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EQUIVARIANT SPECTRAL FLOW AND A LEFSCHETZ THEOREM ON ODD-DIMENSIONAL SPIN MANIFOLDS

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We present a heat kernel proof of an equivariant index theorem on odd-dimensional spin manifolds, stated for Toeplitz operators. A notion of equivariant spectral flow is raised and plays an important role in our proof.

Introduction

Atiyah–Singer index theorems have profound applications and consequences, and can be proved in several ways. Of particular interest is the heat kernel proof, which allows one to obtain refinements such as local index theorems for Dirac operators. See [Berline et al. 1992] for a comprehensive treatment of the heat kernel method on even-dimensional manifolds. The heat kernel method also leads to direct analytic proofs of the equivariant index theorem for Dirac operators on even-dimensional spin manifolds; see [Bismut 1984; Berline and Vergne 1985; Lafferty et al. 1992].

This paper presents a heat kernel proof of an equivariant index theorem on odd-dimensional spin manifolds, stated for Toeplitz operators.

Baum and Douglas [1982] first stated and proved an odd-index theorem for Toeplitz operators using the general Atiyah–Singer index theorem for elliptic pseudodifferential operators. It is known that one can give a heat kernel proof of such a theorem; we describe briefly the basic ideas involved. The first step is to apply a result of Booß and Wojciechowski [1993] to identify the index of the Toeplitz operator to the spectral flow of a certain family of self-dual elliptic operator with positive order. The second step is then to use the relationship between spectral flows and variations of η-invariants to evaluate this spectral flow; see [Getzler 1993].

Our proof of the equivariant odd-index theorem follows the same strategy. We need to introduce a concept of equivariant spectral flow and establish an equivariant version of the Booß–Wojciechowski theorem mentioned above. We then extend the relationship between the spectral flow and variations of η invariants to the

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equivariant setting. Finally, we use the local index techniques to evaluate these variations.

For simplicity we follow the method of Lafferty, Yu and Zhang [1992], but there is no difficulty in applying other methods, such as those of [Bismut 1984] or [Berline and Vergne 1985].

Dai and Zhang [1998] introduced the concept of higher spectral flow and gave a heat kernel treatment to the family index problem for Toeplitz operators. They also proved recently, after a draft of this paper was finished, an index theorem for Toeplitz operators on odd-dimensional manifolds with boundary [Dai and Zhang 2001]. It would be interesting to prove an equivariant version of their theorem, which might result in new fixed-point theorems.

This paper is organized as follows. In Section 1 we review the basic definition of the Toeplitz operators associated to Dirac operators on odd-dimensional spin manifolds and prove the equivariant odd index theorem by using the Baum–Douglas trick [1982] and also the general Atiyah–Singer [1968] Lefschetz fixed-point theorem for elliptic pseudodifferential operators. In Section 2 we introduce the equivariant spectral flow and prove an equivariant extension of the Booß–Wojciechowski theorem [1993]. In Section 3 we establish a relation between the equivariant spectral flow and the variations of equivariant η invariants. This in turn gives a heat kernel formula for the equivariant index of the Toeplitz operators. In Section 4 we evaluate these variations by adopting the local index theorem techniques.

1. Toeplitz operators and a Lefschetz fix point theorem

We begin by fixing notations on odd-dimensional Clifford algebras that are used in this paper. From now on, we fix $n = 2m + 1$, where m is a positive integer.

Let V be a n-dimensional real vector space with a positive inner product and some orthonormal basis $e_1, \ldots, e_n$. Set

$$T(V) = \mathbb{R} \oplus V \oplus V \otimes V \oplus \cdots ,$$

and let I to be the two-sided ideal of $T(V)$ generated by $\{x \otimes x + (x, x)1 : x \in V\}$. The Clifford algebra associated to V is defined as

$$C(V) = T(V)/I,$$

and is also denoted $C(n)$. Clearly $\{c_i = e_i I \in C(V)\}$ is the set of generators of $C(V)$ satisfying the relations

$$c_i c_j + c_j c_i = -2\delta_{ij}.$$

Define the chirality operator of $C(n)\otimes\mathbb{C}$ to be

$$\Gamma = (\sqrt{-1})^{m+1} c_1 \ldots c_n;$$

it lies in the center of $C(n)\otimes\mathbb{C}$. There is a unique irreducible complex $C(n)$-representation S of dimension 2^m such that $\Gamma = \mathrm{Id}_S$ on S.

For future use, we define the symbol map $\sigma : C(n) \to \mathrm{End} \bigwedge^* \mathbb{C}^n$ by

$$\sigma(c_i) = e_i \wedge - \iota(e_i^*), \tag{1–1}$$

with ι denoting contraction with elements of V^*. Thus σ is a complex representation of $C(n)$. For $x \in C(n)$ nonscalar, we have

$$\mathrm{Tr}_S(x) = -\sqrt{-1}\,(-2\sqrt{-1})^m \big(\sigma(x)\,1\big)_{[n]}, \tag{1–2}$$

where $(\cdot)_{[d]}$, for any integer d, denotes the d-dimensional part of an exterior form.

We proceed to define the Toeplitz operator of interest. Throughout this paper we assume M to be a closed oriented spin manifold of dimension $n = 2m+1$, with a fixed spin structure. We also fix a Riemannian metric g_{TM} on M.

Let $S(M)$ be the canonical complex spinor bundle of M; this is also a $C(T^*M)$-module. The canonical Levi-Civita connection induces a natural connection ∇^S on S. Choose a local orthonormal basis $e_1, \ldots, e_n$ is for TM, with dual basis $e^1, \ldots, e^n \in T^*M$. The canonical Dirac operator on S can be defined as

$$D^S = \sum c(e^i)\nabla^S_{e_i}; \tag{1–3}$$

it is a self-adjoint first-order elliptic differential operator acting on $S(M)$, and so it induces a spectral decomposition of $\Gamma_{L^2}(S)$. Denote by $L^2_+(S)$ the direct sum of eigenspaces of D associated to nonnegative eigenvalues, and by P_+ the orthogonal projection operator from $L^2(S)$ to $L^2_+(S)$. Set $P_- = \mathrm{Id} - P_+$ and

$$P = P_+ - P_-. \tag{1–4}$$

Given a trivial complex vector bundle $\mathbb{C}^N$ over M carrying the trivial metric and connection, D and P extend trivially as operators acting on $\Gamma(S\otimes\mathbb{C}^N)$. Let $g : M \to U(N)$ be a smooth map. Then g extends to an action on $S(M)\otimes\mathbb{C}^N$ as $\mathrm{Id}_{S(M)}\otimes g$, still denoted by g.

Definition 1.1. The Toeplitz operator associated to D and g is

$$T_g = (P_+\otimes \mathrm{Id}_{\mathbb{C}^N})g(P_+\otimes \mathrm{Id}_{\mathbb{C}^N}) : L^2_+(S(M)\otimes\mathbb{C}^N) \to L^2_+(S(M)\otimes\mathbb{C}^N).$$

It is a classical fact that T_g is a bounded Fredholm operator between the given Hilbert spaces. If we define Γ_λ to be the eigenspace of D with eigenvalue λ, Γ_λ is of finite dimension for each λ.

We then describe the equivariant index problem for Toeplitz operators.

Consider a compact group H of isometries of M preserving the orientation and spin structure. H also acts on $\Gamma(S(M)\otimes\mathbb{C}^N)$; since its action commutes with the Dirac operator D, it also commutes with P_+ and P. Furthermore each Γ_λ is H-invariant. But to ensure the H-invariance of Toeplitz operator, we need an additional assumption on H:

$$g(hx)=g(x) \quad \text{for any } h\in H \text{ and any } x\in M. \tag{1–5}$$

As a consequence,

$$T_g h_{\Gamma(S(M)\otimes\mathbb{C}^N)} = h_{\Gamma(S(M)\otimes\mathbb{C}^N)} T_g.$$

Definition 1.2. Given T_g and H as above and satisfying (1–5), the equivariant index of T_g associated with H is the following virtual representation of H in $R(H)$, the representation ring of H:

$$\mathrm{Ind}_H(T_g)=\ker\ T_g-\mathrm{coker}\ T_g.$$

We also write, for any $h\in H$,

$$\mathrm{Ind}(h,T_g)=\mathrm{Tr}(h,\mathrm{Ind}_H(T_g)). \tag{1–6}$$

An application of the general Atiyah–Singer index theorem [1968] as in [Baum and Douglas 1982] gives:

Theorem 1.3. *For T_g as above, let F_i's be the fixed, connected submanifolds of M under the action of any $h\in H$, and ν_i the normal bundle of F_i in TM. Then*

$$\mathrm{Ind}(h,T_g)=\sum_i\biggl(\Bigl(\frac{-\sqrt{-1}}{2\pi}\Bigr)^{(1+\dim F_i)/2} \times \hat{A}(F)\,\mathrm{ch}(g)\Bigl[\mathrm{Pf}\bigl(2\sin\bigl(\tfrac12\sqrt{-1}\,(R^\nu(F_i)+\Theta_i)\bigr)\bigr)\Bigr]^{-1}[F_i]\biggr),$$

where, in any local coordinate system, Θ_i is the logarithm of the Jacobian matrix of $h|\nu_i$, $R^\nu(F_i)$ is the curvature matrix of the bundle ν_i, and

$$\mathrm{ch}(g)=\int_0^1\mathrm{Tr}\bigl[g^{-1}dg\exp(u(1-u)(g^{-1}dg)^2)\bigr]\,du \tag{1–7}$$

is the odd Chern character for the differentiable map $g:M\to U(N)$.

In Section 4 we will prove a local version of this theorem.

2. Equivariant spectral flow and equivariant index problem

Set $I=[0,1]$. Let $\{D_u\}_{u\in I}$ be a continuous family of self-dual elliptic operators of positive order on the Hilbert space $\mathcal{H}=L^2(S\otimes\mathbb{C}^N)$. For any fixed $u\in I$, Spec D_u

is discrete, and we denote the corresponding eigenspaces by $\Gamma_{u,\lambda}$ for $\lambda \in \operatorname{Spec} D_u$. For any open $U \subset \mathbb{R}$, define $\Gamma_{u,U} = \bigoplus_{\lambda \in U} \Gamma_{u,\lambda}$.

Recall first the usual (scalar) spectral flow [Atiyah et al. 1976].

Definition 2.1. For $D_u (u \in I)$ a continuous family of self-dual elliptic operator of positive order, consider the graph of $\operatorname{Spec} D_u$:

$$\mathfrak{S} = \bigcup \operatorname{Spec} D_u,$$

which is a closed set of $\mathbb{R} \times I$. The spectral flow $\operatorname{sf}(\{D_u\})$ of $\{D_u\}$ is the intersection number of $\mathfrak{S}$ with the line $\{-\delta\} \times I$ for sufficiently small positive δ. (If both D_0 and D_1 are invertible, we can replace δ by 0 in this definition.)

We would like to extend this notion to the equivariant case. Let H be as in Section 1 and $R(H)$ its representation ring. Assume that each D_u in the preceding discussion is compatible with the action of H. Thus every $\Gamma_{u,\lambda}$ can be viewed as an element of $R(H)$.

The next result is an extension of the continuity of the spectra for a family of self-dual operators.

Lemma 2.2. *Let $\{D_u\}$ be as above. For a fixed $u_0 \in I$ and any $\lambda \in \operatorname{Spec} D_{u_0}$ with $\dim \Gamma_{u_0,\lambda} = k$, we can find a positive ϵ such that for any $u \in I$ satisfying $|u - u_0| < \epsilon$ there is an open set U, containing λ and depending only on u_0, such that*

$$\dim \Gamma_{u,U} = k.$$

Furthermore,

$$\Gamma_{u,U} = \Gamma_{u_0,\lambda} \tag{2–1}$$

as elements of $R(H)$.

Proof. The equality $\dim \Gamma_{u,U} = k$ is actually proved in [Booß-Bavnbek and Wojciechowski 1993, Lemma 17.1]. More precisely, the discussion there shows that there exist a $\epsilon > 0$ and k continuous functions $f_1, \dots, f_k : (u_0 - \epsilon, u_0 + \epsilon) \to \mathbb{R}$ such that $f_i(u_0) = \lambda$; furthermore, for any $u \in (u_0 - \epsilon, u_0 + \epsilon)$, there exists an open set U that depends only on u_0 and contains λ satisfying

$$\{f_j(u)\}_{j=1}^k = \operatorname{Spec} D_u \cap U.$$

Let Q_u, for $u_0 - \epsilon < u < u_0 + \epsilon$, be the orthonormal projections of $\Gamma_{u,U}$ onto $\Gamma_{u_0,U}$. By the continuity of $\{D_u\}$ and the f_j's, $\{Q_u\}$ is a continuous family of self-adjoint projections. Thus it is possible to readjust ϵ if necessary so that

$$\|Q_u - Q_{u_0}\| < 1,$$

for $u_0-\epsilon < u < u_0+\epsilon$. Now using a trick from [Reed and Simon 1978, p. 72], if we define

$$W_u = \big(1-(Q_u-Q_{u_0})^2\big)^{-1/2}\big(Q_u Q_{u_0}+(1-Q_u)(1-Q_{u_0})\big),$$

it is easy to verify that W_u is unitary and

$$W_u^{-1} Q_u W_u = Q_{u_0}.$$

The image of Q_u is $\Gamma_{u,U}$ and the construction above is H-compatible, so (2–1) easily follows. □

We proceed to define the equivariant spectral flow. Given $\{D_u\}$ as above, set

$$\operatorname{Spec}_H D_u = \{(\lambda, \Gamma_{u,\lambda}) : \lambda \in \operatorname{Spec} D_u\}.$$

By Lemma 2.2 and the fact that $R(H)$ has only countable many irreducible elements, there exist $f_j(u) \in C(I)$ and $R_j \in R(H)$, for $j \in \mathbb{N}$, such that

$$\textstyle\bigcup_{u\in I} \operatorname{Spec} D_u = \bigcup_j (f_j(u), R_j).$$

Definition 2.3. Given D_u as above, the equivariant spectral flow of D_u is

$$\mathrm{sf}_H(\{D_u\}) = \sum_j \epsilon(f_j) R_j,$$

where $\epsilon(f_j)$ is the intersection number of the graph f_j with the line $u = -\delta$ for sufficiently small positive δ. Also set

$$\mathrm{sf}(h, \{D_u\}) = \mathrm{Tr}(h, \mathrm{sf}_H(\{D_u\}).$$

Remark 2.4. It is not hard to see that, as in the scalar case, only finite many $\epsilon(f_k)$'s in this definition are nonzero. If both D_0 and D_1 are invertible, δ can be replaced by 0.

Remark 2.5. As in the scalar case, the equivariant spectral flow is a homotopy invariant. In particular, let E_σ be the affine space of all the elliptic, positive-ordered operators in $\mathcal{H}$ with the same symbol σ. Then E_σ is convex, and hence contractible. Therefore, given any two fixed H-compatible points in E_σ, the equivariant spectra of two different H-compatible paths connecting them are the same.

Remark 2.6. Applying the method above, it is not hard to extend the notion of higher spectral flow in the sense of Dai and Zhang [1998] to the equivariant setting. We leave the details to the reader.

For the rest of this paper, we pick a particular family $\{D_u\}$:

$$D_u = (1-u) D^S \otimes \mathrm{Id}_{\mathbb{C}^N} + u g^{-1} (D^S \otimes \mathrm{Id}_{\mathbb{C}^N})\, g, \tag{2–2}$$

where D^S is the Dirac operator defined in (1–3) and $D^S \otimes \mathrm{Id}_{\mathbb{C}^N}$ is its extension to the Hilbert space $\mathcal{H} = L^2(S \otimes \mathbb{C}^N)$. This family $\{D_u\}$ satisfy the conditions in Definitions 2.1 and 2.3. All the D_u's have the same symbol, denoted by σ.

The next theorem clarifies the relation between the equivariant spectral flow and our original index problem. The method used here is from [Booß-Bavnbek and Wojciechowski 1993].

Theorem 2.7. $\mathrm{Ind}(h, T_g) = -\mathrm{sf}(h, \{D_u\})$.

Proof. Set $P_u = (1-u)P + ug^{-1}Pg$, where P is defined in (1–4). Apply the same argument used in the proof of [Booß-Bavnbek and Wojciechowski 1993, Theorem 17.17], which is compatible with our equivariant setting, to conclude that

$$\mathrm{sf}_H(\{D_u\}) = \mathrm{sf}_H(\{P_u\}). \tag{2–3}$$

Straightforward calculation gives

$$\ker T_g = \{u \in P_+\mathcal{H},\, gu \in P_-\mathcal{H}\}, \qquad \operatorname{coker} T_g = \{u \in P_-\mathcal{H},\, gu \in P_+\mathcal{H}\}.$$

Then it is easy to see that

$$P_u(v) = \begin{cases} v & \text{if } v \in P_+\mathcal{H},\ gv \in P_+\mathcal{H}, \\ -v & \text{if } v \in P_-\mathcal{H},\ gv \in P_-\mathcal{H}, \\ (1-2u)v & \text{if } v \in \ker T_g, \\ (2u-1)v & \text{if } v \in \operatorname{coker} T_g. \end{cases}$$

The equality in Theorem 2.7 follows from this and (2–3). □

3. Equivariant spectral flow and equivariant eta functions

Eta invariants first appeared in [Atiyah et al. 1975] and have a close relation with the spectral flow; see [Bismut and Freed 1986; Getzler 1993]. In this section we extend this relation to the equivariant case.

Definition 3.1. Let D be a self-adjoint operator on the Hilbert space $\mathcal{H}$. The eta function associated to D is defined as

$$\eta(s, D) = \sum_{\lambda \neq 0} \operatorname{sign} \lambda \frac{\dim \Gamma_\lambda}{|\lambda|^s},$$

where $\operatorname{Re} s$ is large enough, λ runs over the nonzero eigenvalues of D and Γ_λ is the eigenspace of D with eigenvalue λ.

It is then clear that

$$\eta(s, D) = \frac{1}{\Gamma\left(\frac{1}{2}(s+1)\right)} \int_0^\infty \mathrm{Tr}(De^{-tD^2})\, t^{(s-1)/2}\, dt.$$

By a result from [Bismut and Freed 1986], the eta function of D is analytic for $\operatorname{Re} s > -\frac{1}{2}$; in particular, we write

$$\eta(D) = \eta(0, D).$$

Define the truncated η function, for $\epsilon > 0$, to be

$$\eta_\epsilon(s, D) = \frac{1}{\Gamma\left(\frac{1}{2}(s+1)\right)} \int_\epsilon^\infty \operatorname{Tr}(De^{-tD^2}) t^{(s-1)/2}\, dt$$

and write

$$\eta_\epsilon(D) = \eta_\epsilon(0, D).$$

The equivariant eta function can be defined similarly:

Definition 3.2. Let D be as in Definition 3.1. If there is a compact group H acting on $\mathscr{H}$ and D commutes with the action of H, the equivariant eta function associated to D is defined as

$$\eta(h, s, D) = \frac{1}{\Gamma\left(\frac{1}{2}(s+1)\right)} \int_0^\infty \operatorname{Tr}(hDe^{-tD^2}) t^{(s-1)/2}\, dt$$

for $\operatorname{Re} s$ large enough.

A regularity result [Zhang 1990] allows us to write

$$\eta(h, D) = \eta(h, 0, D).$$

We also define the truncated equivariant eta function, for an $\epsilon > 0$, to be

$$\eta_\epsilon(h, s, D) = \frac{1}{\Gamma\left(\frac{1}{2}(s+1)\right)} \int_\epsilon^\infty \operatorname{Tr}(hDe^{-tD^2}) t^{(s-1)/2}\, dt,$$

and we set

$$\eta_\epsilon(h, D) = \eta_\epsilon(h, 0, D) \quad \text{for any } h \in H.$$

We then consider the variation of equivariant eta functions.

Suppose $\mathscr{F}$ is the real Banach space of all bounded self-adjoint operators on $\mathscr{H}$. Let Φ be the affine space

$$\Phi = \{D^S \otimes \mathrm{Id}_{\mathbb{C}^N} + E : E \in \mathscr{F}\}.$$

It is clear that for any u, the D_u defined in (2–2) lies in Φ.

Theorem 3.3. *For an H-invariant D in Φ and any $h \in H$, define a one-form $\alpha_{\epsilon,h}$ on Φ such that*

$$\alpha_{\epsilon,h}(X)(D) = (\epsilon/\pi)^{1/2} \operatorname{Tr}\left(hXe^{-\epsilon D^2}\right) \quad \textit{for } X \in T_D\Phi = \mathscr{F}.$$

Then $\alpha_{\epsilon,h}$ is closed and

$$d\eta_\epsilon(h, D) = 2\alpha_{\epsilon,h}(D). \tag{3–1}$$

Proof. The proof is almost the same of that of [Getzler 1993, Proposition 2.5], taking into account that h and D commute. □

We can now state the main result of this section:

Theorem 3.4. *For any H-invariant path in $\Phi = \Phi(D_0)$ connecting D_0 and D_1 as in (2–2) and any $h \in H$,*

$$\mathrm{sf}(h, \{D_u\}) = -\int_\gamma \alpha_{\epsilon,h}. \tag{3–2}$$

Proof. A similar formula for the scalar case is demonstrated in [Getzler 1993]; we imitate the method of proof.

By the closedness of $\alpha_{\epsilon,h}$ and Remark 2.5, both sides of (3–2) are independent of the choice of the H-invariant path $\gamma : I \to E_\sigma$ with $\gamma(0) = D_0$ and $\gamma(1) = D_1$. The union $\bigcup \mathrm{Spec}_H(\{\gamma(u)\})$ can be written as $\bigcup_j (f_j(u), R_j)$ as in Section 2, where $R_j \in R(H)$ and $f_j \in C(I)$ for $j \in \mathbb{N}$.

Using a standard transversality argument, we can choose an H-invariant path γ such that the graph of each f_j intersects $\{u = 0\}$ transversally. By Remark 2.4, there are only finitely many nonzero $\epsilon(f_j)$'s; without loss of generality, let them be $f_1, \dots, f_k$. It is easy to check that, for $h \in H$,

$$\mathrm{sf}(h, \{D_u\}) = \mathrm{sf}(h, \gamma) = \sum_{j=1}^{k} \epsilon(f_j)\,\mathrm{Tr}(h, R_j). \tag{3–3}$$

We calculate the truncated equivariant eta function. For any $j \in \{1, \dots, k\}$, the contribution of the $(f_j(u), R_j)$ to $\eta_\epsilon(h, \gamma(u))$ for a certain $h \in H$, now denoted by $S_{u,j}$, is

$$\frac{1}{\sqrt{\pi}}\,\mathrm{Tr}(h, R_j) \int_\epsilon^\infty f_j(u) e^{-tf_j(u)^2} t^{-1/2}\,dt.$$

Now, $\frac{1}{\sqrt{\pi}} \int_\epsilon^\infty \lambda e^{-t\lambda^2} t^{-1/2}\,dt$ tends to ± 1 as $\lambda \to 0\pm$. Hence, if $\tilde{u}$ is any zero of $f_j(u)$ and $\epsilon(\tilde{u})$ is the intersection number of $f_j(u)$ with $\{0\} \times I$ near $\tilde{u}$, we have

$$S_{\tilde{u}+,j} - S_{\tilde{u}-,j} = 2\epsilon(\tilde{u})\,\mathrm{Tr}(h, R_j).$$

Summing over all the zeros of f_j and using the equality

$$\sum_{\{\tilde{u} \in I \,:\, f_j(\tilde{u}) = 0\}} \epsilon(\tilde{u}) = \epsilon(f_j),$$

we have

$$\epsilon(f_j)\,\mathrm{Tr}(h, R_j) = \frac{1}{2} \sum_{\{\tilde{u} \,:\, f_j(\tilde{u}) = 0\}} (S_{\tilde{u}+,j} - S_{\tilde{u}-,j}) = \frac{1}{2}\Big(-\int_\gamma dS_{u,j} + S_{1,j} - S_{0,j}\Big).$$

Summing over all j, we are led to

$$\sum_{j=1}^{k} \epsilon(f_j(u)) \operatorname{Tr}(h, R_j) = \frac{1}{2}\Bigl(-\int_\gamma d\eta_\epsilon(h, \cdot) + \eta_\epsilon(h, D_1) - \eta_\epsilon(h, D_0)\Bigr).$$

Combining this with (3–1) and (3–3) and noticing that $\eta_\epsilon(h, D_1) = \eta_\epsilon(h, D_0)$, we have (3–2). □

Combining Theorem 2.7 with Theorem 3.4, we have:

Theorem 3.5. $\operatorname{Ind}(h, T_g) = \displaystyle\int_0^1 \sqrt{\frac{\epsilon}{\pi}} \operatorname{Tr}\bigl(h \dot{D}_u e^{-\epsilon D_u^2}\bigr)\, du.$

Here the dot denotes differentiation with respect to u.

Remark 3.6. The right-hand side of this equality is independent of the choice of ϵ, so we can use local index technique to calculate the limit of the integrand of when ϵ tends to 0. In this way we obtain a local version of Theorem 1.3.

4. A Lefschetz theorem on odd spin manifolds

We now apply the setting of [Lafferty et al. 1992] to compute the right-hand side of the equality in Theorem 3.5. We start with a Lichnerowicz-type formula for D_u^2.

Lemma 4.1. *We have*

$$D_u^2 = -\Delta + \frac{K}{4} + u^2 c(\omega^2) + u\bigl(-\iota(\omega^*)\nabla^S + c(d\omega) + d^*\omega\bigr),$$

where $\omega = \dot{D}_u = g^{-1} dg$ *and* K *is the scalar curvature of* M.

Proof. This follows easily from [Berline et al. 1992, Proposition 3.45] and the standard Lichnerowicz formula. □

For a fixed $h \in H$, let $F = \{x \in M : hx = x\}$ be the fixed-point set of h. Without loss of generality we assume F is a connected odd-dimensional totally geodesic submanifold and define its dimension to be k. Let ν be the normal bundle of F in TM, with dimension $2s$, and set $\nu(\delta) = \{x \in \nu : \|x\| < \delta\}$. Thus ν is invariant with respect to h_{TM}; moreover $h_{TM}|\nu$ is nondegenerate.

If $P_\epsilon(x, y) : (S \otimes \mathbb{C}^n) \to (S \otimes \mathbb{C}^n)$ is the kernel for the operator

$$O_\epsilon = (\epsilon/\pi)^{1/2} \int_0^1 \dot{D}_u e^{-\epsilon D_u^2},$$

by the standard heat equation argument, we have

$$\operatorname{Tr}(h O_\epsilon) = \int_M \operatorname{Tr}(h P_\epsilon(hx, x)\, d\mathrm{vol}.$$

A routine argument using pseudodifferential operators shows that

$$\lim_{\epsilon\to 0} \operatorname{Tr}(hP_\epsilon(hx,x)) = 0 \quad \text{if } hx \neq x.$$

As a result, we may localize the computation to F.

Both $\dot{D}_u$ and h are bounded operators, so working in a local trivialization of $\nu(\delta)$ yields the following:

Lemma 4.2. *Define, for $x \in F$,*

$$L_{\mathrm{loc}}(x) = \lim_{\epsilon\to 0} \int_{\nu_\delta|x} \operatorname{Tr}(hP_\epsilon(y,hy))\, dy\, d\mathrm{vol}_F.$$

This is well-defined and independent of δ. Furthermore,

$$\operatorname{Ind}(h, T_g) = \int_F L_{\mathrm{loc}}(x).$$

We now calculate $L_{\mathrm{loc}}(x)$ for $x \in F$. Fix any $x_0 \in F$, let $e_1, \dots, e_n \in TM$ be a local coordinate system in a neighborhood $\mathcal{N}$ of x_0 such that the e_i are orthonormal at x_0 and are parallel along geodesics through x_0; moreover, assume $e_1, \dots e_k \in TF$ and $e_{k+1}, \dots, e_n \in \nu(F)$. For any $x \in \mathcal{N}$ such that $hx \in \mathcal{N}$, there is an $n \times n$-matrix $\mathcal{J}(x)$ satisfying

$$h|_{TM}(e_1(x), \dots, e_n(x)) = (e_1(hx), \dots, e_n(hx))\mathcal{J}(x),$$

while

$$\mathcal{J}(x) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \exp\Theta(x) \end{pmatrix} \quad \text{with } \Theta(x) \in so(2s).$$

Denote by R^{TM} the curvature matrix of the Levi-Civita connection on TM with respect to the chosen $\{e_i\}$:

$$(R^{TM})_{ij} = -\frac{1}{2} \sum_{p,q=1}^{n} R_{ijpq} e^k e^l$$

for $1 \le i, j \le n$, where e^k is the dual vector of e_k. If we choose the metrics and connections on TF and $\nu(F)$ to be the restrictions of those of TM, respectively, we have the curvature matrices

$$(R^{TF})_{ij} = -\frac{1}{2}\sum_{p,q=1}^{k} R_{ijpq}e^p e^q \quad \text{for } 1 \le i,\, j \le k \text{ and}$$

$$(R^{\nu(F)})_{ij} = -\frac{1}{2}\sum_{p,q=1}^{k} R_{ijpq}e^p e^q \quad \text{for } k+1 \le i,\, j \le n.$$

It is known that $\mathcal{J}(x)$ is invariant along the fiber of ν [Berline et al. 1992]. Hence, using the abbreviation $Y(a) = \left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right)$, we can fix the e_i in such a way that

$$\Theta(x_0) = \begin{pmatrix} Y(\theta_1) & & \\ & \ddots & \\ & & Y(\theta_s) \end{pmatrix} \quad \text{with } 0 < \theta_i < 2\pi,$$

$$R^{\nu}(x_0) = \begin{pmatrix} Y(v_1) & & \\ & \ddots & \\ & & Y(v_s) \end{pmatrix},$$

$$R^{F}(x_0) = \begin{pmatrix} 0 & & & \\ & Y(u_1) & & \\ & & \ddots & \\ & & & Y(u_{(k-1)/2}) \end{pmatrix},$$

where the u_i's and v_i's are two-forms representing Chern roots.

It is easy to see that the kernel of $\sigma(hO_\epsilon)$ is $\sigma(hP_\epsilon)$, where the symbol map σ is defined in (1–1).

Now we rescale T^*M as in [Berline et al. 1992] to get, for $\epsilon \to 0$,

$$\begin{aligned} L_0 &= \lim_{\epsilon\to 0} \sigma(hO_\epsilon) \\ &= \int_0^1 \frac{h}{\sqrt{\pi}}\omega \exp\biggl(\sum_i \Bigl(\partial_i - \frac{1}{4}\sum_j R^{TM}_{ij} b_j\Bigr)^2 + u(1-u)\omega^2\biggr)\, du, \end{aligned}$$

where the b_i are local coordinate functions on TM with respect to the chosen local charts.

We proceed as in [Lafferty et al. 1992] to get the following expression for $Q_0(x_0, b) = \lim_{\epsilon\to 0} \sigma(hP_\epsilon(x, hx))$:

$$
\begin{aligned}
&Q_0(x_0, b)\\
&= \int_0^1 \frac{1}{\sqrt{\pi}}\big(\omega \exp(u(1-u)\omega^2)\big)\frac{1}{(4\pi)^{n/2}}(-1)^s\\
&\qquad \times \Big(\prod_1^s \sin\frac{\theta_i}{2}\Big) j_V(R^F)\exp\Big(-\frac{1}{4}\sum_1^s \sin(\theta_i)v_i(b_{k+2i-1}^2+b_{k+2i}^2)\Big) j_V(R^\nu)\\
&\qquad \times \exp\Big(\sum_1^s\Big(-\sqrt{-1}\,\frac{v_i}{2}\sin^2\frac{\theta_i}{2}\coth\big(\tfrac{1}{2}\sqrt{-1}\,v_i(b_{k+2i-1}^2+b_{k+2i}^2)\big)\Big)\Big)\,du\\
&= \int_0^1 (-1)^s\frac{1}{2^n}\frac{1}{\pi^{m+1}}\big(\omega\exp(u(1-u)\omega^2)\big) j_V(R^F)(x)\\
&\qquad \times \prod_1^s\Big((-1)\sin\frac{\theta_i}{2}\,\frac{v_i/2}{\sin v_i/2}\\
&\qquad\qquad \exp\Big(-\sin\frac{\theta_i}{2}\,\frac{v_i/2}{\sin v_i/2}\sin\frac{v_i+\theta_i}{2}\,\big(b_{k+2i-1}^2+b_{k+2i}^2\big)\Big)\Big)\,du,
\end{aligned}
$$

where

$$
j_V(R^\nu)=\prod\frac{\sqrt{-1}\,v_i/2}{\sinh(\sqrt{-1}\,v_i/2)} \quad\text{and}\quad j_V(R^F)=\prod\frac{\sqrt{-1}\,u_i/2}{\sinh(\sqrt{-1}\,u_i/2)}.
$$

In order to calculate $L_{\mathrm{loc}}(x_0)$, we first apply (1–2) to $Q_0(x_0, b)$ and take the trace over $\mathbb{C}^N$; then we integrate over $\nu(F)_{x_0}$. Asymptotically, that means integrating over all the b_i, for $k+1 \le i \le n$. Thus we get L_{loc} represented as a k-form on F:

$$
L_{\mathrm{loc}}(x_0) = \Big(\frac{-\sqrt{-1}}{2\pi}\Big)^{m+1}\Big(\mathrm{ch}(g)\, j_V(R^F)\prod_1^s(-\pi)\sin\Big(\frac{v_i+\theta_i}{2}\Big)^{-1}\Big)_{[k]}, \tag{4–1}
$$

where $\mathrm{ch}(g)$ is defined as in (1–7). (Since each θ_i is nonzero, $\sin(\theta_i+v_i)^{-1}$ makes sense as a polynomial expansion.)

It is then easy to get the characteristic class representation from (4–1):

Theorem 4.3. *Let notations be as above. We have*

$$
L_{\mathrm{loc}}(x_0) = \Big(\frac{-\sqrt{-1}}{2\pi}\Big)^{m+1-s}\hat{A}(F)\,\mathrm{ch}(g)\,\Big[\mathrm{Pf}\,\Big(2\sin\big(\tfrac{1}{2}\sqrt{-1}\,(R^\nu+\Theta)\big)\Big)^{-1}\Big]_{[k]}(x_0).
$$

Remark 4.4. The method we have applied can also be used to prove similar local Lefschetz fixed-point formulae for the Toeplitz operators associated to any Dirac-type operators.

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References

[Atiyah and Singer 1968] M. F. Atiyah and I. M. Singer, "The index of elliptic operators, I", *Ann. of Math.* (2) **87** (1968), 484–530. MR 38 #5243 Zbl 0164.24001

[Atiyah et al. 1975] M. F. Atiyah, V. K. Patodi, and I. M. Singer, "Spectral asymmetry and Riemannian geometry, I", *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 43–69. MR 53 #1655a Zbl 0297.58008

[Atiyah et al. 1976] M. F. Atiyah, V. K. Patodi, and I. M. Singer, "Spectral asymmetry and Riemannian geometry, III", *Math. Proc. Cambridge Philos. Soc.* **79**:1 (1976), 71–99. MR 53 #1655c Zbl 0325.58015

[Baum and Douglas 1982] P. Baum and R. G. Douglas, "K homology and index theory", pp. 117–173 in *Operator algebras and applications I* (Kingston, Ont., 1980), edited by R. V. Kadison, Proc. Sympos. Pure Math. **38-I**, Amer. Math. Soc., Providence, R.I., 1982. MR 84d:58075 Zbl 0532.55004

[Berline and Vergne 1985] N. Berline and M. Vergne, "A computation of the equivariant index of the Dirac operator", *Bull. Soc. Math. France* **113**:3 (1985), 305–345. MR 87f:58146 Zbl 0592.58044

[Berline et al. 1992] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grundlehren der Math. Wiss. **298**, Springer, Berlin, 1992. MR 94e:58130 Zbl 0744.58001

[Bismut 1984] J.-M. Bismut, "The Atiyah-Singer theorems: a probabilistic approach, II: The Lefschetz fixed point formulas", *J. Funct. Anal.* **57** (1984), 329–348. MR 86g:58128b Zbl 0556.58027

[Bismut and Freed 1986] J.-M. Bismut and D. S. Freed, "The analysis of elliptic families, II: Dirac operators, eta invariants, and the holonomy theorem", *Comm. Math. Phys.* **107**:1 (1986), 103–163. MR 88h:58110b Zbl 0657.58038

[Booß-Bavnbek and Wojciechowski 1993] B. Booß-Bavnbek and K. P. Wojciechowski, *Elliptic boundary problems for Dirac operators*, Birkhäuser, Boston, 1993. MR 94h:58168 Zbl 0797.58004

[Dai and Zhang 1998] X. Dai and W. Zhang, "Higher spectral flow", *J. Funct. Anal.* **157**:2 (1998), 432–469. MR 99f:58196 Zbl 0932.37062

[Dai and Zhang 2001] X. Dai and W. Zhang, "An index theorem for Toeplitz operators on odd dimensional manifolds with boundary", preprint, 2001. math.DG/0103230

[Getzler 1993] E. Getzler, "The odd Chern character in cyclic homology and spectral flow", *Topology* **32**:3 (1993), 489–507. MR 95c:46118 Zbl 0801.46088

[Lafferty et al. 1992] J. D. Lafferty, Y. L. Yu, and Z. Weiping, "A direct geometric proof of the Lefschetz fixed point formulas", *Trans. Amer. Math. Soc.* **329**:2 (1992), 571–583. MR 92e:58207 Zbl 0747.58016

[Reed and Simon 1978] M. Reed and B. Simon, *Methods of modern mathematical physics, IV: Analysis of operators*, Academic Press, New York, 1978. MR 58 #12429c Zbl 0401.47001

[Zhang 1990] W. P. Zhang, "A note on equivariant eta invariants", *Proc. Amer. Math. Soc.* **108**:4 (1990), 1121–1129. MR 90j:58144 Zbl 0688.58039

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