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Let (M, p) and $(\hat{M}, \hat{p})$ be the germs of real-analytic 1-infinite type hypersurfaces in $\mathbb{C}^2$. We prove that any formal equivalence sending (M, p) into $(\hat{M}, \hat{p})$ is formally parametrized (and hence uniquely determined by) its jet at p of a predetermined order depending only on (M, p). As an application, we use this to examine the local formal transformation groups of such hypersurfaces.

1. Introduction

A *formal* (*holomorphic*) *mapping* $H : (\mathbb{C}^2, p) \to (\mathbb{C}^2, \hat{p})$, with $p, \hat{p} \in \mathbb{C}^2$, is a $\mathbb{C}^2$-valued formal power series

$$H(Z) = \hat{p} + \sum_{|\alpha|\geq 1} c_\alpha (Z-p)^\alpha, \quad c_\alpha \in \mathbb{C}^2, \quad Z = (Z_1, Z_2).$$

The map H is *invertible* if there exists a formal map $H^{-1} : (\mathbb{C}^2, \hat{p}) \to (\mathbb{C}^2, p)$ such that $H(H^{-1}(Z)) \equiv H^{-1}(H(Z)) \equiv Z$ as formal power series; equivalently, if the Jacobian of H is nonvanishing at p. We denote by $J^k(\mathbb{C}^2, \mathbb{C}^2)_{p,\hat{p}}$ the *jet space* of order k of (formal) holomorphic mappings $(\mathbb{C}^2, p) \to (\mathbb{C}^2, \hat{p})$, and by $j_p^k(H) \in J^k(\mathbb{C}^2, \mathbb{C}^2)_{p,\hat{p}}$ the *k-jet* of H at p. (See Section 2 for further details.)

Suppose that (M, p) and $(\hat{M}, \hat{p})$ are (germs of) real-analytic hypersurfaces at p and $\hat{p}$ respectively, given by the real-analytic, real-valued local defining functions $\rho(Z, \bar{Z})$ and $\hat{\rho}(Z, \bar{Z})$. The formal map H is said to *take* (M, p) *into* $(\hat{M}, \hat{p})$ if

$$\hat{\rho}\big(H(Z), \overline{H(Z)}\big) \equiv c(Z, \bar{Z})\rho(Z, \bar{Z})$$

(in the sense of power series) for some formal power series $c(Z, \bar{Z})$; if in addition the formal map is invertible, it is called a *formal equivalence* between (M, p) and $(\hat{M}, \hat{p})$, and the germs themselves are called *formally equivalent*.

We wish to study the parametrization and finite determination of invertible formal holomorphic mappings of $\mathbb{C}^2$ taking one real-analytic hypersurface M into

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another. There is a great deal of literature on this if M is assumed to be *minimal* at p, that is, if there is no complex hypersurface through p in $\mathbb{C}^2$ contained in M; see the remarks at the end of this introduction. In the present paper, however, we shall assume that M is not minimal at p, so that there exists a complex hypersurface $\Sigma \subset \mathbb{C}^2$ with $p \in \Sigma \subset M$. It is well known [Chern and Moser 1974; Baouendi et al. 1999b, Chapter IV] that for any real-analytic hypersurface $M \subset \mathbb{C}^2$ and point $p \in M$ (not necessarily minimal), there exist local holomorphic coordinates $(z, w) \in \mathbb{C} \times \mathbb{C}$, vanishing at p, such that M is defined locally by the equation

$$\operatorname{Im} w = \Theta(z, \bar{z}, \operatorname{Re} w),$$

where $\Theta(z, \bar{z}, s)$ is a real-valued, real-analytic function such that

$$\Theta(z, 0, s) \equiv \Theta(0, \bar{z}, s) \equiv 0.$$

Such coordinates are called *normal coordinates* for M at p, and are not unique. M is said to be of *finite type* at p if $\Theta(z, \bar{z}, 0) \not\equiv 0$; otherwise M is of *infinite type* at p. This definition is equivalent to being of finite type in the sense of [Kohn 1972] and [Bloom and Graham 1977]. For real-analytic hypersurfaces, it is also equivalent to minimality — indeed, if M is of infinite type at p, then (in normal coordinates) M contains the nontrivial complex hypersurface $\Sigma = \{w = 0\}$. (For details see [Baouendi et al. 1999b, Chapter I], for example.)

In this paper, we shall focus our attention on 1-infinite type points p of a real-analytic hypersurface $M \subset \mathbb{C}^2$, i.e., points at which the normal coordinates above satisfy the additional condition that $\Theta_s(z, \bar{z}, 0) \not\equiv 0$. (See Section 2 for precise definitions.) Our main result gives rational dependence of a formal equivalence between 1-infinite type hypersurfaces on its jet of a predetermined order.

Theorem 1.1. *Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface, and suppose $p \in M$ is of 1-infinite type. There exists an integer k such that, given any hypersurface $\hat{M} \subset \mathbb{C}^2$ with $(\hat{M}, \hat{p})$ formally equivalent to (M, p), there exists a formal power series of the form*

$$\Psi(Z; \Lambda) = \sum_{\alpha} \frac{p_\alpha(\Lambda)}{q(\Lambda)^{\ell_\alpha}} (Z - p)^\alpha, \tag{1}$$

where p_α, q are (respectively) $\mathbb{C}^2$- and $\mathbb{C}$-valued polynomials on the jet space $J^k(\mathbb{C}^2, \mathbb{C}^2)_{p,\hat{p}}$ and the ℓ_α are nonnegative integers, such that any formal equivalence $H : (M, p) \to (\hat{M}, \hat{p})$ satisfies

$$q\big(j_p^k(H)\big) = \det\Big(\frac{\partial H}{\partial Z}(p)\Big) \neq 0 \quad \textit{and} \quad H(Z) = \Psi\big(Z; j_p^k(H)\big).$$

Our proof (presented in Section 5) will actually give a constructive process for determining such an k.

Theorem 1.1 has a number of applications. The first states that any formal equivalence between two germs of 1-infinite type hypersurfaces (M, p) and $(\hat{M}, \hat{p})$ is determined by finitely many derivatives at p.

Theorem 1.2. *Let (M, p) and k be as in Theorem 1.1. If $H^1, H^2 : (M, p) \to (\hat{M}, \hat{p})$ are formal equivalences and*

$$\frac{\partial^{|\alpha|} H^1}{\partial Z^\alpha}(p) = \frac{\partial^{|\alpha|} H^2}{\partial Z^\alpha}(p) \quad \textit{for all } |\alpha| \leq k,$$

then $H^1 = H^2$ as power series.

Our second application deals with the structure of jets of formal equivalences in the jet space $J^k(\mathbb{C}^2, \mathbb{C}^2)_{p,\hat{p}}$, or rather in the submanifold $G^k(\mathbb{C}^2)_{p,\hat{p}}$ of jets of invertible maps taking $(\mathbb{C}^2, p)$ to $(\mathbb{C}^2, \hat{p})$. We shall denote by $\mathcal{F}(M, p; \hat{M}, \hat{p})$ the set of formal equivalences taking (M, p) into $(\hat{M}, \hat{p})$.

Theorem 1.3. *Let (M, p) and k be as in Theorem 1.1. Then for any (germ of a) real-analytic hypersurface $(\hat{M}, \hat{p})$ in $\mathbb{C}^2$, the mapping*

$$j_p^k : \mathcal{F}(M, p; \hat{M}, \hat{p}) \to G^k(\mathbb{C}^2)_{p,\hat{p}}$$

is an injection onto a real algebraic submanifold of $G^k(\mathbb{C}^2)_{p,\hat{p}}$.

Of special interest is the case $(\hat{M}, \hat{p}) = (M, p)$, since $\mathcal{F}(M, p; \hat{M}, \hat{p})$ becomes a group under composition, called the *formal stability group* of M at p and denoted by $\mathrm{Aut}(M, p)$. We shall denote by $G^k(\mathbb{C}^2)_p := G^k(\mathbb{C}^2)_{p,p}$ the *k-jet group* of $\mathbb{C}^2$ at p. The following result is then a corollary of Theorem 1.3.

Theorem 1.4. *Let (M, p) and k be as in Theorem 1.1. Then the mapping*

$$j_p^k : \mathrm{Aut}(M, p) \to G^k(\mathbb{C}^2)_p$$

defines an injective group homomorphism onto a real algebraic Lie subgroup of $G^k(\mathbb{C}^2)_p$.

The study of the (formal) transformation groups of hypersurfaces in $\mathbb{C}^N$ has a long history. Its roots can be traced back to E. Cartan, who studied the structure of the local transformation groups of smooth Levi nondegenerate hypersurfaces in $\mathbb{C}^2$ in [Cartan 1932a; 1932b]. These results were later extended to higher dimensions by Chern and Moser in [Chern and Moser 1974], who also proved the finite determination of such equivalences by their 2-jets.

Further results about the transformation groups of various classes of finite type generic submanifolds of $\mathbb{C}^N$ have been obtained more recently by a number of mathematicians. Regarding the parametrization of transformation groups, we mention the work of Zaitsev [1997], and Baouendi, Ebenfelt, and Rothschild [Baouendi et al. 1999a], which presents modified versions of Theorems 1.2–1.4 valid for smooth generic submanifolds M, $\hat{M}$ in $\mathbb{C}^N$ with M of finite type and $\hat{M}$ finitely nondegenerate. Moreover, there exist a number of results concerning the finite determination of local equivalences addressed in Theorem 1.2. We mention the work of Baouendi, Mir, and Rothschild [Baouendi et al. 2002], which gives the best finite determination results to date for the general case of finite type submanifolds in $\mathbb{C}^N$, and Ebenfelt, Lamel, and Zaitsev [Ebenfelt et al. 2003], which addresses the case $\mathbb{C}^2$ specifically, proving that the local equivalences between any two nonflat real-analytic hypersurface are determined by a finite jet. The reader interested in other recent work on these problems is directed to the excellent survey articles [Rothschild 2003] and [Zaitsev 2002].

For the proofs of the four theorems above, it is convenient to work with formal mappings between formal real hypersurfaces. Hence, the results presented here will be reformulated and proved in this more general context. The following section presents the necessary preliminaries and definitions. In what follows, the distinguished points p and $\hat{p}$ on M and $\hat{M}$, respectively, will, for convenience and without loss of generality, be assumed to be 0.

2. Preliminaries and basic definitions

Formal mappings and hypersurfaces. Let $X = (X_1, \dots, X_N)$ be a N-tuple of indeterminates, and let $\mathcal{R}$ denote a commutative ring with unity. We define

- $\mathcal{R}[[X]] :=$ the ring of formal power series in X with coefficients in $\mathcal{R}$;
- $\mathcal{R}[X] :=$ the ring of polynomials in X with coefficients $\mathcal{R}$.

For $\mathcal{R} = \mathbb{C}$, we shall also define

- $\mathbb{C}\{X\} :=$ the ring of convergent power series in X with coefficients in $\mathbb{C}$;
- $\mathcal{O}_\epsilon(X) :=$ the ring of power series in X with coefficients in $\mathbb{C}$ that converge for $X_j \in \mathbb{C},\ |X_j| < \epsilon,\ 1 \le j \le N$.

We have canonical embeddings

$$\mathbb{C}[X] \subset \mathcal{O}_\epsilon(X) \subset \mathbb{C}\{X\} \subset \mathbb{C}[[X]].$$

A power series $\rho \in \mathbb{C}[[Z, \zeta]]$, where $Z = (Z_1, \dots, Z_N)$ and $\zeta = (\zeta_1, \dots, \zeta_N)$, is called *real* if $\rho(Z, \zeta) = \bar{\rho}(\zeta, Z)$, where $\bar{\rho}$ denotes the power series obtained by replacing the coefficients of ρ by their complex conjugates. If, in addition, the power series ρ satisfies the conditions

$$\rho(0) = 0, \quad d\rho(0) \neq 0, \tag{2}$$

we say that ρ defines a *formal real hypersurface M* of $\mathbb{C}^N$ through 0, and we write

$$M = \left\{\rho(Z, \bar{Z}) = 0\right\}$$

and say that the pair $(M, 0)$ is a *formal real hypersurface*. The function ρ is a *formal defining function* for M. The reader should observe that if M is a formal real hypersurface in $\mathbb{C}^N$ with formal defining function ρ, then in general there is no actual point set $M \subset \mathbb{C}^N$.

Suppose that $\hat{\rho}$ is another formal power series (not necessarily real) satisfying conditions (2). If there exists a power series $a(Z, \zeta)$ (necessarily invertible at 0) such that

$$\hat{\rho}(Z, \zeta) = a(Z, \zeta)\,\rho(Z, \zeta),$$

then we say that $\hat{\rho}$ also defines the formal real hypersurface M, and again we write $M = \{\hat{\rho}(Z, \bar{Z}) = 0\}$.

By a formal mapping $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$, denoted $H \in E(\mathbb{C}^N, \mathbb{C}^N)_{0,0}$, we shall mean an element $H \in \mathbb{C}[[Z]]^N$ such that $H(0) = 0$. We say H is a *formal change of coordinates* if it is formally invertible, i.e., if there exists a formal map $H^{-1} : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ such that

$$H(H^{-1}(Z)) \equiv H^{-1}(H(Z)) \equiv Z$$

as formal power series. As noted in the introduction, H is a formal change of coordinates in $\mathbb{C}^N$ if and only if its Jacobian at 0 is nonzero.

Given a formal change of coordinates H in $\mathbb{C}^N$, we define its corresponding *formal holomorphic change of variable* by

$$Z = H(Z'), \qquad \zeta = \overline{H}(\zeta').$$

If $M = \{\rho(Z, \bar{Z}) = 0\}$ is a formal real hypersurface of $\mathbb{C}^N$, we say M is expressed in the Z' coordinates by $\{\rho(H(Z'), \overline{H(Z')}) = 0\}$.

If $\hat{M} = \{\hat{\rho}(Z, \bar{Z}) = 0\}$ is another formal real hypersurface of $\mathbb{C}^N$, then a formal mapping $H \in E(\mathbb{C}^N, \mathbb{C}^N)_{0,0}$ is said to *take M into $\hat{M}$* if there exists a power series $c(Z, \zeta)$ (not necessarily invertible at 0) such that

$$\hat{\rho}\big(H(Z), \overline{H}(\zeta)\big) = c(Z, \zeta)\,\rho(Z, \zeta).$$

In this situation we write as $H : (M, 0) \to (\hat{M}, 0)$. This definition is independent of the power series used to define M and $\hat{M}$.

If $H : (M, 0) \to (\hat{M}, 0)$ is as above and H is invertible, it follows that H^{-1} takes $\hat{M}$ into M. In this case we say that M and $\hat{M}$ are *formally equivalent*, and that H is a *formal equivalence* between them, denoted $H \in \mathcal{F}(M, 0; \hat{M}, 0)$.

The motivation behind these definitions is the following. If the formal series ρ defining the formal real hypersurface M is actually convergent, then the equation $\rho(Z, \bar{Z}) = 0$ defines a real-analytic hypersurface M of $\mathbb{C}^N$ passing through the origin. Moreover, if $H : \mathbb{C}^N \to \mathbb{C}^N$ is a holomorphic mapping with $H(0) = 0$, and M, $\hat{M}$ are both real-analytic hypersurfaces of $\mathbb{C}^N$, then $H(M) \subset \hat{M}$ if and only if the formal mapping H maps the formal real hypersurface M into the formal real hypersurface $\hat{M}$.

For each positive integer k, we denote by $J^k(\mathbb{C}^N, \mathbb{C}^N)_{0,0}$ the jet space of order k of (formal) holomorphic mappings $(\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$, and by $j_0^k : E(\mathbb{C}^N, \mathbb{C}^N) \to J^k(\mathbb{C}^N, \mathbb{C}^N)_{0,0}$ the corresponding jet mapping taking a formal mapping H to its k-jet at 0, $j_0^k(H)$. We denote by $G^k(\mathbb{C}^N)_0 \subset J^k(\mathbb{C}^N, \mathbb{C}^N)_{0,0}$ the collection of k-jets of invertible formal mappings of $(\mathbb{C}^N, 0)$ to itself.

Given coordinates Z and $\hat{Z}$ on $\mathbb{C}^N$, we may identify the jet space $J^k(\mathbb{C}^N, \mathbb{C}^N)_{0,0}$ with the set of degree-k polynomial mappings of $(\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$. The coordinates on $J^k(\mathbb{C}^N, \mathbb{C}^N)_{0,0}$, which we denote by Λ, can then be taken to be the coefficients of these polynomials. Formal changes of coordinates in $\mathbb{C}^N$ yield polynomial changes of coordinates in $J^k(\mathbb{C}^N, \mathbb{C}^N)_{0,0}$.

If M is a formal real hypersurface in $\mathbb{C}^N$, there is a formal change of coordinates $Z = (z, w) \in \mathbb{C}[\![z, w]\!]^N$ with $z = (z_1, \dots, z_{N-1})$, such that M, under the corresponding formal holomorphic change of variable $Z = Z(z, w)$, $\zeta = \bar{Z}(\chi, \tau)\big)$, is defined by

$$\rho(z, w, \chi, \tau) := \left(\frac{w-\tau}{2i}\right) - \Theta\left(z, \chi, \frac{w+\tau}{2}\right) \in \mathbb{C}[\![Z, \zeta]\!],$$

where $\Theta \in \mathbb{C}[\![z, \chi, s]\!]$ is real and satisfies $\Theta(z, 0, s) = \Theta(0, \chi, s) = 0$. Such coordinates are called *normal coordinates* for M; see [Baouendi et al. 1999b, Chapter IV].

Using the formal Implicit Function Theorem to solve for w above, we see that there exists a unique formal power series $Q \in \mathbb{C}[\![z, \chi, \tau]\!]$ with $Q(0, 0, 0) = 0$ such that $\rho\big(z, Q(z, \chi, \tau), \chi, \tau\big) \equiv 0$; moreover, Q is convergent whenever Θ is. This implies that there exists a power series $a(z, w, \chi, \tau)$, nonvanishing at 0, such that $\rho(z, w, \chi, \tau) = a(z, w, \chi, \tau)\big(w - Q(z, \chi, \tau)\big)$; whence we may write (abusing notation)

$$M = \left\{\left(\frac{w-\overline{w}}{2i}\right) = \Theta\left(z, \bar{z}, \frac{w+\overline{w}}{2}\right)\right\} = \big\{w = Q(z, \bar{z}, \overline{w})\big\}. \tag{3}$$

Observe that the normality of the coordinates implies $Q(z, 0, \tau) = Q(0, \chi, \tau) = \tau$.

Given normal coordinates $Z = (z, w)$ for M as above, define the numbers $m, r, L, K \in \{0, 1, 2, \dots\} \cup \{\infty\}$ as follows. Set

$$m := \sup\{q : \Theta_{s^j}(z, \chi, 0) \equiv 0 \text{ for all } j < q\}. \tag{4}$$

If $m = \infty$ (i.e., if $\Theta \equiv 0$), then set $r = L = K = \infty$. Otherwise, set

$$r := \sup\left\{q : \Theta_{z^\alpha \chi^\beta s^m}(0, 0, 0) = 0 \text{ for all } |\alpha| + |\beta| < q\right\}, \tag{5}$$

$$L := \sup\left\{q : \Theta_{\chi^\beta s^m}(z, 0, 0) \equiv 0 \text{ for all } |\beta| < q\right\}, \tag{6}$$

$$K := \sup\left\{q : \Theta_{z^\alpha \chi^\beta s^m}(0, 0, 0) = 0 \text{ for all } |\alpha| < q,\ |\beta| = L\right\}. \tag{7}$$

We shall show in Theorem 2.1 that this 4-tuple of numbers is independent of the normal coordinates used to define them.

We say that M is of *finite type* at 0 if $m = 0$; otherwise M is of *infinite type* at 0. If we wish to emphasize the number $m \geq 1$, we shall say that M is of *m-infinite type* at 0 if $m < \infty$, and is *flat* at 0 if $m = \infty$. We shall further say M is of *finite type r* at 0 if $m = 0$, and is of *m-infinite type r* at 0 if $1 \leq m < \infty$.

We conclude these definitions by stating a few known results concerning these numbers in the case when M is a real-analytic hypersurface in $\mathbb{C}^N$. In this case, it is known that the pair (m, r) is a biholomorphic invariant of M; see [Meylan 1995]. If M is of infinite type at 0, it contains a formal complex hypersurface Σ passing through 0. (In normal coordinates, we may take $\Sigma = \{w = 0\}$.) In fact, $m > 0$ is constant along the complex hypersurface $\Sigma \subset M$ through 0. And while r is not constant along Σ, there exists a proper, real-analytic subvariety $V \subset \Sigma$ outside of which all points are of m-infinite type 2. See [Ebenfelt 2002] for details.

Statement of results. Our first result shows that the 4-tuple (m, r, L, K) (and hence the notion of being m-infinite type r at a point) is in fact a *formal* invariant of a hypersurface.

Theorem 2.1. *Let $(M, 0)$ be a formal real hypersurface of $\mathbb{C}^N$. Then the numbers (m, r, L, K) are independent of the choice of normal coordinates used to define them. Moreover, if $(\hat{M}, 0)$ is formally equivalent to $(M, 0)$ and has the corresponding 4-tuple $(\hat{m}, \hat{r}, \hat{L}, \hat{K})$, then $(m, r, L, K) = (\hat{m}, \hat{r}, \hat{L}, \hat{K})$.*

We shall then focus on the case $N = 2$ and $m = 1$. We may now state the generalizations of Theorems 1.1 through 1.4 valid for formal real hypersurfaces. Our main result is the following.

Theorem 2.2. *Let $(M, 0)$ be a formal real hypersurface in $\mathbb{C}^2$ of 1-infinite type. There exists an integer k such that given any formal real hypersurface $(\hat{M}, 0)$ in $\mathbb{C}^2$ formally equivalent to $(M, 0)$, there exists a formal power series of the form*

$$\Psi(Z; \Lambda) = \sum_{\alpha} \frac{p_\alpha(\Lambda)}{q(\Lambda)^{\ell_\alpha}} Z^\alpha, \tag{8}$$

where p_α, q are (respectively) $\mathbb{C}^2$- and $\mathbb{C}$-valued polynomials on the jet space $J^k(\mathbb{C}^2, \mathbb{C}^2)_{0,0}$ and the ℓ_α are nonnegative integers, such that any formal equivalence $H \in \mathcal{F}(M, 0; \hat{M}, 0)$ satisfies

$$q\big(j_0^k(H)\big) = \det\left(\frac{\partial H}{\partial Z}(0)\right) \neq 0, \qquad H(Z) = \Psi\big(Z; j_0^k(H)\big).$$

It is clear from the remarks made in the previous section that Theorem 2.2 is a more general version of Theorem 1.1 from the introduction. As a consequence of this result, we have the following, from which Theorem 1.2 is derived.

Theorem 2.3. *Let $(M,0)$ be a formal real hypersurface in $\mathbb{C}^2$ of 1-infinite type, and let k be the number described in Theorem 2.2. If $(\hat{M},0)$ is a formal hypersurface formally equivalent to $(M,0)$, and $H^1, H^2 : (M,0) \to (\hat{M},0)$ are formal equivalences such that*

$$\frac{\partial^{|\alpha|} H^1}{\partial Z^\alpha}(0) = \frac{\partial^{|\alpha|} H^2}{\partial Z^\alpha}(0) \quad \textit{for all } |\alpha| \le k,$$

then $H^1 = H^2$ as power series.

We shall then prove the following generalization of Theorem 1.4.

Theorem 2.4. *Let M and k be as in Theorem 2.2. The mapping*

$$j_0^k : \mathrm{Aut}(M,0) \to G^k(\mathbb{C}^2)_0$$

defines an injective group homomorphism onto a real algebraic Lie subgroup of $G^k(\mathbb{C}^2)_0$.

The following generalization of Theorem 1.3 is a consequence of Theorem 2.4.

Theorem 2.5. *Let M and k be as in Theorem 2.2. For any formal real hypersurface $\hat{M}$ in $\mathbb{C}^2$, the mapping*

$$j_0^k : \mathscr{F}(M,0;\hat{M},0) \to J^k(\mathbb{C}^2)_0$$

is an injection onto a real algebraic submanifold of $G^k(\mathbb{C}^2)_0$.

3. Formal invariance of type conditions

In this section, we shall prove Theorem 2.1, or rather a slightly sharper statement of which Theorem 2.1 is an immediate consequence:

Proposition 3.1. *Let $(M,0)$ be a formal real hypersurface in $\mathbb{C}^N$, given in normal coordinates $Z=(z,w)$ by Equation (3). Let $(\hat{M},0)$ be a formal real hypersurface in $\mathbb{C}^N$, given in normal coordinates $\hat{Z}=(\hat{z},\hat{w})$ by the corresponding "hatted" defining functions*:

$$\hat{M} = \left\{ \frac{\hat{w}-\bar{\hat{w}}}{2i} = \hat{\Theta}\Big(\hat{z}, \bar{\hat{z}}, \frac{\hat{w}+\bar{\hat{w}}}{2}\Big) \right\} = \left\{ \hat{w} = \hat{Q}(\hat{z}, \bar{\hat{z}}, \bar{\hat{w}}) \right\}.$$

Define as in Section 2 the 4-tuple (m,r,L,K) for M and the corresponding 4-tuple $(\hat{m},\hat{r},\hat{L},\hat{K})$ for $\hat{M}$. If M and $\hat{M}$ are formally equivalent, then $(m,r,L,K) = (\hat{m},\hat{r},\hat{L},\hat{K})$.

We begin with a useful lemma concerning the form of formal mappings in normal coordinates. It is proved in the same way as [Baouendi et al. 1999b, Lemma 9.4.4].

Lemma 3.2. *Let M, $\hat{M}$ be formal hypersurfaces in $\mathbb{C}^N$ through 0, expressed in normal coordinates as in Proposition 3.1. If $H = (F, G) : (M, 0) \to (\hat{M}, 0)$ is a formal mapping, then $G(z, w) = w\, g(z, w)$ for some $g \in \mathbb{C}[\![z, w]\!]$. Moreover, if H is a formal equivalence, then $F(z, 0) \in \mathbb{C}[\![z]\!]^{N-1}$ is a formal equivalence, and $g(0, 0) \neq 0$.*

As a consequence of this lemma, we shall henceforth write formal equivalences (in suitable normal coordinates) as

$$H(z, w) = \big(f(z, w), w\, g(z, w)\big), \tag{9}$$

with $f = (f^1, \ldots, f^{N-1}) \in \mathbb{C}[\![z, w]\!]^{N-1}$ satisfying $\det f_z(0, 0) \neq 0$ and $g \in \mathbb{C}[\![z, w]\!]$ satisfying $g(0, 0) \neq 0$. Observe that the condition that H map M formally into $\hat{M}$ may be written as

$$Q(z, \chi, \tau)\, g\big(z, Q(z, \chi, \tau)\big) \equiv \hat{Q}\big(f\big(z, Q(z, \chi, \tau)\big), \bar{f}(z, \chi), \tau\, \bar{g}(\chi, \tau)\big). \tag{10}$$

Moreover, for convenience, we shall formally expand f and g as

$$f(z, w) = \sum_{n \geq 0} \frac{f_n(z)}{n!}\, w^n, \qquad g(z, w) = \sum_{n \geq 0} \frac{g_n(z)}{n!}\, w^n. \tag{11}$$

The main technical lemma in the proof of Proposition 3.1 is the following.

Lemma 3.3. *Suppose M, $\hat{M}$ are formal hypersurfaces in $\mathbb{C}^N$ through 0, expressed in normal coordinates as in Proposition 3.1, and assume that $H : (M, 0) \to (\hat{M}, 0)$ is a formal equivalence. Then for every $j \geq 0$, if*

$$\hat{Q}(\hat{z}, \hat{\chi}, 0) \equiv \hat{Q}_{\hat{\tau}}(\hat{z}, \hat{\chi}, 0) - 1 \equiv \hat{Q}_{\hat{\tau}^2}(\hat{z}, \hat{\chi}, 0) \equiv \cdots \equiv \hat{Q}_{\hat{\tau}^j}(\hat{z}, \hat{\chi}, 0) \equiv 0,$$

then

$$Q(z, \chi, 0) \equiv Q_\tau(z, \chi, 0) - 1 \equiv Q_{\tau^2}(z, \chi, 0) \equiv \cdots \equiv Q_{\tau^j}(z, \chi, 0) \equiv 0. \tag{12}$$

Moreover, $g_0(z), g_1(z), \ldots, g_j(z)$ are all real constants (with $g_0(z)$ nonzero), and

$$Q_{\tau^{j+1}}(z, \chi, 0) \equiv g(0)^j\, \hat{Q}_{\hat{\tau}^{j+1}}\big(f_0(z), \bar{f}_0(\chi), 0\big).$$

To prove Lemma 3.3, we make use of two results. The first is a generalization of the Chain Rule due to Faa de Bruno; see [Range 1986], for example:

Lemma 3.4 (Faa de Bruno's Formula). *Suppose that $f = \big(f_1, f_2, \ldots, f_\ell\big) \in \mathbb{C}^\ell[\![z]\!]$ with $z \in \mathbb{C}$ and $f(0) = 0$, and suppose $h(z_1, z_2, \ldots, z_\ell) \in \mathbb{C}[\![z_1, z_2, \ldots, z_\ell]\!]$. Then*

$$\frac{\partial^v}{\partial z^v}\big\{h\big(f(z)\big)\big\} = \sum_{\substack{[\alpha^1]+[\alpha^2]+\cdots\\+[\alpha^\ell]=v}} \frac{v!\, h_{z_1{}^{|\alpha^1|} z_2{}^{|\alpha^2|} \cdots z_\ell{}^{|\alpha^\ell|}}\big(f(z)\big)}{\alpha^1!\, \alpha^2! \cdots \alpha^\ell!} \prod_{\substack{1 \leq q \leq v\\ 1 \leq p \leq \ell}} \left(\frac{f_p{}^{(q)}(z)}{q!}\right)^{\alpha_q^p},$$

where each $\alpha^p = (\alpha_1^p, \dots, \alpha_v^p)$ *denotes an* v*-dimensional multi-index, and*

$$|\alpha^p| = \sum_{q=1}^{v} \alpha_q^p, \quad [\alpha^p] = \sum_{q=1}^{v} q\,\alpha_q^p, \quad \alpha^p! = \prod_{q=1}^{v} (\alpha_q^p)!$$

The proof is a routine induction, and is left to the reader. The other result we shall need gives a second characterization of the number m:

Proposition 3.5 [Baouendi and Rothschild 1991, Proposition 1.7]. *Let* M, m, Θ, *and* Q *be as above. Then*

$$m = \sup\Big\{q : \frac{\partial^j}{\partial\tau^j}\{Q(z,\chi,\tau) - \tau\}\Big|_{\tau=0} \equiv 0 \text{ for all } j < q\Big\}.$$

Furthermore,

$$Q_{\tau^m}(z,\chi,0) = \begin{cases} \dfrac{1 + i\,\Theta_s(z,\chi,0)}{1 - i\,\Theta_s(z,\chi,0)} & \text{if } m = 1, \\ 2i\,\Theta_{s^m}(z,\chi,0) & \text{if } 2 \le m < \infty. \end{cases}$$

Proof of Lemma 3.3. Differentiating identity (10) v times in τ, setting $\tau = 0$, and canceling $v!$ from both sides yields the identity

$$(13)\quad \sum_{k+[\xi]=v} \frac{g_{|\xi|}(z)\,Q_{\tau^k}(z,\chi,0)}{k!\,\xi!} \prod_{p=1}^{v} \left(\frac{Q_{\tau^p}(z,\chi,0)}{p!}\right)^{\xi_p}$$

$$\equiv \sum_{\substack{[\alpha^1]+\cdots+[\alpha^n]+[\beta^1]+\cdots \\ \cdots+[\beta^n]+[\gamma]=v}} \frac{\hat{Q}_{\hat{z}^{(|\alpha^1|,\dots,|\alpha^n|)}\,\hat{\chi}^{(|\beta^1|\cdots,|\beta^n|)}\,\hat{\tau}^{|\gamma|}}\big(f_0(z), \bar{f}_0(\chi), 0\big)}{\alpha^1!\cdots\alpha^n!\,\beta^1!\cdots\beta^n!\,\gamma!}$$

$$\times \prod_{\substack{1\le q\le v \\ 1\le u\le n}} \left(\sum_{[\eta]=q} \frac{f^u_{|\eta|}(z)}{\eta!} \prod_{r=1}^{q} \left(\frac{Q_{\tau^r}(z,\chi,0)}{r!}\right)^{\eta_r}\right)^{\alpha_q^u} \left(\frac{\overline{f_q^u}(\chi)}{q!}\right)^{\beta_q^u} \left(\frac{\overline{g_{q-1}}(\chi)}{(q-1)!}\right)^{\gamma_q}.$$

We now proceed by induction. For $j = 0$, we assume only that $\hat{Q}(\hat{z}, \hat{\chi}, 0) \equiv 0$. Setting $\tau = 0$ in identity (10), we find

$$Q(z,\chi,0)\; g\big(z, Q(z,\chi,0)\big) \equiv \hat{Q}\big(f_0(z), \bar{f}_0(\chi), 0\big) = 0.$$

Since $g(z, Q(z,\chi,0))$ does not vanish at $z = \chi = 0$, we conclude $Q(z,\chi,0) \equiv 0$.

Applying the $v = 1$ case of identity (13), we find

$$Q_\tau(z,\chi,0)\,g_0(z) \equiv \hat{Q}_{\hat{\tau}}\big(f_0(z), \bar{f}_0(\chi), 0\big)\,\bar{g}_0(\chi).$$

Setting $\chi = 0$ yields $g_0(z) \equiv \bar{g}_0(0) = \overline{g_0(0)}$, whence $g_0(z)$ is a real constant r, and since H is invertible, $r \ne 0$ necessarily. Dividing $g_n(z) = \bar{g}_0(\chi) = r \ne 0$ from both sides of the identity above yields

$$Q_\tau(z,\chi,0) \equiv \hat{Q}_{\hat{\tau}}\big(f_0(z), \bar{f}_0(\chi), 0\big),$$

which proves the $j=0$ case.

Now, assume that the lemma holds for some $j-1 \geq 0$; we shall prove it for j. Suppose that (12) holds. By induction, we know that

$$Q(z,\chi,0) \equiv Q_\tau(z,\chi,0) - 1 \equiv Q_{\tau^2}(z,\chi,0) \equiv \cdots \equiv Q_{\tau^{j-1}}(z,\chi,0) \equiv 0,$$

that $g_0, g_1, \ldots, g_{j-1}$ are constant functions, and that

$$Q_{\tau^j}(z,\chi,0) \equiv r^{j-1}\,\hat{Q}_{\hat{\tau}^j}\big(f_0(z), \bar{f}_0(\chi), 0\big).$$

In the $j=1$ case, this implies $Q_\tau(z,\chi,0) \equiv 1$; otherwise it implies $Q_{\tau^j}(z,\chi,0) \equiv 0$, as desired.

Substituting these values into identity (13) (with $v = j+1$), we obtain

$$r\,Q_{\tau^{j+1}}(z,\chi,0) + (j+1)\,g_j(z) \equiv r^{j+1}\,\hat{Q}_{\hat{\tau}^{j+1}}\big(f_0(z), \bar{f}_0(\chi), 0\big) + (j+1)\,\bar{g}_j(\chi).$$

Setting $\chi = 0$ yields

$$(j+1)\,g_j(z) = (j+1)\,\bar{g}_j(0) = (j+1)\overline{g_j(0)},$$

so $g_j(z)$ is a real constant. Subtracting $(j+1)\,g_j(z)$ from both sides and dividing by $r \neq 0$ completes the induction. □

Corollary 3.6. *Let M, $\hat{M}$ be formal real submanifolds of $\mathbb{C}^N$ through 0, given in normal coordinates as in Proposition 3.1. Define m for M and the corresponding $\hat{m}$ for $\hat{M}$. If M and $\hat{M}$ are formally equivalent, then $m = \hat{m}$.*

Proof. Lemma 3.3 implies $m \geq \hat{m}$. Then reverse the roles of M and $\hat{M}$. □

We shall be primarily interested in formal real hypersurfaces which are of infinite type, but nonflat, at 0. That is, formal hypersurfaces of m-infinite type for some positive integer m. In this case, Corollary 3.6 may be strengthened as follows.

Proposition 3.7. *If M is of m-infinite type at 0 and $H \in \mathcal{F}(M,0;\hat{M},0)$, then $\hat{M}$ is of m-infinite type at 0, $g_0, g_1, \ldots, g_{m-1}$ are constant, and*

$$0 \not\equiv \Theta_{s^m}(z,\chi,0) \equiv g_0(0)^{m-1}\,\hat{\Theta}_{\hat{s}^m}\big(f_0(z), \bar{f}_0(\chi), 0\big).$$

Proof. Put together Lemma 3.3, Corollary 3.6, and Proposition 3.5. □

We now have the necessary ingredients to prove Proposition 3.1.

Proof of Proposition 3.1. We have seen that $m = \hat{m}$. If the hypersurfaces are of finite type, then it is well known that the triple (r, L, K) is a formal invariant. (An outline of the proof that r is a formal invariant, for example, may be found in

[Baouendi et al. 1999b, Chapter I].) Similarly, $r=\infty$ if and only if $m=\hat{m}=\infty$, which in turn holds if and only if $\hat{r}=\infty$; and likewise if $L=\infty$ or $K=\infty$. Hence, it suffices to assume that all the numbers in question are positive integers. By Proposition 3.7, we have

$$0 \not\equiv \Theta_{s^m}(z,\chi,0) \equiv g_0(0)^{m-1}\,\hat{\Theta}_{\hat{s}^m}\big(f_0(z),\bar{f}_0(\chi),0\big).$$

A straightforward induction using Faa de Bruno's formula implies that for any multi-indices α and β,

$$\Theta_{z^\alpha\chi^\beta s^m}(z,\chi,0) = g_0(0)^{m-1}\sum_{\substack{|\mu|\le|\alpha|\\|\nu|\le|\beta|}} \hat{\Theta}_{\hat{z}^\mu\hat{\chi}^\nu\hat{s}^m}\big(f_0(z),\bar{f}_0(\chi),0\big) \\ \times P^{\alpha\beta}_{\mu\nu}\Big(\big((f_0^u)_{z^\gamma}(z)\big)_{|\gamma|\le|\mu|},\big((\bar{f}_0^u)_{\chi^\delta}(\chi)\big)_{|\delta|\le|\nu|}\Big),$$

where each $P^{\alpha\beta}_{\mu\nu}$ is a polynomial in its arguments.

This implies that $\Theta_{z^\alpha\chi^\beta s^m}(0,0,0)=0$ whenever $|\alpha|+|\beta|<\hat{r}$, whence $r\ge\hat{r}$ necessarily. Reversing the roles of M and $\hat{M}$ yields $r=\hat{r}$. Similarly, the equality of r and $\hat{r}$ then implies that $\Theta_{\chi^\beta s^m}(z,0,0)\equiv 0$ whenever $|\beta|<\hat{L}$, whence $L\ge\hat{L}$; reversing the roles of the formal hypersurfaces establishes equality. The proof that $K=\hat{K}$ is similar, and is left to the reader. $\square$

4. The 1-infinite type case in $\mathbb{C}^2$

Notation and results. From now on we deal only with formal real hypersurfaces of $\mathbb{C}^2$, and in particular those hypersurfaces that are of 1-infinite type at 0. Suppose that M is such a formal hypersurface. We shall write M in normal coordinates $Z=(z,w)$ as in (3). Since M is of 1-infinite type, this implies that we can write $Q(z,\chi,\tau)=\tau\,S(z,\chi,\tau)$ for some $S\in\mathbb{C}[[z,\chi,\tau]]$, so that

$$M=\left\{\left(\frac{w-\overline{w}}{2i}\right)=\Theta\left(z,\bar{z},\frac{w+\overline{w}}{2}\right)\right\}=\big\{w=\overline{w}\,S(z,\bar{z},\overline{w})\big\}. \tag{14}$$

For convenience, we shall write

$$\theta(z,\chi)=\sum_{j=0}^{\infty}\frac{\theta_j(z)}{j!}\chi^j := \Theta_s(z,\chi,0)\not\equiv 0 \tag{15}$$

Observe that $\theta_j(z)\equiv 0$ if $j<L$ and $\theta_L^{(j)}(0)=0$ if $j<K$,where L, K are defined by equations (6) and (7). It will be useful for later computations to observe that Proposition 3.5 implies

$$S(z,\chi,0)=\frac{1+i\theta(z,\chi)}{1-i\theta(z,\chi)}, \tag{16}$$

whence repeated differentiation in χ yields

$$S_{\chi^j}(z,0,0)=\begin{cases}1 & \text{if } j=0,\\ 0 & \text{if } 1\le j\le L-1,\\ 2i\,\theta_L(z) & \text{if } j=L,\\ 2i\,\theta_{L+1}(z)-4\theta_1(z)^2 & \text{if } j=L+1.\end{cases} \tag{17}$$

We now define a new, rather technical, invariant for 1-infinite type hypersurfaces. Letting δ_k^j denote the Kronecker delta function, we define the number $T\in\{0,1\}$ by

$$T:=\prod_{q=0}^{K-2}\delta^0_{\theta^{(q)}_{L+1}(0)}. \tag{18}$$

That is, $T=1$ if and only if $\theta_{L+1}(z)=O(|z|^{K-1})$; by means similar to the proofs for the numbers r, L, and K, it can be shown that T is a formal invariant. Details are left to the reader.

Now assume that $\hat M$ is a formal real hypersurface of $\mathbb{C}^2$ that is formally equivalent to M, and write it in normal coordinates $\hat Z=(\hat z,\hat w)$ as

$$\hat M=\left\{\frac{\hat w-\bar{\hat w}}{2i}=\hat\Theta\Big(\hat z,\bar{\hat z},\frac{\hat w+\bar{\hat w}}{2}\Big)\right\}=\left\{\hat w=\bar{\hat w}\,\hat S(\hat z,\bar{\hat z},\bar{\hat w})\right\}, \tag{19}$$

We write $\hat\theta(\hat z,\hat\chi):=\hat\Theta_{\hat s}(\hat z,\hat\chi,0)$ as above.

If $H:(M,0)\to(\hat M,0)$ is a formal equivalence, Lemma 3.2 implies that $H(z,w)$ is of the form given by (9), with $f,g\in\mathbb{C}[[z,w]]$ and $f_z(0,0)\,g(0,0)\neq 0$. Observe that identity (10) can be rewritten (after canceling an extra τ from both sides) as

$$\begin{aligned} S(z,\chi,\tau)\,g\big(z,\tau\, S(z,\chi,\tau)\big)&\\ \equiv \bar g(\chi,\tau)\,\hat S\big(f\big(z,\tau\, S(z,\chi,\tau)\big),&\ \bar f(z,\chi),\tau\,\bar g(\chi,\tau)\big).\end{aligned} \tag{20}$$

We shall continue to use the formal Taylor expansions of f and g in w given by equation (11), and shall write

$$f_n(z):=\sum_{k\ge 0}\frac{1}{k!}\,\overline{a_n^k}\,z^k,\qquad g_n(z):=\sum_{k\ge 0}\frac{1}{k!}\,\overline{b_n^k}\,z^k, \tag{21}$$

where the bar denotes complex conjugation. Note that, in particular, $a_0^0=0$, $a_0^1\neq 0$, and $b_0^0=\overline{b_0^0}\neq 0$.

Finally, for $n \geq 0$, define the formal rational mapping $\Upsilon^n : (\mathbb{C}^2, 0) \to (\mathbb{C}^4, 0)$ by

$$\Upsilon_1^n(z,\chi) := K\,\frac{\theta_L(z)}{\theta_L'(z)}\Big(\frac{1+i\,\theta(z,\chi)}{1-i\,\theta(z,\chi)}\Big)^n\theta_z(z,\chi) - L\,\frac{\bar\theta_L(\chi)}{\bar\theta_L'(\chi)}\theta_\chi(z,\chi),$$

$$\Upsilon_2^n(z,\chi) := (1+\theta(z,\chi)^2)\bigg(\Big(\frac{1+i\,\theta(z,\chi)}{1-i\,\theta(z,\chi)}\Big)^n - 1\bigg) - 2i\,n\,\frac{\bar\theta_L(\chi)}{\bar\theta_L'(\chi)}\theta_\chi(z,\chi),$$

$$\begin{aligned}\Upsilon_3^n(z,\chi) := \delta_L^1\,\delta_T^1\Bigg(&\delta_K^1\,\theta_1^{(L)}(0)\,\frac{\theta_\chi(z,\chi,0)}{\bar\theta_1'(\chi)}\\ &+\frac{\theta_1^{(K)}(0)\theta_2^{(K)}(0)-\theta_1^{(K+1)}(0)\theta_2^{(K-1)}(0)}{K\,\theta_1^{(K)}(0)^2}\,\frac{\bar\theta_1(\chi)}{\bar\theta_1'(\chi)}\theta_\chi(z,\chi)\\ &-\Big(\frac{1+i\,\theta(z,\chi)}{1-i\,\theta(z,\chi)}\Big)^n\bigg(\theta_1(z)\big(1+\theta(z,\chi)^2\big)+\Big(\frac{\theta_2(z)}{\theta_1'(z)}-2i\,n\,\frac{\theta_1(z)^2}{\theta_1'(z)}\Big)\theta_z(z,\chi)\bigg)\\ &+\frac{\theta_2^{(K-1)}(0)}{\theta_1^{(K)}(0)}\bigg(\bar\theta_1(\chi)\big(1+\theta(z,\chi)^2\big)+\Big(\frac{\bar\theta_2(\chi)}{\bar\theta_1'(\chi)}+2i\,n\,\frac{\bar\theta_1(\chi)^2}{\bar\theta_1'(\chi)}\Big)\theta_\chi(z,\chi)\bigg)\Bigg),\end{aligned}$$

$$\begin{aligned}\Upsilon_4^n(z,\chi) := \delta_K^1\Bigg(&\frac{\bar\theta_1(\chi)}{\theta_1'(0)}\big(1+\theta(z,\chi)^2\big)-\frac{\theta_z(z,\chi)}{\theta_1'(z)}\Big(\frac{1+i\,\theta(z,\chi)}{1-i\,\theta(z,\chi)}\Big)^n\\ &+\frac{\theta_\chi(z,\chi)}{\theta_1'(0)}\bigg(2i\,n\,\frac{\bar\theta_1(\chi)^2}{\bar\theta_1'(\chi)}+\frac{\bar\theta_2(\chi)}{\bar\theta_1'(\chi)}-\frac{\theta_1''(0)}{\theta_1'(0)}\,\frac{\bar\theta_1(\chi)}{\bar\theta_1'(\chi)}\bigg)\Bigg),\end{aligned}$$

where the θ_j are defined by (15). We shall prove in the next section that these four equations actually define *formal power series* in (z,χ), rather than quotients of formal power series.

Observe that the formal mapping Υ^n depends on the choice of normal coordinates $Z = (z, w)$ for the formal hypersurface M.

We are now able to state the main technical result of the paper, which may be viewed as a sharper version of Theorem 2.2, but with conjugated derivatives.

Theorem 4.1. *Let $(M, 0)$ be a formal real hypersurface in $\mathbb{C}^2$ which is of 1-infinite type, given in normal coordinates $Z = (z, w)$ by equation (14). Define $\Upsilon^n(z,\chi)$ as immediately above. For each $n \in \mathbb{N}$, define the complex vector space*

$$\mathscr{V}^n(M) := \operatorname{span}_{\mathbb{C}}\big\{\upsilon_{s,t}^n := \Upsilon^n_{z^s\chi^t}(0,0) : s, t \in \mathbb{N}\big\} \subset \mathbb{C}^4. \tag{22}$$

Then the dimension of the vector space $\mathscr{V}^n(M)$ is a formal invariant for each n, and the invariant set of integers

$$\mathscr{D}(M) := \big\{n \in \mathbb{N} : \dim_{\mathbb{C}}\mathscr{V}^n(M) < 2+\delta_K^1+\delta_L^1\,\delta_T^1\big\} \tag{23}$$

is always finite.

Furthermore, given a formal real hypersurface $(\hat{M}, 0)$ *in* $\mathbb{C}^2$ *formally equivalent to* $(M, 0)$*, normal coordinates* $\hat{Z} = (\hat{z}, \hat{w})$ *for* $\hat{M}$*, and* $n \in \mathbb{N}$*, there exists a formal power series* $\mathcal{A}_n(z; \Delta, \Lambda) \in \mathbb{C}[\Delta, \Lambda][\![z]\!]^2$*, with* $(z, \Delta, \Lambda) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{4|\mathfrak{D}(M)|}$*, such that*

$$\big(f_n(z), g_n(z)\big) \equiv \mathcal{A}_n\left(z; \frac{1}{a_0^1 b_0^0}, \big(a_j^0, b_j^0, a_j^1, b_j^1\big)_{j \in \mathfrak{D}(M)}\right).$$

for any $H \in \mathcal{F}(M, 0; \hat{M}, 0)$.

Moreover, if M *and* $\hat{M}$ *are convergent, there exists an* $\epsilon > 0$ *such that the map*

$$z \mapsto \mathcal{A}_n\left(z; \frac{1}{a_0^1 b_0^0}, \big(a_j^0, b_j^0, a_j^1, b_j^1\big)_{j \in \mathfrak{D}(M)}\right)$$

lies in $\mathcal{O}_\epsilon(z)^2$ *for every* $H \in \mathcal{F}(M, 0; \hat{M}, 0)$ *and every* $n \in \mathbb{N}$.

Examples. We now use Theorem 4.1 and Proposition 3.7 to calculate the formal transformation groups of various 1-infinite type hypersurfaces.

Example 4.2. Consider the family of 1-infinite type hypersurfaces

$$M_c^j := \left\{(z, w) : \operatorname{Im} w = c \operatorname{Re} w\, |z|^{2j}\right\}, \quad c \in \mathbb{R} \setminus \{0\}, \quad j \geq 1.$$

Observe that $L = K = j$, $T = 1$, and $\theta(z, \chi) = c z \chi$. If $n > 0$, it can be shown that $\{v_{2j,2j}^n, v_{3j,3j}^n\}$ is a basis for $\mathcal{V}^n(M_c^j)$ if $j \geq 2$, and that adding the vectors $\{v_{2,3}^n, v_{3,2}^n\}$ extends this to a basis for $\mathcal{V}^n(M_c^1)$. Hence, in any case, we have $\mathfrak{D}(M_c^j) = \{0\}$, so any formal equivalence with source M_c^j is determined by (a_0^1, b_0^0).

Applying Proposition 3.7 with $M = \hat{M} = M_c^j$ implies $f_0(z) = \varepsilon z$ for some $\varepsilon \in \mathbb{C}$ with $|\varepsilon| = 1$. It thus follows that

$$\operatorname{Aut}(M_c^j, 0) = \{(z, w) \mapsto (\varepsilon z, r w) : \varepsilon \in \mathbb{C},\ |\varepsilon| = 1,\ r \in \mathbb{R} \setminus \{0\}\}.$$

In particular, every formal automorphism converges.

Observe that for $j \neq k$, the hypersurfaces M_c^j and M_b^k are *not* formally equivalent (Theorem 2.1). On the other hand, M_c^j and M_b^j are formally equivalent if and only if $c/b > 0$. In this case, applying Proposition 3.7 implies that $f_0(z) = \alpha z$ for some $\alpha \in \mathbb{C}$ of modulus $(c/b)^{1/2j}$. It thus follows that

$$\mathcal{F}(M_c^j, 0; M_b^j, 0) = \left\{(z, w) \mapsto \left(\frac{c}{b}\right)^{1/2j} \big(\varepsilon z, r w\big) : \varepsilon \in \mathbb{C},\ |\varepsilon| = 1,\ r \in \mathbb{R} \setminus \{0\}\right\}.$$

Hence, the hypersurfaces M_c^j are formally equivalent if and only if they are biholomorphically equivalent if and only if b and c have the same sign.

Example 4.3. Consider the family of 1-infinite type hypersurfaces

$$N_b^j := \left\{(z, w) : \operatorname{Im} w = 2 \operatorname{Re} w \operatorname{Re}(bz\bar{z}^j)\right\}, \quad b \in \mathbb{C} \setminus \{0\},\ j \geq 2.$$

Note $L = 1$, $K = j$, and $\theta(z, \chi) = bz\chi^j + \bar{b}z^j\chi$. If $n > 0$, it can be shown that $\{\upsilon^n_{2,2}, \upsilon^n_{3,2}, \upsilon^n_{3,3}\}$ forms a basis for $\mathcal{V}^n(N_b^j)$, so we again conclude that $\mathfrak{D}(N_b^j) = \{0\}$. Hence, every formal equivalence H with source N_b^j is determined by the values a_0^1 and b_0^0.

Now, Proposition 3.7 applied to the case $M = \hat{M} = N_b^j$ implies that a_0^1 is a $(j-1)$-th root of unity and that $f_0(z) = z/a_0^1$. We conclude that

$$\operatorname{Aut}(N_b^j, 0) = \left\{(z, w) \mapsto (\varepsilon z, r w) : \varepsilon \in \mathbb{C},\ \varepsilon^{j-1} = 1,\ r \in \mathbb{R} \setminus \{0\}\right\}.$$

Note that every formal automorphism converges.

Example 4.4. Consider the hypersurface

$$M := \left\{(z, w) : \operatorname{Im} w = \frac{\operatorname{Re} w\, |z|^2}{1 + \sqrt{1 - |z|^4}}, \quad |z| < 1\right\}.$$

It is easy to check that $L = K = 1$ in this case and that $\mathfrak{D}(M) = \{0, 1, 2\}$. (In fact, $\Upsilon_4^1 \equiv 0$ and $2i\,\Upsilon_1^2 \equiv \Upsilon_2^2$.) A complete calculation of the stability group of this hypersurface is given in [Kowalski 2002b], and reveals it to be a real-analytic hypersurface whose stability group at the origin is determined by 3-jets *but not by 2-jets*.

In fact, this example can be generalized as follows. Define for $k = 2, 3, 4, \ldots$ the set

$$M_k := \left\{(z, w) : \overline{w} = w\big(i|z|^2 + \sqrt{1 - |z|^4}\big)^{2/k}\right\},$$

where the principal branch of $\zeta \mapsto \zeta^{2/k}$ is meant. A straightforward calculation shows that each M_k defines a real hypersurface and that $M_2 = M$ above. It can also be shown that $\mathfrak{D}(M_k) = \{0, k/2, k\} \cap \mathbb{Z}$, and that the stability group of M_k is determined by $(k+1)$-jets, *but not by jets of any lesser order*; for details, see [Kowalski 2002a, Chapter 7]. Hence, even though Theorem 4.1 asserts that $\mathfrak{D}(M)$ is always finite, the integers themselves can be arbitrarily large and, consequently, the required jet-order can be as well.

5. Proofs of the main results

Proof of Theorem 4.1. A basic outline of the proof can be divided into four steps.

(1) Given a fixed set of normal coordinates $Z = (z, w)$, we prove that for each $n \in \mathbb{N}$ the power series $f_n(z)$ and $g_n(z)$ are rationally parametrized by the values (a_ℓ^j, b_ℓ^j) for $\ell = 0, 1$ and $0 \leq j \leq n$.

(2) We prove that under these conditions, if $n \notin \mathcal{D}(M)$, the 4-tuple of complex numbers $(a_n^0, a_n^1, b_n^0, b_n^1)$ is itself a polynomial in $1/(a_0^1 b_0^0)$ and (a_ℓ^j, b_ℓ^j) for $\ell = 0, 1$ and $0 \le j \le n-1$.

(3) We prove that $\mathcal{D}(M)$, defined by these normal coordinates, is always finite.

(4) We show that the dimension of $\mathcal{V}^n(M)$ (and hence the set $\mathcal{D}(M)$) is independent of the normal coordinates used to define it.

To fix notation throughout the proof, we assume that M is always given in normal coordinates $Z = (z, w)$ by (14). We also set $\mathcal{D} = \mathcal{D}(M)$ and $\mathcal{V}^n = \mathcal{V}^n(M)$. Similarly, $\hat{M}$, whenever a target formal hypersurface is needed, will always be given in normal coordinates $\hat{Z} = (\hat{z}, \hat{w})$ by (19). If $H : (M, 0) \to (\hat{M}, 0)$ is a formal equivalence, we set

$$\begin{aligned}
\Delta(H) &:= \frac{1}{a_0^1 b_0^0} \in \mathbb{C} \setminus \{0\},\\
\lambda_2^n(H) &:= \left(a_n^1, b_n^0\right) \in \mathbb{C}^2,\\
\lambda_3^n(H) &:= \left(a_n^1, b_n^0, a_n^0\right) \in \mathbb{C}^3,\\
\lambda_4^n(H) &:= \left(a_n^1, b_n^0, a_n^0, b_n^1\right) \in \mathbb{C}^4,\\
\Lambda_j^n(H) &:= \left(\lambda_j^0(H), \lambda_j^1(H), \ldots, \lambda_j^n(H)\right) \in \mathbb{C}^{j(n+1)}.
\end{aligned}$$

We also use the following conventions for naming various types of polynomials and power series.

- $\mathcal{Q}^d(X; \Lambda) \in \mathbb{C}[X, \Lambda] \equiv \mathbb{C}[\Lambda][X]$ denotes a polynomial in X of degree d whose coefficients are polynomial in Λ.
- $\mathcal{P}(\Lambda; X) \in \mathbb{C}[\![X, \Lambda]\!] \equiv \mathbb{C}[\![X]\!][\Lambda]$ denotes a polynomial in Λ whose coefficients are power series in X.
- $\mathcal{R}(X; \Lambda) \in \mathbb{C}[\![X, \Lambda]\!] \equiv \mathbb{C}[\Lambda][\![X]\!]$ denotes a power series in X whose coefficients are polynomial in Λ.

Assume the normal coordinates Z and $\hat{Z}$ for M and $\hat{M}$ are fixed. We now tackle the first step, the parametrizing of f_n and g_n. We begin with a lemma.

Lemma 5.1. *Let $(M, 0)$ and $(\hat{M}, 0)$ be formally equivalent formal 1-infinite type hypersurfaces as above. There exist unique formal power series $U, V \in \mathbb{C}[\![X, Y]\!]$, vanishing at 0, such that*

$$f_0(z) = U\left(z, \frac{z}{a_0^1}\right), \qquad \bar{f}_0(\chi) = V\left(\chi, a_0^1 \chi\right)$$

for any $H \in \mathcal{F}(M, 0; \hat{M}, 0)$. If both M and $\hat{M}$ are convergent hypersurfaces, then $U, V \in \mathbb{C}\{X, Y\}$.

Proof. Proposition 3.7 implies that

$$\theta(z,\chi) \equiv \hat{\theta}\big(f_0(z), \bar{f}_0(\chi)\big). \tag{24}$$

Differentiating this L times in χ using Faa de Bruno's formula and setting $\chi = 0$ yields the identity

$$\theta_L(z) \equiv (a_0^1)^L \hat{\theta}_L\big(f_0(z)\big). \tag{25}$$

Differentiating this K times in z and setting $z = 0$ yields

$$\theta_L^{(K)}(0) = \big(\overline{a_0^1}\big)^K \big(a_0^1\big)^L \hat{\theta}_L^{(K)}(0). \tag{26}$$

In particular, we find that for any formal equivalence $H \in \mathscr{F}(M, 0; \hat{M}, 0)$,

$$\big|f_0'(0)\big| = \big|a_0^1\big| = \left|\frac{\theta_L^{(K)}(0)}{\hat{\theta}_L^{(K)}(0)}\right|^{1/(L+K)} =: \mu \in \mathbb{R}\setminus\{0\}. \tag{27}$$

We can write

$$\theta_L(z) = \frac{1}{K!}\theta_L^{(K)}(0)\, z^K\, t(z),$$

for some $t \in \mathbb{C}[\![z]\!]$ with $t(0) = 1$. Thus, there exists a unique power series $u(z)$ with $u(0) = 1$ such that $u(z)^K = t(z)$. Similarly, write

$$\hat{\theta}_L(\hat{z}) = \frac{1}{K!}\hat{\theta}_L^{(K)}(0)\, \hat{z}^K\, \hat{u}(\hat{z})^K,$$

with $\hat{u}(0) = 1$. Define the formal power series

$$\iota(\hat{z}, X, Y) := \hat{z}\hat{u}(\hat{z}) - \mu^2 Y u(X).$$

Observe that $\iota(0,0,0) = 0$ and $\iota_{\hat{z}}(0,0,0) = 1$, whence the formal Implicit Function Theorem implies the existence of a unique power series $U(X,Y)$, vanishing at $(0,0)$, such that $\iota\big(U(X,Y), X, Y\big) \equiv 0$.

Now, suppose that $H \in \mathscr{F}(M, 0; \hat{M}, 0)$. Then identity (25) may be written as

$$\frac{1}{K!}\theta_L^{(K)}(0)\big(zu(z)\big)^K \equiv (a_0^1)^L \frac{1}{K!}\hat{\theta}_L^{(K)}(0)\big(f_0(z)\hat{u}(f_0(z))\big)^K.$$

Replacing $\theta_L{}^{(K)}(0)$ by equation (26) and canceling common terms yields the identity

$$\big(\overline{a_0^1}\, z u(z)\big)^K \equiv \big(f_0(z)\hat{u}(f_0(z))\big)^K.$$

Formally extracting K-th roots on both sides, we conclude that the two power series in the brackets differ only by some multiple $\varepsilon \in \mathbb{C}$ with $\varepsilon^K = 1$. However, since

$$\frac{\partial}{\partial z}\big(\overline{a_0^1}\, z u(z)\big)\Big|_{z=0} = \overline{a_0^1} = f_0'(0) = \frac{\partial}{\partial z}\big(f_0(z)\hat{u}(f_0(z))\big)\Big|_{z=0},$$

we conclude that $\varepsilon = 1$ necessarily. Moreover, since $a_0^1 \overline{a_0^1} = \mu^2$, we have

$$\mu^2\left(\frac{z}{a_0^1}\right)u(z) \equiv f_0(z)\,\hat{u}\big(f_0(z)\big).$$

Hence, $\iota\big(f_0(z), z, z/a_0^1\big) \equiv 0$, so by the uniqueness of U, we conclude $f_0(z) = U\big(z, z/a_0^1\big)$. Conjugating this result yields $\bar{f}_0(\chi) = V(\chi, a_0^1\chi)$, where V is defined by $V(X, Y) := \overline{U}(X, Y/\mu^2)$.

Finally, observe that if M and $\hat{M}$ are convergent, then the power series θ (hence also u) and $\hat{\theta}$ (hence $\hat{u}$) are convergent. Thus the holomorphic Implicit Function Theorem implies that U and V are necessarily convergent near $(0, 0) \in \mathbb{C}^2$. □

We can now extend this lemma to show that f_n and g_n are similarly parametrized for any $n \geq 0$.

Proposition 5.2. *Let $(M, 0)$, $(\hat{M}, 0)$ be formally equivalent formal 1-infinite type hypersurfaces as above. Then for every $n \in \mathbb{N}$, there exists a formal power series $\mathscr{B}_n(z; \Delta, \Lambda) \in \mathbb{C}[\Delta, \Lambda][[z]]^2$ such that*

$$\big(f_n(z), g_n(z)\big) = \mathscr{B}_n\big(z; \Delta(H), \Lambda^n_{2+\delta^1_K+\delta^1_T}(H)\big) \tag{28}$$

for any $H \in \mathscr{F}(M, 0; \hat{M}, 0)$. In addition, if $n \geq 1$, then in fact

$$\begin{aligned}\frac{f_n(z)}{f_0'(z)} = T^1_n\big(z; \Delta(H), \Lambda^{n-1}_{2+\delta^1_K+\delta^1_T}(H)\big) - \frac{L}{a_0^1}\left(\frac{\theta_L(z)}{\theta_L'(z)}\right)a_n^1 + \frac{n}{b_0^0}\left(\frac{\theta_L(z)}{\theta_L'(z)}\right)b_n^0 \\ + \frac{i\,\delta^1_K}{2b_0^0}\left(\frac{1}{\theta_1'(z)}\right)b_n^1 + \frac{\delta^1_T}{a_0^1}\left(2in\frac{\theta_1(z)^2}{\theta_L'(z)} - \frac{\theta_{L+1}(z)}{\theta_L'(z)} + \frac{La_0^2}{a_0^1}\frac{\theta_L(z)}{\theta_L'(z)}\right)a_n^0,\end{aligned} \tag{29}$$

$$g_n(z) = T^2_n\big(z; \Delta(H), \Lambda^{n-1}_{2+\delta^1_K+\delta^1_T}(H)\big) + b_n^0 + \frac{2i\,b_0^0\,\delta^1_T}{a_0^1}\big(\theta_1(z)\big)a_n^0 \tag{30}$$

with $T(z; \Delta, \Lambda^{n-1}_{2+\delta^1_K+\delta^1_T}) \in \mathbb{C}^2[\Delta, \Lambda^{n-1}_{2+\delta^1_K+\delta^1_T}][[z]]$.

Moreover, if M and $\hat{M}$ are convergent, there exists an $\epsilon > 0$ such that the map

$$z \mapsto \mathscr{B}_n\big(z; \Delta(H), \Lambda^n_{2+\delta^1_K+\delta^1_T}(H)\big)$$

lies in $\mathcal{O}_\epsilon(z)^2$ for every $n \in \mathbb{N}$ and every $H \in \mathscr{F}(M, 0; \hat{M}, 0)$.

Proof. For convenience, we shall set $\gamma = 2 + \delta^1_K + \delta^1_T$. We proceed by induction. The $n = 0$ case follows immediately from Lemma 5.1 and the fact that $g_0(z) \equiv b_0^0$ (Proposition 3.7), so let us assume that the proposition is true up to some $n - 1 \geq 0$. To prove (28), it suffices to prove that equations (29) and (30) hold.

Suppose that $H : (M, 0) \to (\hat{M}, 0)$ is a formal equivalence.[1] Differentiating identity (20) n times in τ using Faa de Bruno's formula and setting $\tau = 0$ (or, equivalently, substituting $Q(z, \chi, \tau) = \tau S(z, \chi, \tau)$ and $v = n+1$ into (13)) yields

$$
\begin{aligned}
(31) \quad & -S(z, \chi, 0)^{n+1} g_n(z) + b_0^0 \hat{S}_{\hat{z}}\big(f_0(z), \bar{f}_0(\chi), 0\big) S(z, \chi, 0)^n f_n(z) \\
& \qquad + b_0^0 \hat{S}_{\hat{\chi}}\big(f_0(z), \bar{f}_0(\chi), 0\big) \bar{f}_n(\chi) + \hat{S}\big(f_0(z), \bar{f}_0(\chi), 0\big) \bar{g}_n(\chi) \\
& \qquad \equiv \mathcal{P}_n\Big(b_0^0, \big(f_j(z), g_j(z), \bar{f}_j(\chi), \bar{g}_j(\chi)\big)_{j=1}^{n-1}; z, \chi, f_0(z), \bar{f}_0(\chi)\Big),
\end{aligned}
$$

where $\mathcal{P}_n(\Lambda; X)$, with $(\Lambda, X) \in \mathbb{C}^{4n-3} \times \mathbb{C}^4$, depends only on M and $\hat{M}$ and *not* the map H. (An explicit formula for $\mathcal{P}_n$ is given following the proof of Proposition 5.2.) Note that Lemma 3.3 implies $\hat{S}\big(f_0(z), \bar{f}_0(\chi), 0\big) = S(z, \chi, 0)$, whence

$$
\hat{S}_{\hat{z}}\big(f_0(z), \bar{f}_0(\chi), 0\big) = \frac{S_z(z, \chi, 0)}{f_0'(z)}, \quad \hat{S}_{\hat{\chi}}\big(f_0(z), \bar{f}_0(\chi), 0\big) = \frac{S_\chi(z, \chi, 0)}{\bar{f}_0'(\chi).}
$$

If equation (28) holds for some $n \in \mathbb{N}$, then

$$
\begin{aligned}
(32) \qquad \overline{\lambda_4^n(H)} &= \big((\mathcal{B}_n)_z^1, (\mathcal{B}_n)^2, (\mathcal{B}_n)^1, (\mathcal{B}_n)_z^2\big)(0; \Delta(H), \Lambda_\gamma^n(H)) \\
&=: \beta_n(\Delta(H), \Lambda_\gamma^n(H)).
\end{aligned}
$$

Applying the inductive hypothesis to this and substituting this into equation (31) yields

$$
(33) \qquad \big(\bar{f}_j(\chi), \bar{g}_j(\chi)\big) = \overline{\mathcal{B}}_j\Big(\chi; \Big(\frac{a_0^1}{\mu}\Big)^2 \Delta(H), \big(\beta_\ell(\Delta(H), \Lambda_\gamma^\ell(H))\big)_{\ell=0}^j\Big)
$$

for $j < n$, where μ is defined in equation (27). Substituting these values into (31) yields

$$
\begin{aligned}
(34) \quad & -S(z, \chi, 0)^{n+1} g_n(z) + S(z, \chi, 0) \bar{g}_n(\chi) + b_0^0 S_z(z, \chi, 0) S(z, \chi, 0)^n \frac{f_n(z)}{f_0'(z)} \\
& \qquad + b_0^0 S_\chi(z, \chi, 0) \frac{\bar{f}_n(\chi)}{\bar{f}_0'(\chi)} \equiv \mathcal{R}_n(z, \chi; \Delta(H), \Lambda_\gamma^{n-1}(H)),
\end{aligned}
$$

with $\mathcal{R}_n(X; \Lambda)$ independent of the mapping H for each $n \geq 0$.

On one hand, substituting $\chi = 0$ and the identities from equations (16) and (17) into (34) yields

$$
(35) \qquad g_n(z) = \mathcal{R}_n(z, 0; \Delta(H), \Lambda_\gamma^{n-1}(H)) + b_n^0 + \frac{2i\, b_0^0}{a_0^1}\big(\theta_1(z)\big) a_n^0.
$$

On the other hand, differentiating identity (34) L times in χ, setting $\chi = 0$, and using the identities from equations (16) and (17) yields (after rearranging terms)

[1] We remark that the construction given in this section can be carried out if *no* formal equivalence exists between M and $\hat{M}$.

the identity

$$
\begin{aligned}
&\theta_L'(z)\,\frac{f_n(z)}{f_0'(z)}\\
&\qquad\equiv -\frac{i}{2b_0^0}(\mathcal{R}_n)_{\chi^j}\big(z,0;\,\Delta(H),\,\Lambda_\gamma^{n-1}(H)\big)+\frac{n+1}{b_0^0}\theta_L(z)\,g_n(z)+\frac{i}{2b_0^0}b_n^L\\
&\qquad\quad -\frac{1}{b_0^0}\big(\theta_L(z)\big)b_n^0-\frac{L}{a_0^1}\big(\theta_L(z)\big)a_n^1-\frac{1}{a_0^1}\Big(\theta_{L+1}(z)+2i\,\theta_1(z)^2-\frac{L\,a_0^2}{a_0^1}\theta_L(z)\Big)a_n^0.
\end{aligned}
$$

Using the formula for $g_n(z)$ from equation (35) and observing that $(\theta_1)^2=\theta_1\theta_L$ for every $L\geq 1$, we can rewrite this identity as

$$
\begin{aligned}
(36)\quad &\theta_L'(z)\,\frac{f_n(z)}{f_0'(z)}\\
&\qquad\equiv -\frac{i}{2b_0^0}(\mathcal{R}_n)_{\chi^j}\big(z,0;\,\Delta(H),\,\Lambda_\gamma^{n-1}(H)\big)-\frac{n}{b_0^0}\big(\theta_L(z)\big)b_n^0+\frac{i}{2b_0^0}b_n^L\\
&\qquad\quad -\frac{L}{a_0^1}\big(\theta_L(z)\big)a_n^1+\frac{1}{a_0^1}\Big(-\theta_{L+1}(z)+2i\,n\theta_1(z)^2+\frac{L\,a_0^2}{a_0^1}\theta_L(z)\Big)a_n^0.
\end{aligned}
$$

We complete the proof by examining cases.

Case 1. $K=1$. In this case $L=T=1$ necessarily, so $\gamma=4$ and $\theta_L'(z)=\theta_1'(z)$ is a multiplicative unit. Dividing it on both sides of (36) yields (29); equation (30) follows from (35).

Case 2. $K>0$. In this case, setting $z=0$ in (36) yields

$$
0=-\frac{i}{2b_0^0}(\mathcal{R}_n)_{\chi^j}\big(z,0;\,\Delta(H),\,\Lambda_\gamma^{n-1}(H)\big)+\frac{i}{2b_0^0}b_n^L,
$$

whence we may replace b_n^L in identity (36) by $(\mathcal{R}_n)_{\chi^j}\big(z,0;\,\Delta(H),\,\Lambda_\gamma^{n-1}(H)\big)$. Thus, after rearranging the terms again, we may rewrite (36) as

$$
\begin{aligned}
(37)\quad &\theta_L'(z)\,\frac{f_n(z)}{f_0'(z)}\\
&\equiv\sum_{j=0}^{K-2}\bigg(\frac{r_j^n\big(\Delta(H),\Lambda_\gamma^{n-1}(H)\big)}{j!}z^j+\mathcal{R}_n^1\big(z;\,\Delta(H),\Lambda_0^{n-1}(H)\big)\bigg)-\frac{n}{b_0^0}\big(\theta_L(z)\big)b_n^0\\
&\qquad -\frac{L}{a_0^1}\big(\theta_L(z)\big)a_n^1+\frac{1}{a_0^1}\bigg(-\theta_{L+1}(z)+2i\,n\theta_1(z)^2+\frac{L\,a_0^2}{a_0^1}\theta_L(z)\bigg)a_n^0,
\end{aligned}
$$

with the r_j^n polynomials and $\mathcal{R}_n^1(z;\,\Delta,\,\Lambda)$ of order at least $K-1$ in z.

Case 2A. $T=1$. Note that $\gamma=3$. Since $\theta_{L+1}^{(j)}(0)=0$ for $j<K-1$, differentiating (37) in z (up to $K-2$ times) yields the relations

$$r_j^n\big(\Delta(H),\Lambda_3^{n-1}(H)\big)=0,\quad 0\le j\le K-2.$$

This does not imply that the polynomials $r_j^n(\Delta,\Lambda)$ are themselves identically zero; merely that they vanish whenever

$$(\Delta,\Lambda)=\big(\Delta(H),\Lambda_3^{n-1}(H)\big)$$

for some formal equivalence $H\in\mathcal{F}(M,0;\hat{M},0)$.

Consequently, we may remove the first $K-1$ summands of the right-hand expression in identity (37). Observe that all the remaining summands are of order at least $K-1$ in z, and hence can be divided by $\theta_L'(z)$ to form another power series. This division yields (29); (30) follows from (35).

Case 2B. $T=0$. Note that $\gamma=2$. We know there exists some $j_0\in\{1,2,\dots,K-2\}$ such that $\theta_{L+1}^{(j_0)}(0)\neq 0$. Differentiating the identity (37) j_0 times in z and setting $z=0$, we obtain

$$0=r_{j_0}^n\big(\Delta(H),\Lambda_2^{n-1}(H)\big)-\frac{\theta_{L+1}^{(j_0)}(0)}{a_0^1}a_n^0,$$

whence we may replace a_n^0 in (35) and (37) by $\dfrac{a_0^1 r_{j_0}^n(\Delta(H),\Lambda_2^{n-1}(H))}{\theta_{L+1}^{(j_0)}(0)}$ to obtain

$$\theta_L'(z)\frac{f_n(z)}{f_0'(z)}\equiv\sum_{j=0}^{K-2}\left(\frac{\tilde{r}_j^n(\Delta(H),\Lambda_2^{n-1}(H))}{j!}z^j+\mathcal{R}_n^2(z;\Delta(H),\Lambda_2^{n-1}(H))\right)$$
$$-\frac{n}{b_0^0}\big(\theta_L(z)\big)b_n^0-\frac{L}{a_0^1}\big(\theta_L(z)\big)a_n^1,$$
$$g_n(z)=\mathcal{R}_n^3\big(z,0;\Delta(H),\Lambda_2^{n-1}(H)\big)+b_n^0.$$

Thus, (30) holds; arguing as in the proof of Case 2A now yields (29).

The only thing missing from the proof is the convergence statement. Assume now that M and $\hat{M}$ define real-analytic hypersurfaces in $\mathbb{C}^2$ through 0. Hence, there exists a $\delta>0$ such that

$$S(z,\chi,\tau)\in\mathcal{O}_\delta(z,\chi,\tau),\quad \hat{S}\big(\hat{z},\hat{\chi},\hat{\tau}\big)\in\mathcal{O}_\delta\big(\hat{z},\hat{\chi},\hat{\tau}\big).$$

Without loss of generality, we shall assume that δ is chosen small enough such that $\theta_L(z)\neq 0$ for $0<|z|<\delta$, since the zeros of a nonconstant holomorphic function of one variable are isolated.

Similarly, since $U(X,Y) \in \mathbb{C}\{X,Y\}$ vanishes at 0 by Lemma 5.1, there exists an $\eta > 0$ such that $U(X,Y) \in \mathbb{O}_\eta(X,Y)$ and satisfies $\left|U(X,Y)\right| < \delta$ whenever $|X|, |Y| < \eta$.

Choose $\epsilon < \min\{\delta, \eta, \mu\eta\}$, where μ is defined by equation (27). We claim this is the desired $\epsilon > 0$; the proof is by induction. The case $n = 0$ follows from Lemma 5.1. Assuming this choice of ϵ holds up to some $n-1$, then observe that the mapping

$$(z,\chi) \mapsto \mathcal{R}_n\big(z,\chi;\Delta(H),\Lambda_\gamma^{n-1}(H)\big) \equiv \mathcal{P}_n\Big(b_0^0, \big(f_j(z), g_j(z), \bar{f}_j(\chi), \bar{g}_j(\chi)\big)_{j=1}^{n-1}; z, \chi, f_0(z), \bar{f}_0(\chi)\Big)$$

converges if $|z|, |\chi| < \delta$ for any $H \in \mathcal{F}(M, 0; \hat{M}, 0)$. Fix such an H. By equation (35), we conclude $g_n(z)$ converges on the ball $B^1(0,\epsilon) = \{z \in \mathbb{C} : |z| < \epsilon\}$. On the other hand, we have shown that $\theta_L'(z) f_n(z)/f_0'(z) = z^{K-1} q\big(z;\Delta(H),\Lambda_\gamma^{n-1}(H)\big)$, with $q(\,\cdot\,;\Delta(H),\Lambda_\gamma^{n-1}(H))$ convergent on $B^1(0,\epsilon)$. Since $\theta_L'(z)$ converges for $|z| < \epsilon$ and in the ϵ-ball vanishes only at $z = 0$ (of order $K-1$), we conclude that $f_n(z)$ converges on $B^1(0,\epsilon)$ as well, which completes the proof. □

It is of interest to note that as a consequence of Proposition 5.2, we see that if M and $\hat{M}$ are real-analytic hypersurfaces in $\mathbb{C}^2$ and H is a formal equivalence between them, the formal mappings $z \mapsto H_{w^n}(z,0)$ are convergent for every $n \in \mathbb{N}$; moreover, they converge on some common ϵ-neighborhood of $0 \in \mathbb{C}$, with ϵ independent of n and H.

Because it is useful in doing calculations, we now give the explicit formula for $\mathcal{P}_n$. Using Faa de Bruno's formula, we have

$$\mathcal{P}_n\Big(\big(f_j, g_j, \bar{f}_j, \bar{g}_j\big)_{j=0}^{n-1}; z, \chi, \hat{z}, \hat{\chi}\Big) = p_n\Big(\big(f_j, g_j, \bar{f}_j, \bar{g}_j\big)_{0\le j\le n-1}, \big(S_{\tau^j}(z,\chi,0)\big)_{0\le j\le n}, \big(\hat{S}_{\hat{z}^j\hat{\chi}^k\hat{\tau}^\ell}(\hat{z},\hat{\chi},0)\big)_{0\le j+k+\ell\le n}\Big)$$

where p_n is the universal polynomial

$$p_n\Big(\big(f_j, g_j, \bar{f}_j, \bar{g}_j\big)_{0\le j\le n-1}, \big(S_j\big)_{0\le j\le n}, \big(\hat{S}_{(j,k,\ell)}\big)_{0\le j+k+\ell\le n}\Big) \equiv \sum_{\substack{\alpha\in\mathbb{N}^n\\ k+[\alpha]=n\\ |\alpha|<n}} \frac{n!\, g_{|\alpha|} S_k}{k!\,\alpha!} \prod_{p=1}^{n} \left(\frac{S_{p-1}}{(p-1)!}\right)^{\alpha_p} - \sum_{\substack{\alpha,\beta,\gamma\in\mathbb{N}^n\\ k+[\alpha]+[\beta]+[\gamma]=n\\ [\alpha],[\beta],k<n}} \frac{n!\,\bar{g}_k \hat{S}_{(|\alpha|,|\beta|,|\gamma|)}}{k!\,\alpha!\,\beta!\,\gamma!} \times \prod_{p=1}^{n}\left(\sum_{\substack{\xi\in\mathbb{N}^p\\ [\xi]=p}} \frac{f_{|\xi|}}{\xi!} \prod_{q=1}^{n}\left(\frac{S_{q-1}}{(q-1)!}\right)^{\xi_q}\right)^{\alpha_p} \left(\frac{\bar{f}_p}{p!}\right)^{\beta_p} \left(\frac{\bar{g}_{p-1}}{(p-1)!}\right)^{\gamma_p}.$$

In particular, observe that

(38)

$$\mathcal{P}_n\big((0,0,g_0,\bar{g}_0,0,0,\ldots,0);z,\chi,\hat{z},\hat{\chi}\big) = -g_0\,S_{\tau^n}(z,\chi,0) + \bar{g}_0^n\,\hat{S}_{\hat{\tau}^n}(\hat{z},\hat{\chi},0).$$

This completes the first step of the proof. We move on to the second step, which involves parametrizing Λ^n.

Proposition 5.3. *Let* $(M,0)$ *and* $(\hat{M},0)$ *be formal hypersurfaces of* 1*-infinite type which are formally equivalent as above. Then for every* $n \in \mathbb{N}$, *there exists a power series*

$$\mathcal{A}_n(z;\Delta,\Lambda) \in \mathbb{C}[\Delta,\Lambda][[z]]^2$$

such that

$$\big(f_n(z),g_n(z)\big) = \mathcal{A}_n\Big(z;\Delta(H),\big(\lambda^n_{2+\delta^1_K+\delta^1_L\delta^1_T}(H)\big)_{j\in\mathcal{D}(M),\,j\le n}\Big).$$

for any $H \in \mathcal{F}(M,0;\hat{M},0)$. *Moreover, if* M *and* $\hat{M}$ *are convergent, there exists an* $\epsilon > 0$ *such that the map*

$$z \mapsto \mathcal{A}_n\Big(z;\Delta(H),\big(\lambda^n_{2+\delta^1_K+\delta^1_L\delta^1_T}(H)\big)_{j\in\mathcal{D}(M),\,j\le n}\Big)$$

lies in $\mathcal{O}_\epsilon(z)^2$ *for every* $n \in \mathbb{N}$ *and every* $H \in \mathcal{F}(M,0;\hat{M},0)$.

Proof. We continue with the notation from Proposition 5.2; in particular, we shall continue to let γ denote $2+\delta^1_K+\delta^1_T$. Observe that Proposition 5.3 follows immediately from Proposition 5.2 if it can be shown that for every $n \notin \mathcal{D}(M)$, there exists a $\mathbb{C}^\gamma$-valued polynomial $\omega^n(\Delta,\Lambda)$ such that

$$\lambda^n_\gamma(H) = \omega^n\big(\Delta(H),\Lambda^{n-1}_{2+\delta^1_K+\delta^1_L\delta^1_T}(H)\big) \quad \text{for all } H \in \mathcal{F}(M,0;\hat{M},0). \tag{39}$$

To see this, suppose equation (39) holds for every $n \notin \mathcal{D}(M)$. An easy induction shows that for *every* $n \in \mathbb{N}$, there exists a $\mathbb{C}^\gamma$-valued polynomial $\tilde{\omega}^n(\Delta,\Lambda)$ such that

$$\lambda^n_\gamma(H) = \tilde{\omega}^n\Big(\Delta(H),\big(\lambda^j_{2+\delta^1_K+\delta^1_L\delta^1_T}(H)\big)_{j\in\mathcal{D},\,j\le n}\Big).$$

Substituting this into the power series for $\mathcal{B}_n$ given by Proposition 5.2 completes the proof.

Hence, we must show that a relation of the form given in (39) holds for each $n \notin \mathcal{D}(M)$. To this end, define the power series

$$\tilde{\Upsilon}^n : (\mathbb{C}^2,0) \to (\mathbb{C}^4,0)$$

by $\tilde{\Upsilon}^n_j = \Upsilon^n_j$ for $j \neq 3$, and set

$$\begin{aligned}\tilde{\Upsilon}^n_3(z,\chi) := \delta^1_T \Bigg(& \delta^1_K\, \theta_1^{(L)}(0)\, \frac{\theta_\chi(z,\chi)}{\bar{\theta}'_L(\chi)} \\ & + \frac{L\big(\theta_L^{(K)}(0)\theta_{L+1}^{(K)}(0) - \theta_L^{(K+1)}(0)\theta_{L_1}^{(K-1)}(0)\big)}{K\,\theta_1^{(K)}(0)^2}\, \frac{\bar{\theta}_L(\chi)}{\bar{\theta}'_L(\chi)}\theta_\chi(z,\chi) \\ & - \Big(\frac{1+i\,\theta(z,\chi)}{1-i\,\theta(z,\chi)}\Big)^n \Big(\theta_1(z)\big(1+\theta(z,\chi)^2\big) + \Big(\frac{\theta_{L+1}(z)}{\theta'_L(z)} - 2i\,n\,\frac{\theta_1(z)^2}{\theta'_L(z)}\Big)\theta_z(z,\chi)\Big) \\ & + \frac{\theta_{L_1}^{(K-1)}(0)}{\theta_L^{(K)}(0)}\Big(\bar{\theta}_1(\chi)\big(1+\theta(z,\chi)^2\big) + \Big(\frac{\bar{\theta}_{L+1}(\chi)}{\bar{\theta}'_L(\chi)} + 2i\,n\,\frac{\bar{\theta}_1(\chi)^2}{\bar{\theta}'_L(z)}\Big)\theta_\chi(z,\chi)\Big)\Bigg).\end{aligned}$$

Observe that

$$\delta^1_L\, \tilde{\Upsilon}^n_3 = \Upsilon^n_3.$$

Reconsider the identity (34). If we substitute into it the explicit formulas for $f_n(z)$ and $g_n(z)$ given in Proposition 5.2, as well as the corresponding formulas for $\bar{f}_n(\chi)$ and $\bar{g}_n(\chi)$ given by equation (33), we can rewrite this as

$$\tilde{\Upsilon}^n(z,\chi)^t\,\kappa^n\big(\Delta(H),\lambda^0_2(H)\big)\lambda^n_4(H) \equiv W^n\big(z,\chi;\Delta(H),\Lambda^{n-1}_\gamma(H)\big), \tag{40}$$

where the superscript ^{t} denotes the transpose operation, $\kappa^n(\Delta,\lambda)$ is the 4×4 matrix of polynomials defined by

$$\kappa^n(\Delta,\lambda^0_2) := \begin{pmatrix} (L/K)\,\Delta(b^0_0)^2 & -n/K & -\delta^1_T(L/K)\,a^2_0\,\Delta^2(b^0_0)^3 & 0 \\ 0 & -i/2 & 0 & 0 \\ 0 & 0 & -\delta^1_T\,\Delta(b^0_0)^2 & 0 \\ 0 & 0 & 0 & \delta^1_K i/2 \end{pmatrix}$$

(by Lemma 5.1, a^2_0 is a polynomial in a^1_0), and

$$W^n(z,\chi;\Delta,\Lambda) \in \mathbb{C}[\Delta,\Lambda][\![z,\chi]\!].$$

Denote by $\tilde{\kappa}^n$ the 4×4 matrix function

$$\tilde{\kappa}^n(\Delta,\lambda^0_2) := \begin{pmatrix} (K/L)\,\Delta(a^1_0)^2 & 2i\,n/L\,\Delta(a^1_0)^2 & -a^2_0\,\Delta\,a^1_0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & -\delta^1_T\,\Delta(a^1_0)^2 & 0 \\ 0 & 0 & 0 & -\delta^1_K\,2i \end{pmatrix}.$$

Observe that if $a_0^1 b_0^0 \neq 0$, then

$$\kappa^n\left(\frac{1}{a_0^1 b_0^0}, \lambda_2^0\right) \cdot \tilde{\kappa}^n\left(\frac{1}{a_0^1 b_0^0}, \lambda_2^0\right) = \begin{pmatrix} 1 & 0 & \dfrac{L a_0^2}{K a_0^1}(\delta_T^1 - 1) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \delta_T^1 & 0 \\ 0 & 0 & 0 & \delta_K^1 \end{pmatrix}_j,$$

For convenience, we denote by κ_j^n the upper-left $j \times j$ submatrix of κ^n for $1 \leq j \leq 4$; we define $\tilde{\kappa}_j^n$ similarly. We now complete the proof by examining cases.

Case 1. $K = 1$. Observe that $L = T = 1$ necessarily, so $\tilde{\Upsilon}^n = \Upsilon^n$ and $\kappa_4^n, \tilde{\kappa}_4^n$ are matrix inverses for all $n \in \mathbb{N}$. Suppose that $n \notin \mathfrak{D}(M)$, and choose a basis $\{\upsilon_{s_j,t_j}^n\}_{j=1}^4$ for $\mathcal{V}^n$. If Ξ is the 4×4 matrix whose j-th row is υ_{s_j,t_j}^n, then it follows that Ξ is invertible. Now, differentiating (40) s_j times in z, t_j times in χ, and setting $z = \chi = 0$ (for $j = 1, 2, 3, 4$), we obtain the 4×4 linear system of equations of the form

$$\Xi \kappa_4^n\big(\Delta(H), \lambda_0^2(H)\big) \lambda_4^n = w^n\big(\Delta(H), \Lambda_4^{n-1}(H)\big),$$

Thus, we may take

$$\omega^n(\Delta, \Lambda_4^{n-1}) := \tilde{\kappa}_4^n(\Delta, \lambda_0^2)\, \Xi^{-1}\, w^n(\Delta, \Lambda_4^{n-1})$$

to complete the proof.

Case 2. $K > L = 1 = T$. We have $\tilde{\Upsilon}^n = \Upsilon^n = (\Upsilon_1^n, \Upsilon_2^n, \Upsilon_3^n, 0)$ and $\kappa_3^n, \tilde{\kappa}_3^n$ are inverses for all $n \in \mathbb{N}$. Observe too that (40) reduces to

$$\big(\Upsilon_1^n(z,\chi), \Upsilon_2^n(z,\chi), \Upsilon_3^n(z,\chi)\big)^t \kappa_3^n\big(\Delta(H), \lambda_2^0(H)\big) \lambda_3^n(H) \\ \equiv W^n\big(z, \chi; \Delta(H), \Lambda_3^{n-1}(H)\big).$$

The proof now follows the exact same lines as in the previous case.

Case 3. $T = 0$. Since this implies $K > 1$, it follows that $\tilde{\Upsilon}^n = \Upsilon^n = (\Upsilon_1^n, \Upsilon_2^n, 0, 0)$ and $\kappa_2^n, \tilde{\kappa}_2^n$ are inverses for all $n \in \mathbb{N}$. Here, the identity (40) reduces to

(41)
$$\big(\Upsilon_1^n(z,\chi), \Upsilon_2^n(z,\chi)\big)^t \kappa_2^n\big(\Delta(H), \lambda_2^0(H)\big) \lambda_2^n(H) \equiv W^n\big(z, \chi; \Delta(H), \Lambda_2^{n-1}(H)\big).$$

The proof now follows the exact same lines as in the previous two cases.

Case 4. $L > 1 = T$. Observe that identity (40) reduces to

$$(42)\quad \big(\Upsilon_1^n(z,\chi), \Upsilon_2^n(z,\chi), \tilde{\Upsilon}_3^n(z,\chi)\big)^t \kappa_3^n\big(\Delta(H), \lambda_2^0(H)\big) \lambda_3^n(H) \\ \equiv W^n\big(z, \chi; \Delta(H), \Lambda_3^{n-1}(H)\big).$$

We claim that $a_n^0 = \sigma^n(\Delta(H), \Lambda_3^{n-1}(H))$ for every $n \in \mathbb{N}$, where σ^n is a polynomial. Hence, we can write

$$\big(f_n(z), g_n(z)\big) = \mathcal{B}_n\big(z; \Delta(H), \Lambda_3^n(H)\big) = \widetilde{\mathcal{B}}_n\big(z; \Delta(H), \Lambda_2^n(H)\big);$$

that is, $f_n(z)$ and $g_n(z)$ are given by expressions of the same form as in Proposition 5.2, *but without the a_n^0 term*. Hence, identity (40) reduces to identity (41), and the proof proceeds as in Case 3.

To prove the claim, we proceed by induction. For $n = 0$, this is trivial, as $a_0^0 = 0$. For the inductive step, we consider two cases.

Case 4A. $\theta_{L+1}^{(K-1)}(0) = 0$. Then equation (29) implies

$$\overline{a_n^0} = f_n(0) = \overline{a_0^1}\, T_n^1\big(0; \Delta(H), \Lambda_3^{n-1}(H)\big).$$

Conjugating this and applying equation (33) yields $a_n^0 = \tilde{T}(\Delta(H), \Lambda_3^{n-1}(H))$ for some polynomial $\tilde{T}(\Delta, \Lambda)$. But by the inductive hypothesis, $\Lambda_3^{n-1}(H)$ is itself a polynomial in $(\Delta(H), \Lambda_2^{n-1}(H))$, so the induction is complete in this case.

Case 4B. $\theta_{L+1}^{(K-1)}(0) \neq 0$. Differentiating (42) $L-1$ times in χ and setting $\chi = 0$ yields the identity

$$\frac{\big|\theta_{L+1}^{(K-1)}(0)\big|^2}{\big|\theta_L^{(K)}(0)\big|^2}\theta_L(z)\, a_n^0 = W_{\chi^{L-1}}\big(z, 0; \Delta(H), \Lambda_3^{n-1}(H)\big).$$

Differentiating this K times in z and setting $z = 0$ yields $a_n^0 = \tilde{T}(\Delta(H), \Lambda_3^{n-1}(H))$ for some polynomial $\tilde{T}(\Delta, \Lambda)$. But by the inductive hypothesis, $\Lambda_3^{n-1}(H)$ is itself a polynomial in $(\Delta(H), \Lambda_2^{n-1}(H))$, so the induction is complete in this case. $\square$

This completes the second step. We move on to the third step, counting the elements of $\mathcal{D}$.

Proposition 5.4. *Given a fixed set of normal coordinates Z on M, the set $\mathcal{D}(M)$ defined by equation (23) has at most $2(2+\delta_K^1+\delta_L^1\delta_T^1)$ elements.*

Proof. Consider the power series $\Upsilon^n(z, \chi)$ defined on page 120; we must prove that for all but $2(2+\delta_K^1+\delta_L^1\delta_T^1)$ integers $n \in \mathbb{N}$, the set $\mathcal{V}^n(M)$ has dimension $2+\delta_K^1+\delta_L^1\delta_T^1$.

Consider the matrix

$$\xi(n) := \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \upsilon_{2K,2L}^n & \upsilon_{3K,3L}^n & \upsilon_{3K,2L}^n & \upsilon_{2K,3L}^n \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}^t.$$

Our goal will be to show that for all but at most $2(2+\delta_K^1+\delta_L^1\delta_T^1)$ integers $n \in \mathbb{N}$, the first $2+\delta_K^1+\delta_L^1\delta_T^1$ rows are linearly independent, which implies that $n \notin \mathcal{D}(M)$.

Using Faa de Bruno's formula, we compute that

$$(\Upsilon_1^n)_{\chi^{2L}}(z,0) = 2i\frac{(2L)!}{(L!)^2}K\,\theta_L^{(K)}(z)^2 n + \mathfrak{Q}^0\big(n;\,(\partial^\nu\theta(z,0))_{|\nu|<3L+K+1}\big),$$
$$(\Upsilon_1^n)_{\chi^{3L}}(z,0) = -2\frac{(3L)!}{(L!)^3}K\,\theta_L^{(K)}(z)^3 n^2 + \mathfrak{Q}^1\big(n;\,(\partial^\nu\theta(z,0))_{|\nu|<4L+K+1}\big),$$
$$(\Upsilon_2^n)_{\chi^{2L}}(z,0) = -2\frac{(2L)!}{(L!)^2}\theta_L^{(K)}(z)^2 n^2 + \mathfrak{Q}^1\big(n;\,(\partial^\nu\theta(z,0))_{|\nu|<3L+K+1}\big),$$
$$(\Upsilon_2^n)_{\chi^{3L}}(z,0) = -\frac{4i}{3}\frac{(3L)!}{(L!)^3}\theta_L^{(K)}(z)^3 n^3 + \mathfrak{Q}^2\big(n;\,(\partial^\nu\theta(z,0))_{|\nu|<4L+K+1}\big),$$
$$(\Upsilon_3^n)_{\chi^{2}}(z,0) = \delta_L^1\delta_T^1\big(-4\theta_1^{(K)}(z)^3 n^2 + \mathfrak{Q}^1\big(n;\,(\partial^\nu\theta(z,0))_{|\nu|<K+4}\big)\big),$$
$$(\Upsilon_3^n)_{\chi^{3}}(z,0) = \delta_L^1\delta_T^1\big(-16i\,\theta_1^{(K)}(z)^4 n^3 + \mathfrak{Q}^2\big(n;\,(\partial^\nu\theta(z,0))_{|\nu|<K+5}\big)\big),$$
$$(\Upsilon_4^n)_{\chi^{2}}(z,0) = \delta_K^1\big(\mathfrak{Q}^0\big(n;\,(\partial^\nu\theta(z,0))_{|\nu|<5}\big)\big),$$
$$(\Upsilon_4^n)_{\chi^{3}}(z,0) = \delta_K^1\big(12\theta_1(z)^2 n^2 + \mathfrak{Q}^1\big(n;\,(\partial^\nu\theta(z,0))_{|\nu|<5}\big)\big).$$

Setting $\alpha := \theta_L^{(K)}(0)$ it follows, we may write $\xi(n) = C_1(n) + C_2(n)$, with $C_1(n)$ given by

$$\begin{pmatrix} \dfrac{2i\,K(2L)!\,(2K)!\,\alpha^2}{(L!\,K!)^2}n & \dfrac{-2(2L)!\,(2K)!\,\alpha^2}{(L!\,K!)^2}n^2 & 0 & 0 \\ \dfrac{-2K(3L)!\,(3K)!\,\alpha^3}{(L!\,K!)^3}n^2 & \dfrac{-4i(3L)!\,(3K)!\,\alpha^3}{3(L!\,K!)^3}n^3 & 0 & 0 \\ 0 & 0 & \delta_L^1\delta_T^1\dfrac{-4(3K)!\,\alpha^3}{(K!)^3}n^2 & 0 \\ 0 & 0 & 0 & \delta_K^1\,72\alpha^2 n^2 \end{pmatrix}$$

and $C_2(n)$ of the form

$$\begin{pmatrix} \mathfrak{Q}^0(n;\,j_0^{3L+3K+1}\theta) & \mathfrak{Q}^1(n;\,j_0^{3L+3K+1}\theta) & \delta_L^1\delta_T^1\mathfrak{Q}^1(n;\,j_0^{3K+4}\theta) & \delta_K^1\mathfrak{Q}^0(n;\,j_0^{7}\theta) \\ \mathfrak{Q}^1(n;\,j_0^{4L+4K+1}\theta) & \mathfrak{Q}^2(n;\,j_0^{4L+4K+1}\theta) & \delta_L^1\delta_T^1\mathfrak{Q}^2(n;\,j_0^{4K+5}\theta) & \delta_K^1\mathfrak{Q}^2(n;\,j_0^{9}\theta) \\ \mathfrak{Q}^1(n;\,j_0^{3L+4K+1}\theta) & \mathfrak{Q}^2(n;\,j_0^{3L+4K+1}\theta) & \delta_L^1\delta_T^1\mathfrak{Q}^1(n;\,j_0^{4K+4}\theta) & \delta_K^1\mathfrak{Q}^0(n;\,j_0^{8}\theta) \\ \mathfrak{Q}^1(n;\,j_0^{4L+3K+1}\theta) & \mathfrak{Q}^2(n;\,j_0^{4L+3K+1}\theta) & \delta_L^1\delta_T^1\mathfrak{Q}^2(n;\,j_0^{3K+5}\theta) & \delta_K^1\mathfrak{Q}^2(n;\,j_0^{8}\theta) \end{pmatrix}.$$

We shall denote by $\xi_j(n)$ the upper-left $j \times j$ submatrix of $\xi(n)$ for $j = 1, 2, 3, 4$. We complete the proof by examining cases.

Case 1. $K = 1$. In this case $L = T = 1$ as well, whence $2 + \delta_K^1 + \delta_L^1\delta_T^1 = 4$. By examining the matrix $\xi_4(n)$, and in particular the term of highest order in n in each of its entries, we find that

$$\det \xi_4(n) = 110592\alpha^{10} n^8 + \mathcal{Q}^7(n;\, j_0^9\theta).$$

Since $\alpha \neq 0$, this is a nonzero, eighth degree polynomial in n, and hence has at most eight distinct zeros (in the complex plane). If $\det \xi_4(n_0) \neq 0$, then the four rows of $\xi(n_0)$ are linearly independent, which completes the claim.

Case 2. $K > L = T = 1$. In this case, we have $2 + \delta_K^1 + \delta_L^1 \delta_T^1 = 3$. By examining the highest order terms in n as above, we find that

$$\det \xi_3(n) = 64\, K \frac{(2K)!(3K)!^2}{(K!)^8} \alpha^8 n^6 + \mathcal{Q}^5(n;\, j_0^{4K+5}\theta).$$

Arguing as above implies that for all but (at most) six integers n, the matrix $\xi_3(n)$ is invertible, whence the first three rows of $\xi(n)$ are linearly independent. This completes the claim.

Case 3. $L > 1$ or $T = 0$. Since either of these conditions necessarily implies $K > 1$, we conclude that $2 + \delta_K^1 + \delta_L^1 \delta_T^1 = 2$. Since

$$\det \xi_2(n) = -\frac{4}{3} K \frac{(2L)!\,(3L)!\,(2K)!\,(3K)!}{(L!\,K!)^5} \alpha^5 n^4 + \mathcal{Q}^3(n;\, j_0^{4L+4K+1}\theta),$$

the proof is complete by arguments similar to the previous case. □

Note that while $\mathcal{D}(M)$ is always finite, it is also never empty. Indeed, $0 \in \mathcal{D}(M)$ for any 1-infinite type hypersurface M, since it is easy to check that $\Upsilon_2^0(z, \chi) \equiv 0$.

This completes the third step of the proof. We complete the proof by showing that $\mathcal{D}(M)$ is independent of the choice of normal coordinates used to define it. In fact, we prove the following, which completes the proof of Theorem 4.1.

Proposition 5.5. *Suppose that M, $Z = (z, w)$, Υ^n, and $\mathcal{V}^n = \mathcal{V}^n(M)$ are as above. Let $(\hat{M}, 0)$ be formally equivalent to $(M, 0)$, with corresponding power series $\hat{\Upsilon}^n$ and subspaces $\hat{V}^n = \mathcal{V}^n(\hat{M})$ defined using the normal coordinates $\hat{Z} = (\hat{z}, \hat{w})$. Then for every $n \in \mathbb{N}$, the dimensions of $\mathcal{V}^n$ and $\hat{V}^n$ are equal. In particular, the dimension of subspace $\mathcal{V}^n(M) \subset \mathbb{C}^4$ is independent of the choice of normal coordinates used to define it.*

Proof. Let $H(z, w) = \big(f(z, w), w\, g(z, w)\big)$ be a formal equivalence between M and $\hat{M}$. Consider the formal power series

$$(z, \chi) \mapsto \hat{\Upsilon}^n\big(f_0(z),\, \bar{f}_0(\chi)\big) \in \mathbb{C}[[z, \chi]]^4,$$

which may be viewed as the power series $\hat{\Upsilon}^n$ given in the Z coordinates. Using Faa de Bruno's formula and the fact that $f_0 : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a formal change of coordinates, it is straightforward to verify that

$$\operatorname{span}_{\mathbb{C}}\left\{\hat{\upsilon}^n_{s,t} := \frac{\partial^{s+t}}{\partial z^s \partial \chi^t}\hat{\Upsilon}^n\big(f_0(z),\, \bar{f}_0(\chi)\big)\Big|_{\substack{z=0\\ \chi=0}} : s,\, t \in \mathbb{N}\right\} = \hat{V}^n.$$

From (24) we derive

$$\hat{\theta}_{\hat{z}}\big(f_0(z),\, \bar{f}_0(\chi)\big) = \frac{\theta_z(z,\chi)}{f_0'(z)}, \quad \hat{\theta}_{\hat{\chi}}\big(f_0(z),\, \bar{f}_0(\chi)\big) = \frac{\theta_\chi(z,\chi)}{\bar{f}_0'(\chi)},$$

whereas repeated differentiation of this in χ yields

$$\hat{p}_{L+1}\big(f_0(z)\big) = \frac{1}{2(a_0^1)^{L+2}}\big(2a_0^1\, p_{L+1}(z) - (L+1)L\, a_0^2\, p_L(z)\big).$$

From this and identity (25), it follows by an elementary (albeit involved) calculation that

$$\begin{aligned}
\hat{\Upsilon}^n_1\big(f_0(z),\, \bar{f}_0(\chi)\big) &= \Upsilon^n_1(z,\chi),\\
\hat{\Upsilon}^n_2\big(f_0(z),\, \bar{f}_0(\chi)\big) &= \Upsilon^n_2(z,\chi),\\
\hat{\Upsilon}^n_3\big(f_0(z),\, \bar{f}_0(\chi)\big) &= \frac{1}{a_0^1}\Upsilon^n_3(z,\chi) + \frac{\delta^1_T\, a_0^2}{K(a_0^1)^2}\Upsilon^n_1(z,\chi),\\
\hat{\Upsilon}^n_4\big(f_0(z),\, \bar{f}_0(\chi)\big) &= a_0^1\, \Upsilon^n_4(z,\chi).
\end{aligned}$$

Now, suppose that $\{\hat{\upsilon}^n_{s_j,t_j}\}_{j=1}^{\ell_0}$ is any collection of vectors in $\hat{V}^n$; consider the corresponding vectors $\upsilon^n_{s_j,t_j} \in \mathcal{V}^n$. Observe that if $\hat{\Xi}$, Ξ denote the $4\times\ell_0$ matrices whose columns are, respectively, the $\hat{\upsilon}^n_{s_j,t_j}$, $\upsilon^n_{s_j,t_j}$, then in view of the above identities, these matrices necessarily have the same rank. In particular, the columns of $\hat{\Xi}$ are linearly independent if and only if the columns of Ξ are. From this it follows that $\hat{V}^n$ and $\mathcal{V}^n$ have the same dimension. □

The main results. We use Theorem 4.1 to prove the main theorems stated at the end of Section 2. We begin with Theorem 2.2.

Proof. Let M be a formal real hypersurface of 1-infinite type at 0. Observe that the result of Theorem 2.2 is independent of the choice of coordinates Z, so without loss of generality let us take $Z = (z, w)$ to be normal coordinates for M, so that M is given by equation (14). Let $\mathcal{D} = \mathcal{D}(M)$ be as in Theorem 4.1, and set $k := 2 + \max\mathcal{D}$, which exists since $\mathcal{D}$ is a finite set.

To prove this k is sufficient, suppose $\hat{M}$ is a formally equivalent formal real hypersurface. Define the corresponding $\mathcal{A}^n$ as in Theorem 4.1. Fix a formal equivalence $H \in \mathcal{F}(M, 0; \hat{M}, 0)$. Conjugating the formula for (f_n, g_n) implies that

$$\big(\bar{f}_n(\chi),\, \bar{g}_n(\chi)\big) = \bar{\mathcal{A}}^n\left(\frac{1}{\overline{a_0^1}\,\overline{b_0^0}},\, \big(\overline{a_j^0}, \overline{b_j^0}, \overline{a_j^1}, \overline{b_j^1}\big)_{j\in\mathcal{D}}\right),$$

whence

$$(a_n^0, b_n^0, a_n^1, b_n^1) = A_n\left(\frac{1}{\overline{a_0^1}\,\overline{b_0^0}}, \left(\overline{a_j^0}, \overline{b_j^0}, \overline{a_j^1}, \overline{b_j^1}\right)_{j\in\mathcal{D}}\right), \quad n = \mathbb{N},$$

with $A_n \in \mathbb{C}[\Delta, \Lambda]^4$. Substituting this into $\mathcal{A}_n$, and recalling that

$$\Delta(H) = \frac{1}{a_0^1 b_0^0} = \frac{\overline{a_0^1}}{\mu^2 \overline{b_0^0}},$$

where μ is defined by (27), we can write

$$\left(f_n(z), g_n(z)\right) = \Gamma^n\left(z; \frac{1}{\overline{a_0^1}\,\overline{b_0^0}}, \left(\overline{a_j^0}, \overline{b_j^0}, \overline{a_j^1}, \overline{b_j^1}\right)_{j\in\mathcal{D}}\right),$$

with $\Gamma^n(z; \Delta, \Lambda) \in \mathbb{C}[\Delta, \Lambda][\![z]\!]^2$. Write

$$\Gamma^n_{z^j}\left(0; \frac{1}{\overline{a_0^1}\,\overline{b_0^0}}, \left(\overline{a_j^0}, \overline{b_j^0}, \overline{a_j^1}, \overline{b_j^1}\right)_{j\in\mathcal{D}}\right) =: \frac{c_j^n\left(\left(\overline{a_j^0}, \overline{b_j^0}, \overline{a_j^1}, \overline{b_j^1}\right)_{j\in\mathcal{D}}\right)}{\left(\overline{a_0^1}\,\overline{b_0^0}\right)^{\ell_j^n}},$$

with $\ell_j^n \in \mathbb{N}$ and c_j^n a $\mathbb{C}^2$-valued polynomial.

Now, observe that

$$\frac{\partial^{\ell+j} H}{\partial z^\ell \partial w^j}(0,0) = \left(\overline{a_j^\ell}, j\overline{b_{j-1}^\ell}\right).$$

In particular, $\overline{a_j^0}$ is a term in (the coordinates of) $j_0^k(H)$, a_j^1 and b_j^0 are terms in $j_0^{k+1}(H)$, and b_j^1 is a term in $j_0^{j+2}(H)$. Hence, c_n^j is a polynomial in $j_0^{2+\max\mathcal{D}}(H) = j_0^k(H)$ and

$$0 \neq \overline{a_0^1}\,\overline{b_0^0} = \det\left(\frac{\partial H}{\partial Z}(0,0)\right) =: q\left(j_0^k(H)\right),$$

so the proof is complete in view of equation (11). □

By inspecting Propositions 5.2 through 5.5, we see that we can replace the k given in the proof by $k := 1 + \delta_K^1 + \max\mathcal{D}$ to get a better bound in the $K > 1$ case, and if $\mathcal{D} = \{0\}$, then we may take $k = 1$ since $b_0^1 = 0$ by Proposition 3.7.

We now use this result to prove Theorem 2.3.

Proof. Let M, k be as in Theorem 2.2. Suppose that $\hat{M}$ is formally equivalent to M, and let Ψ be the formal power series from Theorem 2.2. If $H^1, H^2 : (M, 0) \to (\hat{M}, 0)$ are two formal equivalences that satisfy

$$\frac{\partial^{|\alpha|} H^1}{\partial Z^\alpha}(0) = \frac{\partial^{|\alpha|} H^2}{\partial Z^\alpha}(0) \quad \text{for all } |\alpha| \leq k,$$

it follows that $j_0^k(H^1) = j_0^k(H^2)$. If we call this common jet Λ_0, it follows from Theorem 2.2 that $H^1(Z) \equiv \Psi(Z; \Lambda_0) \equiv H^2(Z)$, as desired. □

We now tackle the two applications of Theorem 2.2 mentioned in Section 2. First we prove Theorem 2.4.

Proof. Let M, k be as in Theorem 2.2, and let Ψ be the formal power series defined in accord with that theorem with $\hat{M} = M$. That the mapping

$$j_0^k : \mathrm{Aut}(M, 0) \to J_0^k(\mathbb{C}^2, \mathbb{C}^2)_{0,0}$$

is injective follows from Theorem 2.3. Observe that $\Lambda_0 \in J^k(\mathbb{C}^2, \mathbb{C}^2)_{0,0}$ is in the image of j_0^k if and only if $q(\Lambda_0) \neq 0$ — so that $\Lambda_0 \in G^k(\mathbb{C}^2)_0)$ — and

$$\Lambda_0 = j_0^k\big(\Psi(\,\cdot\,, \Lambda_0)\big), \tag{43}$$

$$\rho\big(\Psi(Z, \Lambda_0), \overline{\Psi}\zeta, \overline{\Lambda}0)\big) = a(Z, \zeta)\rho(Z, \zeta) \tag{44}$$

for some multiplicative unit $a(Z, \zeta) \in \mathbb{C}[\![Z, \zeta]\!]$, where ρ is a defining power series for M. In view of equation (8), (43) is a finite set of polynomial equations in Λ_0, whereas (44) is a (possibly countably infinite) set of polynomial equations in $(\Lambda_0, \overline{\Lambda}0)$. Hence, the image of the mapping j_0^k is a locally closed subgroup of the Lie group $G^k(\mathbb{C}^2)_0$, and so is a Lie subgroup. □

And as a corollary, we have Theorem 2.5.

Proof. Let M, k be as in Theorem 2.2, and let $(\hat{M}, 0)$ be formally equivalent to $(M, 0)$. Injectivity of the jet map again follows from Theorem 2.3. Now, fix a formal equivalence $H_0 : (M, 0) \to (\hat{M}, 0)$; then any other formal equivalence is of the form $H := H_0 \circ A$, where $A \in \mathrm{Aut}(M, 0)$. In particular,

$$\begin{aligned} j_0^k\big(\mathcal{F}(M, 0; \hat{M}, 0)\big) &= \big\{ j_0^k(H_0 \circ A) : A \in \mathrm{Aut}(M, 0)\big\} \\ &= \big\{ j_0^k(H_0) \cdot j_0^k(A) : A \in \mathrm{Aut}(M, 0)\big\} \\ &= j_0^k(H_0) \cdot j_0^k\big(\mathrm{Aut}(M, 0)\big). \end{aligned}$$

Hence, the image of $\mathcal{F}(M, 0; \hat{M}, 0)$ is merely a coset of the algebraic Lie subgroup $j_0^k\big(\mathrm{Aut}(M, 0)\big)$ in the Lie group $G^k(\mathbb{C}^2)_0$, and so is itself a real-algebraic submanifold of $G^k(\mathbb{C}^2)_0$. □

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