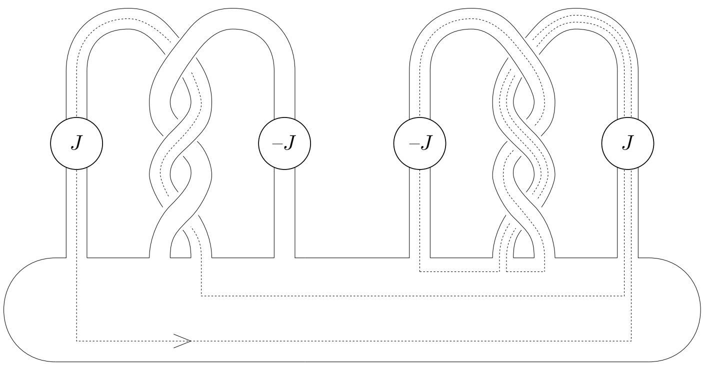
We denote by L_J the connected sum of K_J with the reverse of $-K_J$:

$$L_J = K_J \mathbin{\#} -K_J^r.$$

As observed by Kearton, L_J is a mutant of the slice knot $K_J \mathbin{\#} -K_J$.

Theorem 4.1. *For any choice of J, we have $g_4(L_J) \leq 1$ and thus $g_4(nL_J) \leq n$.*

Proof. Here is an illustration of L_J, showing also a simple closed curve on the genus-2 Seifert surface F. This curve has self-linking number 0 and represents the



slice knot $J \mathbin{\#} -J$. Thus F can be surgered in the 4-ball to reduce its genus to 1, showing that L_J bounds a surface of genus 1 in the 4-ball, as desired. □

The homology of the 3-fold branched cover of L_J, N_3, naturally splits as

$$(\mathbb{Z}_7 \oplus \mathbb{Z}_7) \oplus (\mathbb{Z}_7 \oplus \mathbb{Z}_7),$$

with a 2-eigenspace spanned by the vectors $e_2 \oplus 0$ and $0 \oplus e_2'$, which we abbreviate simply by e_2 and e_2'. Similarly for the 4-eigenspace. We denote the corresponding $\mathbb{Z}_7$-valued characters given by linking with e_2 and e_2' by χ_2 and χ_2', respectively.

Theorem 4.2. *The Casson–Gordon invariants of L_J are given by*

$$\sigma(L_J,\ a\chi_2 + b\chi_2') = \epsilon(a)s_7(J) + \epsilon(b)s_7(J),$$
$$\sigma(L_J,\ a\chi_4 + b\chi_4') = -\big(\epsilon(a)s_7(J) + \epsilon(b)s_7(J)\big),$$

where $\epsilon(x) = 0$ or 1 depending on whether $x = 0$ or $x \neq 0$ modulo 7.

Proof. This follows from the additivity of Casson–Gordon invariants; see [Litherland 1984] or [Gilmer 1983]. The only unexpected aspect of the formula is that, since we are dealing with $K_J \# -K_J^r$, it might have been anticipated that the difference $\epsilon(a)s_7(J) - \epsilon(b)s_7(J)$ would appear rather than the sum. This switch occurs because the connected sum involves the mirror image of the reverse, rather than simply the mirror image; thus the role of J and $-J$ are reversed in the second summand. □

5. Proof of Theorem 1.2

As observed by Kearton, for any knots L_1 and L_2, the connected sums $L_1 \# -L_2$ and $L_1 \# -L_2^r$ are mutants of each other. It follows immediately that for $m < n$, the knot nL_J is a mutant of $mL_J \# (n-m)(K_J \# -K_J)$. Since $K_J \# -K_J$ is slice, this second knot is concordant to, and hence of the same 4-genus as, mL_J. To prove Theorem 1.2 we show that for each positive integer n there exists a knot J such that $g_4(mL_J) = m$ for all $m \leq n$.

Fix a positive integer n and select an arbitrary m with $1 \leq m \leq n$. The knot J will be chosen as its necessary properties become apparent.

Suppose that mL_J bounds a surface F in the 4-ball with genus $g(F) = k < m$. Let V_3 denote the 3-fold branched cover of B^4 branched over F having for boundary the m-fold connected sum mN_3. Also, abbreviate by D the image of $\operatorname{Tor} H_2(V_3, mN_3, \mathbb{Z})$ in $H_1(mN_3, \mathbb{Z})$. An examination of the proof of Gilmer's theorem in [Gilmer 1982] reveals that this D is the metabolizer given in our statement of the result, Theorem 2.1. Thus $|\sigma(mL_J, \chi)| \leq 6k$ for any χ corresponding to an element in D.

With $\mathbb{Z}_7$-coefficients, $H_1(mN_3, \mathbb{Z})$ has dimension $4m$, so by Gilmer's theorem we have $\dim H_1(mN_3, \mathbb{Z}) - 2\dim D \leq 2(3-1)k = 4k$. Hence D is nontrivial, since $k < m$.

Observe that by its construction, D is equivariant with respect to the deck transformation and hence contains an eigenvector. Assume that it is a 2-eigenvector. If we write $H_1(mN_3, \mathbb{Z}) = \oplus_m H_1(N_3, \mathbb{Z})$, the 2-eigenvectors are naturally denoted $e_{2,i}$ and $e'_{2,i}$, with $1 \leq i \leq m$, where $e_{2,i}$ and $e'_{2,i}$ are the 2-eigenvectors in the i-th summand. A nontrivial 2-eigenvector in D will be of the form $\sum_i a_i e_{2,i} + \sum_i b_i e'_{2,i}$. Using additivity, the Casson–Gordon invariant corresponding to the dual character is given by:

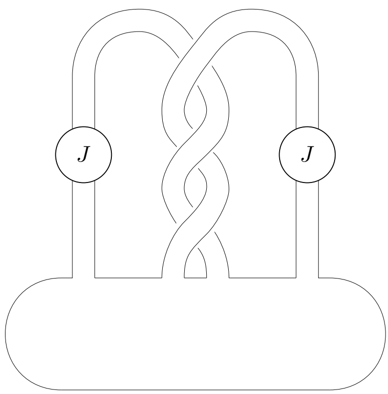
$$\left(\sum_i \epsilon(a_i)\right) s_7(J) + \left(\sum_i \epsilon(b_i)\right) s_7(J).$$

To complete the proof, observe that this sum is greater than or equal to $s_7(J)$, so that if J is chosen so that $s_7(J) > 6n$ a contradiction is achieved. Notice that the choice of J depends only on n and not m.

A similar argument applies if D contains only a 4-eigenvector.

6. The growth of $g_4(nK)$ for algebraically slice knots K

For a general knot K one has $g_4(nK) \leq ng_4(K)$ but one does not usually have an equality. In the case of a knot T, such as the trefoil, for which the 4-genus is detected by a classical (additive) invariant, such as the signature, one can sometimes demonstrate that $g_4(nT) = ng_4(T)$. But for algebraically slice knots with $g_4(K) \neq 0$ such arguments are not possible. In fact, it is unknown whether in the topological category there is such an algebraically slice knot for which the equality holds for all n. (In the smooth setting, Livingston [2003] has constructed an algebraically slice knot K for which $g_4(K) = \tau(K) = 1$, where τ is the invariant defined in [Ozsváth and Szabó 2003]. Since τ is additive and bounds g_4, it follows that $g_4(nK) = ng_4(K)$ for all n.) We will here observe that one can come quite close for the knot T_J, where T_J is the knot illustrated below, built as K_J is, only



with J tied in both bands rather than J in one band and $-J$ in the other. (Similar results hold for K_J and L_J but the proof would require the continued use of 3-fold covers rather than the 2-fold cover for which the estimates are simpler.)

Theorem 6.1. *For all ϵ with $0 < \epsilon < 1$, there is a knot J such that $g_4(nT_J) > (1-\epsilon)ng_4(T_J)$ for all $n > 0$.*

Proof. Our proof builds upon Gilmer's original argument [1982]. Observe first that $g_4(T_J) \leq 1$. For the 2-fold branched cover we have that $H_1(M_2, \mathbb{Z}) = \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and the $\mathbb{Z}_3$-dimension satisfies $\dim H_1(nM_2, \mathbb{Z}_3) = 2n$.

If nT_J bounds a surface in the 4-ball of genus k at most $(1-\epsilon)n$, then by Gilmer's theorem there exists a self-annihilating summand D with

$$\dim H_1(nM_2, \mathbb{Z}_3) - 2\dim D \leq 2k$$

and such that $|\sigma(nK_J, \chi)| \leq 4k$ for all characters χ dual to elements in D.

One computes that $\dim D \geq n-k$. A linear algebra argument, basically Gauss–Jordan elimination, now implies that some element of D will be of the form $\bigoplus_i \chi_i$ with at least $n-k$ of the χ_i nontrivial, and for each of these χ_i the corresponding Casson–Gordon invariant is at least $2\sigma_{1/3}(J)$. Thus we have the equation

$$\left|(n-k)2\sigma_{1/3}(J)\right| \leq 4k.$$

Since $k \leq (1-\epsilon)n$, this reduces to $\left|\epsilon n 2\sigma_{1/3}(J)\right| \leq 4(1-\epsilon)n$, which is to say

$$\left|\sigma_{1/3}(J)\right| \leq \frac{2(1-\epsilon)}{\epsilon}.$$

The proof is completed by noting that for any ϵ one can select a J for which this inequality does not hold. □

7. Mutation and algebraic concordance

In this section we develop a crossing change formula for the algebraic concordance class of a knot in order to prove Theorem 1.1: mutation preserves the algebraic concordance class of a knot. Certain knot invariants, such as the Alexander polynomial and Tristram–Levine signatures, provide algebraic concordance invariants, and these have been shown to be mutation invariants (see for instance [Cooper and Lickorish 1999; Lickorish and Millett 1987]), but the general question of whether mutation can change the algebraic concordance class has remained open. We note that changing a knot to its orientation reverse is a very special case of mutation and reversal does not change the algebraic concordance class of a knot, as follows from [Long 1984]. (More directly, it can be shown that the complete set of algebraic concordance invariants defined by Levine [1969a] are unchanged by matrix transposition, the operation on Seifert matrices induced by reversal.)

We will first present a proof that the normalized Alexander polynomial is invariant under mutation; this argument is not new but must be presented to set up the needed notation for the analysis of algebraic concordance that follows. This is followed by a review of the theory and algebra of Levine's [1969a] algebraic concordance group $\mathcal{G}$. In the last part of the section we present a crossing change formula for the algebraic concordance class of a knot and use this to prove the mutation invariance of this class.

The Alexander and Conway polynomial. For an oriented link L, a choice of connected Seifert surface F for L, and a choice of basis for $H_1(F,\mathbb{Z})$ there is a Seifert matrix $V(L)$, say of dimension $r \times r$. The (normalized) Alexander polynomial $\Delta_L(t)$ of L can be defined by setting

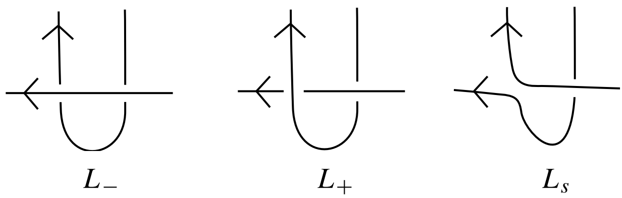
$$V_t(L) = (1-t)V + (1-\bar{t})V^t \qquad \text{and} \qquad \Delta_L(t) = \frac{1}{z^r}\det V_t(L),$$

where V^t denotes the transpose, $\bar{t} = t^{-1}$ and $z = t^{-1/2} - t^{1/2}$. (Recall that $\Delta_L(t)$ can be expressed as a polynomial in z, $\Delta_L(t) = C_L(z) \in \mathbb{Z}[z]$, and this defines the Conway polynomial [1970].) Notice that $z^2 = -(1-\bar{t})(1-t)$, so that if r is even (for instance, when L is connected, so r is twice the genus of F), we have $\Delta_L \in \mathbb{Z}[\bar{t}, t]$ and elementary algebraic manipulations lead to the usual normalized Alexander polynomial,

$$\Delta_L(t) = t^{-r/2} \det(V - tV^t).$$

(This polynomial is clearly independent of change of basis and an observation below will show that it is an S-equivalence invariant [Trotter 1973] and thus depends only on K.)

Here is a local picture of link diagrams for links L_-, L_+, and L_s, with the



diagrams identical outside the local picture. Any crossing change and smoothing can be achieved using this local change. In the diagram for L_- a Reidemeister move eliminates the two crossings. If Seifert's algorithm is used to construct a Seifert surface F_0 for L_- using this simplified diagram, the corresponding Seifert matrix will be denoted A. The Seifert surfaces for the links L_- and L_+ that arise from Seifert's algorithm applied to the given diagrams are formed from F_0 by adding two twisted bands. From this we have that $V(L_\pm)$ is given by a $(r+2) \times (r+2)$ matrix of the form

$$V(L_\pm) = \begin{pmatrix} & & & a_1 & 0 \\ & A & & \vdots & \vdots \\ & & & a_r & 0 \\ a_1 & \cdots & a_r & b & 1 \\ 0 & \cdots & 0 & 0 & \epsilon_\pm \end{pmatrix},$$

where all entries are identical in these two matrices except that $\epsilon_- = 0$ and $\epsilon_+ = -1$. $V(L_s)$ is given by the same matrix, with the last row and column deleted.

A few consequences of these calculations follow quickly.

Theorem 7.1. *The normalized Alexander polynomial is an S-equivalence invariant and hence is a knot invariant.*

Proof. S-equivalence is generated by the operation on Seifert matrices that takes a matrix A and replaces it with the matrix denoted $V(L_-)$ above. That this doesn't change the Alexander polynomial is easily checked: expand the relevant determinant along the last column and then along the last row. □

Theorem 7.2 (The Conway skein relation). *The Alexander polynomial satisfies* $\Delta_{L_+} - \Delta_{L_-} = z\Delta_{L_s}$.

Proof. This again is a simple exercise in algebra, expanding the determinant along the last column and then last row. □

Theorem 7.3. *The Alexander polynomials of mutant knots are the same.*

Proof. In the construction of the mutant K^*, if the intersection of K with the ball B that is being taken out and replaced via an involution is invariant under the extension of that involution to the 3-ball, then $K^* = K$ and the polynomials are the same. In general, a series of crossing changes and smoothings converts $K \cap B$ into invariant tangles, so, via the Conway skein relation, the polynomial of K^* is the same as that for K. □

If K is a knot, the Alexander polynomial satisfies $\Delta_K(1) = 1$ and in particular $\Delta_K(t)$ is nontrivial. Hence, in the matrices above, working now with K instead of L, A_t is nonsingular. Thus, for $V_t(K_\pm)$ the same set of row and column operations can be used to eliminate the entries corresponding to the a_i in V. There results the following matrix $W_t(K_\pm)$, where the entries are rational functions in t and the matrix is hermitian with respect to the involution induced by the map $t \to \bar{t}$:

$$W_t(K_\pm) = \begin{pmatrix} & & & 0 & 0 \\ & A_t & & \vdots & \vdots \\ 0 & \cdots & 0 & c(t) & 1-t \\ 0 & \cdots & 0 & 1-\bar{t} & \epsilon_\pm(1-t)(1-\bar{t}) \end{pmatrix}.$$

Lemma 7.4. *The ratio* $\Delta_{K_+}/\Delta_{K_-}$ *is equal to* $c(t)+1$.

Proof. This follows from a calculation of the relevant determinants. □

Algebraic concordance. An algebraic Seifert matrix is a square integral matrix V satisfying $\det(V - V^t) = \pm 1$. Such a matrix is called metabolic if it is congruent to a matrix of the form

$$\begin{pmatrix} 0 & A \\ B & C \end{pmatrix},$$

with A, B, and C square. Levine defined the algebraic concordance group $\mathcal{G}$ to be the set of equivalence classes of algebraic Seifert matrices, with V_1 and V_2 equivalent if $V_1 \oplus -V_2$ is metabolic. The group operation is induced by direct sum.

A rational algebraic concordance group $\mathcal{G}^{\mathbb{Q}}$ can be similarly defined, where now it is required that $\det\big((V-V^t)(V+V^t)\big) \neq 0$. Levine [1969a] proved that the inclusion $\mathcal{G} \to \mathcal{G}^{\mathbb{Q}}$ is injective.

Consider next the set of nonsingular hermitian matrices with coefficients in the field $\mathbb{Q}(t)$, where $\mathbb{Q}(t)$ has the involution $t \to \bar{t}$. In this case the equivalence relation generated by congruence to metabolic matrices results in the Witt group of $\mathbb{Q}(t)$, denoted $W(\mathbb{Q}(t))$.

Theorem 7.5. *The map*

$$V \to V_t = (1-t)V + (1-\bar{t})V^t$$

induces an injection $\mathcal{G} \to W(\mathbb{Q}(t))$.

Proof. A proof is given in [Litherland 1984] for $\mathcal{G}^{\mathbb{Q}}$ (denoted there by $W_S(\mathbb{Q}, -)$), and the theorem follows from the injectivity of the inclusion $\mathcal{G} \to \mathcal{G}^{\mathbb{Q}}$. In defining $\mathcal{G}^{\mathbb{Q}}$, Litherland restricts to nonsingular matrices, but as he notes, Levine proved that every class in $\mathcal{G}$ has a nonsingular representative. To simplify notation, we will use $W_t(K)$ to denote both the matrix and the Witt class represented by the matrix when the meaning is clear in context. □

Crossing changes and algebraic concordance. From the calculations and notation above, if a crossing change is performed on a knot K, the difference of Witt classes associated to the Seifert forms is given by

$$W_t(K_+) - W_t(K_-) = (A_t \oplus C_+) \oplus -(A_t \oplus C_-),$$

where

$$C_\pm = \begin{pmatrix} c(t) & 1-t \\ 1-\bar{t} & \epsilon_\pm(1-t)(1-\bar{t}) \end{pmatrix}.$$

Since $A_t \oplus -A_t$ is Witt trivial, as is C_-, only C_+ contributes to the difference of Witt classes. Diagonalization, the identification of $c(t)+1$ with $\Delta_{L_+}/\Delta_{K_-}$, and a final multiplication of a basis element (by Δ_{K_-}) yields the following theorem.

Theorem 7.6. $W_t(K_+) - W_t(K_-)$ *is represented by the matrix*

$$\begin{pmatrix} \Delta_{K_+}(t)\Delta_{K_-}(t) & 0 \\ 0 & -1 \end{pmatrix},$$

and thus the difference is determined by the Alexander polynomials of the knots.

The special case of $\omega = -1$ in the following corollary is a result from [Murasugi 1965]. The proof of the corollary follows from Theorem 7.6 by setting $t = \omega$ and induction on the number of crossing changes needed to reduce K to an unknot. To avoid the matrix being nonsingular, we must restrict to prime power roots of unity.

Corollary 7.7. *For ω a prime power root of unity,* $\operatorname{sign}(\Delta_K(\omega)) = (-1)^{\sigma_\omega(K)/2}$.

We now have the main result of this section, the following corollary of Theorem 7.6, a restatement of Theorem 1.1.

Corollary 7.8. *The algebraic concordance class of a knot is invariant under mutation; that is, $W_t(K) = W_t(K^*)$ for any knot K and its mutant K^*.*

Proof. A sequence of crossing changes in the tangle in K that is being mutated converts it into a tangle that is invariant under mutation. Thus we have a sequence of knots

$$K = K_0,\ K_1,\ \ldots,\ K_n = K_n^*,\ K_{n-1}^*,\ \ldots,\ K_0^* = K^*,$$

where $K_n = K_n^*$. By the previous theorem and the mutation invariance of the Alexander polynomial, each pair of successive differences is equal:

$$W_t(K_i) - W_t(K_{i+1}) = W_t(K_i^*) - W_t(K_{i+1}^*).$$

Thus $W_t(K) - W_t(K_n) = W_t(K^*) - W_t(K_n^*)$. Since $K_n = K_n^*$, the proof is complete. □

8. Generalized Mutation

Cooper and Lickorish [1999] studied the effect of a generalization of mutation, called *genus-2 mutation*, on the Seifert form of a knot. Here we deduce from their result an alternative proof of Theorem 1.1. In fact, since they demonstrate that generalized mutation generates a finer relation than mutation, a stronger result than Theorem 1.1 is in fact achieved.

Genus-2 mutation consists of removing a solid handlebody of genus 2 that contains a knot K from S^3 and replacing it via an involution of the boundary. The involution is selected to extend to the solid handlebody so that it has three fixed arcs. The resulting knot is called K^*. According to [Cooper and Lickorish 1999] there are Seifert matrices for K and K^* of the form

$$V = \begin{pmatrix} A & B^t \\ B & C \end{pmatrix} \qquad \text{and} \qquad V^* = \begin{pmatrix} A & B^t \\ B & C^t \end{pmatrix},$$

respectively, where A and C are square and B is of the form $(0 \mid b)$ for some single column b. Since V is a Seifert matrix and $V - V^t = (A - A^t) \oplus (C - C^t)$, we see that A and C are also algebraic Seifert matrices. Note that

$$V_t = \begin{pmatrix} A_t & -z^2 B^t \\ -z^2 B & C_t \end{pmatrix} \qquad \text{and} \qquad V_t^* = \begin{pmatrix} A_t & -z^2 B^t \\ -z^2 B & (C^t)_t \end{pmatrix}$$

where $z = t^{-1/2} - t^{1/2}$ and $z^2 = -(1-t)(1-\bar{t}) = -(1-t) - (1-\bar{t})$.

Since A is a Seifert matrix, A_t is nonsingular and hermitian. Let

$$P = \begin{pmatrix} I & z^2 (A_t)^{-1} B^t \\ 0 & I \end{pmatrix}.$$

Then V_t and V_t^* are congruent to $\bar{P}^t V_t P$ and $\bar{P}^t V_t^* P$, respectively, which in turn are seen, after a simple computation, to equal

$$\begin{pmatrix} A_t & 0 \\ 0 & C_t - z^4 B(A_t)^{-1} B^t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_t & 0 \\ 0 & (C^t)_t - z^4 B(A_t)^{-1} B^t \end{pmatrix}.$$

Suppose that A is an $m \times m$ matrix. Let $\alpha(t) \in \mathbb{Q}(t)$ be the (m, m) entry of $(A_t)^{-1}$ and recall that $B = (0 \mid b)$ for some single column b with integral entries. It is easy to see that

$$B(A_t)^{-1} B^t = \alpha(t) b b^t.$$

In particular, it is symmetric. For simplicity, let $E = C_t - z^4 B(A_t)^{-1} B^t$. Then $E^t = (C^t)_t - z^4 B(A_t)^{-1} B^t$ and we have that V_t and V_t^* are congruent to $A_t \oplus E$ and $A_t \oplus E^t$, respectively. The difference of Witt classes of V_t and V_t^* is given by

$$(A_t \oplus E) \oplus -(A_t \oplus E^t).$$

Since $A_t \oplus -A_t$ is Witt trivial, only $E \oplus -E^t$ contributes to the difference of Witt classes. Observe that E is a nonsingular hermitian matrix since $A_t \oplus E$ and A_t are. There is a nonsingular matrix Q such that $F = \overline{Q}^t E Q$ is diagonal. This implies that $F = F^t = Q^t E^t \overline{Q}$. Using congruence by base change $Q \oplus \overline{Q}$, we see $E \oplus -E^t$ is congruent to $F \oplus -F$, which is Witt trivial. Thus, $V_t = V_t^*$ in $W(\mathbb{Q}(t))$ and K and K^* are algebraically concordant since $\mathcal{G} \to W(\mathbb{Q}(t))$ is injective.

9. Strongly positive amphicheiral knots

A knot K is called strongly positive amphicheiral if it is invariant under an orientation-reversing involution of S^3 that preserves the orientation of K. This is easily seen to be equivalent to the statement that K, when viewed as a knot in $\mathbb{R}^3 \subset S^3$, is isotopic to a knot, again denoted by K, that is invariant under the involution $\tau : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\tau(x) = -x$, where $x \in \mathbb{R}^3$.

Hartley and Kawauchi [1979] proved that if K is strongly positive amphicheiral then $\Delta_K(t) = (F(t))^2$ for some Alexander polynomial F. Long [1984] proved that strongly positive amphicheiral knots are algebraically slice. Here we demonstrate that Long's theorem is in fact a corollary of the Hartley–Kawauchi theorem and the crossing change formula for the algebraic concordance class.

A bit of notation will be helpful: for a strongly amphicheiral knot that is invariant under the involution τ, τ defines a pairing of the crossing points in a diagram of K. A *paired crossing change* on such a K consists of changing both of a pair of crossings. Notice that since τ is orientation-reversing, the two crossings will be of opposite sign, so we denote the original knot K_{+-} and the knot formed by making the paired crossing changes K_{-+}.

Lemma 9.1. *A sequence of paired crossing changes converts a strongly positive amphicheiral knot into the unknot.*

Proof. Since an involution of S^1 cannot have one fixed point, K misses the origin in $\mathbb{R}^3$ and thus projects to a knot $\overline{K}$ in the quotient $\mathbb{R}^3 - \{0\}/\tau \equiv \mathbb{RP}^2 \times \mathbb{R}$. Since $\overline{K}$ lifts to a single component in the cover, it is homotopic to standard generator of $\pi_1(\mathbb{RP}^2 \times \mathbb{R})$, whose lift is an unknot in the cover. That homotopy can be carried out by a sequence of crossing changes, each of which lifts to a pair of crossing changes in the cover. □

Theorem 9.2 [Long 1984]. *A strongly positive amphicheiral knot is algebraically slice.*

Proof. Let K be the knot. By the previous lemma we need only show that $W_t(K_{+-}) - W_t(K_{-+})$ represents 0 in $W(\mathbb{Q}(t))$.

Working in the Witt group we can write

$$W_t(K_{+-}) - W_t(K_{-+}) = \big(W_t(K_{+-}) - W_t(K_{--})\big) - \big(W_t(K_{-+}) - W_t(K_{--})\big).$$

Applying Theorem 7.6, this is represented by the difference

$$\begin{pmatrix} \Delta_{K_{+-}}(t)\Delta_{K_{--}}(t) & 0 \\ 0 & -1 \end{pmatrix} \oplus -\begin{pmatrix} \Delta_{K_{-+}}(t)\Delta_{K_{--}}(t) & 0 \\ 0 & -1 \end{pmatrix}$$

Applying the Hartley–Kawauchi theorem, we write

$$\Delta_{K_{+-}}(t) = F(t)^2 \quad \text{and} \quad \Delta_{K_{-+}}(t) = G(t)^2,$$

and then cancel the (-1) summands to arrive at the difference

$$\begin{pmatrix} F(t)^2\Delta_{K_{--}}(t) & 0 \\ 0 & -G(t)^2\Delta_{K_{--}}(t) \end{pmatrix}.$$

This form has a metabolizer generated by the vector $(G(t), F(t)) \in \mathbb{Q}(t)^2$, and hence it is trivial in the Witt group, as desired. □

10. The Hartley–Kawauchi Theorem

Here we present a combinatorial proof of the theorem that for strongly positive amphicheiral knots the Alexander polynomial is a square of an Alexander polynomial. The proof also gives an alternative, though longer, route to Long's theorem than was given in the previous section. We begin by considering the existence of an equivariant Seifert surface for such a knot.

If Seifert's algorithm for constructing a Seifert surface is applied to a diagram for a strongly amphicheiral knot that is invariant under τ, the resulting surface will be invariant. In addition, τ restricted to this surface is orientation-preserving since τ preserves the orientation of the knot that is the boundary of the surface.

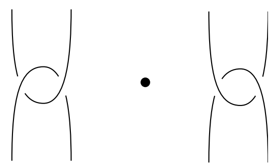
However τ reverses the positive normal direction since it reverses the orientation of R^3. Thus:

Lemma 10.1. *Let K be a strongly positive amphicheiral knot with involution τ. A Seifert surface F of K can be constructed so that F is invariant under τ and its Seifert form θ satisfies*

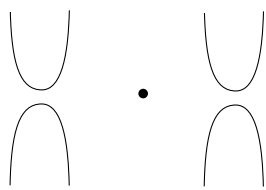
$$\theta(\tau u, \tau v) = -\theta(v, u)$$

for all $u, v \in H_1(F)$.

To understand the effect of crossing changes, we consider two figures. The first represents a portion of a symmetric diagram of a strongly amphicheiral knot, say K_{+-}:



The dot in center of the figure represents the origin in $\mathbb{R}^3$, the center of symmetry. For the knot K_{-+} the diagram will be the same, only a symmetric pair of crossing changes has been made. Thus, for K_{-+} the clasps pull apart, leaving a knot, denoted K', with diagram as follows:



Suppose that K' has an equivariant Seifert surface F_0 given by Seifert's algorithm and $H_1(F_0)$ has symplectic basis $w_1, \ldots, w_r$. Then an equivariant Seifert surface F for K_{+-} is given by adding four bands to F_0. The basis for $H_1(F_0)$ can be naturally extended to symplectic one for $H_1(F)$, $w_1, \ldots, w_r, x, y, \tau x, \tau y$, where y has trivial Seifert pairing with all elements other than x and itself, and x has trivial Seifert pairing with τy.

Let A be the Seifert matrix of F_0 with respect to $w_1, \ldots, w_r$ and let T denote the matrix representing the action of τ on $H_1(F_0)$. Then Lemma 10.1 applied to F_0 can be rewritten in terms of matrices: $T^t A T = -A^t$. After hermitianizing and taking inverses, we have

$$T(A_t)^{-1}T^t = -(A_t^t)^{-1} = \overline{-(A_t)^{-1}}.$$

To find the Seifert matrix for F with respect to the basis above, a couple of things have to be clarified. First, note that

$$\theta(x, \tau x) = -\theta(\tau\tau x, \tau x) = -\theta(x, \tau x),$$

and hence $\theta(x, \tau x) = 0$. Similarly, $\theta(\tau x, x) = 0$. Next, let

$$a = \begin{pmatrix} \theta(w_1, x) \\ \vdots \\ \theta(w_r, x) \end{pmatrix} \qquad \text{and} \qquad T = (t_{ij})_{1 \le i,j \le r}.$$

Then

$$\begin{pmatrix} \theta(w_1, \tau x) \\ \vdots \\ \theta(w_r, \tau x) \end{pmatrix} = \begin{pmatrix} -\theta(x, \tau w_1) \\ \vdots \\ -\theta(x, \tau w_r) \end{pmatrix} = \begin{pmatrix} -\sum_j t_{j1}\theta(x, w_j) \\ \vdots \\ -\sum_j t_{jr}\theta(x, w_j) \end{pmatrix}$$

$$= -T^t \begin{pmatrix} \theta(x, w_1) \\ \vdots \\ \theta(x, w_r) \end{pmatrix} = -T^t a.$$

It follows readily that the Seifert matrix for K_{+-} is the $(r+4) \times (r+4)$ matrix

$$V^\epsilon = \begin{pmatrix} A & a & 0 & -T^t a & 0 \\ a^t & b & 1 & 0 & 0 \\ 0 & 0 & \epsilon & 0 & 0 \\ -a^t T & 0 & 0 & -b & 0 \\ 0 & 0 & 0 & -1 & -\epsilon \end{pmatrix}, \qquad \text{where } \epsilon = -1.$$

Similarly, for K_{-+} the same matrix arise, only in this case $\epsilon = 0$. After hermitianizing we get

$$V_t^\epsilon = \begin{pmatrix} A_t & -z^2 a & 0 & z^2 T^t a & 0 \\ -z^2 a^t & -z^2 b & 1-t & 0 & 0 \\ 0 & 1-\bar{t} & -z^2\epsilon & 0 & 0 \\ z^2 a^t T & 0 & 0 & z^2 b & -(1-\bar{t}) \\ 0 & 0 & 0 & -(1-t) & z^2\epsilon \end{pmatrix},$$

where $z = t^{-1/2} - t^{1/2}$. Let

$$P = \begin{pmatrix} I & z^2(A_t)^{-1}a & 0 & -z^2(A_t)^{-1}T^t a & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $W_t^\epsilon = \bar{P}^t V_t^\epsilon P$. Then

$$W_t^\epsilon = \begin{pmatrix} A_t & 0 & 0 & 0 & 0 \\ 0 & -z^2b - z^4a^t(A_t)^{-1}a & 1-t & z^4a^t(A_t)^{-1}T^ta & 0 \\ 0 & 1-\bar{t} & -z^2\epsilon & 0 & 0 \\ 0 & z^4a^tT(A_t)^{-1}a & 0 & z^2b - z^4a^tT(A_t)^{-1}T^ta & -(1-\bar{t}) \\ 0 & 0 & 0 & -(1-t) & z^2\epsilon \end{pmatrix}.$$

Let $c(t) = -z^2b - z^4a^t(A_t)^{-1}a$. Since W_t^ϵ is hermitian, $c(t) = \overline{c(t)}$. The (1, 1)-entry of the lower right 2×2 submatrix of W_t^ϵ is

$$z^2b - z^4a^t\left(T(A_t)^{-1}T^t\right)a = \overline{z^2b + z^4a^t(A_t)^{-1}a} = \overline{-c(t)} = -c(t).$$

Let $d(t) = z^4a^t(A_t)^{-1}T^ta$. Then the 1×1 matrix $d(t)$ is equal to its transpose

$$z^4a^tT(A_t^t)^{-1}a = z^4a^tT\left(-T(A_t)^{-1}T^t\right)a = -z^4a^t(A_t)^{-1}T^ta = -d(t),$$

and hence $d(t)=0$. Also, note that $z^4a^tT(A_t)^{-1}a = \overline{d(t)} = 0$ since W_t^ϵ is hermitian.

Thus V_t^ϵ is congruent, by base change P, to

$$A_t \oplus C \oplus -C^t,$$

where

$$C = \begin{pmatrix} c(t) & 1-t \\ 1-\bar{t} & -z^2\epsilon \end{pmatrix}.$$

Since $\det P = 1$,

$$\Delta_{K_{+-}} = (c(t)+1)^2\frac{1}{z^r}\det A_t = (c(t)+1)^2\Delta_{K_{-+}},$$

where $c(t) = c(\bar{t})$. This proves the Hartley–Kawauchi theorem.

Next, to prove Long's theorem, we will show that $V_t(K_{+-})$, A_t, and $V_t(K_{-+})$ are all Witt-equivalent. It suffices to show that $C \oplus -C^t$ is Witt-trivial. Observe that C is nonsingular and hermitian since $A_t \oplus C \oplus -C^t$ and A_t are. There is a nonsingular matrix Q such that $D = \bar{Q}^tCQ$ is diagonal. This implies that

$$D = D^t = Q^tC^t\bar{Q}.$$

Using congruence by base change $Q \oplus \bar{Q}$, we see that $C \oplus -C^t$ is congruent to $D \oplus -D$, which is Witt trivial. Thus, K_{+-} and K_{-+} are algebraically concordant. This proves Long's theorem.

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